Notes of Microeconomic Theory

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Contents

Part One: Individual Decision Making

1. Preference and Choice

1.1 Preference-based Approach

Definition 1.1.1 — Preference Relation. Preference relation $\succsim$ is a binary relation defined on the set of alternatives X.

Definition 1.1.2 — Rational. The preference relation $\succsim$ is *rational* if it possesses the following two properties:

1. Completeness: $\forall x, y \in X,\ x \succsim y$ or $y \succsim x$.
2. Transitivity: $\forall x, y, z \in X,\ x \succsim y$ and $y \succsim z \implies x \succsim z$.

Definition 1.1.3 — Strict Preference Relation. Define the *strict preference relation* $\succ$ as follows:

$$x \succ y \iff x \succsim y \text{ but not } y \succsim x.$$

Definition 1.1.4 — Indeference Relation. Define the *indifference relation* $\sim$ as follows:

$$x \sim y \iff x \succsim y \text{ and } y \succsim x.$$

Definition 1.1.5 — Utility Function. A function $u : X \to \mathbb{R}$ is a *utility function representing preference relation* $\succsim$ if

$$\forall x, y \in X,\ x \succ y \iff u(x) \geq u(y).$$

Proposition 1.1.1 If there is a utility function that represents a preference relation $\succsim$, then $\succsim$ must be rational.

1.2 Choice-based Approach

Definition 1.2.1 — Choice Structure. A *choice structure* is a binary tuple $(\mathscr{B}, C(\cdot))$, where the set $\mathscr{B} \subset 2^X / \{\varnothing\}$ is a family of nonempty subsets of X and the mapping $C : \mathscr{B} \to \mathscr{B}$ assigns a nonempty set of chosen elements $C(B) \subset B$ for every budget set $B \in \mathscr{B}$.

Definition 1.2.2 — Revealed Preference Relation. Given a choice structure $(\mathscr{B}, C(\cdot))$ the *revealed preference relation* $\succsim^*$ is a binary relation on X defined by

$$x \succsim^* y \Longleftrightarrow \exists B \in \mathscr{B},\ x, y \in B \text{ and } x \in C(B).$$

R For convenience we define a binary relation $\succ^*$ on X informally as follows

$$x \succ^* y \Longleftrightarrow \exists B \in \mathscr{B},\ x, y \in B \text{ and } x \in C(B) \text{ and } y \notin C(B).$$

We will say "x is revealed preferred to y" if $x \succ^* y$.

Definition 1.2.3 — Weak Axiom of Revealed Preference. The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the *weak axiom of revealed preference* if the following property

$$B, B' \in \mathscr{B} \text{ and } x, y \in B \text{ and } x, y \in B' \text{ and } x \in C(B) \text{ and } y \in C(B') \implies x \in C(B')$$

or equivalently

$$\forall x, y \in X,\ x \succsim^* y \implies \text{not } y \succ^* x$$

holds.

1.3 The Relationship between Preference Relations and Choice Rules

1.3.1 Rationality Implies WARP

Definition 1.3.1 Define the correspondence $C^*(\cdot, \succsim) : \mathscr{B} \to 2^X$ as

$$C^*(B, \succsim) = \{x \in B \mid \forall y \in B, x \succsim y\}.$$

We say the preference $\succsim$ *generates* the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$ if $C^*(B, \succsim) \neq \varnothing$ for all $B \in \mathscr{B}$.

R If X is finite, then $C^*(B, \succsim)$ will be nonempty. From now on, we will consider only preferences $\succsim$ and families of budget sets $\mathscr{B}$ such that $C^*(B, \succsim)$ is nonempty for all $B \in \mathscr{B}$.

Proposition 1.3.1 — Rationality Implies WARP. Suppose that $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot, \succsim))$, satisfies the weak axiom of revealed preference.

1.3.2 WARP does NOT Implies Rationality

Definition 1.3.2 — Rationalization. Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ *rationalizes* $C(\cdot)$ relative to $\mathscr{B}$ if

$$\forall B \in \mathscr{B},\ C^*(B, \succsim) = C(B),$$

that is, if the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$ generated by $\succsim$ is identical with $(\mathscr{B}, C(\cdot))$.

