Notes of Microeconomic Theory

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Part One: Individual Decision Making

1. Preference and Choice

1.1 Preference-based Approach

Definition 1.1.1 — Preference Relation. Preference relation $\succsim$ is a binary relation defined on the set of alternatives X.

Definition 1.1.2 — Rational. The preference relation $\succsim$ is *rational* if it possesses the following two properties:

1. Completeness: $\forall x, y \in X,\ x \succsim y$ or $y \succsim x$.
2. Transitivity: $\forall x, y, z \in X,\ x \succsim y$ and $y \succsim z \implies x \succsim z$.

Definition 1.1.3 — Strict Preference Relation. Define the *strict preference relation* $\succ$ as follows:

$$x \succ y \iff x \succsim y \text{ but not } y \succsim x.$$

Definition 1.1.4 — Indeference Relation. Define the *indifference relation* $\sim$ as follows:

$$x \sim y \iff x \succsim y \text{ and } y \succsim x.$$

Definition 1.1.5 — Utility Function. A function $u : X \to \mathbb{R}$ is a *utility function representing preference relation* $\succsim$ if

$$\forall x, y \in X,\ x \succ y \iff u(x) \geq u(y).$$

Proposition 1.1.1 If there is a utility function that represents a preference relation $\succsim$, then $\succsim$ must be rational.

1.2 Choice-based Approach

Definition 1.2.1 — Choice Structure. A *choice structure* is a binary tuple $(\mathscr{B}, C(\cdot))$, where the set $\mathscr{B} \subset 2^X/\{\varnothing\}$ is a family of nonempty subsets of X and the mapping $C : \mathscr{B} \to \mathscr{B}$ assigns a nonempty set of chosen elements $C(B) \subset B$ for every budget set $B \in \mathscr{B}$.

Definition 1.2.2 — Revealed Preference Relation. Given a choice structure $(\mathscr{B}, C(\cdot))$ the *revealed preference relation* $\succsim^*$ is a binary relation on X defined by

$$x \succsim^* y \iff \exists B \in \mathscr{B},\ x, y \in B \text{ and } x \in C(B).$$

R For convenience we define a binary relation $\succ^*$ on X informally as follows

$$x \succ^* y \iff \exists B \in \mathscr{B},\ x, y \in B \text{ and } x \in C(B) \text{ and } y \notin C(B).$$

We will say "x is revealed preferred to y" if $x \succ^* y$.

Definition 1.2.3 — Weak Axiom of Revealed Preference. The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the *weak axiom of revealed preference* if the following property

$$B, B' \in \mathscr{B} \text{ and } x, y \in B \text{ and } x, y \in B' \text{ and } x \in C(B) \text{ and } y \in C(B') \implies x \in C(B')$$

or equivalently

$$\forall x, y \in X,\ x \succsim^* y \implies \text{not } y \succ^* x$$

holds.

1.3 The Relationship between Preference Relations and Choice Rules

1.3.1 Rationality Implies WARP

Definition 1.3.1 Define the correspondence $C^*(\cdot, \succsim) : \mathscr{B} \to 2^X$ as

$$C^*(B, \succsim) = \{x \in B \mid \forall y \in B, x \succsim y\}.$$

We say the preference $\succsim$ *generates* the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$ if $C^*(B, \succsim) \neq \varnothing$ for all $B \in \mathscr{B}$.

R If X is finite, then $C^*(B, \succsim)$ will be nonempty. From now on, we will consider only preferences $\succsim$ and families of budget sets $\mathscr{B}$ such that $C^*(B, \succsim)$ is nonempty for all $B \in \mathscr{B}$.

Proposition 1.3.1 — Rationality Implies WARP. Suppose that $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot, \succsim))$, satisfies the weak axiom of revealed preference.

1.3.2 WARP does NOT Implies Rationality

Definition 1.3.2 — Rationalization. Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ *rationalizes* $C(\cdot)$ relative to $\mathscr{B}$ if

$$\forall B \in \mathscr{B},\ C^*(B, \succsim) = C(B),$$

that is, if the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$ generated by $\succsim$ is identical with $(\mathscr{B}, C(\cdot))$.

Proposition 1.3.2 If $(\mathscr{B}, C(\cdot))$ is a choice structure such that
(i) the weak axiom is satisfied,
(ii) $\mathscr{B}$ includes all subsets of X of up to three elements,
then there is a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$; that is, $C^*(B, \succsim) = C(B)$ for all $B \in \mathscr{B}$. Furthermore, this rational preference relation is the only preference relation that does so.

