CSE340 FALL 2020 - HOMEWORK 3
SOLUTION

Due: Monday October 26, 2020 by 11:59 PM on Canvas

All submissions should be typed, no exceptions.

1. **(Lambda calculus: binding)**. For each of the following expressions. determine for each variable x the λx. it is bound to. I have numbered the variables and the abstractions. The variables are numbered using Arabic numerals and the abstractions are numbered using roman numerals and you should specify which variable numbers are bound to which abstraction number. For example, if variables 4 and 7 are bound to abstraction I, your answer should be of the form I ➞ 4 7 to indicate that abstraction I has variables 4, and 7 bound by it. If variable 3 is free, then your answer should have the form free ➞ 3.

Example (λy. x y λx. y) x
I 1 2 II 3 4

Answer

free ➞ 1 4

I ➞ 2 3

II ➞

Your answers need not be colored

1.1. (λy. x z)
I 1 2

free ➞ 1 2

I ➞

1.2. (λy. x y (λy. y) (λz. y))
I 1 2 II 3 III 4

free ➞ 1

I ➞ 2 4

II ➞ 3

III ➞

1.3. (λy. x z (λx. y (λy. x y λx. y) y)) x
I 1 2 II 3 III 4 5 IV 6 7 8

free ➞ 1 2 8

I ➞ 3 7

II ➞ 4

III ➞ 5 6

IV ➞

1.4. ((λx. x (λx. x)) x (λx. x x (λx. x)) x) x
I 1 II 2 3 III 4 5 IV 6 7 8

free → 3 7 8

I → 1

II → 2

III → 4 5

IV → 6

1.5. (λx. (λx. (λx. x (λx. x (λx. x x) (λx. x)))) x x)
I II III 1 IV 2 V 3 4 VI 5 6 7

free →

I → 6 7

II →

III → 1

IV → 2

V → 3 4

VI → 5

2. (**Reducible Expressions**) For each of the following, identify all the reducible expressions by <u>highlighting</u> the λx. , the t and the t' of the reducible expressions. If there is more than one redex in the given expression, you should identify all the redexes, each one on a separate line. If there is no reducible expression, you should say so in your answer.

Example 1 (λx. λx. x x)
Answer no redex

Example 2 ((λx. x x) (λx. x x)) ((λx. x x) (λx. x x))
Answer **(** (λx. x x) (λx. x x)) **(** (λx. x x) (λx. x x))
((λx. x x) (λx. x x)) **(** (λx. x x) (λx. x x))

2.1. (λx. x (λx. x) x) (x x) (λx. x x) x

(((λx. (x (λx. x)) x) (x x)) (λx. x x)) x 1 Redex

2.2. x (λx. (λx. x) x) x

(x (λx. (λx. x) x)) x 1 Redex

2.3. (λx. (λx. (λx. x) y) z) λx. w (λx. x) x

(λx. (λx. (λx. x) y) z) (λx. w (λx. x))
(λx. (λx. (λx. x) y) z) λx. w (λx. x) x
(λx. (λx. (λx. x) y) z) λx. w (λx. x) x 3 Redexes

2.4. λz. (λx. (λy. z y) x) y

(λz. (λx. (λy. z y) x) y)
(λz. (λx. (λy. z y) x) y) 2 Redexes

2.5. λx. (λy. (λz. x y) x) λx. (λx. w) (y z)

(λx. (λy. (λz. x y) x) (λx. (λx. w) (y z)))
(λx. (λy. (λz. x y) x) (λx. (λx. w) (y z)))
(λx. (λy. (λz. x y) x) (λx. (λx. w) (y z))) 3 Redexes

2.6. (λx. (λy. (λz. x y) x)) y z

((λx. (λy. (λz. x y) x)) y) z
((λx. (λy. (λz. x y) x)) y) z 2 Redexes

3. (**Alpha Renaming**) For each of the following redexes, do alpha renaming if it is needed. If renaming is not needed, you should say so. <u>You should highlight the renamed variable(s) in your answer</u>. You should not do beta reduction.

