Measure Theory and Probability Theory

Contents

Chapter 1

Measures

1.1 Class of sets

If 2^Ω denotes the power set of the set Ω and a collection of subsets $\mathcal{C} \subset 2^\Omega$, we say $\mathcal{C}$ is defined on Ω.

For convenience, let's define some operations on a collection of sets $\mathcal{C} \in 2^\Omega$.

- Given $f : \Sigma \to \Omega$, $f^{-1}(\mathcal{C}) := \{f^{-1}(A) : A \in \mathcal{C}\}$,
- Given $B \subset \Omega$, $B \cap \mathcal{C} := \{B \cap A : A \in \mathcal{C}\}$.

Definition 1.1.1 (semiring) a nonempty family of sets $\mathcal{S}$ is called a *semiring (of sets)* if the following three statements are true for all sets A and B,

1. $\varnothing \in S$
2. $A, B \in \mathcal{S}$ implies $A \cap B \in \mathcal{S}$
3. $A, B \in \mathcal{S}$ implies $A - B = \cup_{i=1}^n C_i$, where $\{C_i\}_{i=1}^n$ is a collection of pairwise disjoint sets in $\mathcal{S}$.

Definition 1.1.2 (ring) a nonempty family of sets $\mathcal{R}$ is called a *ring (of sets)* if it is closed under finite union and difference. That is, the following two statements are true for all sets A and B,

1. $A, B \in \mathcal{R}$ implies $A \cup B \in \mathcal{R}$
2. $A, B \in \mathcal{R}$ implies $A - B \in \mathcal{R}$.

A ring of sets $\mathcal{R}$ is a ring in the context of abstract algebra with the operations intersection(multiplication) and symmetric difference(addition), where the multiplicative identity is the set $\cup_{A \in \mathcal{R}} A$.

Definition 1.1.3 (algebra) a ring $\mathcal{F}$ defined on Ω is called an *algebra (of sets)* if it contains Ω.

Proposition 1.1.1 Every ring is a semiring.

Proof. Let $\mathcal{R}$ be a ring. First we see $\varnothing = A - A \in \mathcal{R}$. Second, it is straightforward to check

$$A \cap B = (A \cup B - (A - B)) - (B - A) \in \mathcal{R}.$$

Finally, it is a trivial fact that $A - B = C_1 \in \mathcal{R}$. Thus we show the ring $\mathcal{R}$ is also a semiring.

The following equivalent definitions of algebra are common to see.

Definition 1.1.4 (algebra) A collection of sets $\mathcal{F} \subset 2^{\Omega}$ is called an *algebra (of sets)* if

(a) $\Omega \in \mathcal{F}$,

(b) $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$,

(c) $A, B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$.

An algebra of sets is an associative algebra over a field in the context of abstract algebra: it is a ring and also a vector space over the field F_2 of 2 elements, which makes it an F_2-algebra.

Proposition 1.1.2 Every algebra is a ring.

Proof. Note it holds that

$$A - B = A \cap B^c = (A^c \cup B)^c.$$

Example 1.1.1 Let Ω be a nonempty set, and let $\#A$ denote the number of elements of a set $A \subset \Omega$. Then

$$\mathcal{F} = \{A \subset \Omega : \text{ either } \#A \text{ is finite or } \#A^c \text{ is finite}\}$$

is an algebra defined on Ω.

Definition 1.1.5 (σ-algebra) A class $\mathcal{F} \subset 2^{\Omega}$ is called a *σ-algebra* if it is an algebra and if it satisfies

(d) $A_n \in \mathcal{F}$ for $n \geq 1 \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Under the assumption that $\mathcal{F}$ is an algebra, condition (d) can be replaced by a weaker condition (d'), which leads to the following equivalent definition.

