# Linear Algebra

H.C.

Latest Update: January 19, 2022

# Contents

**1 Determinant** **3**
1.1 Calculation Techniques . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3

**2 Matrix** **6**
2.1 Properties . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6

# Chapter 1

# Determinant

## 1.1 Calculation Techniques

**Example 1.1.1**

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} = (x + (n-1)a)\,(x-a)^{n-1}\,.$$

*Proof.* **Upper triangularization**. There are two possible ways to achieve this.

1. Multiple subtraction. Subtract row 1 from row 2, row 3, ... , row $n$ to get a shape like

$$\begin{vmatrix} * & * & * & \cdots & * \\ * & * & & & \\ * & & * & & \\ \vdots & & & \ddots & \\ * & & & & * \end{vmatrix}$$

and then eliminate the first column.

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} \xlongequal[\mathtt{r_n -= r_1}]{\substack{\mathtt{r_2 -= r_1} \\ \cdots}} \begin{vmatrix} x & a & \cdots & a \\ a-x & x-a & & \\ \vdots & & \ddots & \\ a-x & & & x-a \end{vmatrix}$$

$$\xlongequal{\mathtt{c_1 += c_2 + \cdots + c_n}} \begin{vmatrix} x+(n-1)a & a & \cdots & a \\ & x-a & & \\ & & \ddots & \\ & & & x-a \end{vmatrix}.$$

2. Accumulation. Add row 2, row 3, ... , row $n$ to row 1.

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix} \xlongequal{\mathtt{r_1 += r_2 + \cdots + r_n}} \begin{vmatrix} x+(n-1)a & x+(n-1)a & \cdots & x+(n-1)a \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix}$$

$$= (x+(n-1)a) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x \end{vmatrix}$$

$$\xlongequal[\mathtt{r_n} -= a * \mathtt{r_1}]{\mathtt{r_2} -= a * \mathtt{r_1} \\ \cdots} (x+(n-1)a) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ & x-a & & \\ & & \ddots & \\ & & & x-a \end{vmatrix}.$$

□

**Example 1.1.2** Suppose $x_i \neq a_i$ $(i = 1, 2, \cdots, n)$.

$$D_n = \begin{vmatrix} x_1 & a_2 & \cdots & a_n \\ a_1 & x_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & x_n \end{vmatrix} = \left(1 + \sum_{i=1}^{n} \frac{a_i}{x_i - a_i}\right) \prod_{i=1}^{n} (x_i - a_i).$$

*Proof.* **Increase order**. Consider

$$D_n = \begin{vmatrix} x_1 & a_2 & \cdots & a_n \\ a_1 & x_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & x_n \end{vmatrix} = \left|\begin{array}{c:c} 1 & a_1 \quad \cdots \quad a_n \\ \hdashline 0 & D_n \end{array}\right|.$$

Then we can use the multiple subtraction method in Example 1.1.1 to get an upper triangular determinant.

$$D_n = \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & x_1 - a_1 & & & \\ -1 & & x_2 - a_2 & & \\ \vdots & & & \ddots & \\ -1 & & & & x_n - a_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 + \sum_{i=1}^{n} \frac{a_i}{x_i - a_i} & a_1 & a_2 & \cdots & a_n \\ & x_1 - a_1 & & & \\ & & x_2 - a_2 & & \\ & & & \ddots & \\ & & & & x_n - a_n \end{vmatrix}.$$

□

**Example 1.1.3** Suppose $a \neq b$.

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ b & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & x \end{vmatrix} = \frac{a(x-b)^n - b(x-a)^n}{a-b}.$$

*Proof.* **Recurrence**. By splitting the first column we get

$$
\begin{aligned}
D_n &= \frac{a}{b}\begin{vmatrix} b & b & \cdots & b \\ b & x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & x \end{vmatrix} + \begin{vmatrix} x-a & a & \cdots & a \\ 0 & x & \cdots & a \\ 0 & \vdots & \ddots & \vdots \\ 0 & b & \cdots & x \end{vmatrix} \\
&= \frac{a}{b}\begin{vmatrix} b & b & \cdots & b \\ & x-b & \cdots & a-b \\ & & \ddots & \vdots \\ & & & x-b \end{vmatrix} + \left|\begin{array}{c:c} x-a & a \quad \cdots \quad a \\ \hdashline 0 & D_{n-1} \end{array}\right| \\
&= a(x-b)^{n-1} + (x-a)D_{n-1}.
\end{aligned}
$$

Transpose it to obtain another recurrence relation

$$D_n = D_n^T = b(x-a)^{n-1} + (x-b)D_{n-1}.$$

Solve $D_n$ from the two recurrence equations.

□

# Chapter 2

# Matrix

## 2.1 Properties

**Proposition 2.1.1 (properties of square matrices)** Let $A, B$ be $n \times n$ matrices.

1. If $A, B$ is invertible,

$$(AB)^{-1} = B^{-1}A^{-1}, \quad (kA)^{-1} = \frac{1}{k}A^{-1}(k \neq 0), \quad |A^n| = |A|^n \ (n \in \mathbb{Z}).$$

2.

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}, \quad (kA)^{\mathrm{T}} = kA^{\mathrm{T}}, \quad \left|A^{\mathrm{T}}\right| = |A|, (A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}.$$

3.

$$\mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A), \quad \mathrm{adj}(kA) = k^{n-1}\mathrm{adj}(A), \quad |\mathrm{adj}(A)| = |A|^{n-1}, \quad \mathrm{adj}(\mathrm{adj}(A)) = |A|^{n-2}A.$$

4. The mappings ${}^{-1}$, ${}^{\mathrm{T}}$, adj are commutable.

**Proposition 2.1.2 (properties of block matrices)** Suppose $A = (A_{ij})$.

1.

$$A^{\mathrm{T}} = (A_{ji}^{\mathrm{T}}).$$

2. If $A, B$ is invertible,

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix}$$

**Proposition 2.1.3 (properties of block diagonal matrices)** Suppose $A = \mathrm{diag}(A_1, A_2, \cdots, A_n)$, $A_i (i = 1, 2, \cdots, n)$ are square matrices.

1.

$$\mathrm{diag}(A_1, A_2, \cdots, A_n)^k = \mathrm{diag}(A_1^k, A_2^k, \cdots, A_n^k) \quad (k \in \mathbb{Z}_{\geq 0}).$$

Furthermore, if $A_i (i = 1, 2, \cdots, n)$ are invertible,

$$\mathrm{diag}(A_1, A_2, \cdots, A_n)^k = \mathrm{diag}(A_1^k, A_2^k, \cdots, A_n^k) \quad (k \in \mathbb{Z}).$$

2.

$$|\mathrm{diag}(A_1, A_2, \cdots, A_n)| = |A_1| |A_n| \cdots |A_n|.$$

# Appendix

1.