Linear Algebra

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Latest Update: January 20, 2022

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Chapter 1

Determinant

1.1 Calculation Techniques

Example 1.1.1

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} = (x + (n-1)a)\,(x-a)^{n-1}.$$

Proof. **Upper triangularization**. There are two possible ways to achieve this.

1. Multiple subtraction. Subtract row 1 from row 2, row 3, ... , row n to get a shape like

$$\begin{vmatrix} * & * & * & \cdots & * \\ * & * & & & \\ * & & * & & \\ \vdots & & & \ddots & \\ * & & & & * \end{vmatrix}$$

and then eliminate the first column.

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} \xlongequal[\substack{\cdots \\ r_n\,-=\,r_1}]{r_2\,-=\,r_1} \begin{vmatrix} x & a & \cdots & a \\ a-x & x-a & & \\ \vdots & & \ddots & \\ a-x & & & x-a \end{vmatrix}$$

$$\xlongequal{c_1\,+=\,c_2+\cdots+c_n} \begin{vmatrix} x+(n-1)a & a & \cdots & a \\ & x-a & & \\ & & \ddots & \\ & & & x-a \end{vmatrix}.$$

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2. Accumulation. Add row 2, row 3, ... , row n to row 1.

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} \xlongequal{\mathbf{r_1}+=\mathbf{r_2}+\cdots+\mathbf{r_n}} \begin{vmatrix} x+(n-1)a & x+(n-1)a & \cdots & x+(n-1)a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix}$$

$$= (x+(n-1)a) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix}$$

$$\xlongequal[\cdots]{\substack{\mathbf{r_2}\,-=\,a*\mathbf{r_1} \\ \mathbf{r_n}\,-=\,a*\mathbf{r_1}}} (x+(n-1)a) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ & x-a & & \\ & & \ddots & \\ & & & x-a \end{vmatrix}.$$

$\square$

Example 1.1.2 Suppose $x_i \neq a_i$ $(i = 1, 2, \cdots, n)$.

$$D_n = \begin{vmatrix} x_1 & a_2 & \cdots & a_n \\ a_1 & x_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & x_n \end{vmatrix} = \left(1 + \sum_{i=1}^{n} \frac{a_i}{x_i - a_i}\right) \prod_{i=1}^{n} (x_i - a_i).$$

Proof. **Increase order**. Consider

$$D_n = \begin{vmatrix} x_1 & a_2 & \cdots & a_n \\ a_1 & x_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & x_n \end{vmatrix} = \begin{vmatrix} 1 & a_1 & \cdots & a_n \\ \hline 0 & & D_n & \\ \end{vmatrix}.$$

Then we can use the multiple subtraction method in Example 1.1.1 to get an upper triangular determinant.

$$D_n = \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & x_1-a_1 & & & \\ -1 & & x_2-a_2 & & \\ \vdots & & & \ddots & \\ -1 & & & & x_n-a_n \end{vmatrix}$$

$$= \begin{vmatrix} 1+\sum_{i=1}^{n} \frac{a_i}{x_i-a_i} & a_1 & a_2 & \cdots & a_n \\ & x_1-a_1 & & & \\ & & x_2-a_2 & & \\ & & & \ddots & \\ & & & & x_n-a_n \end{vmatrix}.$$

$\square$

Example 1.1.3 Suppose $a \neq b$.

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ b & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & x \end{vmatrix} = \frac{a(x-b)^n - b(x-a)^n}{a-b}.$$

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Proof. **Recurrence**. By splitting the first column we get

$$
D_n = \frac{a}{b}
\begin{vmatrix}
b & b & \cdots & b \\
b & x & \cdots & a \\
\vdots & \vdots & \ddots & \vdots \\
b & b & \cdots & x
\end{vmatrix}
+
\begin{vmatrix}
x-a & a & \cdots & a \\
0 & x & \cdots & a \\
0 & \vdots & \ddots & \vdots \\
0 & b & \cdots & x
\end{vmatrix}
$$

$$
= \frac{a}{b}
\begin{vmatrix}
b & b & \cdots & b \\
 & x-b & \cdots & a-b \\
 & & \ddots & \vdots \\
 & & & x-b
\end{vmatrix}
+
\begin{vmatrix}
x-a & a & \cdots & a \\
\hline
0 & & D_{n-1} &
\end{vmatrix}
$$

$$
= a(x-b)^{n-1} + (x-a)D_{n-1}.
$$

Transpose it to obtain another recurrence relation

$$
D_n = D_n^T = b(x-a)^{n-1} + (x-b)D_{n-1}.
$$

Solve D_n from the two recurrence equations.

$\square$

Chapter 2

Matrix

2.1 Properties

Proposition 2.1.1 (properties of square matrices) Let A, B be $n \times n$ matrices.

1. If A, B is invertible,

$$(AB)^{-1} = B^{-1}A^{-1}, \quad (kA)^{-1} = \frac{1}{k}A^{-1}(k \neq 0), \quad |A^n| = |A|^n \ (n \in \mathbb{Z}).$$

2.