2. Consumer Choice

2.1 Preference-based Approach

3. Classical Demand Theory

3.1 Preference Relation: Basic Properties

To start with, let's introduce some notations for convenience. Supposing that $x, y \in \mathbb{R}^L_+$, then x and y can have the following relations:

1. $y \gg x \iff y_i > x_i \ (i = 1, 2, \cdots, L)$;
2. $y \geqslant x \iff y_i \geqslant x_i \ (i = 1, 2, \cdots, L)$;
3. $y > x \iff y \geqslant x$ and $y \neq x$.

Given a preference relation $\succsim$ on consumption set X, these sets are common to meet:

1. upper contour set: $R(x) = \{y \in X | y \succsim x\}$;
2. lower contour set: $R^{-1}(x) = \{y \in X | x \succsim y\}$;
3. $P(x) = \{y \in X | y \succ x\}$;
4. $P^{-1}(x) = \{y \in X | x \succ y\}$.

We assume throughout that the preference relation $\succsim$ is rational in the sense introduced in Section 1.1, that is, $\succsim$ is complete and transitive.

Definition 3.1.1 — Local Nonsatiation. A preference relation $\succsim$ on X is *locally nonsatiated* if

$$\forall x \in X, \ \forall \varepsilon > 0, \ \exists y \in X, \ \|y - x\| \leq \varepsilon \text{ and } y \succ x$$

or equivalently

$$\forall x \in X, \ \forall \varepsilon > 0, \ B(x, \varepsilon) \cap P(x) \neq \varnothing.$$

Definition 3.1.2 — Monotonicity. A preference relation $\succsim$ on X is *monotone* if

$$\forall x, y \in X, \ y \gg x \implies y \succ x$$

or equivalently

$$\forall x \in X,\ \{y \in X | y \gg x\} \subset P(x).$$

Definition 3.1.3 — Strong Monotonicity. A preference relation $\succsim$ on X is *strongly monotone* if

$$\forall x, y \in X,\ y \geqslant x \text{ and } y \neq x \implies y \succ x$$

or equivalently

$$\forall x \in X,\ \{y \in X | y > x\} \subset P(x).$$

Proposition 3.1.1 Let $\succsim$ be a preference relationon on X.

1. If $\succsim$ is strongly monotone, then it is monotone.
2. If $\succsim$ is monotone, then it is locally nonsatiated.

Definition 3.1.4 — Convexity. A preference relation $\succsim$ on X is *convex* if for all $x, y, z \in X$,

$$y \succsim x \text{ and } z \succsim x \implies \forall \alpha \in [0,1],\ \alpha y + (1-\alpha)z \succsim x$$

or equivalently

$$\forall x \in X,\ R(x) \text{ is a convex set.}$$

Definition 3.1.5 — Strict Convexity. A preference relation $\succsim$ on X is *strictly convex* if for all $x, y, z \in X$,

$$y \succsim x \text{ and } z \succsim x \text{ and } y \neq z \implies \forall \alpha \in (0,1),\ \alpha y + (1-\alpha)z \succ x$$

or equivalently for all $x \in X$,

$$\forall y, z \in R(x),\ y \neq z \implies \forall \alpha \in (0,1),\ \alpha y + (1-\alpha)z \in P(x).$$

Proposition 3.1.2 Let $\succsim$ be a preference relationon on X. If $\succsim$ is strictly convex, then it is convex.

Definition 3.1.6 — Homothetic Preference. A monotone preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is *homothetic* if for all $x, y \in X$,

$$x \sim y \implies \forall \alpha \geqslant 0,\ \alpha x \sim \alpha y.$$

Definition 3.1.7 — Quasilinear Preference. The preference relation $\succsim$ on $X = (-\infty, +\infty) \times \mathbb{R}^{L-1}_+$ is *quasilinear* with respect to commodity 1 (called, in this case, the numeraire commodity) if

1. All the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \cdots, 0)$ and any $\alpha \in \mathbb{R}$.
2. Commodity 1 is desirable; that is, $x + \alpha e \succ x$ for all $x \in X$ and $a > 0$.

3.2 Preference and Utility

Definition 3.2.1 — Continuity. The preference relation $\succsim$ on X is *continuous* if it is preserved under limits, that is, for any sequence of pairs $\{(x_n, y_n)\}_{n=1}^{\infty}$ with $x_n \succsim y_n$ for all $n \in \mathbb{N}^*$,

$$\left.\begin{array}{l}\lim\limits_{n\to\infty} x_n = x \\ \lim\limits_{n\to\infty} y_n = y\end{array}\right\} \implies x \succsim y,$$

or alternatively

$$\forall x \in X,\ R(x) \text{ and } R^{-1}(x) \text{ are closed set.}$$

Theorem 3.2.1 Suppose that the rational preference relation $\succsim$ on $X = \mathbb{R}_+^L$ is continuous. Then there is a continuous utility function $u(x)$ that represents $\succsim$.