Definition 1.1.6 (σ-algebra) A class $\mathcal{F} \subset 2^{\Omega}$ is called a *σ-algebra* if it is an algebra and if it satisfies

(d') $A_n \in \mathcal{F},\ A_n \subset A_{n+1}$ for $n \geq 1 \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Example 1.1.2 Let Ω be a nonempty set. Then

$$\mathcal{F} = \{A \subset \Omega : \text{ either } A \text{ is a countable set or } A^c \text{ is a countable set}\}$$

is a σ-algebra defined on Ω.

It is easy to check the intersection of a family of σ-algebras on Ω is also a σ-algebra. However, the union of two σ-algebras may not even be an algebra. In many instances, given an arbitrary collection of subsets of Ω, one would like to extend it to a σ-algebra by adding extra subsets of Ω as few as possible. This leads to the following definition.

Definition 1.1.7 If $\mathcal{A} \subset 2^{\Omega}$, then the *$\sigma$-algebra generated by* $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$, is defined as

$$\sigma(\mathcal{A}) = \bigcap_{\mathcal{F} \in \mathcal{I}(\mathcal{A})} \mathcal{F}$$

where $\mathcal{I}(\mathcal{A}) = \{\mathcal{F} : \mathcal{A} \subset \mathcal{F} \text{ and } \mathcal{F} \text{ is a } \sigma\text{-algebra defined on } \Omega\}$ is the collection of all σ-algebras containing the class $\mathcal{A}$.

Note that since the power set 2^Ω contains $\mathcal{A}$ and is itself a σ-algebra defined on Ω, the collection $\mathcal{I}(\mathcal{A})$ is not empty and hence, the intersection in the above definition is well defined.

Definition 1.1.8 (Borel σ-algebra) The *Borel σ-algebra* on a topological space $\mathbb{S}$ is defined as the σ-algebra generated by the collection of open sets in $\mathbb{S}$.

Example 1.1.3 Let $\mathcal{B}(\mathbb{R}^k)$ denote the Borel σ-algebra on the Euclidean space $\mathbb{R}^k$ with standard topology, $1 \le k < +\infty$. Then, $\mathcal{B}(\mathbb{R}^k) = \sigma(\{A : A \text{ is an open subset of } \mathbb{R}^k\})$ is also generated by each of the following classes of sets

$$
\begin{aligned}
\mathcal{O}_1 &= \{(a_1, b_1) \times \cdots \times (a_k, b_k) : -\infty \le a_i < b_i \le +\infty, 1 \le i \le k\};\\
\mathcal{O}_2 &= \{(-\infty, x_1) \times \cdots \times (-\infty, x_k) : x_i \in \mathbb{R}, 1 \le i \le k\};\\
\mathcal{O}_3 &= \{(a_1, b_1) \times \cdots \times (a_k, b_k) : a_i, b_i \in \mathbb{Q}, a_i < b_i, 1 \le i \le k\};\\
\mathcal{O}_4 &= \{(-\infty, x_1) \times \cdots \times (-\infty, x_k) : x_i \in \mathbb{Q}, 1 \le i \le k\}.
\end{aligned}
$$

Definition 1.1.9 (π-system) A class $\mathcal{C}$ of subsets of Ω is a π-system or a π-class if

$$A, B \in \mathcal{C} \implies A \cap B \in \mathcal{C}.$$

Example 1.1.4 The classes $\mathcal{O}_i$ $(i = 1, 2, 3, 4)$ in Example 1.1.3 are all π-systems.

Definition 1.1.10 (λ-system) A class $\mathcal{L}$ of subsets of Ω is a λ-system or a λ-class if

(a) $\Omega \in \mathcal{L}$,

(b) $A, B \in \mathcal{L}, A \subset B \implies B \setminus A \in \mathcal{L}$,

(c) $A_n \in \mathcal{L}, A_n \subset A_{n+1}$ for all $n \ge 1 \implies A_n \in \mathcal{L}$ for $n \ge 1$.

Example 1.1.5 Every σ-algebra is a λ-system.

Theorem 1.1.1 If $\mathcal{C}$ is a π-system, then $\lambda(\mathcal{C}) = \sigma(\mathcal{C})$.

