Notes of Calculus

Analysis I

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Part One

1. Text Chapter

1.1 Paragraphs of Text

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1.2 Citation

This statement requires citation [2]; this one is more specific [1, page 122].

1.3 Lists

Lists are useful to present information in a concise and/or ordered way[1].

1.3.1 Numbered List

1. The first item
2. The second item
3. The third item

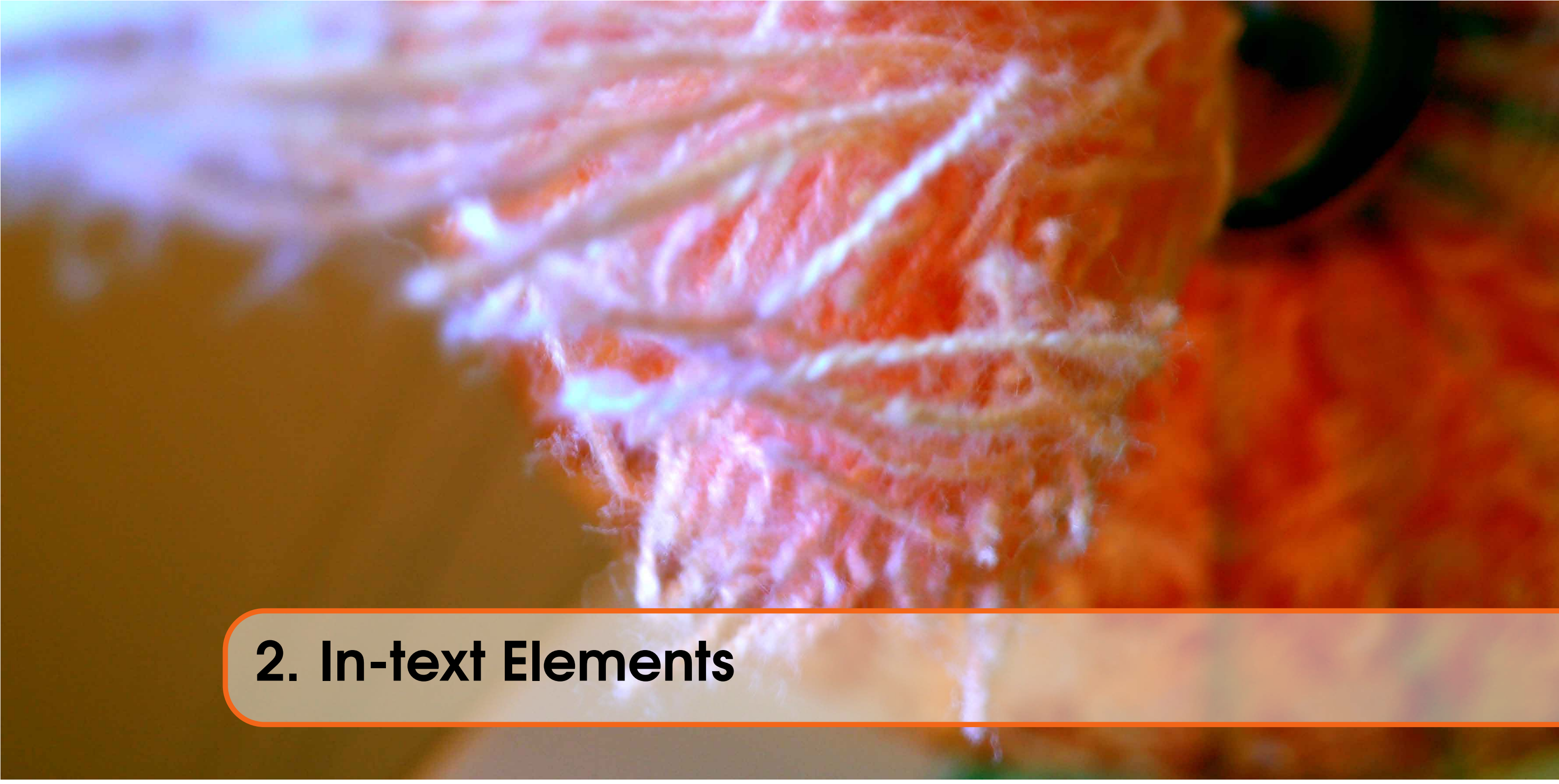
1.3.2 Bullet Points

- The first item
- The second item
- The third item

1.3.3 Descriptions and Definitions

Name Description
Word Definition
Comment Elaboration

[1] Footnote example...