Definition 3.2.2 — Quasiconcavity. The utility function $u(x)$ is *quasiconcave* if

$$\forall x \in \mathbb{R}_+^L,\ \{y \in \mathbb{R}_+^L \,|\, u(y) \geqslant u(x)\} \text{ is convex}$$

or alternatively

$$\forall x, y \in \mathbb{R}_+^L,\ \forall \alpha \in [0,1],\ u(\alpha x + (1-\alpha)y) \geqslant \min\{u(x), u(y)\}.$$

Proposition 3.2.2 The utility function $u(\cdot)$ is quasiconcave if and only if it represents a convex preference.

Definition 3.2.3 — Strict Quasiconcavity. The utility function $u(x)$ is *strictly quasiconcave* if

$$\forall x \in \mathbb{R}_+^L,\ \{y \in \mathbb{R}_+^L \,|\, u(y) \geqslant u(x)\} \text{ is strictly convex}$$

or alternatively

$$\forall x, y \in \mathbb{R}_+^L,\ \forall \alpha \in (0,1),\ x \neq y \implies u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\}.$$

Proposition 3.2.3 The utility function $u(\cdot)$ is strictly quasiconcave if and only if it represents a strictly convex preference.

Proposition 3.2.4 If the utility function $u(\cdot)$ is strictly quasiconcave, then it is quasiconcave.

3.3 The Utility Maximization Problem

Assume there exists a rational, continuous and locally nonsatiated preference relation $\succsim$ on $X = \mathbb{R}_+^L$, which is represented by a continuous utility function $u(x)$.

The consumer's problem of choosing her most preferred consumption bundle given prices $p \gg 0$ and wealth level $w > 0$ can now be stated as the following utility maximization problem (UMP):

$$\max_{x\geq 0} \quad u(x)$$
$$\text{s.t.} \;\; p\cdot x \leq w.$$

Proposition 3.3.1 If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.

Definition 3.3.1 — Walrasian Demand Correspondence/Function. The rule that assigns the set of optimal consumption vectors in the UMP to each price wealth situation $(p,w) \gg 0$ is denoted by $x(p,w)$ and is known as the *Walrasian demand correspondence*. That is

$$x(\cdot,\cdot):\ (p,w) \mapsto \{x^* \in \mathbb{R}^L_+ | u(x^*) = \max_{x\geq 0} u(x) \text{ s.t. } p\cdot x \leq w\}$$

When $x(p,w)$ is single-valued for all (p,w), we refer to it as the *Walrasian demand function*.

Proposition 3.3.2 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p,w)$ possesses the following properties:

1. Homogeneity of degree zero in (p,w): $x(\alpha p, \alpha w) = x(p,w)$ for any p,w and scalar $\alpha > 0$.
2. Walras' law: $p\cdot x = w$ for all $x \in x(p,w)$.
3. Convexity/uniqueness:
   If $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p,w)$ is a convex set.
   If $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p,w)$ consists of a single element.

Definition 3.3.2 — Indirect Utility Function. For each $(p,w) \gg 0$, the utility value of the UMP is denoted $v(p,w) \in \mathbb{R}$. The function $v(p,w)$ is called the *indirect utility function*. It is equal to $u(x^*)$ for any $x^* \in x(p,w)$, which is

$$v(p,w) = u(x(p,w)) = \max_{x\geq 0} u(x) \text{ s.t. } p\cdot x \leq w.$$

Proposition 3.3.3 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The indirect utility function $v(p,w)$ is

1. Homogeneous of degree zero.
2. Strictly increasing in w and nonincreasing in p_l for any l.
3. Quasiconvex: the set $\{(p,w)|v(p,w) \leq \bar{v}\}$ is convex for any $\bar{v}$.
4. Continuous in p and w.

Theorem 3.3.4 — Kuhn Tucker Conditions in UMP. if $x^* \in x(p,w)$ is a solution to the UMP and $u(x)$ is continuously differentiable, then there exists a Lagrange multiplier

$\lambda \geq 0$ such that for all $l = 1, \cdots, L$:

$$\frac{\partial u(x^*)}{\partial x_l} \leq \lambda p_l, \quad \text{with equality if } x_l^* > 0,$$

or equivalently in matrix notation, where $\nabla u(x) = \left[\frac{\partial u(x)}{\partial x_l}, \frac{\partial u(x)}{\partial x_l}, \cdots, \frac{\partial u(x)}{\partial x_L}\right]^{\mathrm{T}}$,

$$\nabla u(x^*) \leq \lambda p,$$
$$x_l^* \left(\frac{\partial u(x^*)}{\partial x_l} - \lambda p_l\right) = 0.$$