1.2 Measurable Space

Definition 1.2.1 (measurable space) Given a nonempty set Ω and a σ-algebra $\mathcal{F}$ on Ω, the tuple $(\Omega, \mathcal{F})$ is called a *measurable space*.

Definition 1.2.2 (measurable map) Given two measurable spaces $(\Omega, \mathcal{F})$ and $(S, \mathcal{S})$, the map $f : \Omega \to S$ is *$\mathcal{F}/\mathcal{S}$-measurable* if

$$f^{-1}(B) \in \mathcal{F}, \quad \forall B \in \mathcal{S}.$$

We say f is measurable if the underlying σ-algebras $\mathcal{F}$ and $\mathcal{S}$ are self-evident.

Note f^{-1} preserves σ-algebra, which leads to the following definition.

Definition 1.2.3 (σ-algebra generated by measurable map) Assume f is a measurable map from $(\Omega, \mathcal{F})$ to $(S, \mathcal{S})$. The σ-algebra generated by f is defined as

$$\sigma(f) := \sigma\left(f^{-1}(\mathcal{S})\right) = f^{-1}(\mathcal{S}).$$

Definition 1.2.4 (tensor product σ-algebra) Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. Denote by $\mathcal{F}_1 \otimes \mathcal{F}_2$ the σ-algebra on the Cartesian product $\Omega_1 \times \Omega_2$ generated by subsets of the form $B_1 \times B_2$, where $B_1 \in \mathcal{F}_1$ and $B_2 \in \mathcal{F}_2$, i.e.,

$$\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma\left(\{B_1 \times B_2 : B_1 \in \mathcal{F}_1, B_2 \in \mathcal{F}_2\}\right).$$

Definition 1.2.5 (product measurable space) Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. The *product measurable space* of $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ is defined as $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$.

Definition 1.2.6 (infinite product measurable space) Let $\{(\Omega_i, \mathcal{F}_i)\}_{i\in T}$ be a collection of measurable spaces. The *infinite product measurable space* of $\{(\Omega_i, \mathcal{F}_i)\}_{i\in T}$ is defined as $\left(\prod_{i\in T} \Omega_i, \bigotimes_{i\in T} \mathcal{F}_i\right)$, where

$$\mathcal{MC}(\{\mathcal{F}_i\}_{i\in T}) := \left\{\prod_{i\in T} B_i : B_i \in \mathcal{F}_i,\ B_i = \Omega_i \text{ for all but a finite number of } i \in T\right\}$$

is called the collection of *measurable cylinders* and

$$\bigotimes_{i\in T} \mathcal{F}_i = \sigma\left(\mathcal{MC}\left(\{\mathcal{F}_i\}_{i\in T}\right)\right).$$

Product measurable space is the product in the category consisting of all measurable spaces and measurable functions and satisfies the following universal property: for every cone consisting of a measurable space Y and a family of measurable functions $\{f_i : Y \to \Omega_i\}_{i\in T}$, there exists a unique measurable function

$$f : Y \longrightarrow \prod_{i\in T} \Omega_i, \qquad y \longmapsto (f_i(y))_{i\in T}$$

denoted by $\prod_{i\in T} f_i$ such that the following diagrams commute for all i in T,

$$\begin{tikzcd} Y \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_i"'] & \prod_{i\in T} \Omega_i \arrow[d, "\pi_i"] \\ & \Omega_i \end{tikzcd}$$

where $\pi_i : (\omega_k)_{k\in T} \mapsto \omega_i$ is the projection map. Thus we see that $\bigotimes_{i\in T} \mathcal{F}_i$ is the smallest σ-algebra on $\prod_{i\in T} \Omega_i$ such that every projection map π_i is measurable, that is

$$\bigotimes_{i\in T} \mathcal{F}_i = \sigma\left(\bigcup_{i\in T} \sigma\left(\pi_i\right)\right).$$

Generally, we have the following proposition.