Proposition 1.3.2 If $(\mathscr{B}, C(\cdot))$ is a choice structure such that

(i) the weak axiom is satisfied,

(ii) $\mathscr{B}$ includes all subsets of X of up to three elements,

then there is a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$; that is, $C^*(B,\succsim) = C(B)$ for all $B \in \mathscr{B}$. Furthermore, this rational preference relation is the only preference relation that does so.

2. Consumer Choice

2.1 Preference-based Approach

3. Classical Demand Theory

3.1 Preference Relation: Basic Properties

To start with, let's introduce some notations for convenience. Supposing that $x, y \in \mathbb{R}_+^L$, then x and y can have the following relations:

1. $y \gg x \iff y_i > x_i \ (i = 1, 2, \cdots, L)$;
2. $y \geqslant x \iff y_i \geqslant x_i \ (i = 1, 2, \cdots, L)$;
3. $y > x \iff y \geqslant x$ and $y \neq x$.

Given a preference relation $\succsim$ on consumption set X, these sets are common to meet:

1. upper contour set: $R(x) = \{y \in X | y \succsim x\}$;
2. lower contour set: $R^{-1}(x) = \{y \in X | x \succsim y\}$;
3. $P(x) = \{y \in X | y \succ x\}$;
4. $P^{-1}(x) = \{y \in X | x \succ y\}$.

We assume throughout that the preference relation $\succsim$ is rational in the sense introduced in Section 1.1, that is, $\succsim$ is complete and transitive.

Definition 3.1.1 — Local Nonsatiation. A preference relation $\succsim$ on X is *locally nonsatiated* if

$$\forall x \in X,\ \forall \varepsilon > 0,\ \exists y \in X,\ \|y - x\| \leq \varepsilon \text{ and } y \succ x$$

or equivalently

$$\forall x \in X,\ \forall \varepsilon > 0,\ B(x, \varepsilon) \cap P(x) \neq \varnothing.$$

Definition 3.1.2 — Monotonicity. A preference relation $\succsim$ on X is *monotone* if

$$\forall x, y \in X,\ y \gg x \implies y \succ x$$

or equivalently

$$\forall x \in X,\ \{y \in X | y \gg x\} \subset P(x).$$

Definition 3.1.3 — Strong Monotonicity. A preference relation $\succsim$ on X is *strongly monotone* if

$$\forall x, y \in X,\ y \geqslant x \text{ and } y \neq x \implies y \succ x$$

or equivalently

$$\forall x \in X,\ \{y \in X | y > x\} \subset P(x).$$

Proposition 3.1.1 Let $\succsim$ be a preference relationon on X.
1. If $\succsim$ is strongly monotone, then it is monotone.
2. If $\succsim$ is monotone, then it is locally nonsatiated.

Definition 3.1.4 — Convexity. A preference relation $\succsim$ on X is *convex* if for all $x, y, z \in X$,

$$y \succsim x \text{ and } z \succsim x \implies \forall \alpha \in [0,1],\ \alpha y + (1-\alpha) z \succsim x$$

or equivalently

$$\forall x \in X,\ R(x) \text{ is a convex set.}$$

Definition 3.1.5 — Strict Convexity. A preference relation $\succsim$ on X is *strictly convex* if for all $x, y, z \in X$,

$$y \succsim x \text{ and } z \succsim x \text{ and } y \neq z \implies \forall \alpha \in (0,1),\ \alpha y + (1-\alpha) z \succ x$$

or equivalently for all $x \in X$,

$$\forall y, z \in R(x),\ y \neq z \implies \forall \alpha \in (0,1),\ \alpha y + (1-\alpha) z \in P(x).$$

Proposition 3.1.2 Let $\succsim$ be a preference relationon on X. If $\succsim$ is strictly convex, then it is convex.

Definition 3.1.6 — Homothetic Preference. A monotone preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is *homothetic* if for all $x, y \in X$,

$$x \sim y \implies \forall \alpha \geqslant 0,\ \alpha x \sim \alpha y.$$

Definition 3.1.7 — Quasilinear Preference. The preference relation $\succsim$ on $X = (-\infty, +\infty) \times \mathbb{R}^{L-1}_+$ is *quasilinear* with respect to commodity 1 (called, in this case, the numeraire commodity) if

1. All the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \cdots, 0)$ and any $\alpha \in \mathbb{R}$.
2. Commodity 1 is desirable; that is, $x + \alpha e \succ x$ for all $x \in X$ and $a > 0$.