The Lagrange multiplier λ in the first-order conditions gives the shadow value of relaxing the constraint in the UMP. It therefore equals the consumer's marginal utility value of wealth at the optimum. To see this directly, consider for simplicity the case where $x(p,w)$ is a differentiable function and $x(p,w) \gg 0$. Thus we have

$$D_w v(p,w) = D_w u(x(p,w)) = \nabla u(x(p,w)) \cdot D_w x(p,w) = \lambda p \cdot D_w x(p,w) = \lambda,$$

where the last equality follows because $p \cdot x(p,w) = w$ holds for all w and differentiate it with respect to w shows $p \cdot D_w x(p,w) = 1$.

3.4 The Expenditure Minimization Problem

In this section, we study the following expenditure minimization problem (EMP) for $p \gg 0$ and $u > u(0)$:

$$\min_{x \geq 0} \quad p \cdot x$$
$$\text{s.t.} \quad u(x) \geq u.$$

Definition 3.4.1 — Hicksian Demand Correspondence/Function. The set of optimal commodity vectors in the EMP is denoted $h(p,u)$ and is known as the Hicksian demand correspondence, or function if single-valued. That is

$$h(\cdot,\cdot): \ (p,u) \mapsto \{x^* \in \mathbb{R}_+^L \mid p \cdot x^* = \min_{x \geq 0} p \cdot x \ \text{ s.t. } u(x) \geq u\}$$

Proposition 3.4.1 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}_+^L$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p,u)$ possesses the following properties:

1. Homogeneity of degree zero in p: $h(\alpha p, u) = h(p,u)$ for any p, u and $\alpha > 0$.
2. No excess utility: For any $x \in h(p,u)$, $u(x) = u$.
3. Convexity/uniqueness: If $\succsim$ is convex, then $h(p,u)$ is a convex set; and if $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then there is a unique element in $h(p,u)$.

Definition 3.4.2 — Expenditure Function. Given prices $p \gg 0$ and required utility level $u > u(0)$, the value of the EMP is denoted $e(p,u)$. The function $e(p,u)$ is called the *expenditure function*. It is equal to $p \cdot h^*$ for any $h^* \in h(p,u)$, which is

$$e(p,u) = p \cdot h(p,u) = \min_{x \geq 0} p \cdot x \ \text{ s.t. } u(x) \geq u.$$

Proposition 3.4.2 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$:. The expenditure function $e(p,u)$ is

1. Homogeneous of degree one in p.
2. Strictly increasing in u and nondecreasing in p_l, for any l.
3. Concave in p.
4. Continuous in p and u.

Theorem 3.4.3 — Kuhn Tucker Conditions in EMP. if $x^* \in h(p,w)$ is a solution to the EMP and $u(x)$ is continuously differentiable, then there exists a Lagrange multiplier $\lambda \geq 0$ such that for all $l = 1, \cdots, L$:

$$p_l \geq \lambda \frac{\partial u(x^*)}{\partial x_l}, \quad \text{with equality if } x_l^* > 0,$$

or equivalently in matrix notation, where $\nabla u(x) = \left[\frac{\partial u(x)}{\partial x_l}, \frac{\partial u(x)}{\partial x_l}, \cdots, \frac{\partial u(x)}{\partial x_L}\right]^{\mathrm{T}}$,

$$p \geq \lambda \nabla u(x^*),$$
$$x_l^* \left(p_l - \lambda \frac{\partial u(x^*)}{\partial x_l}\right) = 0.$$

3.5 Relationships between Demand, Indirect Utility, and Expenditure Functions

Proposition 3.5.1 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p \gg 0$. We have

1. If x^* is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this EMP is exactly w. It follows that $\forall p \gg 0,\ w > 0,\ u > u(0)$,

$$h(p, v(p,w)) = x(p,w),$$
$$e(p, v(p,w)) = w.$$

2. If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximized utility level in this UMP is exactly u. It follows that $\forall p \gg 0,\ w > 0,\ u > u(0)$,

$$x(p, e(p,u)) = h(p,u),$$
$$v(p, e(p,u)) = u.$$

4. Aggregate Demand

5. Production

5.1 Production Set

A production vector (also known as an input-output, or netput, vector, or as a production plan) is a vector $y = (y_1, \cdots, y_L) \in \mathbb{R}^L$ that describes the (net) outputs of the L commodities from a production process. We adopt the convention that positive numbers denote outputs and negative numbers denote inputs. Some elements of a production vector may be zero; this just means that the process has no net output of that commodity.

Definition 5.1.1 — Production Set. The set of all production vectors that constitute feasible plans for the firm is known as the *production set* and is denoted by $Y \subset \mathbb{R}^L$.

