1 Basic notation

Definition 1.1 (independent increments) A stochastic process $(X_t)_{t\in T}$ has *independent increments* if for every $n \in \mathbb{N}_+$ and any $t_1 \le t_2 \le \cdots \le t_n$, the increment $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

Definition 1.2 (strictly stationary process) Let $(X_t)_{t\in T}$ be a stochastic process and let $F_X(x_{t_1+\tau}, \ldots, x_{t_k+\tau})$ represent the distribution function of the joint distribution of $(X_t)_{t\in T}$ at times $t_1+\tau, \ldots, t_k+\tau$. Then, $(X_t)_{t\in T}$ is said to be strictly stationary if, for all k, for all τ, and for all $t_1, \ldots, t_k$,

$$F_X(x_{t_1+\tau}, \ldots, x_{t_k+\tau}) = F_X(x_{t_1}, \ldots, x_{t_k}).$$

2 Poisson process

Definition 2.1 (Poisson process (I)) A stochastic process $(N_t)_{t\ge 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is said to be a *Poisson process* with rate $\lambda > 0$ if

(i) $N_0 = 0$;

(ii) $(N_t)_{t\ge 0}$ has independent increments: for any $n \in \mathbb{N}_+$ and any $0 \le t_1 \le t_2 \le \cdots \le t_n$, the increment $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \cdots, N_{t_n} - N_{t_{n-1}}$ are independent;

(iii) for any $0 \le s < t$, $N_t - N_s \sim Pois(\lambda(t-s))$, that is

$$\mathrm{P}(N_t - N_s = k) = e^{-\lambda(t-s)}\frac{\lambda(t-s)^k}{k!} \quad (k = 0, 1, 2, \cdots).$$

Definition 2.2 (counting process) A *counting process* is a stochastic process $(N_t)_{t\ge 0}$ with values that are non-negative, integer, and non-decreasing:

(i) $N_t \ge 0$;

(ii) N_t is an integer;

(iii) If $0 \le s \le t$, then $N_s \le N_t$.

For any $0 \le s < t$, the counting process $N_t - N_s$ represents the number of events that occurred on $(s, t]$.

Definition 2.3 (Poisson process (II)) A counting process $(N_t)_{t\ge 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is said to be a *Poisson process* with rate $\lambda > 0$ if

(i) $N_0 = 0$;

(ii) $(N_t)_{t\ge 0}$ has independent increments;

(iii) For all $t \ge 0$, $\mathrm{P}(N_{t+h} - N_t = 1) = \lambda h + o(h)$ when $h \to 0$;

(iv) For all $t \ge 0$, $\mathrm{P}(N_{t+h} - N_t \ge 2) = o(h)$ when $h \to 0$;

Definition 2.4 (Poisson process (III)) A stochastic process $(N_t)_{t\ge 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is said to be a *Poisson process* with rate $\lambda > 0$ if

$$N_t = \sum_{n=1}^{\infty} n I_{[T_n, T_{n+1})}(t),$$

where $T_n = X_1 + X_2 + \cdots + X_n$ and X_i i.i.d $\sim Exp(\lambda)$ (Here the pdf of $Exp(\lambda)$ is taken as $\lambda e^{-\lambda x} I_{(0,+\infty)}(x)$).

Proposition 2.1 Definition 2.1, 2.3, 2.4 are equivalent definitions of Poisson process.

Proof.

- Definition 2.1 $\implies$ Definition 2.3

  Here we are only to show the implication of Definition 2.3(iii) and Definition 2.3(iv). Since $N_{t+h} - N_t \sim Pois(\lambda h)$, when $h \to 0$ we have

$$\begin{aligned}
\mathrm{P}(N_{t+h} - N_t = 1) &= e^{-\lambda h}\lambda h = (1 - \lambda h + o(h))\lambda h = \lambda h + o(h), \\
\mathrm{P}(N_{t+h} - N_t \geq 2) &= 1 - \mathrm{P}(N_{t+h} - N_t = 0) - \mathrm{P}(N_{t+h} - N_t = 1) \\
&= 1 - e^{-\lambda h} - e^{-\lambda h}\lambda h \\
&= 1 - (1 - \lambda h + o(h)) - (\lambda h + o(h)) \\
&= o(h).
\end{aligned}$$