3.2 Preference and Utility

Definition 3.2.1 — Continuity. The preference relation $\succsim$ on X is *continuous* if it is preserved under limits, that is, for any sequence of pairs $\{(x_n, y_n)\}_{n=1}^{\infty}$ with $x_n \succsim y_n$ for all $n \in \mathbb{N}^*$,

$$\left.\begin{aligned} \lim_{n\to\infty} x_n = x \\ \lim_{n\to\infty} y_n = y \end{aligned}\right\} \implies x \succsim y,$$

or alternatively

$$\forall x \in X,\ R(x) \text{ and } R^{-1}(x) \text{ are closed set.}$$

Theorem 3.2.1 Suppose that the rational preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is continuous. Then there is a continuous utility function $u(x)$ that represents $\succsim$.

Definition 3.2.2 — Quasiconcavity. The utility function $u(x)$ is *quasiconcave* if

$$\forall x \in \mathbb{R}^L_+,\ \{y \in \mathbb{R}^L_+ \,|\, u(y) \geqslant u(x)\} \text{ is convex}$$

or alternatively

$$\forall x, y \in \mathbb{R}^L_+,\ \forall \alpha \in [0,1],\ u(\alpha x + (1-\alpha)y) \geqslant \min\{u(x), u(y)\}).$$

Proposition 3.2.2 The utility function $u(\cdot)$ is quasiconcave if and only if it represents a convex preference.

Definition 3.2.3 — Strict Quasiconcavity. The utility function $u(x)$ is *strictly quasiconcave* if

$$\forall x \in \mathbb{R}^L_+,\ \{y \in \mathbb{R}^L_+ \,|\, u(y) \geqslant u(x)\} \text{ is strictly convex}$$

or alternatively

$$\forall x, y \in \mathbb{R}^L_+,\ \forall \alpha \in (0,1),\ x \neq y \implies u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\}.$$

Proposition 3.2.3 The utility function $u(\cdot)$ is strictly quasiconcave if and only if it represents a strictly convex preference.

Proposition 3.2.4 If the utility function $u(\cdot)$ is strictly quasiconcave, then it is quasiconcave.

3.3 The Utility Maximization Problem

II

Part X

4. Text Chapter

4.1 Paragraphs of Text

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4.2 Citation

This statement requires citation [2]; this one is more specific [1, page 122].

4.3 Lists

Lists are useful to present information in a concise and/or ordered way[1].

