

STOCHASTIC PROCESS

1 Preliminaries

Definition 1.1 (stochastic process) For a given probability space (Ω, \mathcal{F}, P) and a measurable space (S, \mathcal{E}) , a *stochastic process* is a collection of S -valued random variables on (Ω, \mathcal{F}, P) indexed by some set T , which can be written as $X = \{X(t) : t \in T\}$ or $X = (X_t)_{t \in T}$ or $X : \Omega \times T \rightarrow S$. This mathematical space S is called its state space.

For convenience, we always assume T is a totally ordered set and denote the collection of all finite subsets of T by \mathcal{I}_T , namely

$$\mathcal{I}_T = \{\{t_1, t_2, \dots, t_n\} : t_1, \dots, t_n \in T, n \geq 1\}.$$

Definition 1.2 (independent increments) A stochastic process $(X_t)_{t \in T}$ has *independent increments* if for every $n \in \mathbb{N}_+$ and any $t_1 \leq t_2 \leq \dots \leq t_n$, the increments $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.

Definition 1.3 (independent increments) A stochastic process $(X_t)_{t \in T}$ has *independent increments* if for every $n \in \mathbb{N}_+$ and any $t_1 \leq t_2 \leq \dots \leq t_n$, the increments $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.

Definition 1.4 (strictly stationary process) Let $(X_t)_{t \in T}$ be a stochastic process and let $F_X(x_{t_1+\tau}, \dots, x_{t_k+\tau})$ represent the distribution function of the joint distribution of $(X_t)_{t \in T}$ at times $t_1 + \tau, \dots, t_k + \tau$. Then, $(X_t)_{t \in T}$ is said to be strictly stationary if, for all k , for all τ , and for all t_1, \dots, t_k ,

$$F_X(x_{t_1+\tau}, \dots, x_{t_k+\tau}) = F_X(x_{t_1}, \dots, x_{t_k}).$$

2 Poisson Process

Definition 2.1 (Poisson process (I)) A stochastic process $(N_t)_{t \geq 0}$ defined on a probability space (Ω, \mathcal{F}, P) is said to be a *Poisson process* with rate $\lambda > 0$ if

- (i) $N_0 = 0$;
- (ii) $(N_t)_{t \geq 0}$ has independent increments: for any $n \in \mathbb{N}_+$ and any $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$, the increment $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \dots, N_{t_n} - N_{t_{n-1}}$ are independent;
- (iii) for any $0 \leq s < t$, $N_t - N_s \sim \text{Pois}(\lambda(t-s))$, that is

$$P(N_t - N_s = k) = e^{-\lambda(t-s)} \frac{\lambda(t-s)^k}{k!} \quad (k = 0, 1, 2, \dots).$$

Definition 2.2 (counting process) A *counting process* is a stochastic process $(N_t)_{t \geq 0}$ with values that are non-negative, integer, and non-decreasing:

- (i) $N_t \geq 0$;
- (ii) N_t is an integer;
- (iii) If $0 \leq s \leq t$, then $N_s \leq N_t$.

For any $0 \leq s < t$, the counting process $N_t - N_s$ represents the number of events that occurred on $(s, t]$.

Definition 2.3 (Poisson process (II)) A counting process $(N_t)_{t \geq 0}$ defined on a probability space (Ω, \mathcal{F}, P) is said to be a *Poisson process* with rate $\lambda > 0$ if

- (i) $N_0 = 0$;
- (ii) $(N_t)_{t \geq 0}$ has independent increments;
- (iii) For all $t \geq 0$, $P(N_{t+h} - N_t = 1) = \lambda h + o(h)$ when $h \rightarrow 0$;
- (iv) For all $t \geq 0$, $P(N_{t+h} - N_t \geq 2) = o(h)$ when $h \rightarrow 0$;

Definition 2.4 (Poisson process (III)) A stochastic process $(N_t)_{t \geq 0}$ defined on a probability space (Ω, \mathcal{F}, P) is said to be a *Poisson process* with rate $\lambda > 0$ if

$$N_t = \sum_{n=1}^{\infty} n I_{[T_n, T_{n+1})}(t),$$

where $T_n = X_1 + X_2 + \cdots + X_n$ and X_i i.i.d $\sim \text{Exp}(\lambda)$ (Here the pdf of $\text{Exp}(\lambda)$ is taken as $\lambda e^{-\lambda x} I_{(0, +\infty)}(x)$).