- Definition 2.3 $\implies$ Definition 2.1

  Only Definition 2.1(iii) needs to be derived. Given the Laplace transform of the nonnegative random variables N_t and N_{t+h}

$$L_{N_t}(u) = \mathrm{E}[e^{-uN_t}], \quad L_{N_{t+h}}(u) = \mathrm{E}[e^{-uN_{t+h}}], \quad u \geq 0,$$

  according to Definition 2.3(ii) we can obtain

$$\begin{aligned}
L_{N_{t+h}}(u) &= \mathrm{E}[e^{-uN_{t+h}}] \\
&= \mathrm{E}[e^{-uN_t}e^{-u(N_{t+h}-N_t)}] \\
&= \mathrm{E}[e^{-uN_t}]\mathrm{E}[e^{-u(N_{t+h}-N_t)}] \\
&= L_{N_t}(u)\mathrm{E}[e^{-u(N_{t+h}-N_t)}].
\end{aligned}$$

  Note that

$$\begin{aligned}
&\mathrm{E}[e^{-u(N_{t+h}-N_t)}] \\
&= e^0\mathrm{P}(N_{t+h} - N_t = 0) + e^{-u}\mathrm{P}(N_{t+h} - N_t = 1) + \sum_{j=2}^{\infty} e^{-un}\mathrm{P}(N_{t+h} - N_t = j) \\
&= 1 - \lambda h + o(h) + e^{-u}(\lambda h + o(h)) + o(h) \\
&= 1 - \lambda h + e^{-u}\lambda h + o(h) \quad (h \to 0).
\end{aligned}$$

  Denote $g(t+h) = L_{N_{t+h}}(u)$ and $g(t) = L_{N_t}(u)$ for some fixed u and then we get

$$\frac{g(t+h) - g(t)}{h} = \frac{g(t)(1 - \lambda h + e^{-u}\lambda h + o(h)) - g(t)}{h} = g(t)\lambda(e^{-u} - 1) + \frac{o(h)}{h}.$$

  Letting $h \to 0$ yields the differential equation

$$g'(t) = g(t)\lambda(e^{-u} - 1).$$

  The initial condition $g(0) = \mathrm{E}[e^{-uN_0}] = 1$ determines a special solution of the equation

$$g(t) = L_{N_t}(u) = e^{\lambda t(e^{-u}-1)},$$

  which coincides with the Laplace transform of Poisson distribution $Pois(\lambda t)$. Since Laplace transform uniquely determines the distribution, we can thus conclude $N_t \sim Pois(\lambda t)$. Given any $r \geq 0$, define a stochastic process $N'_t = N_{r+t} - N_r$ and we can check that $(N'_t)_{t\geq 0}$ is also a counting process satisfying all the contitions in Definition 2.3. Hence by repeating the proof above we can show $N'_t \sim Pois(\lambda t)$, which is equivalent to Definition 2.1(iii).

- <u>Definition 2.1 $\implies$ Definition 2.4</u>

  Let $T_n = \inf\{t \geq 0 : N_t = n\}$ for $n \in \mathbb{N}_+$. Note that given any $t \geq 0$, $\{N_t = n\} = \{T_n \leq t < T_{n+1}\}$. Thus we have

$$N_t = \sum_{n=1}^{\infty} n I_{N_t=n} = \sum_{n=1}^{\infty} n I_{[T_n, T_{n+1})}(t).$$

  Let $X_1 = T_1, X_n = T_n - T_{n-1} (n \geq 2)$. Since $\mathrm{P}(X_1 > t) = \mathrm{P}(N_t = 0) = e^{-\lambda t}$, we see $X_1 \sim Exp(\lambda)$. Since

$$\mathrm{P}(X_2 > t | X_1 = t_1) = \mathrm{P}(X_2 > t | X_1 = t_1)$$

  When $n \geq 2$, since

$$\begin{aligned}
&\mathrm{P}(X_n > t | X_{n-1} = t_{n-1}, \cdots, X_1 = t_1) \\
&= \mathrm{P}(T_n - T_{n-1} > t | T_{n-1} - T_{n-2} = t_{n-1}, \cdots, T_1 = t_1) \qquad (\text{let } s_n = t_n + \cdots + t_1) \\
&= \mathrm{P}(T_n > s_{n-1} + t | T_{n-1} = s_{n-1}, \cdots, T_1 = s_1) \\
&= \mathrm{P}(N_{s_{n-1}+t} = n - 1 | N_{s_{n-1}} = n - 1) \qquad (\text{memoryless property of } (N_t)) \\
&= \mathrm{P}(N_{s_{n-1}+t} - N_{s_{n-1}} = 0 | N_s = n - 1) \\
&= \mathrm{P}(N_{s_{n-1}+t} - N_{s_{n-1}} = 0) \\
&= e^{-\lambda t},
\end{aligned}$$