4.3.1 Numbered List

1. The first item
2. The second item
3. The third item

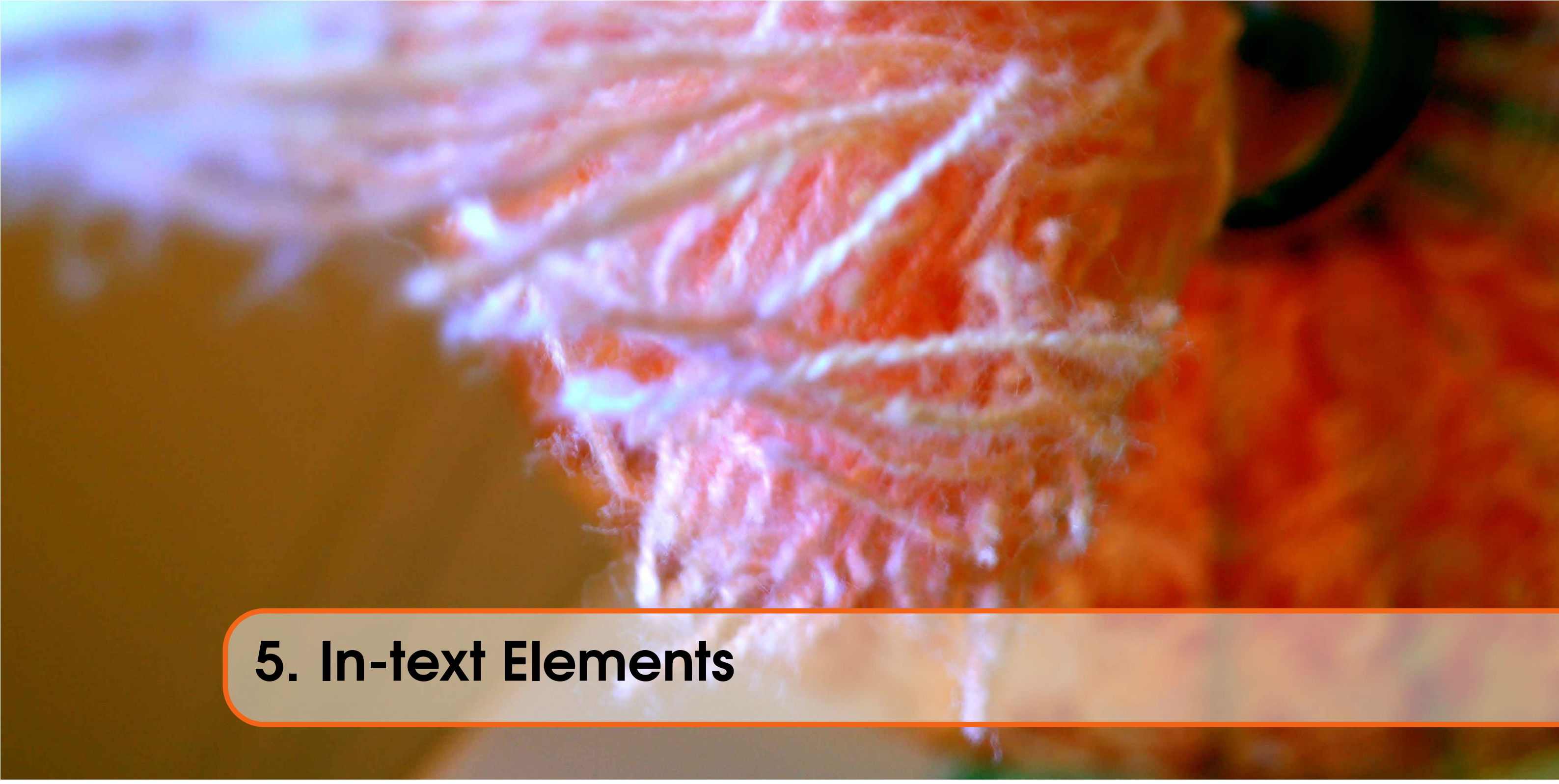
4.3.2 Bullet Points

- The first item
- The second item
- The third item

4.3.3 Descriptions and Definitions

Name Description
Word Definition
Comment Elaboration

[1] Footnote example...



5. In-text Elements

5.1 Theorems

This is an example of theorems.

5.1.1 Several equations

This is a theorem consisting of several equations.

Theorem 5.1.1 — Name of the theorem. In $E = \mathbb{R}^n$ all norms are equivalent. It has the properties:

$$\big| ||\mathbf{x}|| - ||\mathbf{y}|| \big| \leq ||\mathbf{x} - \mathbf{y}|| \tag{5.1}$$

$$||\sum_{i=1}^{n} \mathbf{x}_i|| \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{5.2}$$

5.1.2 Single Line

This is a theorem consisting of just one line.

Theorem 5.1.2 A set $\mathscr{D}(G)$ in dense in $L^2(G)$, $|\cdot|_0$.

5.2 Definitions

This is an example of a definition. A definition could be mathematical or it could define a concept.

Definition 5.2.1 — Definition name. Given a vector space E, a norm on E is an application, denoted $||\cdot||$, E in $\mathbb{R}^+ = [0, +\infty[$ such that:

$$||\mathbf{x}|| = 0 \Rightarrow \mathbf{x} = \mathbf{0} \tag{5.3}$$

$$||\lambda \mathbf{x}|| = |\lambda| \cdot ||\mathbf{x}|| \tag{5.4}$$

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \tag{5.5}$$

5.3 Notations

Notation 5.1. *Given an open subset G of $\mathbb{R}^n$, the set of functions φ are:*

1. *Bounded support G;*
2. *Infinitely differentiable;*

a vector space is denoted by $\mathscr{D}(G)$.

