Linear Algebra

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Chapter 1

Determinant

1.1 Calculation Techniques

Example 1.1.1

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} = (x + (n-1)a)\,(x-a)^{n-1}\,.$$

Proof. **Upper triangularization**. There are two possible ways to achieve this.

1. Multiple subtraction. Subtract row 1 from row 2, row 3, ... , row n to get a shape like

$$\begin{vmatrix} * & * & * & \cdots & * \\ * & * & & & \\ * & & * & & \\ \vdots & & & \ddots & \\ * & & & & * \end{vmatrix}$$

and then eliminate the first column.

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} \overset{\substack{\mathtt{r_2 -= r_1} \\ \cdots \\ \mathtt{r_n -= r_1}}}{=\!=\!=} \begin{vmatrix} x & a & \cdots & a \\ a-x & x-a & & \\ \vdots & & \ddots & \\ a-x & & & x-a \end{vmatrix}$$

$$\overset{\mathtt{c_1 += c_2 + \cdots + c_n}}{=\!=\!=} \begin{vmatrix} x+(n-1)a & a & \cdots & a \\ & x-a & & \\ & & \ddots & \\ & & & x-a \end{vmatrix}.$$

2. Accumulation. Add row 2, row 3, ... , row n to row 1.

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} \xlongequal{\mathtt{r_1 += r_2 + \cdots + r_n}} \begin{vmatrix} x+(n-1)a & x+(n-1)a & \cdots & x+(n-1)a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix}$$

$$= (x+(n-1)a) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix}$$

$$\xlongequal[\mathtt{r_n} -= a * \mathtt{r_1}]{\substack{\mathtt{r_2} -= a*\mathtt{r_1} \\ \cdots}} (x+(n-1)a) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ & x-a & & \\ & & \ddots & \\ & & & x-a \end{vmatrix}.$$

□

Example 1.1.2 Suppose $x_i \neq a_i$ $(i = 1, 2, \cdots, n)$.

$$D_n = \begin{vmatrix} x_1 & a_2 & \cdots & a_n \\ a_1 & x_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & x_n \end{vmatrix} = \left(1 + \sum_{i=1}^{n} \frac{a_i}{x_i - a_i}\right) \prod_{i=1}^{n} (x_i - a_i).$$

Proof. **Increase order**. Consider

$$D_n = \begin{vmatrix} x_1 & a_2 & \cdots & a_n \\ a_1 & x_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & x_n \end{vmatrix} = \left|\begin{array}{c:ccc} 1 & a_1 & \cdots & a_n \\ \hdashline 0 & & D_n & \end{array}\right|.$$

Then we can use the multiple subtraction method in Example 1.1.1 to get an upper triangular determinant.

$$D_n = \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & x_1 - a_1 & & & \\ -1 & & x_2 - a_2 & & \\ \vdots & & & \ddots & \\ -1 & & & & x_n - a_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 + \sum_{i=1}^{n} \frac{a_i}{x_i - a_i} & a_1 & a_2 & \cdots & a_n \\ & x_1 - a_1 & & & \\ & & x_2 - a_2 & & \\ & & & \ddots & \\ & & & & x_n - a_n \end{vmatrix}.$$

□

Example 1.1.3 Suppose $a \neq b$.

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ b & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & x \end{vmatrix} = \frac{a(x-b)^n - b(x-a)^n}{a-b}.$$

Proof. **Recurrence**. By splitting the first column we get

$$\begin{aligned}
D_n &= \frac{a}{b}\begin{vmatrix} b & b & \cdots & b \\ b & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & x \end{vmatrix} + \begin{vmatrix} x-a & a & \cdots & a \\ 0 & x & \cdots & a \\ 0 & \vdots & \ddots & \vdots \\ 0 & b & \cdots & x \end{vmatrix} \\
&= \frac{a}{b}\begin{vmatrix} b & b & \cdots & b \\ & x-b & \cdots & a-b \\ & & \ddots & \vdots \\ & & & x-b \end{vmatrix} + \left|\begin{array}{c:c} x-a & a \quad \cdots \quad a \\ \hdashline 0 & D_{n-1} \end{array}\right| \\
&= a(x-b)^{n-1} + (x-a)D_{n-1}.
\end{aligned}$$

Transpose it to obtain another recurrence relation

$$D_n = D_n^T = b(x-a)^{n-1} + (x-b)D_{n-1}.$$

Solve D_n from the two recurrence equations.