2. In-text Elements

2.1 Theorems

This is an example of theorems.

2.1.1 Several equations

This is a theorem consisting of several equations.

Theorem 2.1.1 — Name of the theorem. In $E=\mathbb{R}^n$ all norms are equivalent. It has the properties:

$$\big|||\mathbf{x}||-||\mathbf{y}||\big| \leq ||\mathbf{x}-\mathbf{y}|| \tag{2.1}$$

$$||\sum_{i=1}^{n}\mathbf{x}_i|| \leq \sum_{i=1}^{n}||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{2.2}$$

2.1.2 Single Line

This is a theorem consisting of just one line.

Theorem 2.1.2 A set $\mathscr{D}(G)$ in dense in $L^2(G)$, $|\cdot|_0$.

2.2 Definitions

This is an example of a definition. A definition could be mathematical or it could define a concept.

Definition 2.2.1 — Definition name. Given a vector space E, a norm on E is an application, denoted $||\cdot||$, E in $\mathbb{R}^+=[0,+\infty[$ such that:

$$||\mathbf{x}||=0 \Rightarrow \mathbf{x}=\mathbf{0} \tag{2.3}$$

$$||\lambda\mathbf{x}||=|\lambda|\cdot||\mathbf{x}|| \tag{2.4}$$

$$||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}||+||\mathbf{y}|| \tag{2.5}$$

2.3 Notations

Notation 2.1. *Given an open subset G of $\mathbb{R}^n$, the set of functions φ are:*

1. *Bounded support G;*
2. *Infinitely differentiable;*

a vector space is denoted by $\mathscr{D}(G)$.

2.4 Remarks

This is an example of a remark.

R The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K} = \mathbb{R}$, however, established properties are easily extended to $\mathbb{K} = \mathbb{C}$.

2.5 Corollaries

This is an example of a corollary.

Corollary 2.5.1 — Corollary name. The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K} = \mathbb{R}$, however, established properties are easily extended to $\mathbb{K} = \mathbb{C}$.

2.6 Propositions

This is an example of propositions.

2.6.1 Several equations

Proposition 2.6.1 — Proposition name. It has the properties:

$$\left| ||\mathbf{x}|| - ||\mathbf{y}|| \right| \leq ||\mathbf{x} - \mathbf{y}|| \tag{2.6}$$

$$||\sum_{i=1}^{n} \mathbf{x}_i|| \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{2.7}$$

2.6.2 Single Line

Proposition 2.6.2 Let $f, g \in L^2(G)$; if $\forall \varphi \in \mathscr{D}(G)$, $(f, \varphi)_0 = (g, \varphi)_0$ then $f = g$.

2.7 Examples

This is an example of examples.

2.7.1 Equation and Text

■ **Example 2.1** Let $G = \{x \in \mathbb{R}^2 : |x| < 3\}$ and denoted by: $x^0 = (1, 1)$; consider the function:

$$f(x) = \begin{cases} e^{|x|} & \text{si } |x - x^0| \leq 1/2 \\ 0 & \text{si } |x - x^0| > 1/2 \end{cases} \tag{2.8}$$

The function f has bounded support, we can take $A = \{x \in \mathbb{R}^2 : |x - x^0| \leq 1/2 + \varepsilon\}$ for all $\varepsilon \in]0; 5/2 - \sqrt{2}[$. ■

2.7.2 Paragraph of Text

■ **Example 2.2 — Example name.** Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris. ■

2.8 Exercises

This is an example of an exercise.