Definition 5.1.2 — Transformation Function. It is somtimes convenient to describe the production set Y using a function $F(\cdot)$, called the *transformation function*. The transformation function $F(\cdot)$ has the property that $Y = \{y \in \mathbb{R}^L | F(y) \leq 0\}$ and $F(y) = 0$ if and only if y is an element of the boundary of Y. The set of boundary points of $Y = \{y \in \mathbb{R}^L | F(y) \leq 0\}$, is known as the *transformation frontier*.

Definition 5.1.3 — Marginal Rate of Transformation. If $F(\cdot)$ is differentiable, and if the production vector $\bar{y}$ satisfies $F(\bar{y}) = 0$, then for any commodities l and k, the ratio

$$MRT_{lk}(\bar{y}) = \frac{\frac{\partial F(\bar{y})}{\partial y_l}}{\frac{\partial F(\bar{y})}{\partial y_k}}$$

is called the marginal rate of transformation (MRT) of good l for good k at $\bar{y}$.

Definition 5.1.4 — Production Function. One of the most frequently encountered production models is that in which there is a single output. A single-output technology is commonly described by means of a *production function* $f(z)$ that gives the maximum amount q of output that can be produced using input amounts $(z_1, z_2, \cdots, z_{L-1}) \geq 0$. For example, if the output is good L, then (assuming that output can be disposed of at no cost) the production function $f(\cdot)$ gives rise to the production set:

$$Y = \{(-z_1, -z_2, \cdots, -z_{L-1}, q) \in \mathbb{R}^L \mid q - f(z_1, z_2, \cdots, z_{L-1}) \leq 0, (z_1, z_2, \cdots, z_{L-1}) \geq 0\}.$$

Definition 5.1.5 — Input Requirement Set. Given the production function $f(\cdot)$ and the required output q, the corresponding *input requirement set* is

$$V(q) = \{(z_1, z_2, \cdots, z_{L-1}) \in \mathbb{R}_+^{L-1} \mid f(z_1, z_2, \cdots, z_{L-1}) \geq q\}.$$

Proposition 5.1.1 If f is a production function and $V(q)$ is the corresponding input requirement set, then

$$f \text{ is quasi-concave} \iff V(q) \text{ is convex.}$$

Proposition 5.1.2 If f is a production function and Y is the corresponding production set, then

$$f \text{ is concave} \iff Y \text{ is convex.}$$

Definition 5.1.6 — Marginal Rate of Technical Substitution. Holding the level of output fixed, we can define the *marginal rate of technical substitution* (MRTS) of input l for input k at $\bar{z}$ as

$$MRTS_{lk}(\bar{z}) = \frac{\frac{\partial f(\bar{z})}{\partial z_l}}{\frac{\partial f(\bar{z})}{\partial z_k}} = -\frac{dz_k}{dz_l}.$$

Definition 5.1.7 — Elasticity of Substitution. Elasticity of substitution is the elasticity of the ratio of two inputs function with respect to the ratio of their marginal products, that is

$$\varepsilon_{lk} = \frac{d \ln \frac{z_k}{z_l}}{d \ln MRTS_{lk}} = \frac{d \ln \frac{z_k}{z_l}}{d \ln \frac{f_l(z_l)}{f_k(z_k)}}.$$

Definition 5.1.8 — Increasing Returns to Scale. The production process exhibits *nondecreasing returns to scale* if for any $y \in Y$, we have $\alpha y \in Y$ for any scale $\alpha > 1$.

The production function f exhibits *increasing returns to scale* (IRTS) if for all $z \in \mathbb{R}_+^{L-1}$ and $t > 1$ we have

$$f(tz) > tf(z).$$

Definition 5.1.9 — Decreasing Returns to Scale. The production process exhibits *nonincreasing returns to scale* if for any $y \in Y$, we have $\alpha y \in Y$ for any scale $0 \leq \alpha \leq 1$.

The production function f exhibits *decreasing returns to scale* (DRTS) if for all

$z \in \mathbb{R}_+^{L-1}$ and $t > 1$ we have

$$f(tz) < tf(z).$$

Definition 5.1.10 — Constant Returns to Scale. The production process exhibits *constant returns to scale* if for any $y \in Y$, we have $\alpha y \in Y$ for any scale $\alpha \geq 0$.