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}, \quad (kA)^{\mathrm{T}} = kA^{\mathrm{T}}, \quad \left|A^{\mathrm{T}}\right| = |A|, (A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}.$$

3.

$$\mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A), \quad \mathrm{adj}(kA) = k^{n-1}\mathrm{adj}(A), \quad |\mathrm{adj}(A)| = |A|^{n-1}, \quad \mathrm{adj}(\mathrm{adj}(A)) = |A|^{n-2}A.$$

4. The mappings $^{-1}$, $^{\mathrm{T}}$, adj are commutable.

Proposition 2.1.2 (properties of block matrices) Suppose $A = (A_{ij})$.

1.

$$A^{\mathrm{T}} = (A_{ji}^{\mathrm{T}}).$$

2. If A, B is invertible,

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix}$$

Proposition 2.1.3 (properties of block diagonal matrices) Suppose $A = \mathrm{diag}(A_1, A_2, \cdots, A_n)$, $A_i(i = 1, 2, \cdots, n)$ are square matrices.

1.

$$\mathrm{diag}(A_1, A_2, \cdots, A_n)^k = \mathrm{diag}(A_1^k, A_2^k, \cdots, A_n^k) \quad (k \in \mathbb{Z}_{\geq 0}).$$

Furthermore, if $A_i(i = 1, 2, \cdots, n)$ are invertible,

$$\mathrm{diag}(A_1, A_2, \cdots, A_n)^k = \mathrm{diag}(A_1^k, A_2^k, \cdots, A_n^k) \quad (k \in \mathbb{Z}).$$

2.

$$|\mathrm{diag}(A_1, A_2, \cdots, A_n)| = |A_1| |A_n| \cdots |A_n|.$$

2.2 Linear Equations

2.2.1 Homogeneous Linear Equations

Let $A_{m \times n} = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in M_{m,n}(F)$. There are two possible solution sets for the equation $Ax = 0$.

1. $Ax = 0$ has unique solution $x = 0$.

2. $Ax = 0$ has infinitely many solutions.

Row Echelon Form
Suppose A is equivalent to the following row echelon form

$$\begin{pmatrix} 1 & * & \cdots & * & * & \cdots & * & * & * & \cdots & * \\ & 1 & * & \cdots & & * & * & * & \cdots & * \\ & & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & & 1 & * & \cdots & * \\ & & & 0_{(m-r) \times n} \end{pmatrix}.$$

We see $r(A)$ equals the number of the nonzero columns r.

1. $Ax = 0$ has unique solution $x = 0 \iff r(A) = n$.

2. $Ax = 0$ has infinitely many solutions $\iff r(A) < n$.

Spanned Space of Column Vectors
If we see A as $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, where $\alpha_j \in F^m$ $(j = 1, 2 \cdots, n)$, then

$$Ax = 0 \iff x_1 \alpha_1 + x_2 \alpha_2 + \cdots + x_n \alpha_n = 0.$$

1. $Ax = 0$ has unique solution $x = 0 \iff \alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent $\iff$ $\dim \operatorname{span}(\alpha_1, \alpha_2, \cdots, \alpha_n) = n$.

2. $Ax = 0$ has infinitely many solutions $x = 0 \iff \alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly dependent $\iff$ $\dim \operatorname{span}(\alpha_1, \alpha_2, \cdots, \alpha_n) < n$.

Linear Mapping
If we see A as a linear mapping in $\operatorname{Hom}_{\mathsf{Vect}_F}(F^n, F^m)$, that is

$$A : F^n \longrightarrow F^m$$
$$x \longmapsto Ax,$$

then the solution space of $Ax = 0$ is $\ker A$. Since $\operatorname{im} A = \operatorname{span}(\alpha_1, \alpha_2, \cdots, \alpha_n)$, we have

$$r(A) = \dim \operatorname{span}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \dim \operatorname{im} A.$$

1. $Ax = 0$ has unique solution $x = 0 \iff A$ is injective $\iff \ker A = 0 \iff \operatorname{im} A = F^m$.

2. $Ax = 0$ has infinitely many solutions $x = 0 \iff A$ is not injective $\iff \dim \ker A > 0 \iff$ $\dim \operatorname{im} A < n$.

Orthogonal Complement
If we see A as $(\beta_1, \beta_2, \cdots, \beta_n)$, where $\beta_i \in F^n$ $(i = 1, 2 \cdots, m)$, then

$$Ax = 0 \iff \langle \beta_i^T, x \rangle = \beta_i x = 0, \ i = 1, 2 \cdots, m.$$

The solution space of $Ax = 0$ is

$$\bigcap_{i=1}^{m} \operatorname{span} \left(\beta_i^T \right)^{\perp}.$$

2.2.2 Nonhomogeneous Linear Equations

Let $A_{m \times n} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. There are two possible results for the equation $Ax = b$.

1. has unique solution $x = 0$.

2. has infinitely many solution.

Matrix theory. Suppose $(A|b)$ is equivalent to the following row echelon form

$$
\left(
\begin{array}{ccccccccccc|c}
1 & * & \cdots & * & * & \cdots & * & * & * & \cdots & * & 0 \\
 & & & 1 & * & \cdots & * & * & * & \cdots & * & 0 \\
 & & & & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & & & & 1 & * & \cdots & * & & 0 \\
 & & & & 0_{(m-r) \times n} & & & & & & & 0_{(m-r) \times 1}
\end{array}
\right)
$$

1. has unique solution $x = 0$:

2. has infinitely many solution.

Appendix

1.