  it is plain to show that $\{X_i\}$ is sequence of independent random variable. Furthermore, we have

$$\mathrm{P}(X_n > t) = \mathrm{E}[\mathrm{P}(X_n > t | X_{n-1}, \cdots, X_1)] = e^{-\lambda t},$$

  which implies X_i i.i.d $\sim Exp(\lambda)$,

- <u>Definition 2.4 $\implies$ Definition 2.1</u>

  Clearly $N_0 = 0$ holds. Since $T_n = X_1 + X_2 + \cdots + X_n$ and X_i i.i.d $\sim Exp(\lambda)$, we can deduce the jointly probability density function of $(T_1, T_2, \cdots, T_m)$

$$\begin{aligned}
f_S(y_1, y_2, \cdots, y_m) &= f_X(y_1, y_2 - y_1, \cdots, y_m - y_{m-1}) \left| \frac{\partial(x_1, \cdots, x_m)}{\partial(y_1, \cdots, y_m)} \right| \\
&= \lambda^m e^{-\lambda y_m} I_{\{0 \leq y_1 < \cdots < y_m\}}.
\end{aligned}$$

  Thus for any $1 \leq j_1 < j_2 < \cdots < j_n$, the jointly probability density function of $(T_{j_1}, T_{j_2}, \cdots, T_{j_n})$ is

$$\frac{y_1^{j_1-1}}{(j_1-1)!} \frac{(y_2-y_1)^{j_2-j_1-1}}{(j_2-j_1-1)!} \cdots \frac{(y_n-y_{n-1})^{j_n-j_{n-1}-1}}{(j_n-j_{n-1}-1)!} \lambda^{j_n} e^{-\lambda y_n} I_{\{0 \leq y_1 < \cdots < y_n\}}$$

  Note

$$N_t = \sum_{n=1}^{\infty} n I_{[T_n, T_{n+1})}(t)$$

implies $\{N_t = n\} = \{T_n \le t < T_{n+1}\}$. For any $n \in \mathbb{N}_+$ and any $0 \le t_1 < t_2 < \cdots < t_n$, we have