Proposition 2.1 Definition 2.1, 2.3, 2.4 are equivalent definitions of Poisson process.

Proof.

- Definition 2.1 \implies Definition 2.3

Here we are only to show the implication of [Definition 2.3\(iii\)](#) and [Definition 2.3\(iv\)](#). Since $N_{t+h} - N_t \sim \text{Pois}(\lambda h)$, when $h \rightarrow 0$ we have

$$\begin{aligned} P(N_{t+h} - N_t = 1) &= e^{-\lambda h} \lambda h = (1 - \lambda h + o(h)) \lambda h = \lambda h + o(h), \\ P(N_{t+h} - N_t \geq 2) &= 1 - P(N_{t+h} - N_t = 0) - P(N_{t+h} - N_t = 1) \\ &= 1 - e^{-\lambda h} - e^{-\lambda h} \lambda h \\ &= 1 - (1 - \lambda h + o(h)) - (\lambda h + o(h)) \\ &= o(h). \end{aligned}$$

- Definition 2.3 \implies Definition 2.1

Only [Definition 2.1\(iii\)](#) needs to be derived. Given the Laplace transform of the nonnegative random variables N_t and N_{t+h}

$$L_{N_t}(u) = E[e^{-uN_t}], \quad L_{N_{t+h}}(u) = E[e^{-uN_{t+h}}], \quad u \geq 0,$$

according to Definition 2.3(ii) we can obtain

$$\begin{aligned} L_{N_{t+h}}(u) &= E[e^{-uN_{t+h}}] \\ &= E[e^{-uN_t} e^{-u(N_{t+h} - N_t)}] \\ &= E[e^{-uN_t}] E[e^{-u(N_{t+h} - N_t)}] \\ &= L_{N_t}(u) E[e^{-u(N_{t+h} - N_t)}]. \end{aligned}$$

Note that

$$\begin{aligned}
& \mathbb{E}[e^{-u(N_{t+h}-N_t)}] \\
&= e^0 \mathbb{P}(N_{t+h}-N_t=0) + e^{-u} \mathbb{P}(N_{t+h}-N_t=1) + \sum_{j=2}^{\infty} e^{-uj} \mathbb{P}(N_{t+h}-N_t=j) \\
&= 1 - \lambda h + o(h) + e^{-u}(\lambda h + o(h)) + o(h) \\
&= 1 - \lambda h + e^{-u} \lambda h + o(h) \quad (h \rightarrow 0).
\end{aligned}$$

Denote $g(t+h) = L_{N_{t+h}}(u)$ and $g(t) = L_{N_t}(u)$ for some fixed u and then we get

$$\frac{g(t+h) - g(t)}{h} = \frac{g(t)(1 - \lambda h + e^{-u} \lambda h + o(h)) - g(t)}{h} = g(t) \lambda (e^{-u} - 1) + \frac{o(h)}{h}.$$

Letting $h \rightarrow 0$ yields the differential equation

$$g'(t) = g(t) \lambda (e^{-u} - 1).$$

The initial condition $g(0) = \mathbb{E}[e^{-uN_0}] = 1$ determines a special solution of the equation

$$g(t) = L_{N_t}(u) = e^{\lambda t(e^{-u}-1)},$$

which coincides with the Laplace transform of Poisson distribution $Pois(\lambda t)$. Since Laplace transform uniquely determines the distribution, we can thus conclude $N_t \sim Pois(\lambda t)$. Given any $r \geq 0$, define a stochastic process $N'_t = N_{r+t} - N_r$ and we can check that $(N'_t)_{t \geq 0}$ is also a counting process satisfying all the conditions in Definition 2.3. Hence by repeating the proof above we can show $N'_t \sim Pois(\lambda t)$, which is equivalent to Definition 2.1(iii).