Example 1 (λx. x x) x
Answer no renaming needed

Example 2 (λx. λy. λz. x y λw. z) (x y z w)
Answer (λx. λu. λv. x u λw. v) (x y z w)

Example 3 (λx. λy. λz. x y λw. z) (x z) (z w)
Answer (λx. λy. λv. x y λw. v) (x z) (z w)

3.1. (λx. λy. λz. y z) (y z)
No renaming needed

3.2. (λx. λy. x λz. y z) (y z)
(λx. λw. x λz. w z) (y z)

3.3. (λx. (λy. x (λx. (λz. x y z)))) (y z)
(λx. λw. x λx. λz. x w z) (y z)

3.4. (λx. (λy. x (λx. (λz. x y z)))) (λy. y z)
No renaming needed

3.5. (λy. x λz. x y z) (λy. y z)
(λy. x λu. x y u) (λy. y z)

3.6. (λx. (λy. x (λx. (λz. x y z)))) (y z)
(λx. λw. x λx. λz. x w z) (y z)

4. **(Beta reductions)** For each of the following expressions, identify the redexes and, for each redex, do <u>only one</u> beta reduction step to reduce the redex. If renaming is needed you should do renaming first before doing the beta reduction.

// The examples below clearly show that you should answer by highlighting the redexes and also highlight the result of the
// reduction (and the renaming if applicable). You should follow the same format as the examples below

Example 1 (λx. x x) x
Answer (λx. x x) x $\rightarrow^{\beta}$ x x

Example 2 (λx. λy. λz. x y λw. z) (x y z w)
Answer (λx. λy. λz. x y λw. z) (x y z w) $\rightarrow^{\alpha}$ (λx. λu. λv. x u λw. v) (x y z w)
(λx. λu. λv. x u λw. v) (x y z w) $\rightarrow^{\beta}$ λu. λv. (x y z w) u λw. v

Example 3 (λx. (λy. x) y) z
Answer // This expression has two redexes you should show one beta reduction for each redex separately
(λx. (λy. x) y) z $\rightarrow^{\beta}$ (λy. z) y // beta reduction for first redex
(λx. (λy. x) y) z $\rightarrow^{\beta}$ (λx. x) z // beta reduction for second redex

4.1. (λx. (λy. λz. y) z) (y z)
(λx. (λy. λz. y) z) (y z) $\rightarrow^{\beta}$ (λy. λz. y) z
(λx. (λy. λz. y) z)(y z) $\rightarrow^{\alpha}$ (λx. (λy. λu. y) z) (y z)
(λx. (λy. λu. y) z)(y z) $\rightarrow^{\beta}$ (λx. (λu. (z))) (y z)

4.2. (λx. (λy. x) (λz. y z)) (y z)
(λx. (λy. x) (λz. y z)) (y z) $\rightarrow^{\alpha}$ (λx. (λw. x) (λz. y z)) (y z)
(λx. (λw. x) (λz. y z)) (y z) $\rightarrow^{\beta}$ (λw. (y z)) (λz. y z))
(λx. (λy. x) (λz. y z)) (y z) $\rightarrow^{\beta}$ (λx. (x)) (y z)

4.3. (λx. λy. x λx. λz. x y z) (y z)
(λx. (λy. x (λx. (λz. x y z)))) (y z) $\rightarrow^{\alpha}$ (λx. (λw. x (λx. (λz. x w z)))) (y z)
(λx. (λw. x (λx. (λz. x w z)))) (y z) $\rightarrow^{\beta}$ (λw. (y z) (λx. (λz. x w z)))

4.4. (λx. λx. x λx. λz. x y z) ((λy. y z) z)
(λx. (λx. x (λx. (λz. x y z)))) ((λy. y z) z) $\rightarrow^{\beta}$ (λx. x (λx. (λz. x y z)))
(λx. (λx. x (λx. (λz. x y z)))) ((λy. y z) z) $\rightarrow^{\beta}$ (λx. (λx. x (λx. (λz. x y z)))) ((z z))