Proposition 1.2.1

$$\sigma\left(\prod_{i\in T} f_i\right) = \sigma\left(\bigcup_{i\in T} \sigma\left(f_i\right)\right).$$

Proposition 1.2.2 (product and Borel σ-fields) Let $\{S_i\}_{i=1}^{\infty}$ be a sequence of separable metric spaces. Then

$$\mathcal{B}\left(\prod_{i=1}^{\infty} S_i\right) = \bigotimes_{i=1}^{\infty} \mathcal{B}\left(S_i\right)$$

In particular, $\mathcal{B}\left(\mathbb{R}^d\right) = (\mathcal{B}(\mathbb{R}))^d$.

Proposition 1.2.3 (convergence and limits) Let $f_1, f_2, \ldots$ be measurable functions from a measurable space $(\Omega, \mathcal{F})$ into some metric space (S, ρ). Then

(i) $\{\omega \in \Omega : f_n(\omega) \text{ converges}\} \in \mathcal{A}$ if S is complete;

(ii) $f_n \to f$ on Ω implies that f is measurable.

1.3 Measure

Definition 1.3.1 (measure) Given a measurable space $(\Omega, \mathcal{F})$, a set function $\mu : \mathcal{F} \to [0, +\infty]$ is called a *measure* if

(a) $\mu(\varnothing) = 0$;

(b) σ-additivity: for any countable collection $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{F}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Definition 1.3.2 (measure space) A *measure space* is a triple $(\Omega, \mathcal{F}, \mu)$ where Ω is a set, $\mathcal{F}$ is a σ-algebra on Ω and $\mu : \mathcal{F} \to [0, +\infty]$ is a measure.

Definition 1.3.3 (pre-measure) Let $\mathcal{S}$ be a semiring on Ω. A set function $\mu_0 : \mathcal{S} \to [0, +\infty]$ is called a *pre-measure* if

(a) $\mu_0(\varnothing) = 0$;

(b) σ-additivity: for any countable collection $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{S}$,

$$\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Clearly we see every measure is a pre-measure, since every σ-algebra is a semiring.

Definition 1.3.4 ($\boldsymbol{\sigma}$-finite measure) Let $(\Omega, \mathcal{F})$ be a measurable space and μ a measure on it. The measure μ is called a σ-finite measure if the set Ω can be covered with at most countably many measurable sets with finite measure, which means that there are sets $A_1, A_2, \ldots \in \mathcal{F}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}_+$ that satisfy

$$\bigcup_{n=1}^{\infty} A_n = \Omega.$$

σ-finite pre-measure can be defined in a similar way.

Definition 1.3.5 (finite measure) Let $(\Omega, \mathcal{F})$ be a measurable space and μ a measure on it. The measure μ is called a *finite measure* if $\mu(\Omega) < \infty$.

Definition 1.3.6 (probability measure) Let $(\Omega, \mathcal{F})$ be a measurable space and μ a measure on it. The measure μ is called a *probability measure* if $\mu(\Omega) = 1$. In this case, $(\Omega, \mathcal{F}, \mu)$ is called a *probability measure space* or *probability space* in short.

Theorem 1.3.1 (Carathéodory's extension theorem) Let $\mathcal{R}$ be a ring on Ω and let $\mu_0 : \mathcal{R} \to [0, +\infty]$ be a pre-measure. There exists a measure $\mu : \sigma(\mathcal{R}) \to [0, +\infty]$ such that μ is an extension of μ_0, That is, $\mu|\mathcal{R} = \mu_0$. Moreover, if μ_0 is σ-finite then the extension μ is unique and also σ-finite.

Definition 1.3.7 (complete measure) A measure μ defined on the measurable space $(\Omega, \mathcal{F})$ is called *complete* if for any $A \in \mathcal{F}$ with $\mu(A) = 0$, $2^A \subset \mathcal{F}$. In this case, the measure space $(\Omega, \mathcal{F}, \mu)$ is called a complete measure space.