• Definition 2.1 \implies Definition 2.4

Let $T_n = \inf\{t \geq 0 : N_t = n\}$ for $n \in \mathbb{N}_+$. Note that given any $t \geq 0$, $\{N_t = n\} = \{T_n \leq t < T_{n+1}\}$. Thus we have

$$N_t = \sum_{n=1}^{\infty} n I_{N_t=n} = \sum_{n=1}^{\infty} n I_{[T_n, T_{n+1})}(t).$$

Let $X_1 = T_1, X_n = T_n - T_{n-1} (n \geq 2)$. Since $\mathbb{P}(X_1 > t) = \mathbb{P}(N_t = 0) = e^{-\lambda t}$, we see $X_1 \sim Exp(\lambda)$. Since

$$\mathbb{P}(X_2 > t | X_1 = t_1) = \mathbb{P}(X_2 > t | X_1 = t_1)$$

When $n \geq 2$, since

$$\begin{aligned}
& \mathbb{P}(X_n > t | X_{n-1} = t_{n-1}, \dots, X_1 = t_1) \\
&= \mathbb{P}(T_n - T_{n-1} > t | T_{n-1} - T_{n-2} = t_{n-1}, \dots, T_1 = t_1) \quad (\text{let } s_n = t_n + \dots + t_1) \\
&= \mathbb{P}(T_n > s_{n-1} + t | T_{n-1} = s_{n-1}, \dots, T_1 = s_1) \\
&= \mathbb{P}(N_{s_{n-1}+t} = n-1 | N_{s_{n-1}} = n-1) \quad (\text{memoryless property of } (N_t)) \\
&= \mathbb{P}(N_{s_{n-1}+t} - N_{s_{n-1}} = 0 | N_s = n-1) \\
&= \mathbb{P}(N_{s_{n-1}+t} - N_{s_{n-1}} = 0) \\
&= e^{-\lambda t},
\end{aligned}$$

it is plain to show that $\{X_i\}$ is sequence of independent random variable. Furthermore, we have

$$\mathbb{P}(X_n > t) = \mathbb{E}[\mathbb{P}(X_n > t | X_{n-1}, \dots, X_1)] = e^{-\lambda t},$$

which implies X_i i.i.d $\sim Exp(\lambda)$,

• Definition 2.4 \implies Definition 2.1

Clearly $N_0 = 0$ holds. Since $T_n = X_1 + X_2 + \dots + X_n$ and X_i i.i.d $\sim \text{Exp}(\lambda)$, we can deduce the jointly probability density function of (T_1, T_2, \dots, T_m)

$$\begin{aligned} f_S(y_1, y_2, \dots, y_m) &= f_X(y_1, y_2 - y_1, \dots, y_m - y_{m-1}) \left| \frac{\partial(x_1, \dots, x_m)}{\partial(y_1, \dots, y_m)} \right| \\ &= \lambda^m e^{-\lambda y_m} I_{\{0 \leq y_1 < \dots < y_m\}}. \end{aligned}$$

Thus for any $1 \leq j_1 < j_2 < \dots < j_n$, the jointly probability density function of $(T_{j_1}, T_{j_2}, \dots, T_{j_n})$ is

$$\frac{y_1^{j_1-1}}{(j_1-1)!} \frac{(y_2 - y_1)^{j_2-j_1-1}}{(j_2-j_1-1)!} \dots \frac{(y_n - y_{n-1})^{j_n-j_{n-1}-1}}{(j_n-j_{n-1}-1)!} \lambda^{j_n} e^{-\lambda y_n} I_{\{0 \leq y_1 < \dots < y_n\}}$$

