Monday, October 18, 2021 11:53 AM

Project 2.

203 submissions

86 median

64 mean

95 ≥ 90

109 ≥ 80

Exam 1 Grades : wednesday

Project 3 : Tomorrow

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last time - β - reductions

- booleans

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today. - β - reductions with renaming

- arithmetic
- pairs (data structure)

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renaming in C

```
int f (int x)                    int f (int y)
{ return x + 1;         ≡       {
}                                   return y + 1;
                                 }
```

```
int f ( int x )                  int f(int y)
{                                { int w; w = y;
     int y;   y = x;
```

needs not

```
    int y;   y = x;                    int w;  w = y;
    { int x;                     ≡     { int x;   needs not to be changed
        x = 2 * y;                         x = 2 * w;
        return x;                          return x;
    }                                  }
}                                    }
```

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renaming in λ-calculus

$$\equiv \begin{array}{l} (\lambda x.\ x)\ y \longrightarrow y \\ (\lambda w.\ w)\ y \longrightarrow y \end{array}$$

$$\underbrace{(\lambda x.\ z)}_{\text{constant function in } x}\ y \longrightarrow z$$

$\equiv$

$$(\lambda w.\ z)\ y \longrightarrow z$$

$$(\lambda x.\ \lambda y.\ a)\ y \longrightarrow \lambda y.\ a$$

$$\equiv \qquad\qquad \equiv$$

$$(\lambda x\ \lambda w.\ a)\ y \longrightarrow \lambda w.\ a$$

---

free, bound

$$(\underline{\lambda x}.\ \lambda y.\ \boxed{x})\ y \ \cancel{\longrightarrow}\ \lambda y.\ y \qquad ①$$

$$\equiv \qquad\qquad\qquad \neq$$

$$(\lambda x.\ \lambda w.\ x)\ y \longrightarrow \lambda w.\ y$$

① : we say that y is captured by the substitution.

β - reductions should avoid having variables that are captured by the substitution

We want capture - avoiding substitution.

α - renaming : replace an abstraction variable and all instances of the name that are bond to it with a new name while avoiding name clashes.

Examples. 1

free

$$(\lambda z.\ \underline{\lambda y.\ x\ z})\ (\underline{\lambda x.\ y}) \not\to$$

$$(\lambda x.\quad t\quad)\quad t'$$

bond

$$\lambda z.\ \lambda y.\ x\ (\lambda x.\ y)$$

---

we need to rename

$$(\lambda z.\ \lambda y.\ x\ z)\ (\lambda x.\ y) \xrightarrow{\alpha}$$

we rename the local y not the outer y

$$(\lambda z.\ \lambda w.\ x\ z)\ (\lambda x.\ y) \xrightarrow{\beta}$$

free

$$\lambda w.\ x\ (\lambda x.\ y)$$

free

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(2)

$$(\lambda y.\ \lambda z.\ y\ (\lambda x.\ x)\ z)\ z \xrightarrow{\alpha}$$

free

$$(\lambda y.\ \lambda w.\ y\ (\lambda x.\ x)\ w)\ z \longrightarrow$$

free

$$\lambda w.\ z\ (\lambda x.\ x)\ w$$

free

(3)

$$((\lambda x.\ \lambda y.\ \lambda z.\ x\ y\ z)\ y)\ z \xrightarrow{\alpha}$$

$$(\lambda x.\ \lambda w.\ \lambda z.\ x\ w\ z)\ y\ z \xrightarrow{\beta}$$

$$(\lambda w.\ \lambda z.\ y\ w\ z)\ z \xrightarrow{\alpha}$$

$$(\lambda w.\ \lambda v.\ y\ w\ v)\ z \xrightarrow{\beta}$$

$$\lambda v.\ y\ z\ v$$

Church's numerals

$$0 = \lambda s.\ \lambda z.\ z$$

$$1 = \lambda s.\ \lambda z.\ s\ z$$

$$2 = \lambda s.\ \lambda z.\ s\ (s\ z)$$

parentheses important

$$3 = \lambda s.\ \lambda z.\ s\ (s\ (s\ z))$$

$$4 = \lambda s.\ \lambda z.\ s\ (s\ (s\ (s\ z)))$$

$$4 = \lambda s.\ \lambda z.\ s\ (s\ (s\ (s\ z)))$$

$\vdots$

How do these numerals behave?