$$
\begin{aligned}
&\mathrm{P}(N_{t_1} = j_1, N_{t_1} - N_{t_2} = j_2, \cdots, N_{t_n} - N_{t_{n-1}} = j_n) \\
=& \mathrm{P}(N_{t_1} = j_1, N_{t_1} = j_1 + j_2, \cdots, N_{t_n} = j_1 + \cdots + j_n) \quad (\text{let } k_n = j_1 + \cdots j_n) \\
=& \mathrm{P}(T_{k_1} \le t_1, T_{k_1+1} > t_1, T_{k_2} \le t_2, T_{k_2+1} > t_2, \cdots, T_{k_n} \le t_n, T_{k_n+1} > t_n) \\
=& \int_{y_{k_1} \le t_1 < y_{k_1+1} < \cdots < y_{k_n} \le t_n < y_{k_n+1}} \frac{y_{k_1}^{k_1-1}}{(k_1-1)!} \cdots \frac{(y_{k_n} - y_{k_{n-1}+1})^{k_n - k_{n-1} - 2}}{(k_n - k_{n-1} - 2)!} \lambda^{k_n+1} e^{-\lambda y_{k_n+1}} dy_1 \cdots dy_{k_n+1} \\
=& \int_{y_{k_1} \le t_1 < y_{k_1+1} < \cdots < y_{k_n} \le t_n} \frac{y_{k_1}^{k_1-1}}{(k_1-1)!} \cdots \frac{(y_{k_n} - y_{k_{n-1}+1})^{k_n - k_{n-1} - 2}}{(k_n - k_{n-1} - 2)!} \lambda^{k_n} dy_{k_1} \cdots dy_{k_n} \int_{t_n}^{\infty} -de^{-\lambda y_{k_n+1}} \\
=& \int_{y_{k_1} \le t_1 < y_{k_1+1} < \cdots < t_{n-1} < y_{k_{n-1}+1} < t_n} \\
& \frac{y_{k_1}^{k_1-1}}{(k_1-1)!} \cdots \frac{(y_{k_{n-1}} - y_{k_{n-2}+1})^{k_{n-1} - k_{n-2} - 2}}{(k_n - k_{n-1} - 2)!} \lambda^{k_n} e^{-\lambda t_n} dy_1 \cdots dy_{k_{n-1}+1} \int_{y_{k_{n-1}+1}}^{t_n} d\frac{(y_{k_n} - y_{k_{n-1}+1})^{k_n - k_{n-1} - 1}}{(k_n - k_{n-1} - 1)!} \\
=& \int_{y_{k_1} \le t_1 < y_{k_1+1} < \cdots < t_{n-1} < y_{k_{n-1}+1} < t_n} \\
& \frac{y_{k_1}^{k_1-1}}{(k_1-1)!} \cdots \frac{(y_{k_{n-1}} - y_{k_{n-2}+1})^{k_{n-1} - k_{n-2} - 2}}{(k_n - k_{n-1} - 2)!} \lambda^{k_n} e^{-\lambda t_n} dy_1 \cdots dy_{k_{n-1}} \int_{t_{n-1}}^{t_n} -d\frac{(t_n - y_{k_{n-1}})^{k_n - k_{n-1}}}{(k_n - k_{n-1})!} \\
=& \int_{y_{k_1} \le t_1 < y_{k_1+1} < \cdots < t_{n-1} < y_{k_{n-1}+1} < t_n} \\
& \frac{y_{k_1}^{k_1-1}}{(k_1-1)!} \cdots \frac{(y_{k_{n-1}} - y_{k_{n-2}+1})^{k_{n-1} - k_{n-2} - 2}}{(k_n - k_{n-1} - 2)!} \lambda^{k_n} e^{-\lambda t_n} dy_1 \cdots dy_{k_{n-1}} \frac{(t_n - t_{n-1})^{k_n - k_{n-1}}}{(k_n - k_{n-1})!} \\
=& \cdots \\
=& \lambda^{k_n} e^{-\lambda t_n} \frac{t_1^{k_1}}{k_1!} \frac{(t_2 - t_1)^{k_2 - k_1}}{(k_2 - k_1)!} \cdots \frac{(t_n - t_{n-1})^{k_n - k_{n-1}}}{(k_n - k_{n-1})!} \\
=& e^{-\lambda t_1} \frac{(\lambda t_1)^{j_1}}{j_1} e^{-\lambda(t_2 - t_1)} \frac{(\lambda(t_2 - t_1))^{j_2}}{j_2} \cdots e^{-\lambda(t_n - t_{n-1})} \frac{(\lambda(t_n - t_{n-1}))^{j_n}}{j_n!}
\end{aligned}
$$

Therefore, we conclude $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \cdots, N_{t_n} - N_{t_{n-1}}$ are independent and for any $0 \le s < t$, $N_t - N_s \sim Pois(\lambda(t-s))$.

□

Proposition 2.2 Let $(N_t)_{t \ge 0}$ be a Poisson process.

1. $N_t \sim Pois(\lambda t)$, $\mathrm{E}[N_t] = \mathrm{Var}(N_t) = \lambda t$.

Appendix

Distribution	pmf $\mathrm{P}(X=k)$	Support	Mean	Variance
Bernoulli $B(1,p)$	$p^k(1-p)^{1-k}$	$\{0,1\}$	p	$p(1-p)$
Binomial $B(n,p)$	$\binom{n}{k} p^k(1-p)^{n-k}$	$\{0,\cdots,n\}$	np	$np(1-p)$
Negative Binomial $NB(r,p)$	$\binom{k+r-1}{k}(1-p)^r p^k$	$\mathbb{N}$	$\dfrac{pr}{1-p}$	$\dfrac{pr}{(1-p)^2}$
Poisson $Pois(\lambda)$	$\dfrac{\lambda^k e^{-\lambda}}{k!}$	$\mathbb{N}$	λ	λ
Geometric $Geo(p)$	$(1-p)^{k-1}p$	$\mathbb{N}_+$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$
Hypergeometric $H(N,K,n)$	$\dfrac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$	$\{0,\cdots,\min(n,K)\}$	$n\dfrac{K}{N}$	$n\dfrac{K}{N}\dfrac{N-K}{N}\dfrac{N-n}{N-1}$
Uniform (discrete) $DU(a,b)$	$\dfrac{1}{n}$	$\{a,a+1,\dots,b\}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a+1)^2-1}{12}$