Note

$$N_t = \sum_{n=1}^{\infty} n I_{[T_n, T_{n+1})}(t)$$

implies $\{N_t = n\} = \{T_n \leq t < T_{n+1}\}$. For any $n \in \mathbb{N}_+$ and any $0 \leq t_1 < t_2 < \dots < t_n$, we have

$$\begin{aligned} &P(N_{t_1} = j_1, N_{t_1} - N_{t_2} = j_2, \dots, N_{t_n} - N_{t_{n-1}} = j_n) \\ &= P(N_{t_1} = j_1, N_{t_1} = j_1 + j_2, \dots, N_{t_n} = j_1 + \dots + j_n) \quad (\text{let } k_n = j_1 + \dots + j_n) \\ &= P(T_{k_1} \leq t_1, T_{k_1+1} > t_1, T_{k_2} \leq t_2, T_{k_2+1} > t_2, \dots, T_{k_n} \leq t_n, T_{k_n+1} > t_n) \\ &= \int_{y_{k_1} \leq t_1 < y_{k_1+1} < \dots < y_{k_n} \leq t_n < y_{k_n+1}} \frac{y_{k_1}^{k_1-1}}{(k_1-1)!} \dots \frac{(y_{k_n} - y_{k_{n-1}+1})^{k_n-k_{n-1}-2}}{(k_n-k_{n-1}-2)!} \lambda^{k_n+1} e^{-\lambda y_{k_n+1}} dy_1 \dots dy_{k_n+1} \\ &= \int_{y_{k_1} \leq t_1 < y_{k_1+1} < \dots < y_{k_n} \leq t_n} \frac{y_{k_1}^{k_1-1}}{(k_1-1)!} \dots \frac{(y_{k_n} - y_{k_{n-1}+1})^{k_n-k_{n-1}-2}}{(k_n-k_{n-1}-2)!} \lambda^{k_n} dy_{k_1} \dots dy_{k_n} \int_{t_n}^{\infty} -de^{-\lambda y_{k_n+1}} \\ &= \int_{y_{k_1} \leq t_1 < y_{k_1+1} < \dots < t_{n-1} < y_{k_{n-1}+1} < t_n} \frac{y_{k_1}^{k_1-1}}{(k_1-1)!} \dots \frac{(y_{k_{n-1}} - y_{k_{n-2}+1})^{k_{n-1}-k_{n-2}-2}}{(k_n-k_{n-1}-2)!} \lambda^{k_n} e^{-\lambda t_n} dy_1 \dots dy_{k_{n-1}+1} \int_{y_{k_{n-1}+1}}^{t_n} d \frac{(y_{k_n} - y_{k_{n-1}+1})^{k_n-k_{n-1}-1}}{(k_n-k_{n-1}-1)!} \\ &= \int_{y_{k_1} \leq t_1 < y_{k_1+1} < \dots < t_{n-1} < y_{k_{n-1}+1} < t_n} \frac{y_{k_1}^{k_1-1}}{(k_1-1)!} \dots \frac{(y_{k_{n-1}} - y_{k_{n-2}+1})^{k_{n-1}-k_{n-2}-2}}{(k_n-k_{n-1}-2)!} \lambda^{k_n} e^{-\lambda t_n} dy_1 \dots dy_{k_{n-1}} \int_{t_{n-1}}^{t_n} -d \frac{(t_n - y_{k_{n-1}})^{k_n-k_{n-1}}}{(k_n-k_{n-1})!} \\ &= \int_{y_{k_1} \leq t_1 < y_{k_1+1} < \dots < t_{n-1} < y_{k_{n-1}+1} < t_n} \frac{y_{k_1}^{k_1-1}}{(k_1-1)!} \dots \frac{(y_{k_{n-1}} - y_{k_{n-2}+1})^{k_{n-1}-k_{n-2}-2}}{(k_n-k_{n-1}-2)!} \lambda^{k_n} e^{-\lambda t_n} dy_1 \dots dy_{k_{n-1}} \frac{(t_n - t_{n-1})^{k_n-k_{n-1}}}{(k_n-k_{n-1})!} \\ &= \dots \\ &= \lambda^{k_n} e^{-\lambda t_n} \frac{t_1^{k_1}}{k_1!} \frac{(t_2 - t_1)^{k_2-k_1}}{(k_2-k_1)!} \dots \frac{(t_n - t_{n-1})^{k_n-k_{n-1}}}{(k_n-k_{n-1})!} \\ &= e^{-\lambda t_1} \frac{(\lambda t_1)^{j_1}}{j_1!} e^{-\lambda(t_2-t_1)} \frac{(\lambda(t_2-t_1))^{j_2}}{j_2!} \dots e^{-\lambda(t_n-t_{n-1})} \frac{(\lambda(t_n-t_{n-1}))^{j_n}}{j_n!} \end{aligned}$$