The production function f exhibits *constant returns to scale* (CRTS) if for all $z \in \mathbb{R}_+^{L-1}$ and $t \geq 0$ we have

$$f(tz) = tf(z).$$

Definition 5.1.11 — Homogeneity. The production function $f(z)$ is *homogeneous* of degree k if and only if $\forall t > 0$,

$$f(tz) = t^k f(z).$$

Proposition 5.1.3 If the production function $f(z)$ is homogeneous of degree k, then

- $k > 1 \iff$ IRTS
- $k < 1 \iff$ DRTS
- $k = 1 \iff$ CRTS

Proposition 5.1.4 If $f(z)$ is homogeneous of degree k, then $f_i(z) = \frac{\partial f(z)}{\partial z_i}$ is homogeneous of degree $k-1$.

Proposition 5.1.5 If $f(z)$ is homogeneous of degree k, then a ray from the origin cuts all isoquants at points with the same slope.

Proposition 5.1.6 If the production $f(z)$ is concave, it must exhibit decreasing returns to scale. However, the converse is not true.

Definition 5.1.12 — Elasticity of Scale. If we scale input vector by $t > 0$, output is scaled to $q(t) = f(tz)$. Elasticity of scale is

$$e(z) = \frac{t}{q(t)} \frac{dq(t)}{dt}\bigg|_{t=1}$$

Definition 5.1.13 — Local Returns to Scale. The production function $f(z)$ exhibits

- locally IRTS if $e(z) > 1$
- locally DRTS if $e(z) < 1$
- locally CRTS if $e(z) = 1$

Definition 5.1.14 — Homotheticity. $f(z)$ is *homothetic* if it can be written as $f(z) = g(h(z))$, where $g : R \to R$ is a strictly increasing function, and $h : R_+^L \to R_+$ is a function that is homogeneous of degree one.

5.2 Profit Maximization Problem

Given a price vector $p \gg 0$ and a production vector $y \in R^L$, the profit generated by implementing y is $p \cdot y$. By the sign convention, this is precisely the total revenue minus the total cost. Given the technological constraints represented by its production set Y, the firm's profit maximization problem (PMP) is then

$$\max_{y} \quad p \cdot y \qquad \text{s.t.} \quad y \in Y.$$

Using a transformation function $F(\cdot)$ to describe Y, we can equivalently state the PMP as

$$\max_{y} \quad p \cdot y \qquad \text{s.t.} \quad F(y) \leq 0.$$

Definition 5.2.1 — Supply Correspondence. The set of optimal production vectors in the PMP is denoted $y(p)$ and is known as the supply correspondence. That is

$$y(\cdot): \ p \mapsto \{y^* \in \mathbb{R}^L_+ | p \cdot y^* = \max_y p \cdot y \ \text{ s.t. } F(y) \leq 0\}.$$

Definition 5.2.2 — Profit Function. Given a production set Y, the firm's profit function $\pi(p)$ associates the value of the solution to the PMP to every p. That is

$$\pi(p) = p \cdot y(p) = \max_y p \cdot y \ \text{ s.t. } F(y) \leq 0.$$

Proposition 5.2.1 If the production set Y exhibits nondecreasing returns to scale, then either $\pi(p) \leq 0$ or $\pi(p) = +\infty$.

Theorem 5.2.2 — Kuhn Tucker Conditions in PMP. if $y^* \in y(p)$ is a solution to the PMP and $F(y)$ is continuously differentiable, then there exists a Lagrange multiplier $\lambda \geq 0$ such that for all $l = 1, \cdots, L$:

$$p_l = \lambda \frac{\partial F(y^*)}{\partial y_l}$$

or equivalently in matrix notation, where $\nabla F(x) = \left[\frac{\partial F(x)}{\partial y_l}, \frac{\partial F(x)}{\partial y_l}, \cdots, \frac{\partial F(x)}{\partial y_L}\right]^{\mathrm{T}}$,

$$p = \lambda \nabla F(y^*).$$

When Y corresponds to a single-output technology with differentiable production function $f(z)$, we can view the firm's decision as simply a choice over its input levels z. In this special case, we shall let the scalar $p > 0$ denote the price of the firm's output and the vector $w \gg 0$ denote its input prices. Then PMP can be stated as

$$\max_{z,q} \quad pq - w \cdot z \qquad \text{s.t.} \quad q \leq f(z), \ z \geq 0.$$

Theorem 5.2.3 — Kuhn Tucker Conditions in PMP. if (z^*, q^*) is a solution to the PMP and $f(z)$ is continuously differentiable, then for all $l = 1, \cdots, L$:

$$p\frac{\partial f(z^*)}{\partial y_l} \leq w_i, \text{ with equality if } z_l^* > 0,$$

or equivalently in matrix notation, where $\nabla f(z) = \left[\frac{\partial f(x)}{\partial z_l}, \frac{\partial f(z)}{\partial z_l}, \cdots, \frac{\partial f(z)}{\partial z_L}\right]^{\mathrm{T}}$,

$$p\nabla f(z^*) \leq w,$$
$$(p\nabla f(z^*) - w) \cdot z^* = 0.$$

II

Part X

6. Text Chapter

6.1 Paragraphs of Text

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6.2 Citation

This statement requires citation [**book_key**]; this one is more specific [**article_key**].