Definition 1.3.8 (outer measure) A set function $\mu^* : 2^\Omega \to [0, +\infty]$ is called a *outer measure* on Ω if

(a) $\mu^*(\varnothing) = 0$;

(b) $A \subset B \subset \Omega \implies \mu^*(A) \leq \mu^*(B)$;

(b) countable subadditivity: for any countable collection $\{A_i\}_{i=1}^\infty$ of sets in 2^Ω,

$$\mu^*\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu^*(A_n).$$

Definition 1.3.9 Given a pre-measure μ_0 on a semiring $\mathcal{S} \subset 2^\Omega$, the *outer measure induced by* μ_0 is the set function $\mu_0^* : 2^\Omega \to [0, +\infty]$, as

$$\mu_0^*(A) = \inf\left\{\sum_{n=1}^\infty \mu_0(A_n) : \{A_n\}_{n\geq 1} \subset \mathcal{S}, A \subset \bigcup_{n\geq 1} A_n\right\}.$$

Especially, given a measure μ on a measurable space $(\Omega, \mathcal{F})$, the *outer measure induced by* μ is the set function $\mu^* : 2^\Omega \to [0, +\infty]$, as

$$\mu^*(A) = \inf\{\mu(C) : C \in \mathcal{F}, A \subset C\}.$$

Definition 1.3.10 Suppose that μ_0^* be the outer measure induced by pre-measure μ_0 on a semiring $\mathcal{S} \subset 2^\Omega$. A set A is said to be μ_0^* -measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subset \Omega$$

Definition 1.3.11 (product measure) Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces. A measure μ on the measurable space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ is said to be a *product measure* of μ_1 and μ_2 if it satisfies the property

$$(\mu)(B_1 \times B_2) = \mu_1(B_1)\mu_2(B_2)$$

for all $B_1 \in \Omega_1$, $B_2 \in \Omega_2$. In this case, the measure space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu)$ is called the product measure space of $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$.

Proposition 1.3.1 Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces. If the measure μ_1 and μ_2 are σ-finite, the product measure of μ_1 and μ_2 uniquely exists and denoted by $\mu_1 \times \mu_2$.

Definition 1.3.12 (product probability measure) Let $\{(\Omega_i, \mathcal{F}_i, \mathrm{P}_i)\}_{i\in T}$ be a collection of probability measure spaces. The map

$$\begin{aligned} \mathrm{P}_0 : \mathcal{MC}(\{\mathcal{F}_i\}_{i\in T}) &\longrightarrow [0,1] \\ \prod_{i\in T} B_i &\longmapsto \prod_{i\in T:\mathrm{P}_i(B_i)<1} \mathrm{P}_i(B_i) \end{aligned}$$

can be uniquely extended to a probability measure P on the product measurable space $\left(\prod_{i\in T}\Omega_i, \bigotimes_{i\in T}\mathcal{F}_i\right)$. The *product probability space* of $\{(\Omega_i, \mathcal{F}_i, \mathrm{P}_i)\}_{i\in T}$ is defined as $\left(\prod_{i\in T}\Omega_i, \bigotimes_{i\in T}\mathcal{F}_i, \mathrm{P}\right)$.

Chapter 2

Integration

Assume that all the functions in this chapter are from Ω to $\mathbb{R}$.

2.1 The definition of integration

2.1.1 The integration of nonnegative simple functions

Definition 2.1.1 (simple function) Given a measurable space $(\Omega, \mathcal{F})$, a simple function is a function of the form

$$f(\omega) = \sum_{i=1}^{m} a_i \mathbf{1}_{A_i}(\omega), \quad a_i \in \mathbb{R},$$

where A_i are disjoint subsets of Ω that belong to $\mathcal{F}$.