Therefore, we conclude $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \dots, N_{t_n} - N_{t_{n-1}}$ are independent and for any $0 \leq s < t$, $N_t - N_s \sim \text{Pois}(\lambda(t-s))$.

□

Proposition 2.2 Let $(N_t)_{t \geq 0}$ be a Poisson process.

1. $N_t \sim \text{Pois}(\lambda t)$, $E[N_t] = \text{Var}(N_t) = \lambda t$.
2. For $0 \leq s \leq t$, $E[N_t N_s] = \lambda^2 ts + \lambda s$, $\text{Cov}(N_t, N_s) = \lambda s$.
3. For $0 \leq s \leq t$, $E[N_t | N_s] = N_s + \lambda(t - s)$. So Poisson process is a submartingale.
4. Poisson process is a Markov process. For $0 \leq t_1 < t_2 < \dots < t_n$ and $0 \leq k_1 \leq k_2 \leq \dots \leq k_n$,

$$\begin{aligned} & P(N_{t_n} = k_n | N_{t_{n-1}} = k_{n-1}, \dots, N_{t_1} = k_1) \\ &= P(N_{t_n} = k_n | N_{t_{n-1}} = k_{n-1}) \\ &= P(N_{t_n} - N_{t_{n-1}} = k_n - k_{n-1}) \\ &= e^{-\lambda(t_n - t_{n-1})} \frac{(\lambda(t_n - t_{n-1}))^{k_n - k_{n-1}}}{(k_n - k_{n-1})!}. \end{aligned}$$

Proof. Apply [Definition 2.1\(ii\)](#) and it is straightforward to show the properties. □

3 Compound Poisson Process

Definition 3.1 (compound Poisson distribution) Suppose that $N \sim \text{Pois}(\lambda)$ and that Z_1, Z_2, Z_3, \dots are i.i.d. random variables independent of N with a probability measure $v(dy)$ on \mathbb{R} . Then the probability distribution of the sum of N i.i.d. random variables

$$Y = \sum_{n=1}^N Z_n$$

is a *compound Poisson distribution*.

Definition 3.2 (compound Poisson process) A *compound Poisson process*, parameterised by a rate $\lambda > 0$ and jump size distribution $v(dy)$, is a process $(Y_t)_{t \geq 0}$ given by

$$Y_t = \sum_{n=1}^{N_t} Z_n,$$

where $(N_t)_{t \geq 0}$ is a Poisson process with rate λ , and $(Z_n)_{n \in \mathbb{N}_+}$ are independent and identically distributed random variables with distribution $v(dy)$, which are also independent of $(N_t)_{t \geq 0}$.

Proposition 3.1 Let $(Y_t)_{t \geq 0}$ be a compound Poisson process with a rate λ and jump $(Z_n)_{n \in \mathbb{N}_+}$. For convenience, assume $Z_n \stackrel{d}{=} Z$ and $E[Z^2] < +\infty$.

1. $E[Y_t] = \lambda t E[Z]$.
2. $\text{Var}(Y_t) = \lambda t E[Z^2]$.
3. The moment generating function $M_{Y_t}(a) = E[e^{aY_t}] = e^{\lambda t(E[e^{aZ}] - 1)} = e^{\lambda t(M_Z(a) - 1)}$

Proof.