5.4 Remarks

This is an example of a remark.

R The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K} = \mathbb{R}$, however, established properties are easily extended to $\mathbb{K} = \mathbb{C}$.

5.5 Corollaries

This is an example of a corollary.

Corollary 5.5.1 — Corollary name. The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K} = \mathbb{R}$, however, established properties are easily extended to $\mathbb{K} = \mathbb{C}$.

5.6 Propositions

This is an example of propositions.

5.6.1 Several equations

Proposition 5.6.1 — Proposition name. It has the properties:

$$\left| ||\mathbf{x}|| - ||\mathbf{y}|| \right| \leq ||\mathbf{x} - \mathbf{y}|| \tag{5.6}$$

$$||\sum_{i=1}^{n} \mathbf{x}_i|| \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{5.7}$$

5.6.2 Single Line

Proposition 5.6.2 Let $f, g \in L^2(G)$; if $\forall \varphi \in \mathscr{D}(G)$, $(f, \varphi)_0 = (g, \varphi)_0$ then $f = g$.

5.7 Examples

This is an example of examples.

5.7.1 Equation and Text

■ **Example 5.1** Let $G = \{x \in \mathbb{R}^2 : |x| < 3\}$ and denoted by: $x^0 = (1,1)$; consider the function:

$$f(x) = \begin{cases} e^{|x|} & \text{si } |x - x^0| \leq 1/2 \\ 0 & \text{si } |x - x^0| > 1/2 \end{cases} \tag{5.8}$$

The function f has bounded support, we can take $A = \{x \in \mathbb{R}^2 : |x - x^0| \leq 1/2 + \varepsilon\}$ for all $\varepsilon \in]0; 5/2 - \sqrt{2}[$. ■

5.7.2 Paragraph of Text

■ **Example 5.2 — Example name.** Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris. ■

5.8 Exercises

This is an example of an exercise.

Exercise 5.1 This is a good place to ask a question to test learning progress or further cement ideas into students' minds. ■

5.9 Problems

Problem 5.1 What is the average airspeed velocity of an unladen swallow?

5.10 Vocabulary

Define a word to improve a students' vocabulary.
Vocabulary 5.1 — Word. Definition of word.

6. Presenting Information

6.1 Table

Treatments	Response 1	Response 2
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table 6.1: Table caption