Distribution	pdf	Mean	Variance
Degenerate δ_a	$I_{\{a\}}(x)$	a	0
Uniform (continuous) $U(a,b)$	$\dfrac{1}{b-a}I_{[a,b]}(x)$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
Exponential $Exp(\lambda)=\Gamma(1,\lambda)$	$\lambda e^{-\lambda x}I_{[0,+\infty)}(x)$	λ^{-1}	λ^{-2}
Normal $N(\mu,\sigma^2)$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Gamma $\Gamma(\alpha,\beta)$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}I_{(0,+\infty)}(x)$	$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$
Chi-squared $\chi_k^2=\Gamma(\frac{k}{2},\frac{1}{2})$	$\dfrac{1}{2^{k/2}\Gamma(k/2)}\ x^{k/2-1}e^{-x/2}I_{(0,+\infty)}(x)$	k	$2k$
Student's t_ν	$\dfrac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\dfrac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$	0	$\dfrac{\nu}{\nu-2}$ for $\nu>2$

Distribution	Moment-generating function	Characteristic function
Degenerate δ_a	e^{ta}	e^{ita}
Bernoulli $B(1,p)$	$1-p+pe^t$	$1-p+pe^{it}$
Geometric $Geo(p)$	$\frac{pe^t}{1-(1-p)e^t}$, $t<-\ln(1-p)$	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Binomial $B(n,p)$	$(1-p+pe^t)^n$	$(1-p+pe^{it})^n$
Negative Binomial $NB(r,p)$	$\frac{(1-p)^r}{(1-pe^t)^r}$	$\frac{(1-p)^r}{(1-pe^{it})^r}$
Poisson $Pois(\lambda)$	$e^{\lambda(e^t-1)}$	$e^{\lambda(e^{it}-1)}$
Uniform (continuous) $U(a,b)$	$\begin{cases} \frac{\mathrm{e}^{tb}-\mathrm{e}^{ta}}{t(b-a)} & \text{for } t\neq 0 \\ 1 & \text{for } t=0 \end{cases}$	$\begin{cases} \frac{\mathrm{e}^{itb}-\mathrm{e}^{ita}}{it(b-a)} & \text{for } t\neq 0 \\ 1 & \text{for } t=0 \end{cases}$
Uniform (discrete) $DU(a,b)$	$\frac{e^{at}-e^{(b+1)t}}{(b-a+1)\,(1-e^t)}$	$\frac{e^{it\mu}}{(b-a+1)\,(1-e^{it})}$
Laplace $L(\mu,b)$	$\frac{e^{t\mu}}{1-b^2t^2}$, $\lvert t\rvert<1/b$	$\frac{e^{it\mu}}{1+b^2t^2}$
Normal $N(\mu,\sigma^2)$	$e^{t\mu+\frac{1}{2}\sigma^2t^2}$	$e^{it\mu-\frac{1}{2}\sigma^2t^2}$
Chi-squared χ^2_k	$(1-2t)^{-\frac{k}{2}}$	$(1-2it)^{-\frac{k}{2}}$
Noncentral chi-squared $\chi^2_k(\lambda)$	$e^{\lambda t/(1-2t)}(1-2t)^{-\frac{k}{2}}$	$e^{i\lambda t/(1-2it)}(1-2it)^{-\frac{k}{2}}$
Gamma $\Gamma(\alpha,\beta)$	$\left(1-\frac{t}{\beta}\right)^{-\alpha}$, $t<\beta$	$\left(1-\frac{it}{\beta}\right)^{-\alpha}$
Exponential $Exp(\lambda)$	$\frac{\lambda}{\lambda-t}, t<\lambda$	$\frac{\lambda}{\lambda-it}$
Multivariate normal $N(\boldsymbol{\mu},\boldsymbol{\Sigma})$	$e^{\mathbf{t}^T\left(\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{\Sigma}\mathbf{t}\right)}$	$e^{\mathbf{t}^T\left(i\boldsymbol{\mu}-\frac{1}{2}\boldsymbol{\Sigma}\mathbf{t}\right)}$
Cauchy $Cauchy(\mu,\theta)$	Does not exist	$e^{it\mu-\theta\lvert t\rvert}$
Multivariate Cauchy $MultiCauchy(\boldsymbol{\mu},\boldsymbol{\Sigma})$	Does not exist	$e^{i\mathbf{t}^{\mathrm{T}}\boldsymbol{\mu}-\sqrt{\mathbf{t}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{t}}}$