6.3 Lists

Lists are useful to present information in a concise and/or ordered way[1].

6.3.1 Numbered List

1. The first item
2. The second item
3. The third item

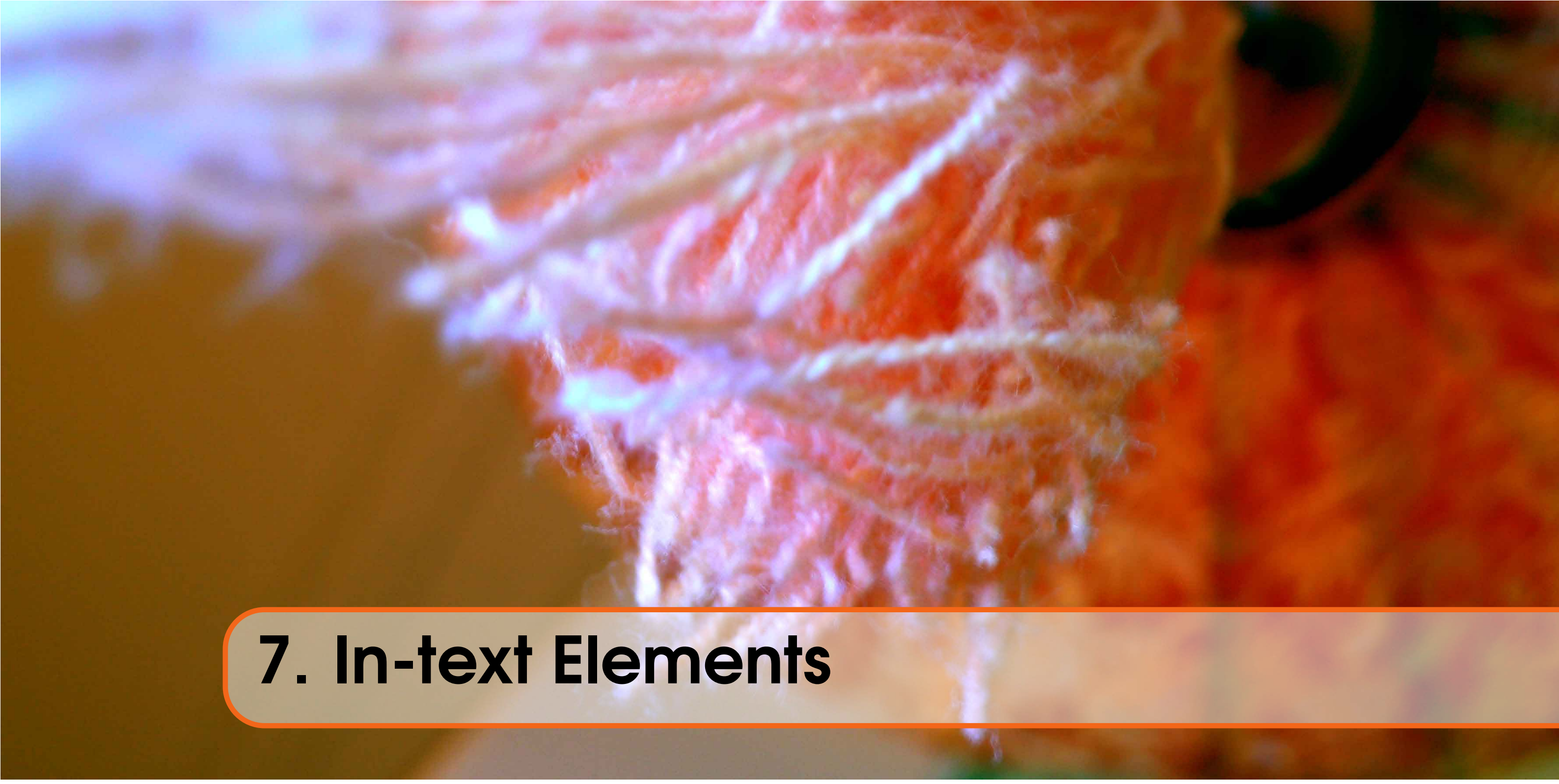
[1] Footnote example...

6.3.2 Bullet Points

- The first item
- The second item
- The third item

6.3.3 Descriptions and Definitions

Name Description
Word Definition
Comment Elaboration



7. In-text Elements

7.1 Theorems

This is an example of theorems.

7.1.1 Several equations

This is a theorem consisting of several equations.