4.5. (λy. y λz. x y z) (λy. (λy. y) z)
(λy. y λz. x y z) (λy. (λy. y) z) $\rightarrow^{\alpha}$ (λy. y λw. x y w) (λy. (λy. y) z)
(λy. y λw. x y w) (λy. (λy. y) z) $\rightarrow^{\beta}$ ((λy. (λy. y) z) (λw. x (λy. (λy. y) z) w))
(λy. y λz. x y z) (λy. (λy. y) z) $\rightarrow^{\beta}$ (λy. y λz. x y z) (λy. (z))

4.6. (λx. λy. x λx. λz. x y z) (y z)
(λx. (λy. x (λx. (λz. x y z)))) (y z) $\rightarrow^{\alpha}$ (λx. (λw. x (λx. (λz. x w z)))) (y z)
(λx. (λw. x (λx. (λz. x w z)))) (y z) $\rightarrow^{\beta}$ (λw. (y z) (λx. (λz. x w z)))

// 4.3 and 4.6 are the same so you better have the same answers for them. If you have different answers for 4.3 and
// 4.6, you will not get credit for them

5. **(Call by value)** repeat problem 5 but do beta reductions according to call by value.

// This is asking you to repeat problem 4. The answer to this question should have the same format as the answer to problem 4.

5.1 (λx. (λy. λz. y) z) (y z)

There are two redexes:

1. (λx. (λy. λz. y) z) (y z) and
2. (λx. (λy. λz. y) z) (y z)

Redex 2 is under abstraction, so it cannot be reduced under call by value evaluation strategy. Redex 1 is not under abstraction and its argument is (y z) which is a value, so only redex 1 can be reduced under call by value evaluation strategy.

(λx. (λy. λz. y) z) (y z) $\to^{\beta}$ (λy. λz. y) z

5.2 (λx. (λy. x) (λz. y z)) (y z)

There are two redexes:

1. (λx. (λy. x) (λz. y z)) (y z) and
2. (λx. (λy. x) (λz. y z)) (y z)

Redex 2 is under abstraction, so it cannot be reduced in call by value. Redex 1 is not under abstraction and its argument is (y z) which is a value, so only redex 1 can be reduced under call by value evaluation strategy

(λx. (λy. x) (λz. y z)) (y z) $\to^{\alpha}$ (λx. (λw. x) (λz. y z)) (y z)
(λx. (λw. x) (λz. y z)) (y z) $\to^{\beta}$ (λw. (y z)) (λz. y z))

5.3 (λx. λy. x λx. λz. x y z) (y z)

There is only one redex which is not under abstraction and whose argument is a value. So, the redex can be reduced under call by value evaluation strategy.

(λx. (λy. x (λx. (λz. x y z)))) (y z) $\to^{\alpha}$ (λx. (λw. x (λx. (λz. x w z)))) (y z)
(λx. (λw. x (λx. (λz. x w z)))) (y z) $\to^{\beta}$ (λw. (y z) (λx. (λz. x w z)))

5.4 (λx. λx. x λx. λz. x y z) ((λy. y z) z)

There are two redexes:

1. (λx. (λx. x (λx. (λz. x y z)))) ((λy. y z) z) and
2. (λx. (λx. x (λx. (λz. x y z)))) ((λy. y z) z)

Both redexes are not under abstraction. The argument of redex 1 is z which is a value, so redex 1 can be reduced under call by value evaluation strategy. The argument of redex 2 is ((λy. y z) z) which is not a value, so redex 2 cannot be reduced under call by value evaluation strategy.

(λx. (λx. x (λx. (λz. x y z)))) ((λy. y z) z) $\to^{\beta}$ (λx. (λx. x (λx. (λz. x y z)))) (z z)
(λx. (λx. x (λx. (λz. x y z)))) (z z) $\to^{\beta}$ (λx. x (λx. (λz. x y z)))

5.5 (λy. y λz. x y z) (λy. (λy. y) z)

There are two redexes:

1. (λy. y (λz. x y z)) (λy. (λy. y) z) and
2. (λy. y (λz. x y z)) (λy. (λy. y) z)

Redex 1 is under abstraction, so it cannot be reduced under call by value evaluation strategy. Redex 2 is not under abstraction and it argument is (λy. (λy. y) z) which is a value, so redex 2 can be reduced under call by value evaluation strategy.