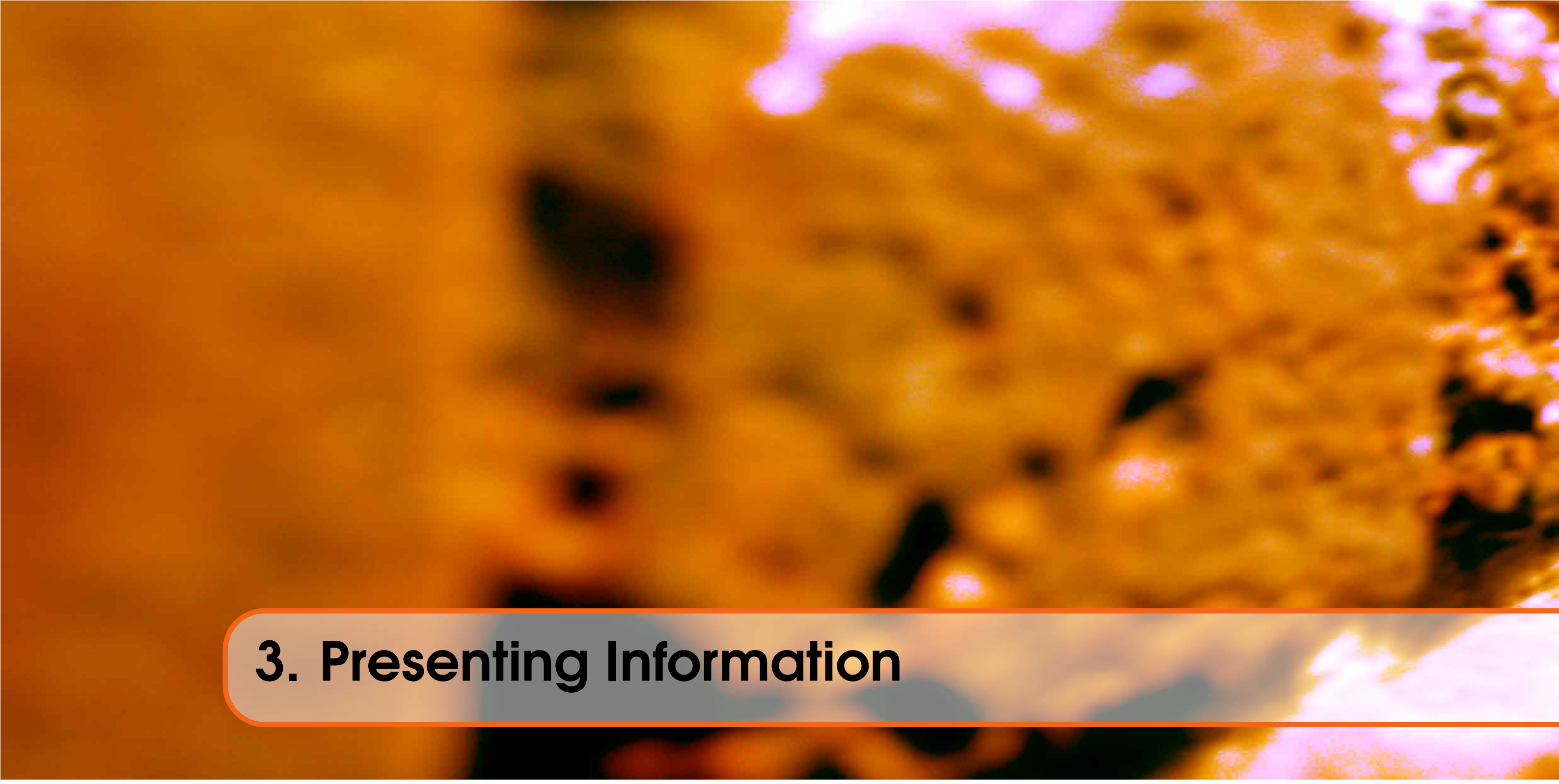
Exercise 2.1 This is a good place to ask a question to test learning progress or further cement ideas into students' minds. ■

2.9 Problems

Problem 2.1 What is the average airspeed velocity of an unladen swallow?

2.10 Vocabulary

Define a word to improve a students' vocabulary.
Vocabulary 2.1 — Word. Definition of word.