□

Chapter 2

Matrix

2.1 Properties

Proposition 2.1.1 (properties of square matrices) Let A, B be $n \times n$ matrices.

1. If A, B is invertible,
$$(AB)^{-1} = B^{-1}A^{-1}, \quad (kA)^{-1} = \frac{1}{k}A^{-1}(k \neq 0), \quad |A^n| = |A|^n \ (n \in \mathbb{Z}).$$

2.
$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}, \quad (kA)^{\mathrm{T}} = kA^{\mathrm{T}}, \quad \left|A^{\mathrm{T}}\right| = |A|, (A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}.$$

3.
$$\mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A), \quad \mathrm{adj}(kA) = k^{n-1}\mathrm{adj}(A), \quad |\mathrm{adj}(A)| = |A|^{n-1}, \quad \mathrm{adj}(\mathrm{adj}(A)) = |A|^{n-2} A.$$

4. The mappings $^{-1}$, $^{\mathrm{T}}$, adj are commutable.

Proposition 2.1.2 (properties of block matrices) Suppose $A = (A_{ij})$.

1.
$$A^{\mathrm{T}} = (A_{ji}^{\mathrm{T}}).$$

2. If A, B is invertible,
$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix}$$

Proposition 2.1.3 (properties of block diagonal matrices) Suppose $A = \mathrm{diag}(A_1, A_2, \cdots, A_n)$, $A_i (i = 1, 2, \cdots, n)$ are square matrices.

1.
$$\mathrm{diag}(A_1, A_2, \cdots, A_n)^k = \mathrm{diag}(A_1^k, A_2^k, \cdots, A_n^k) \quad (k \in \mathbb{Z}_{\geq 0}).$$
Furthermore, if $A_i (i = 1, 2, \cdots, n)$ are invertible,
$$\mathrm{diag}(A_1, A_2, \cdots, A_n)^k = \mathrm{diag}(A_1^k, A_2^k, \cdots, A_n^k) \quad (k \in \mathbb{Z}).$$

2.
$$|\mathrm{diag}(A_1, A_2, \cdots, A_n)| = |A_1| |A_n| \cdots |A_n|.$$

2.2 Linear Equations

2.2.1 Homogeneous Linear Equations

Let $A_{m\times n} = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in M_{m,n}(F)$. There are two possible solution sets for the equation $Ax = 0$.

1. $Ax = 0$ has unique solution $x = 0$.
2. $Ax = 0$ has infinitely many solutions.

Rank

1. $Ax = 0$ has unique solution $x = 0 \iff r(A) = n$.
2. $Ax = 0$ has infinitely many solutions $\iff r(A) < n$.

Row Echelon Form
Suppose A is equivalent to the following row echelon form

$$\begin{pmatrix} 1 & * & \cdots & * & * & \cdots & * & * & * & \cdots & * \\ & & & 1 & * & \cdots & * & * & * & \cdots & * \\ & & & & & & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & & & & & & 1 & * & \cdots & * \\ \hline & & & & & 0_{(m-r)\times n} & & & & & \end{pmatrix}.$$

r denotes the number of the nonzero rows.

1. $Ax = 0$ has unique solution $x = 0 \iff r = n$.
2. $Ax = 0$ has infinitely many solutions $\iff r < n$.

Spanned Space of Column Vectors
If we see A as $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, where $\alpha_j \in F^m$ $(j = 1, 2 \cdots, n)$, then

$$Ax = 0 \iff x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = 0.$$

1. $Ax = 0$ has unique solution $x = 0 \iff \alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent $\iff \dim \operatorname{span}(\alpha_1, \alpha_2, \cdots, \alpha_n) = n$.
2. $Ax = 0$ has infinitely many solutions $x = 0 \iff \alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly dependent $\iff \dim \operatorname{span}(\alpha_1, \alpha_2, \cdots, \alpha_n) < n$.

Linear Mapping
If we see A as a linear mapping $\mathcal{A} \in \operatorname{Hom}_{\mathsf{Vect}_F}(F^n, F^m)$, that is

$$\begin{aligned} \mathcal{A} : F^n &\longrightarrow F^m \\ x &\longmapsto Ax, \end{aligned}$$

then the solution space of $Ax = 0$ is $\ker \mathcal{A}$. Note

$$y \in \operatorname{im} \mathcal{A} \iff \exists x \in F^n, y = Ax \iff \exists x \in F^n, y = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n \iff y \in \operatorname{span}(\alpha_1, \alpha_2, \cdots, \alpha_n).$$

We have $\operatorname{im} \mathcal{A} = \operatorname{span}(\alpha_1, \alpha_2, \cdots, \alpha_n)$, which implies

$$r(A) = \dim \operatorname{span}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \dim \operatorname{im} \mathcal{A}.$$

1. $Ax = 0$ has unique solution $x = 0 \iff \mathcal{A}$ is injective $\iff \ker \mathcal{A} = 0 \iff \operatorname{im} \mathcal{A} = F^m$.