$(\lambda y.\ y\ \lambda z.\ x\ y\ z\ )\ (\lambda y. (\lambda y.\ y)\ z) \rightarrow^{\alpha} (\lambda y.\ y\ \lambda w.\ x\ y\ w\ )\ (\lambda y. (\lambda y.\ y)\ z\ )$
$(\lambda y.\ y\ \lambda w.\ x\ y\ w\ )\ (\lambda y. (\lambda y.\ y)\ z) \rightarrow^{\beta} (\ (\lambda y. (\lambda y.\ y)\ z\ )\ (\lambda w.\ x\ (\lambda y. (\lambda y.\ y)\ z\ )\ w\ ))$

5.6 Same as 5.3

6. (**Normal Order**) repeat problem 5 but do beta reductions according to normal order evaluation

// This is asking you to repeat problem 4. The answer to this question should have the same format as the answer to problem 4.

Under normal order evaluation strategy, the leftmost outermost redex is reduced first.

6.1 $(\lambda x.\ (\lambda y. \lambda z.\ y\ )\ z\ )\ (\ y\ z\ )$

There are two redexes:

1. $(\lambda x.\ (\lambda y. \lambda z.\ y\ )\ z\ )\ (\ y\ z\ )$ and
2. $(\lambda x.\ (\lambda y. \lambda z.\ y\ )\ z\ )\ (\ y\ z\ )$

Redex 1 is the leftmost outermost redex. we get:

$(\lambda x.\ (\lambda y. \lambda z.\ y\ )\ z\ )\ (\ y\ z\ ) \rightarrow^{\beta} (\lambda y. \lambda z.\ y\ )\ z$

6.2 $(\lambda x.\ (\lambda y.\ x)\ (\lambda z.\ y\ z)\ )\ (\ y\ z\ )$

The are two redexes:

1. $(\lambda x.\ (\lambda y.\ x)\ (\lambda z.\ y\ z)\ )\ (\ y\ z\ )$ and
2. $(\lambda x.\ (\lambda y.\ x)\ (\lambda z.\ y\ z)\ )\ (\ y\ z\ )$

Redex 1 is the leftmost outermost redex. We get:

$(\lambda x.\ (\lambda y.\ x)\ (\lambda z.\ y\ z)\ )\ (\ y\ z\ ) \rightarrow^{\alpha} (\lambda x.\ (\lambda w.\ x)\ (\lambda z.\ y\ z)\ )\ (\ y\ z\ )$
$(\lambda x.\ (\lambda w.\ x)\ (\lambda z.\ y\ z)\ )\ (\ y\ z\ ) \rightarrow^{\beta} (\lambda w.\ (y\ z)\ )\ (\lambda z.\ y\ z))$

6.3 $(\lambda x.\ \lambda y.\ x\ \lambda x.\ \lambda z.\ x\ y\ z\ )\ (\ y\ z\ )$

There is only one redex which is the leftmost outermost redex, We get the following:

$(\lambda x.\ (\lambda y.\ x\ (\lambda x.\ (\lambda z.\ x\ y\ z\ )))\ )\ (\ y\ z\ ) \rightarrow^{\alpha} (\lambda x.\ (\lambda w.\ x\ (\lambda x.\ (\lambda z.\ x\ w\ z\ ))))\ (\ y\ z\ )$
$(\lambda x.\ (\lambda w.\ x\ (\lambda x.\ (\ \lambda z.\ x\ w\ z\ )))\ )\ (\ y\ z\ ) \rightarrow^{\beta} (\ \lambda w.\ (y\ z)\ (\lambda x.\ (\ \lambda z.\ x\ w\ z))\ )$