Definition 2.1.2 (integration of nonnegative simple function) Given a measure space $(\Omega, \mathcal{F}, \mu)$, the integral of the nonnegative simple function $f(x) = \sum_{i=1}^{m} a_i \mathbf{1}_{A_i}(\omega)$ with respect to μ is defined as

$$\int_\Omega f d\mu := \sum_{i=1}^{m} a_i \mu_{A_i}.$$

Note that if $f(\omega) = \sum_{j=1}^{m} b_j \mathbf{1}_{B_j}(\omega)$ where B_i are disjoint subsets of Ω that belong to $\mathcal{F}$, there must be

$$A_i \cap B_j \neq 0 \implies a_i = b_j.$$

By σ-additivity of the measure μ, we have

$$\begin{aligned}\sum_{i=1}^{m} a_i \mu\left(A_i\right) &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \mu\left(A_i \cap B_j\right) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} b_j \mu\left(A_i \cap B_j\right) = \sum_{j=1}^{n} b_j \mu\left(B_j\right).\end{aligned}$$

Hence $\int_\Omega f d\mu$ does not depend on the representation of f used in its definition.

2.1.2 The integration of nonnegative measurable functions

Proposition 2.1.1 Let $\{f_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ be two sequences of nonnegative simple functions on $(\Omega, \mathcal{F}, \mu)$ to such that if as $n \to \infty$, $f_n(\omega) \uparrow f(\omega)$ and $g_n(\omega) \uparrow f(\omega)$ for all $\omega \in \Omega$, then

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \lim_{n\to\infty} \int_\Omega g_n d\mu.$$

Definition 2.1.3 (integration of nonnegative measurable function) Given a measure space $(\Omega, \mathcal{F}, \mu)$, the integral of the nonnegative measurable function f with respect to μ is defined as

$$\int_\Omega f d\mu := \lim_{n\to\infty} \int_\Omega f_n d\mu$$

where $\{f_n\}_{n\geq 1}$ is any sequence of nonnegative simple functions such that $f_n(\omega) \uparrow f(\omega)$ for all $\omega \in \Omega$.

2.1.3 The integration of measurable functions

Definition 2.1.4 (integration of measurable function) Given a measure space $(\Omega, \mathcal{F}, \mu)$. Let $f^+ = f\mathbf{1}_{\{f\geq 0\}}$ and $f^- = -f\mathbf{1}_{\{f<0\}}$. The integral of f with respect to μ is defined as

$$\int_\Omega f d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu$$

provided that at least one of the integrals on the right side is finite.

Definition 2.1.5 (integrable function) Given a measure space $(\Omega, \mathcal{F}, \mu)$, a measurable function f is integrable with respect to μ if

$$\int_\Omega |f| d\mu$$

exists, or equivalently both $\int_\Omega f^+ d\mu$ and $\int_\Omega f^- d\mu$ are finite.

2.2 L^p-space

Definition 2.2.1 (linear space $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $0 < p \leq \infty$. Let

$$\|f\|_p := \begin{cases} \left(\int_\Omega |f|^p d\mu\right)^{\min\left\{\frac{1}{p},1\right\}}, & 0 < p < \infty, \\ \inf\left\{M : |f| < M \text{ a.e.}\right\}, & p = \infty. \end{cases}$$

Then

$$\mathcal{L}^p(\Omega, \mathcal{F}, \mu) := \{f : \|f\|_p < \infty\}$$

is a linear space.

Definition 2.2.2 (Banach space $L^p(\Omega, \mathcal{F}, \mu)$) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then

$$\mathcal{N} := \{f : f = 0 \text{ a.e.}\} = \ker\left(\|\cdot\|_p\right).$$

is a subspace of $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ and

$$L^p(\Omega, \mathcal{F}, \mu) := \mathcal{L}^p(\Omega, \mathcal{F}, \mu)/\mathcal{N}$$

is a Banach space with the norm

$$\begin{aligned} \|\cdot\|_p : L^p(\Omega, \mathcal{F}, \mu) &\longrightarrow [0, \infty), \\ f + \mathcal{N} &\longmapsto \|f\|_p. \end{aligned}$$

For simplicity, we can identify f with its equivalence class $f + \mathcal{N}$.