Theorem 7.1.1 — Name of the theorem. In $E = \mathbb{R}^n$ all norms are equivalent. It has the properties:

$$\big| ||\mathbf{x}|| - ||\mathbf{y}|| \big| \leq ||\mathbf{x} - \mathbf{y}|| \tag{7.1}$$

$$|| \sum_{i=1}^{n} \mathbf{x}_i || \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{7.2}$$

7.1.2 Single Line

This is a theorem consisting of just one line.

Theorem 7.1.2 A set $\mathscr{D}(G)$ in dense in $L^2(G)$, $|\cdot|_0$.

7.2 Definitions

This is an example of a definition. A definition could be mathematical or it could define a concept.

Definition 7.2.1 — Definition name. Given a vector space E, a norm on E is an application, denoted $||\cdot||$, E in $\mathbb{R}^+ = [0, +\infty[$ such that:

$$||\mathbf{x}|| = 0 \ \Rightarrow \ \mathbf{x} = \mathbf{0} \tag{7.3}$$

$$||\lambda\mathbf{x}|| = |\lambda| \cdot ||\mathbf{x}|| \tag{7.4}$$

$$||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \tag{7.5}$$

7.3 Notations

Notation 7.1. *Given an open subset G of $\mathbb{R}^n$, the set of functions φ are:*

1. *Bounded support G;*
2. *Infinitely differentiable;*

a vector space is denoted by $\mathscr{D}(G)$.

7.4 Remarks

This is an example of a remark.

R The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K} = \mathbb{R}$, however, established properties are easily extended to $\mathbb{K} = \mathbb{C}$.

7.5 Corollaries

This is an example of a corollary.

Corollary 7.5.1 — Corollary name. The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K} = \mathbb{R}$, however, established properties are easily extended to $\mathbb{K} = \mathbb{C}$.

7.6 Propositions

This is an example of propositions.

7.6.1 Several equations

Proposition 7.6.1 — Proposition name. It has the properties:

$$\big| ||\mathbf{x}|| - ||\mathbf{y}|| \big| \leq ||\mathbf{x}-\mathbf{y}|| \tag{7.6}$$

$$||\sum_{i=1}^{n} \mathbf{x}_i|| \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{7.7}$$

7.6.2 Single Line

Proposition 7.6.2 Let $f, g \in L^2(G)$; if $\forall \varphi \in \mathscr{D}(G)$, $(f, \varphi)_0 = (g, \varphi)_0$ then $f = g$.

7.7 Examples

This is an example of examples.

7.7.1 Equation and Text

■ **Example 7.1** Let $G = \{x \in \mathbb{R}^2 : |x| < 3\}$ and denoted by: $x^0 = (1,1)$; consider the function:

$$f(x) = \begin{cases} e^{|x|} & \text{si } |x - x^0| \leq 1/2 \\ 0 & \text{si } |x - x^0| > 1/2 \end{cases} \tag{7.8}$$

The function f has bounded support, we can take $A = \{x \in \mathbb{R}^2 : |x - x^0| \leq 1/2 + \varepsilon\}$ for all $\varepsilon \in]0;5/2 - \sqrt{2}[$. ■

7.7.2 Paragraph of Text

■ **Example 7.2 — Example name.** Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

■

7.8 Exercises

This is an example of an exercise.

Exercise 7.1 This is a good place to ask a question to test learning progress or further cement ideas into students' minds. ■

7.9 Problems

Problem 7.1 What is the average airspeed velocity of an unladen swallow?

7.10 Vocabulary

Define a word to improve a students' vocabulary.
Vocabulary 7.1 — Word. Definition of word.

8. Presenting Information

8.1 Table

Treatments	**Response 1**	**Response 2**
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table 8.1: Table caption

8.2 Figure

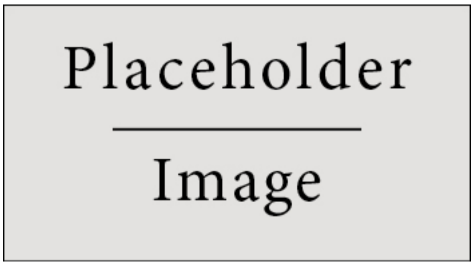


Figure 8.1: Figure caption

Part Two

9. Limit of function

9.1 Equivalent Infinitesimal

Definition 9.1.1 If the relation $f(x) = \gamma(x)g(x)$ holds ultimately over $\mathscr{B}$ where $\lim\limits_{\mathscr{B}} \gamma(x) = 1$, we say that *the function f behaves asymptotically like g over $\mathscr{B}$*, or, more briefly, that *f is equivalent to g over $\mathscr{B}$*.

IV

Part N

Bibliography

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