6.2 Figure

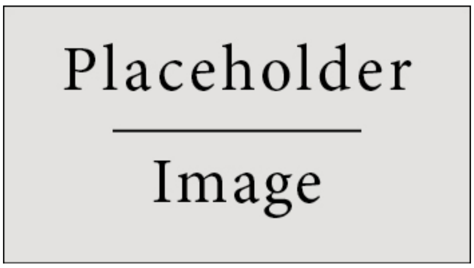
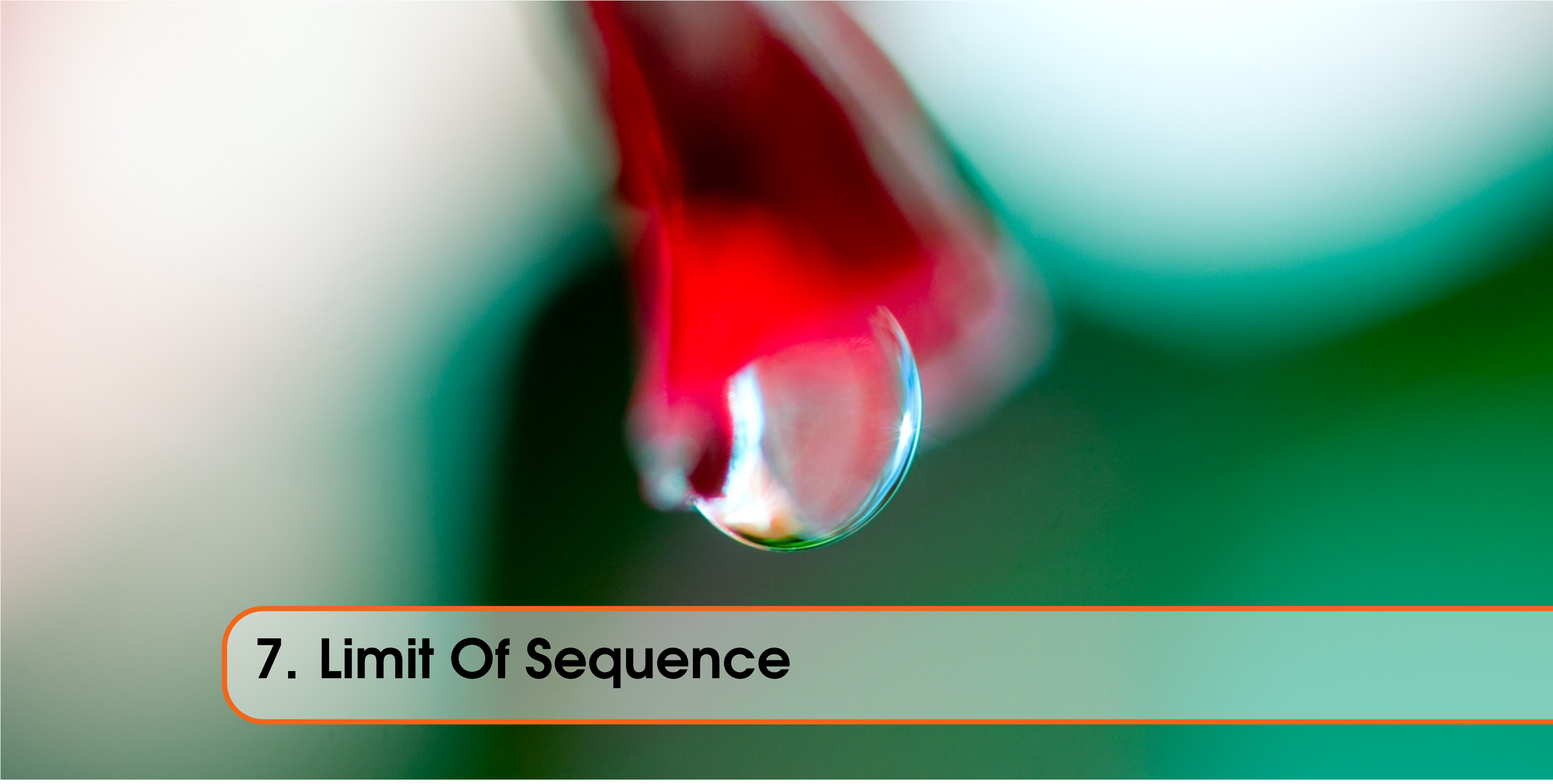


Figure 6.1: Figure caption

III Part Two



7. Limit Of Sequence

7.1 Cauchy proposition

Theorem 7.1.1 If a sequence $\{x_n\}$ converges to l, then its arithmetic mean of the preceding n terms also converges to l, namely

$$\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = \lim_{n\to\infty} x_n = l. \tag{7.1}$$

Proof. According to the condition $\lim\limits_{n\to\infty} x_n = l$, given $\varepsilon > 0$, there exists a positive number N such that $|x_n - l| < \varepsilon$ for all $n > N$. Assume $n > N$ and in this case we can make an estimation as follows:

$$\begin{aligned}\left|\frac{x_1+x_2+\cdots+x_n}{n} - l\right| &= \frac{|(x_1-l)+(x_2-l)+\cdots+(x_n-l)|}{n} \\ &\leqslant \frac{|(x_1-l)+\cdots+(x_N-l)|}{n} + \frac{|(x_{N+1}-l)+\cdots+(x_n-l)|}{n} \\ &< \frac{M}{n} + \frac{n-N}{n}\varepsilon,\end{aligned}$$

where $M = |(x_1-l)+\cdots+(x_N-l)|$ is a finite number. Thus we can see that if let

$$N_1 = \max\{N, \left[\frac{M}{\varepsilon}\right]\},$$

then for all $n > N_1$ it follows that

$$\left|\frac{x_1+x_2+\cdots+x_n}{n} - l\right| < 2\varepsilon.$$

It clearly implies $\lim\limits_{n\to\infty} \dfrac{x_1+x_2+\cdots+x_n}{n} = l$. ■

R

1. If x_n approaches positive(or negative) infinity, Cauchy proposition still holds. In fact, from $x_n \to +\infty (n \to \infty)$ we see x_n can be greater than any given positive number X when n is large enough. Similarly we can separate the arithmetic mean into two parts and show the second part $(1-N/n)X$ greater than $X/2$ for a sufficiently large n.