3. Presenting Information

3.1 Table

Treatments	Response 1	Response 2
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table 3.1: Table caption

3.2 Figure

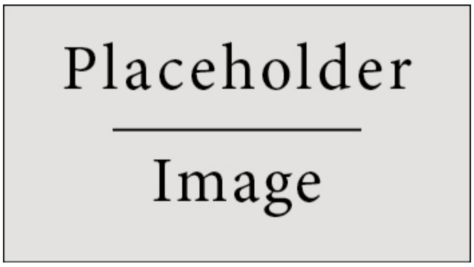


Figure 3.1: Figure caption

II

Part Two

4. Limit Of Sequence

4.1 Cauchy proposition

Theorem 4.1.1 If a sequence $\{x_n\}$ converges to l, then its arithmetic mean of the preceding n terms also converges to l, namely

$$\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = \lim_{n\to\infty} x_n = l. \tag{4.1}$$

Proof. According to the condition $\lim\limits_{n\to\infty} x_n = l$, given $\varepsilon > 0$, there is a positive number N such that $|x_n - l| < \varepsilon$ for all $n > N$. Just assume $n > N$ and we can make an estimation as follows

$$\begin{aligned}
\left|\frac{x_1+x_2+\cdots+x_n}{n} - l\right| &= \frac{|(x_1-l)+(x_2-l)+\cdots+(x_n-l)|}{n} \\
&\leqslant \frac{|(x_1-l)+\cdots+(x_N-l)|}{n} + \frac{|(x_{N+1}-l)+\cdots+(x_n-l)|}{n} \\
&< \frac{M}{n} + \frac{n-N}{n}\varepsilon,
\end{aligned}$$

where $M = |(x_1-l)+\cdots+(x_N-l)|$ is a finite number. Thus we can see that if let

$$N_1 = \max\{N, \left[\frac{M}{\varepsilon}\right]\},$$

then for all $n > N_1$ it follows that

$$\left|\frac{x_1+x_2+\cdots+x_n}{n} - l\right| < 2\varepsilon.$$

It clearly implies $\lim\limits_{n\to\infty} \dfrac{x_1+x_2+\cdots+x_n}{n} = l.$

■

R

1. If x_n approaches positive(or negative) infinity, Cauchy proposition still holds. In fact, from $x_n \to +\infty (n \to \infty)$ we see x_n can be greater than any given positive number X when n is large enough. Similarly we can separate the arithmetic mean into two parts and show the second part $(1-N/n)X$ greater than $X/2$ for a sufficiently large n.

2. The converse of Cauchy proposition is generally not true. A trivial example is $x_n = (-1)^n$. Then

$$\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = 0$$

while x_n has no limit.

Corollary 4.1.2 If a positive term sequence $\{x_n\}$ converges to l, then its geometric mean of the preceding n terms also converges to l, namely

$$\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n} = \lim_{n\to\infty} x_n = l. \tag{4.2}$$

Proof. Applying the mean inequality we have

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} \leqslant \sqrt[n]{x_1 x_2 \cdots x_n} \leqslant \frac{x_1+x_2+\cdots+x_n}{n}.$$

Notice that $\lim\limits_{n\to\infty} x_n = l$ implies $\lim\limits_{n\to\infty} \frac{1}{x_n} = \frac{1}{l}$. From Theorem 4.1.1 we can see

$$\lim_{n\to\infty} \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} = \lim_{n\to\infty} \left(\frac{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}{n} \right)^{-1} = \left(\lim_{n\to\infty} \frac{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}{n} \right)^{-1} = l.$$

And it has been shown that

$$\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = l.$$

According to squeeze theorem, we know the limit $\lim\limits_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n}$ exists and equals l. ■

Proposition 4.1.3 If $x_n > 0 (n = 1, 2, \cdots)$ and the limit $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists, the limit $\lim\limits_{n\to\infty} \sqrt[n]{x_n}$ also exists and

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Proof. Assume $x_0 = 1$ and by Corollary 4.1.2 we immediately get

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\frac{x_1}{x_0} \frac{x_2}{x_1} \cdots \frac{x_n}{x_{n-1}}} = \lim_{n\to\infty} \frac{x_n}{x_{n-1}} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

■

■ **Example 4.1** Find the limit $\lim\limits_{n\to\infty} \frac{n}{\sqrt[n]{n!}}$. ■