1. Since Z_n is independent of N_t , we have

$$E[Y_t] = E[E[Y_t | N_t]] = E\left[E\left[\sum_{n=1}^{N_t} Z_n \middle| N_t\right]\right] = E\left[\sum_{n=1}^{N_t} E[Z_n | N_t]\right] = E[N_t Z] = E[N_t]E[Z] = \lambda t E[Z].$$

2. Since Z_n is independent of N_t , by the law of total variance $\text{Var}(Y_t)$ can be calculated as

$$\begin{aligned}\text{Var}(Y_t) &= \mathbb{E}[\text{Var}(Y_t|N_t)] + \text{Var}(\mathbb{E}[Y_t|N_t]) \\ &= \mathbb{E}[N_t \text{Var}(Z)] + \text{Var}(N_t \mathbb{E}[Z]) \\ &= \text{Var}(Z) \mathbb{E}[N_t] + \mathbb{E}[Z]^2 \text{Var}(N_t) \\ &= \lambda t \text{Var}(Z) + \lambda t \mathbb{E}[Z]^2 \\ &= \lambda t \mathbb{E}[Z^2].\end{aligned}$$

3. Make similar use of the dependence of $(Z_n)_{n \in \mathbb{N}_+}$ and N_t to get

$$\begin{aligned}\mathbb{E}[e^{aY_t}] &= \mathbb{E}\left[e^{a(Z_1+Z_2+\dots+Z_{N_t})}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{a(Z_1+Z_2+\dots+Z_{N_t})} \middle| N_t\right]\right] \\ &= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{a(Z_1+Z_2+\dots+Z_{N_t})} \middle| N_t = n\right] \mathbb{P}(N_t = n) \\ &= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{a(Z_1+Z_2+\dots+Z_n)} \middle| N_t = n\right] e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{aZ_1} e^{aZ_2} \dots e^{aZ_n}\right] e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{aZ}\right]^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t \mathbb{E}[e^{aZ}])^n}{n!} \\ &= e^{\lambda t (\mathbb{E}[e^{aZ}] - 1)}.\end{aligned}$$

□

Let $Y = (Y_t)_{t \geq 0}$ be a compound Poisson process with a rate λ and jump $(Z_n)_{n \in \mathbb{N}_+}$. Let $T_n = \inf\{t \geq 0 : N_t = n\}$ be the time when the n th event happens. Then the Itô integral of a stochastic process K with respect to Y is

$$\int_0^t K dY = \sum_{n=0}^{N_t} K_{T_n} (Y_{T_n} - Y_{T_n-}) = \sum_{n=0}^{N_t} K_{T_n} Z_n.$$

4 Markov Chain

4.1 Discrete-time Markov Chain

Definition 4.1 (discrete-time Markov chain) A *discrete-time Markov chain* on a countable state space S is a sequence of random variables X_1, X_2, X_3, \dots with the Markov property, namely that $\forall n \geq 1, \forall j, i_1, \dots, i_n \in S$,

$$\mathbb{P}(X_{n+1} = j \mid X_1 = i_1, X_2 = i_2, \dots, X_n = i_n) = \mathbb{P}(X_{n+1} = j \mid X_n = i_n),$$

if both conditional probabilities are well defined, i.e., if $\mathbb{P}(X_1 = i_1, \dots, X_n = i_n) > 0$.

Markov property may be interpreted as stating that the probability of moving to the next state depends only on the present state and not on the previous states.

A discrete-time Markov chain $(X_n)_{n \geq 1}$ is *time-homogeneous* if

$$P(X_{n+2} = j \mid X_{n+1} = i) = P(X_{n+1} = j \mid X_n = i)$$

for all $n \geq 1$ and all $i, j \in S$. We only focus on a discrete-time time-homogeneous Markov chain on a countable space if nothing is specified.