2.3 Convergence of measurable function

Definition 2.3.1 (measurable function) Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be measurable spaces, meaning that Ω_1 and Ω_2 are sets equipped with respective σ-algebras Σ_1 and Σ_2. A function $f : \Omega_1 \to \Omega_2$ is said to be a *measurable function*, if for every $E \in \Sigma_2$, the preimage of E under f is measurable in Σ_1, i.e.

$$f^{-1}(E) := \{\omega \in \Omega_1 | \ f(\omega) \in E\} \in \Sigma_1, \ \ \forall E \in \Sigma_2.$$

If $f : \Omega_1 \to \Omega_2$ is measurable, we sometimes write

$$f\colon (\Omega_1, \Sigma_1) \to (\Omega_2, \Sigma_2)$$

to emphasize the dependency on the σ-algebras Σ_1 and Σ_2. In this section, we focus on a sort of special measurable functions:

$$f\colon (\Omega, \Sigma) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})).$$

Definition 2.3.2 (convergence in measure) Let f, f_n $(n \in \mathbb{N}) : \Omega \to \mathbb{R}$ be measurable functions on a measure space (Ω, Σ, μ). The sequence f_n is said to converge in measure to f if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\left(\{\omega \in \Omega : |f(\omega) - f_n(\omega)| \geq \varepsilon\}\right) = 0,$$

Chapter 3

Differentiation

3.1 Signed measures

Definition 3.1.1 (signed measure) Given a measurable space $(\Omega, \mathcal{F})$, a set function $\mu : \mathcal{F} \to [-\infty, +\infty]$ is called a *signed measure* if

(a) $\mu(\varnothing) = 0$;

(b) σ-additivity: for any countable collection $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{F}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

A signed measure can attain at most one of the values $+\infty$ and $-\infty$.

Definition 3.1.2 (finite signed measure) A signed measure μ on measurable space $(\Omega, \mathcal{F})$ is *finite* if $\mu(\mathcal{F}) \in (-\infty, +\infty)$.

Example 3.1.1 Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, let f belong to $L^1(\Omega, \mathcal{F}, \mu)$, and define a function ν on $\mathcal{F}$ by $\nu(A) = \int_A f d\mu$. Then the linearity of the integral and the dominated convergence theorem imply that ν is a signed measure on $(\Omega, \mathcal{F})$. Note that such a signed measure is the difference of the positive measures ν_1 and ν_2 defined by $\nu_1(A) = \int_A f^+ d\mu$ and $\nu_2(A) = \int_A f^- d\mu$.

Theorem 3.1.1 (Hahn Decomposition Theorem) Let $(\Omega, \mathcal{F})$ be a measurable space, and let μ be a signed measure on $(\Omega, \mathcal{F})$. Then there are disjoint subsets P and N of Ω such that P is a positive set for μ, N is a negative set for μ, and $\Omega = P \cup N$. Let

$$\begin{aligned} \mu^+ : \mathcal{F} &\longrightarrow [0, \infty], \\ A &\longmapsto \mu(A \cap P) \end{aligned}$$

$$\begin{aligned} \mu^- : \mathcal{F} &\longrightarrow [0, \infty], \\ A &\longmapsto -\mu(A \cap N) \end{aligned}$$

μ is the difference of two measures, that is

$$\mu = \mu^+ - \mu^-,$$

at least one of which is finite.

The *variation* of the signed measure μ is the measure $|\mu|$ defined by $|\mu| = \mu^+ + \mu^-$. It is easy to check that

$$|\mu(A)| \leq |\mu|(A), \quad \forall A \in \mathcal{F}.$$

Definition 3.1.3 (absolutely continuous) Let $(\Omega, \mathcal{F})$ be a measurable space, and let μ and ν be signed measures on $(\Omega, \mathcal{F})$. Then ν is *absolutely continuous* with respect to μ if for all $A \in \mathcal{F}$,

$$\mu(A) = 0 \implies \nu(A) = 0.$$

One sometimes writes $\nu \ll \mu$ to indicate that ν is absolutely continuous with respect to μ.