4.2 Stolz–Cesàro theorem

Theorem 4.2.1 — Stolz–Cesàro theorem with form of $\frac{0}{0}$. Assume $\{a_n\}$ and $\{b_n\}$ are two infinitesimal sequences of real numbers and $\{a_n\}$ is a strictly decreasing. If

$$\lim_{n\to\infty}\frac{b_{n+1}-b_n}{a_{n+1}-a_n}=l,$$

where l is finite or $\pm\infty$, then we have

$$\lim_{n\to\infty}\frac{b_n}{a_n}=l.$$

Proof. We only prove it for a finite number l. According to the assumption, for every positive number ε, there exists another positive number N such that

$$\left|\frac{b_{n+1}-b_n}{a_{n+1}-a_n}-l\right|<\varepsilon$$

for all $n>N$. Since $a_n>a_{n+1}$ for all $n\in\mathbb{N}^*$, we have

$$(l-\varepsilon)(a_n-a_{n+1})<b_n-b_{n+1}<(l+\varepsilon)(a_n-a_{n+1}).$$

Given $m>n$, replace n by $n+1,n+2,\cdots,m-1$. Then add up the $m-n$ inequalities and we obtain

$$(l-\varepsilon)(a_n-a_m)<b_n-b_m<(l+\varepsilon)(a_n-a_m),$$

or

$$\left|\frac{b_n-b_m}{a_n-a_m}-l\right|<\varepsilon.$$

Note that $\lim\limits_{m\to\infty}a_m=\lim\limits_{m\to\infty}b_m=0$. When $m\to\infty$, for all $n>N$ it follows that

$$\left|\frac{b_n}{a_n}-l\right|\leqslant\varepsilon.$$

With the arbitrariness of selection of ε, this implies $\lim\limits_{n\to\infty}\frac{b_n}{a_n}=l$. ■

Theorem 4.2.2 — Stolz–Cesàro theorem with form of $\frac{*}{\infty}$. Asume $\{a_n\}$ is a strictly increasing sequence such that $\lim a_n=\infty$. If

$$\lim_{n\to\infty}\frac{b_{n+1}-b_n}{a_{n+1}-a_n},$$

where l is finite or $\pm\infty$, then we have

$$\lim_{n\to\infty}\frac{b_n}{a_n}=l.$$

Proof. We just consider a finite l. For every positive number ε, there exists $N_1\in\mathbb{N}^*$ such that

$$\left|\frac{b_{n+1}-b_n}{a_{n+1}-a_n}-l\right|<\varepsilon$$

for all $n > N_1$. Since $a_{n+1} > a_n$ for all $n \in \mathbb{N}^*$, we have

$$(l-\varepsilon)(a_{n+1}-a_n) < b_{n+1}-b_n < (l+\varepsilon)(a_{n+1}-a_n).$$

Given N_1, replace n by $N_1, N_1+1, \cdots, n-1$. Then add up the $n-N$ inequalities and we obtain

$$(l-\varepsilon)(a_n-a_{N_1}) < b_n-b_{N_1} < (l+\varepsilon)(a_n-a_{N_1}),$$

or

$$\left|\frac{b_n-b_{N_1}}{a_n-a_{N_1}}-l\right| < \varepsilon.$$

In order to estimate the value of $\left|\frac{b_n}{a_n}-l\right|$, consider the following identity

$$\frac{b_n}{a_n}-l=\left(1-\frac{a_{N_1}}{a_n}\right)\left(\frac{b_n-b_{N_1}}{a_n-a_{N_1}}-l\right)+\frac{b_n-la_{N_1}}{a_n}.$$

Since $\lim\limits_{n\to\infty} a_n=+\infty$, there exsits a positive number N_2 such that for all $n > N_2$

$$0<\left|\frac{a_N}{a_n}\right|<1 \Leftrightarrow 0<1-\frac{a_N}{a_n}<2$$

and

$$\left|\frac{b_n-la_{N_1}}{a_n}\right| < \varepsilon.$$

Thus for all $n > \max\{N_1, N_2\}$ we have

$$\left|\frac{b_n}{a_n}-l\right| < 3\varepsilon,$$

which indicates $\lim\limits_{n\to\infty}\frac{b_n}{a_n}=l$.