One can consider the *transition matrix* $P = (p_{ij})_{i,j \in S}$, where

$$p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

is called *one-step transition probability*. By the law of total probability it is clear to see

$$\sum_{j \in S} p_{ij} = 1, \quad \forall i \in S.$$

Similarly we can define *n-step transition matrix* $P^{(n)} = (p_{ij}^{(n)})_{i,j \in S}$, where *n-step transition probabilities* $p_{ij}^{(n)}$ is the probability that a process in state i will be in state j after n additional transitions. That is,

$$p_{ij}^{(n)} = P(X_{k+n} = j \mid X_k = i).$$

Proposition 4.1 (Chapman–Kolmogorov equation) Let $(X_n)_{n \geq 1}$ be a discrete-time Markov chain on a countable state space S . The *Chapman–Kolmogorov equation* states that

$$p_{ij}^{(n+m)} = \sum_{k \in S} p_{ik}^{(n)} p_{kj}^{(m)}, \quad \forall i, j \in S,$$

or alternatively

$$P^{(n+m)} = P^{(n)} P^{(m)}.$$

Of course $P^{(1)} = P$. Thus by iteration we show $P^{(n)}$ coincides with P^n .

4.2 Continuous-time Markov Chain

Definition 4.2 (continuous-time Markov chain) A *continuous-time Markov chain* on a countable state space S is a stochastic $(X_t)_{t \geq 0}$ with the Markov property: for all $n \geq 0$, all $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$, and all $j, i_0, \dots, i_n \in S$,

$$P(X_{t_{n+1}} = j \mid X_{t_0} = i_0, X_{t_1} = i_1, \dots, X_{t_n} = i_n) = P(X_{t_{n+1}} = j \mid X_{t_n} = i_n),$$

if both conditional probabilities are well defined, i.e., if $P(X_1 = i_1, \dots, X_n = i_n) > 0$.

A continuous-time Markov chain $(X_t)_{t \geq 0}$ is *time-homogeneous* if

$$P(X_{s+t} = j \mid X_s = i) = P(X_t = j \mid X_0 = i) = p_{ij}(t)$$

for all $s, t \geq 0$ and all $i, j \in S$. We only focus on a continuous-time time-homogeneous Markov chain on a countable space if nothing is specified.

5 Martingale

6 Brownian Motion

Definition 6.1 A stochastic process $(B_t)_{t \geq 0}$ is called a *Brownian motion* if

1. $B_0 = 0$.
2. $(B_t)_{t \geq 0}$ has continuous path, that is $t \mapsto B_t$ is almost surely continuous.
3. $(B_t)_{t \geq 0}$ has independent and stationary increments.
4. For $t > 0$, $B_t \sim N(0, t)$.

Appendix

Distribution	pmf $P(X = k)$	Support	Mean	Variance
Bernoulli $B(1, p)$	$p^k(1 - p)^{1-k}$	$\{0, 1\}$	p	$p(1 - p)$
Binomial $B(n, p)$	$\binom{n}{k} p^k(1 - p)^{n-k}$	$\{0, \dots, n\}$	np	$np(1 - p)$
Negative Binomial $NB(r, p)$	$\binom{k+r-1}{k} (1 - p)^r p^k$	\mathbb{N}	$\frac{pr}{1 - p}$	$\frac{pr}{(1 - p)^2}$
Poisson $Pois(\lambda)$	$\frac{\lambda^k e^{-\lambda}}{k!}$	\mathbb{N}	λ	λ
Geometric $Geo(p)$	$(1 - p)^{k-1} p$	\mathbb{N}_+	$\frac{1}{p}$	$\frac{1 - p}{p^2}$
Hypergeometric $H(N, K, n)$	$\frac{\binom{K}{k} \binom{N-K}{n-k}}{\binom{N}{n}}$	$\{0, \dots, \min(n, K)\}$	$n \frac{K}{N}$	$n \frac{K}{N} \frac{N - K}{N} \frac{N - n}{N - 1}$
Uniform (discrete) $DU(a, b)$	$\frac{1}{n}$	$\{a, a + 1, \dots, b\}$	$\frac{a + b}{2}$	$\frac{(b - a + 1)^2 - 1}{12}$

