1 Basic notation

Definition 1.1 (Independent increments) A stochastic process$(X_t)_{t\in T}$ has *independent increments* if for every $n \in \mathbb{N}_+$ and any $t_1 \leq t_2 \leq \cdots \leq t_n$, the increment $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent;

2 Poisson process

Definition 2.1 (Poisson process (I)) A stochastic process $(N_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is said to be a *Poisson process* with rate $\lambda > 0$ if

1. $N_0 = 0$;
2. $(N_t)_{t\geq 0}$ has independent increments: for any $n \in \mathbb{N}_+$ and any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the increment $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \cdots, N_{t_n} - N_{t_{n-1}}$ are independent;
3. for any $0 \leq s \leq t$, $N_t - N_s \sim \mathrm{Pois}(\lambda(t-s))$, that is

$$\mathrm{P}(N_t - N_s = k) = \mathrm{e}^{-\lambda(t-s)} \frac{\lambda(t-s)^k}{k!} \quad (k = 0, 1, 2, \cdots).$$

Definition 2.2 (Counting process) A *counting process* is a stochastic process $(N_t)_{t\geq 0}$ with values that are non-negative, integer, and non-decreasing:

1. $N_0 \geq 0$;
2. N_t is an integer;
3. If $0 \leq s \leq t$, then $N_s \leq N_t$.

For any $0 \leq s < t$, the counting process $N_t - N_s$ represents the number of events that occurred on $(s, t]$.

Definition 2.3 (Poisson process (II)) A counting process $(N_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is said to be a *Poisson process* with rate $\lambda > 0$ if

1. $N_0 = 0$;
2. $(N_t)_{t\geq 0}$ has stationary and independent increments;
3. For all $t \geq 0$, $\mathrm{P}(N_{t+h} - N_t = 1) = \lambda h + o(h)$ when $h \to 0$;
4. For all $t \geq 0$, $\mathrm{P}(N_{t+h} - N_t \geq 2) = o(h)$ when $h \to 0$;