2. The converse of Cauchy proposition is generally not true. A trivial example is $x_n = (-1)^n$. Then

$$\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = 0$$

while x_n has no limit.

Corollary 7.1.2 If a positive term sequence $\{x_n\}$ converges to l, then its geometric mean of the preceding n terms also converges to l, namely

$$\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n} = \lim_{n\to\infty} x_n = l. \tag{7.2}$$

Proof. Applying the mean inequality we have

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} \leqslant \sqrt[n]{x_1 x_2 \cdots x_n} \leqslant \frac{x_1+x_2+\cdots+x_n}{n}.$$

Notice that $\lim\limits_{n\to\infty} x_n = l$ implies $\lim\limits_{n\to\infty} \frac{1}{x_n} = \frac{1}{l}$. From Theorem 4.1.1 we can see

$$\lim_{n\to\infty} \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} = \lim_{n\to\infty} \left(\frac{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}{n} \right)^{-1} = \left(\lim_{n\to\infty} \frac{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}{n} \right)^{-1} = l.$$

And it has been shown that

$$\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = l.$$

According to squeeze theorem, we can assert that the limit $\lim\limits_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n}$ exists and also equals l. ■

Proposition 7.1.3 If $x_n > 0 (n = 1, 2, \cdots)$ and the limit $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists, the limit $\lim\limits_{n\to\infty} \sqrt[n]{x_n}$ also exists and

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Proof. Assume $x_0 = 1$ and by Corollary 4.1.2 we immediately get

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\frac{x_1}{x_0}\frac{x_2}{x_1}\cdots\frac{x_n}{x_{n-1}}} = \lim_{n\to\infty} \frac{x_n}{x_{n-1}} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

■

■ **Example 7.1** Find the limit $\lim\limits_{n\to\infty} \frac{n}{\sqrt[n]{n!}}$. ■

7.2 Stolz–Cesàro theorem

Theorem 7.2.1 — Stolz–Cesàro theorem with form of $\frac{0}{0}$. Assume $\{a_n\}$ and $\{b_n\}$ are two infinitesimal sequences of real numbers and $\{a_n\}$ is a strictly decreasing. If

$$\lim_{n\to\infty}\frac{b_{n+1}-b_n}{a_{n+1}-a_n}=l,$$

where l is finite or $\pm\infty$, then we have

$$\lim_{n\to\infty}\frac{b_n}{a_n}=l.$$

Proof. Here we only prove it for a finite number l. According to the assumption, for every positive number ε, there exists another positive number N such that

$$\left|\frac{b_{n+1}-b_n}{a_{n+1}-a_n}-l\right|<\varepsilon$$

for all $n>N$. Since $a_n>a_{n+1}$ for all $n\in\mathbb{N}^*$, we have

$$(l-\varepsilon)(a_n-a_{n+1})<b_n-b_{n+1}<(l+\varepsilon)(a_n-a_{n+1}).$$

Given $m>n$, replace n by $n+1,n+2,\cdots,m-1$. Then add up the $m-n$ inequalities and we obtain

$$(l-\varepsilon)(a_n-a_m)<b_n-b_m<(l+\varepsilon)(a_n-a_m),$$

or

$$\left|\frac{b_n-b_m}{a_n-a_m}-l\right|<\varepsilon.$$

Note that $\lim\limits_{m\to\infty}a_m=\lim\limits_{m\to\infty}b_m=0$. When $m\to\infty$, for all $n>N$ it follows that

$$\left|\frac{b_n}{a_n}-l\right|\leqslant\varepsilon.$$

With the arbitrariness of selection of ε, this implies $\lim\limits_{n\to\infty}\frac{b_n}{a_n}=l$. ■