Distribution	pdf	Mean	Variance
Degenerate δ_a	$I_{\{a\}}(x)$	a	0
Uniform (continuous) $U(a, b)$	$\frac{1}{b - a} I_{[a, b]}(x)$	$\frac{a + b}{2}$	$\frac{(b - a)^2}{12}$
Exponential $Exp(\lambda) = \Gamma(1, \lambda)$	$\lambda e^{-\lambda x} I_{[0, +\infty)}(x)$	λ^{-1}	λ^{-2}
Normal $N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Gamma $\Gamma(\alpha, \beta)$	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} I_{(0, +\infty)}(x)$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Chi-squared $\chi_k^2 = \Gamma(\frac{k}{2}, \frac{1}{2})$	$\frac{1}{2^{k/2} \Gamma(k/2)} x^{k/2-1} e^{-x/2} I_{(0, +\infty)}(x)$	k	$2k$
Student's t t_ν	$\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi} \Gamma(\frac{\nu}{2})} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$	0	$\frac{\nu}{\nu - 2}$ for $\nu > 2$

Distribution	Moment-generating function	Characteristic function
Degenerate δ_a	e^{ta}	e^{ita}
Bernoulli $B(1, p)$	$1 - p + pe^t$	$1 - p + pe^{it}$
Geometric $Geo(p)$	$\frac{pe^t}{1 - (1 - p)e^t},$ $t < -\ln(1 - p)$	$\frac{pe^{it}}{1 - (1 - p)e^{it}}$
Binomial $B(n, p)$	$(1 - p + pe^t)^n$	$(1 - p + pe^{it})^n$
Negative Binomial $NB(r, p)$	$\frac{(1 - p)^r}{(1 - pe^t)^r}$	$\frac{(1 - p)^r}{(1 - pe^{it})^r}$
Poisson $Pois(\lambda)$	$e^{\lambda(e^t - 1)}$	$e^{\lambda(e^{it} - 1)}$
Uniform (continuous) $U(a, b)$	$\begin{cases} \frac{e^{tb} - e^{ta}}{t(b - a)} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$	$\begin{cases} \frac{e^{itb} - e^{ita}}{it(b - a)} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$
Uniform (discrete) $DU(a, b)$	$\frac{e^{at} - e^{(b+1)t}}{(b - a + 1)(1 - e^t)}$	$\frac{e^{it\mu}}{(b - a + 1)(1 - e^{it})}$
Laplace $L(\mu, b)$	$\frac{e^{t\mu}}{1 - b^2 t^2}, t < 1/b$	$\frac{e^{it\mu}}{1 + b^2 t^2}$
Normal $N(\mu, \sigma^2)$	$e^{t\mu + \frac{1}{2}\sigma^2 t^2}$	$e^{it\mu - \frac{1}{2}\sigma^2 t^2}$
Chi-squared χ_k^2	$(1 - 2t)^{-\frac{k}{2}}$	$(1 - 2it)^{-\frac{k}{2}}$
Noncentral chi-squared $\chi_k^2(\lambda)$	$e^{\lambda t/(1-2t)}(1 - 2t)^{-\frac{k}{2}}$	$e^{i\lambda t/(1-2it)}(1 - 2it)^{-\frac{k}{2}}$
Gamma $\Gamma(\alpha, \beta)$	$\left(1 - \frac{t}{\beta}\right)^{-\alpha}, t < \beta$	$\left(1 - \frac{it}{\beta}\right)^{-\alpha}$
Exponential $Exp(\lambda)$	$\frac{\lambda}{\lambda - t}, t < \lambda$	$\frac{\lambda}{\lambda - it}$
Multivariate normal $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	$e^{\mathbf{t}^T(\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{\Sigma}\mathbf{t})}$	$e^{\mathbf{t}^T(i\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{\Sigma}\mathbf{t})}$
Cauchy $Cauchy(\mu, \theta)$	Does not exist	$e^{it\mu - \theta t }$
Multivariate Cauchy $MultiCauchy(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	Does not exist	$e^{i\mathbf{t}^T\boldsymbol{\mu} - \sqrt{\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}}}$