$$0\ a\ b \to (\lambda s.\ \lambda z.\ z)\ a\ b$$
$$\to (\lambda z.\ z)\ b$$
$$\to b$$

$$1\ a\ b \to (((\lambda s.\ \lambda z.\ s\ z)\ a)\ b)$$
$$\to (\lambda z.\ a\ z)\ b$$
$$\to a\ b$$

$$2\ a\ b \to a\ (a\ b)$$

$$3\ a\ b \to a\ (a\ (a\ b))$$

$\vdots$

$$n\ a\ b \to \underbrace{a\ (a\ (\cdots\ (a}_{n\ a\text{'s}}\ b)\ \cdots))$$

scc = + 1

$$\boxed{scc = \lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)}$$

$$\boxed{scc = \lambda n. \lambda s. \lambda z. s\ (n\ s\ z)}$$

Example

$$scc\ 3 = (\lambda n. \lambda s. \lambda z. s\ (n\ s\ z))\ 3$$

$$\xrightarrow{\beta} \lambda s. \lambda z. s\ (3\ s\ z)$$

$$\longrightarrow \underbrace{\lambda s. \lambda z. s\ (s\ (s\ (s\ z)))}_{4}$$

$$\boxed{plus = \lambda m. \lambda n. m\ scc\ n}$$

Example.

$$plus\ 3\ 4 = ((\lambda m. \lambda n.\ m\ scc\ n)\ 3)\ 4$$

$$\longrightarrow (\lambda n. 3\ scc\ n)\ 4$$

$$\longrightarrow 3\ scc\ 4$$

$$\longrightarrow \underbrace{scc\ \underbrace{(scc\ (\underbrace{scc\ 4}_{5}))}_{6}}_{7}$$

$$\underline{\hspace{6em}}$$

3 times

$$3\ a\ b = a(a(a\ b))$$

$$\boxed{\text{times} = \lambda m.\ \lambda n.\ m\ (\text{plus}\ n)\ 0}$$

Example

$$\text{times}\ 3\ 4 = (\lambda m.\ \lambda n.\ m\ (\text{plus}\ n)\ 0)\ 3\ 4$$

$$\xrightarrow{\beta} (\lambda n.\ 3\ (\text{plus}\ n)\ 0)\ 4$$

$$\xrightarrow{\beta} 3\ (\text{plus}\ 4)\ 0$$

$$\underbrace{(\text{plus}\ 4)\ \underbrace{(\ (\text{plus}\ 4)\ \underbrace{(\ (\text{plus}\ 4)\ 0\ )}_{4}\ )}_{8}}_{12}$$

plus 4 0 $\equiv$
(plus 4) 0

$$\boxed{\text{Exp} = \lambda n.\ \lambda m.\ n\ (\text{times}\ m)\ 1}$$

$$m^n = \underbrace{m \times m \times m\ -\ .}_{n\ \text{times}}$$

$$\boxed{\text{pair} = \lambda f.\ \lambda s.\ \lambda b.\ b\ f\ s}$$

$$\underline{\text{fst}}\ (\text{pair}\ x\ y) = x \quad ?$$
$$\underline{\text{snd}}\ (\text{pair}\ x\ y) = y \quad ?$$

$$\text{pair}\ x\ y = (\lambda f.\ \lambda s.\ \lambda b.\ b\ f\ s)\ x\ y$$
$$\longrightarrow \lambda b.\ b\ x\ y$$

$$\boxed{\text{fst} = \lambda p.\ p\ \text{tru}}$$
$$\boxed{\text{snd} = \lambda p.\ p\ \text{fls}}$$

$$\text{fst}\ (\text{pair}\ x\ y) = \text{fst}\ (\lambda b.\ b\ x\ y)$$
$$\longrightarrow (\lambda p.\ p\ \text{tru})\ (\underline{\lambda b.\ b\ x\ y})$$
$$\longrightarrow (\lambda b.\ b\ x\ y)\ \text{tru}$$
$$\longrightarrow \text{tru}\ x\ y$$
$$\longrightarrow x$$

$$3t = \lambda m.\lambda n.\lambda p.\ \text{pair}\ m\ (\text{pair}\ n\ p)$$

$$3t = \lambda m.\lambda n.\lambda p.\ pair\ m\ (pair\ n\ p)$$

$$1st = \lambda t.\ fst\ t$$

$$2nd = \lambda t.\ fst\ (snd\ t)$$

$$3rd = \lambda t.\ snd\ (snd\ t)$$

$$swap = \lambda p.\ pair\ (snd\ p)\ (fst\ p)$$

$$init = pair\ tru\ fls$$

$$even = \lambda n.\ fst\ (n\ swap\ init)$$