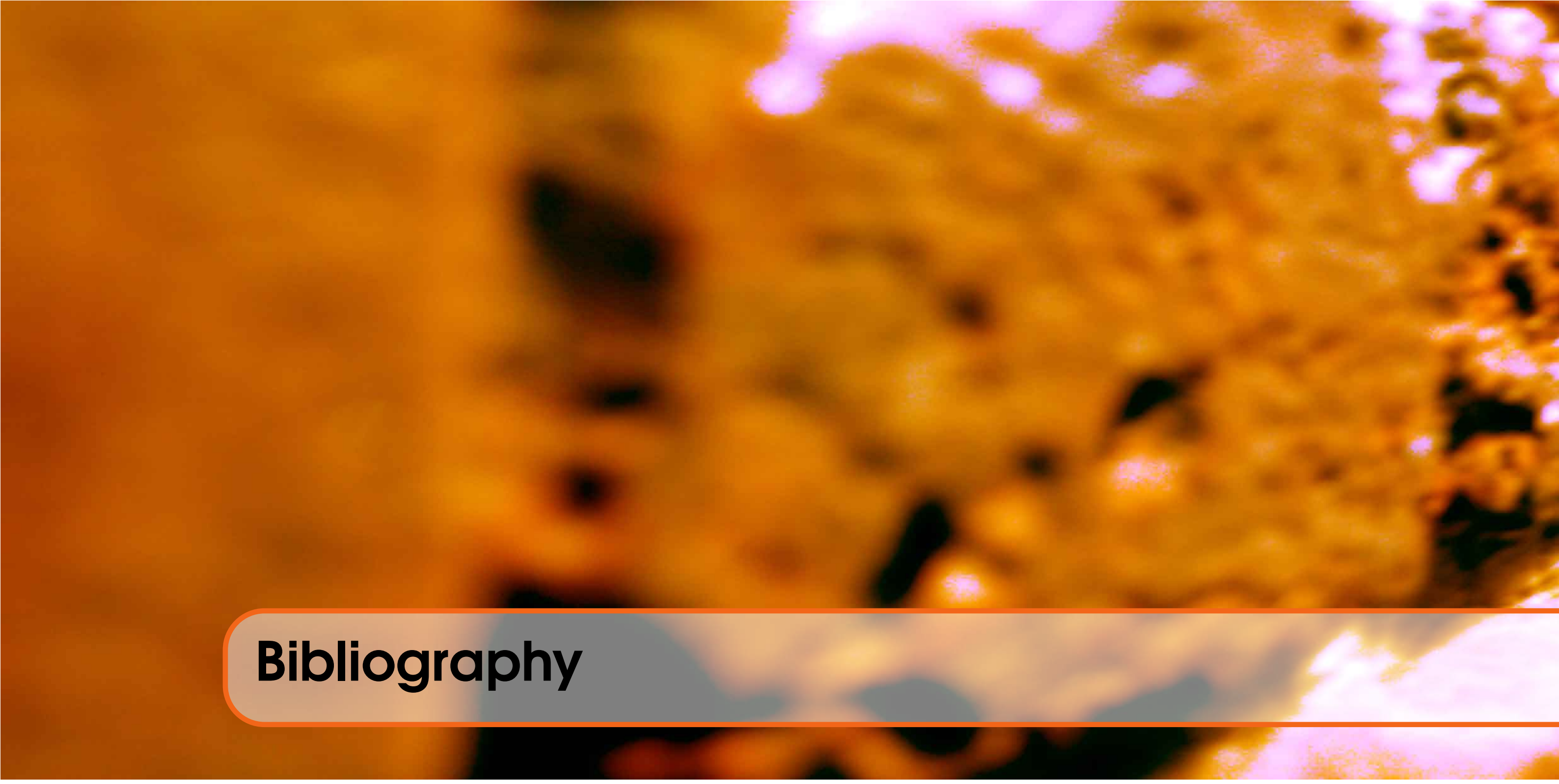
■

5. Limit of function

5.1 Equivalent Infinitesimal

Definition 5.1.1 If the relation $f(x) = \gamma(x)g(x)$ holds ultimately over $\mathscr{B}$ where $\lim\limits_{\mathscr{B}} \gamma(x) = 1$, we say that *the function f behaves asymptotically like g over $\mathscr{B}$*, or, more briefly, that *f is equivalent to g over $\mathscr{B}$*.

Part N



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