Theorem 7.2.2 — Stolz–Cesàro theorem with form of $\frac{*}{\infty}$. Asume $\{a_n\}$ is a strictly increasing sequence such that $\lim a_n=\infty$. If

$$\lim_{n\to\infty}\frac{b_{n+1}-b_n}{a_{n+1}-a_n},$$

where l is finite or $\pm\infty$, then we have

$$\lim_{n\to\infty}\frac{b_n}{a_n}=l.$$

Proof. We just consider a finite l. For every positive number ε, there exists $N_1\in\mathbb{N}^*$ such that

$$\left|\frac{b_{n+1}-b_n}{a_{n+1}-a_n}-l\right|<\varepsilon$$

for all $n > N_1$. Since $a_{n+1} > a_n$ for all $n \in \mathbb{N}^*$, we have

$$(l-\varepsilon)(a_{n+1}-a_n) < b_{n+1}-b_n < (l+\varepsilon)(a_{n+1}-a_n).$$

Given N_1, replace n by $N_1, N_1+1, \cdots, n-1$. Then add up the $n-N$ inequalities and we obtain

$$(l-\varepsilon)(a_n-a_{N_1}) < b_n-b_{N_1} < (l+\varepsilon)(a_n-a_{N_1}),$$

or

$$\left|\frac{b_n-b_{N_1}}{a_n-a_{N_1}}-l\right| < \varepsilon.$$

In order to estimate the value of $\left|\frac{b_n}{a_n}-l\right|$, consider the following identity

$$\frac{b_n}{a_n}-l = \left(1-\frac{a_{N_1}}{a_n}\right)\left(\frac{b_n-b_{N_1}}{a_n-a_{N_1}}-l\right)+\frac{b_n-la_{N_1}}{a_n}.$$

Since $\lim\limits_{n\to\infty} a_n = +\infty$, there exsits a positive number N_2 such that for all $n > N_2$

$$0 < \left|\frac{a_N}{a_n}\right| < 1 \Leftrightarrow 0 < 1-\frac{a_N}{a_n} < 2$$

and

$$\left|\frac{b_n-la_{N_1}}{a_n}\right| < \varepsilon.$$

Thus for all $n > \max\{N_1, N_2\}$ we have

$$\left|\frac{b_n}{a_n}-l\right| < 3\varepsilon,$$

which indicates $\lim\limits_{n\to\infty}\frac{b_n}{a_n} = l$.

■

7.3 Subsequence

Proposition 7.3.1 Every real sequence $\{a_n\}$ has a monotonic subsequence.

Proof. Define $X = \{n \in \mathbb{N}^* | \forall k \geq n, a_k \geq a_n\}$, which is a subset of the index set of the sequence $\{a_n\}$.
If X is an infinite set, we can find an arrangement of the elements in X: $n_1 < n_2 < \cdots < n_i < \cdots (n_i \in X)$. Thus we get an increasing subsequence of $\{a_n\}$: $a_{n_1}, a_{n_2}, \cdots, a_{n_i}, \cdots$.
If X is a finite set, denote $N = \max X$. For every $n > N$, there exists a number $k > n$ such that $a_k < a_n$. Let $n_1 = N+1$ and there exists a number $n_2 > n_1$ such that $a_{n_2} < a_{n_1}$. Since n_2 is also greater than N, there exists $n_3 > n_2$ such that $a_{n_3} < a_{n_2}$. Repeating this process will come to a decreasing subsequence of $\{a_n\}$. ■

8. Limit of function

8.1 Equivalent Infinitesimal

Definition 8.1.1 If the relation $f(x) = \gamma(x)g(x)$ holds ultimately over $\mathscr{B}$ where $\lim_{\mathscr{B}} \gamma(x) = 1$, we say that *the function f behaves asymptotically like g over $\mathscr{B}$*, or, more briefly, that *f is equivalent to g over $\mathscr{B}$*.

IV

Part N

Bibliography

Books

[Smi12] John Smith. *Book title*. 1st edition. Volume 3. 2. City: Publisher, Jan. 2012, pages 123–200 (cited on page 20).

Articles

[Smi13] James Smith. “Article title”. In: 14.6 (Mar. 2013), pages 1–8 (cited on page 20).

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