last time. β-reduction with renaming

                  Arithmetic

                  pair

Today   prd function - 1 ✓
                  $\leq$, $\geq$, $=$ ✓
                  recursion

Example.   Function to test if a number is even.

        pair tru fls
             ⇓ swap
        pair fls tru
             ⇓ swap
        pair tru fls

swap = $\lambda p.$ pair (snd p) (fst p)

init = pair tru fls

Even = $\lambda n.$ fst (n swap init)

3 swap init = swap (swap (swap init))
                              _____
                          pair fls tru
              _____

pair tru fls

pair fls tru

prd -1 ?

$$x = 0$$
$$y = 0$$
$$\text{for } i = 1 \text{ to } n$$
$$\{$$
$$x = y$$
$$y = y + 1$$
$$\}$$
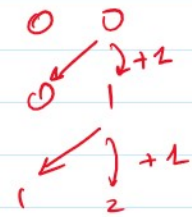
$$zz = \text{pair } 0 \ 0$$

$$next = \lambda p. \ \text{pair } (snd \ p) \ (scc \ (snd \ p))$$

$$\boxed{prd = \lambda n. \ fst \ (n \quad next \quad zz)}$$

$$n = 3$$

$$prd \ 3 = (\lambda n. \ fst \ (n \ next \ zz)) \ 3 =$$
$$fst \ (3 \ next \ zz) \ =$$
$$fst \ (next \ (next \ (next \ zz)))_1 \ =$$

pair 0 1

pair 1 2

$$\overbrace{}^{\text{pair } 2 \; 2}$$

$$\overbrace{\text{pair} \quad 2 \quad 3}$$

$$2 \quad = \quad \text{prd} \; 3$$

iszero

$$\text{iszero} \;=\; \lambda n. \; n \; \underline{? \quad \text{tru}}$$

If $n = 0$
$$\text{iszero} \; 0 = (\lambda n. \; n \; ? \; \text{tru}) \, 0$$
$$= \quad 0 \; ? \; \text{tru}$$
$$= \quad \text{tru}$$
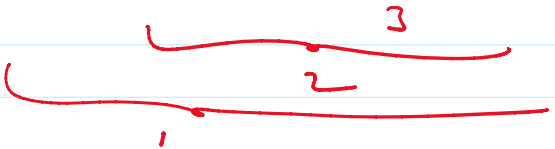
if $n > 0$
$$\text{iszero} \; n = (\lambda n. \; n \; ? \; \text{tru}) \; n$$
$$= \underbrace{(? \; (? \; (? \; \cdots \; (? \; \text{tru})))}_{n \text{ times}}$$

$$? = \lambda x. \; \text{fls}$$

$$\boxed{\text{iszero} \;=\; \lambda n. \; n \; (\lambda x. \; \text{fls}) \; \text{tru}}$$

$$\boxed{\text{minus} \;=\; \lambda m. \lambda n. \; n \; \text{prd} \; m}$$

$$\text{minus} \; 4 \; 3 \;=\; ((\lambda m \, \lambda n. \; n \; \text{prd} \; m) \, 4 \,) \, 3$$

$$= \quad 3 \text{ prd } 4 \quad =$$

$$= \quad \text{prd} \, (\, \text{prd} \, (\, \text{prd} \quad 4 \,) \,)$$

$$\underbrace{\underbrace{\underbrace{}_{3}}_{2}}_{1}$$

gteq $= \lambda m. \lambda n.$ iszero $(\underbrace{\text{minus} \quad n \quad m}_{})$

$$= 0 \text{ if } m \geqslant n$$
$$\neq 0 \text{ if } m < n$$

equal $= \lambda m. \lambda n.$ AND $(\text{gteq } n \, m) \, (\text{gteq } m \, n)$