2. $Ax = 0$ has infinitely many solutions $x = 0 \iff \mathcal{A}$ is not injective $\iff \dim\ker\mathcal{A} > 0 \iff \dim\operatorname{im}\mathcal{A} < n$.

Orthogonal Complement
If we see A as $(\beta_1, \beta_2, \cdots, \beta_n)^T$, where $\beta_i \in F^n$ $(i = 1, 2\cdots, m)$, then

$$Ax = 0 \iff \langle \beta_i, x\rangle = \beta_i^T x = 0,\ (i = 1, 2\cdots, m)\,.$$

The solution space of $Ax = 0$ is

$$\bigcap_{i=1}^{m} \operatorname{span}\left(\beta_i^T\right)^{\perp}.$$

Determinant
If A is a square matrix, then

1. $Ax = 0$ has unique solution $x = 0 \iff |A| \neq 0$.
2. $Ax = 0$ has infinitely many solutions $\iff |A| = 0$.

2.2.2 Nonhomogeneous Linear Equations

Let $A_{m\times n} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. There are two possible results for the equation $Ax = b$:

1. $Ax = b$ has no solution.
2. $Ax = b$ has at least one solution.
   (a) $Ax = b$ has a unique solution.
   (b) $Ax = b$ has infinitely many solutions.

Rank

1. $Ax = b$ has no solution $\iff r\,(A \mid b) = r(A) + 1$.
2. $Ax = b$ has at least one solution $\iff r\,(A \mid b) = r(A)$.
   (a) $Ax = b$ has a unique solution $\iff r(A) = n$.
   (b) $Ax = b$ has infinitely many solutions $\iff r(A) < n$..

Row Echelon Form
Suppose $(A \mid b)$ is equivalent to the following row echelon form

$$\left(\begin{array}{cccccccccccc|c}
1 & * & \cdots & * & * & \cdots & * & * & * & \cdots & * & & d_1 \\
 & & & 1 & * & \cdots & * & * & * & \cdots & * & & d_2 \\
 & & & & & & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots \\
 & & & & & & & 1 & * & \cdots & * & & d_r \\
\hline
\multicolumn{12}{c|}{0_{1\times n}} & d_{r+1} \\
\multicolumn{12}{c|}{0_{(m-r-1)\times n}} & 0_{(m-r-1)\times 1}
\end{array}\right)$$

r denotes the number of the nonzero rows.

1. $Ax = b$ has no solution $\iff d_{r+1} \neq 0$.
2. $Ax = b$ has at least one solution $\iff d_{r+1} = 0$..
   (a) $Ax = b$ has a unique solution $\iff r = n$.
   (b) $Ax = b$ has infinitely many solutions $\iff r < n$..

Linear Mapping
If we see A as a linear mapping $\mathcal{A} \in \mathrm{Hom}_{\mathsf{Vect}_F}(F^n, F^m)$, that is

$$\begin{aligned} \mathcal{A} : F^n &\longrightarrow F^m \\ x &\longmapsto Ax, \end{aligned}$$

then the solution space of $Ax = b$ is $\mathcal{A}^{-1}(b) + \ker \mathcal{A}$.

1. $Ax = b$ has no solution $\iff b \notin \operatorname{im} \mathcal{A}$.
2. $Ax = b$ has at least one solution $\iff b \in \operatorname{im} \mathcal{A}$.
   (a) $Ax = b$ has a unique solution $\iff \mathcal{A}$ is injective $\iff \ker \mathcal{A} = 0 \iff \operatorname{im} \mathcal{A} = F^m$.
   (b) $Ax = b$ has infinitely many solutions $x = 0 \iff \mathcal{A}$ is not injective $\iff \dim \ker \mathcal{A} > 0 \iff \dim \operatorname{im} \mathcal{A} < n$.

Determinant
If A is a square matrix, then

1. $Ax = b$ has unique solution $\iff |A| \neq 0$.
2. $Ax = b$ has no solution or have infinitely many solutions $\iff |A| = 0$.

Appendix

1.