6.4 $(\lambda x.\ \lambda x.\ x\ \lambda x.\ \lambda z.\ x\ y\ z\ )\ ((\lambda y.\ y\ z)\ z\ )$

There are two redexes:

1. $(\lambda x.\ \lambda x.\ x\ \lambda x.\ \lambda z.\ x\ y\ z\ )\ ((\lambda y.\ y\ z)\ z\ )$ and
2. $(\lambda x.\ \lambda x.\ x\ \lambda x.\ \lambda z.\ x\ y\ z\ )\ ((\lambda y.\ y\ z)\ z\ )$

Redex 1 is the leftmost outermost redex, so redex 1 is the one that should be reduced under normal order evaluation strategy.

$(\lambda x.\ (\lambda x.\ x\ (\lambda x.\ (\lambda z.\ x\ y\ z\ )))\ )\ ((\lambda y.\ y\ z)\ z\ ) \rightarrow^{\beta} (\ \lambda x.\ x\ (\lambda x.\ (\lambda z.\ x\ y\ z\ )))$

6.5 $(\lambda y.\ y\ \lambda z.\ x\ y\ z)\ (\lambda y.\ (\lambda y.\ y)\ z)$

There are two redexes:

1. $(\lambda y.\ y\ \lambda z.\ x\ y\ z)\ (\lambda y.\ (\lambda y.\ y)\ z)$ and
2. $(\lambda y.\ y\ \lambda z.\ x\ y\ z)\ (\lambda y.\ (\lambda y.\ y)\ z)$

Redex 1 is the leftmost outermost redex, so it is reduced first under normal order evaluation strategy.

$(\lambda y.\ y\ \lambda z.\ x\ y\ z)\ (\lambda y.\ (\lambda y.\ y)\ z) \to^{\alpha} (\lambda y.\ y\ \lambda w.\ x\ y\ w)\ (\lambda y.\ (\lambda y.\ y)\ z)$
$(\lambda y.\ y\ \lambda w.\ x\ y\ w)\ (\lambda y.\ (\lambda y.\ y)\ z) \to^{\beta} ((\lambda y.\ (\lambda y.\ y)\ z)\ (\lambda w.\ x\ (\lambda y.\ (\lambda y.\ y)\ z)\ w))$

6.6 Same as 6.3

7. (**Evaluating Expressions**) Evaluate each of the following expressions

7.1. tru fls tru tru fls

Answer		tru fls tru tru fls
	➔	(tru fls tru) tru fls
	➔	fls tru fls
	➔	fls

7.2. 2 (plus 2) 3

Answer		2 (plus 2) 3
	➔	(plus 2) ((plus 2) 3))
	➔	(plus 2) 5
	➔	7

7.3. 2 fst (pair (pair 4 5) (pair (1 3))) // this had a missing parenthesis that I added

Answer		2 fst (pair (pair 4 5) (pair (1 3)))
	➔	fst (fst (pair (pair 4 5) (pair (1 3))))
	=	fst (fst (**pair** (pair 4 5) (pair (1 3))))
	➔	fst (pair 4 5)
	➔	4

7.4. 2 fst (pair (pair (plus 2) 5) (pair 1 3)) 5 // this had an extra parenthesis that I removed and a missing “pair” that I added

Answer		2 fst (pair (pair (plus 2) 5) (pair 1 3)) 5
	➔	fst (fst (pair (pair (plus 2) 5) (pair 1 3))) 5
	=	fst (fst **(pair** (pair (plus 2) 5) (pair 1 3)) **)** 5
	➔	fst (pair (plus 2) 5) 5
	➔	(plus 2) 5
	➔	7

7.5. snd (pair (plus 2) (plus 3)) (tru 4 5)

Answer snd (**pair** (plus 2) (plus 3)) (true 4 5)

→ plus 3 (tru 4 5)

→ plus 3 4

→ 7

8. (**Operator precedence parsing**) No question on this homework assignment. I will ask about it in the next homework assignment, closer to the second exam!