Definition 3.1.4 (mutually singular) Let $(\Omega, \mathcal{F})$ be a measurable space. A positive measure μ on $(\Omega, \mathcal{F})$ is concentrated on the $\mathcal{F}$-measurable set E if $\mu\left(E^c\right) = 0$. A signed or complex measure μ on $(\Omega, \mathcal{F})$ is concentrated on the $\mathcal{F}$-measurable set E if the variation $|\mu|$ of μ is concentrated on E, or equivalently, if each $\mathcal{F}$-measurable subset A of E^c satisfies $\mu(A) = 0$. Now suppose that μ and ν are measures (or signed measures) on $(\Omega, \mathcal{F})$. Then μ and v are *mutually singular* if there is an $\mathcal{F}$-measurable set E such that μ is concentrated on E and ν is concentrated on E^c. One sometimes writes $\mu \perp \nu$ to indicate that μ and ν are mutually singular.

Theorem 3.1.2 (Lebesgue Decomposition Theorem) . Let $(\Omega, \mathcal{F})$ be a measurable space, let μ be a positive measure on $(\Omega, \mathcal{F})$, and let v be a finite signed, complex, or σ -finite positive measure on $(\Omega, \mathcal{F})$. Then there are unique finite signed, complex, or positive measures v_a and v_s on $(\Omega, \mathcal{F})$ such that (a) v_a is absolutely continuous with respect to μ, (b) v_s is singular with respect to μ, and (c) $v = v_a + v_s$.

Chapter 4

Probability Space

1.Conditional Expectation

Definition 4.0.1 (conditional expectation) Let X be a $\mathcal{F}$-measurable random variable on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ such that $\mathrm{E}[|X|] < \infty$. Given a σ-algebra $\mathcal{G} \subset \mathcal{F}$, a random variable Z that is $\mathcal{G}$-measurable and satisfies

$$\mathrm{E}[XI_A] = \mathrm{E}[ZI_A] \quad \text{for all } A \in \mathcal{G}$$

is called the *conditional expectation* of Y given $\mathcal{G}$ and is written as $\mathrm{E}(X|\mathcal{G})$.

In probability theory, we show that conditional expectation exists and is unique up to absolutely surely equality. If not pointed out explicitly, all equalities and inequalities involving conditional expectation are considered to hold absolutely surely.

Proposition 4.0.1 (projection in $L^2(\Omega, \mathcal{F}, \mathrm{P})$) Assume $X \in L^2(\Omega, \mathcal{F}, \mathrm{P})$ and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. The set

$$S_{\mathcal{G}} = \{Y \in L^2(\Omega, \mathcal{F}, \mathrm{P}) : \sigma(Y) \in \mathcal{G}\}$$

is a closed subspace of $L^2(\Omega, \mathcal{F}, \mathrm{P})$. The conditional expectation mapping

$$\begin{aligned} \mathrm{E}(\cdot|\mathcal{G}) : L^2(\Omega, \mathcal{F}, \mathrm{P}) &\longrightarrow S_{\mathcal{G}} \\ X &\longmapsto \mathrm{E}(X|\mathcal{G}) \end{aligned}$$

is a projection to the subspace $S_{\mathcal{G}}$ in $L^2(\Omega, \mathcal{F}, \mathrm{P})$.

Proposition 4.0.2 Let X, Y, X_n be integrable $\mathcal{F}$-measurable random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Assume $\mathcal{G}, \mathcal{H}$ are σ-algebras such that $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$.

1. If a, b are constants, then $\mathrm{E}[aX + bY|\mathcal{G}] = a\mathrm{E}[X|\mathcal{G}] + b\mathrm{E}[Y|\mathcal{G}]$.
2. If X equals a constant a, then $\mathrm{E}[X|\mathcal{G}] = a$.
3. If $X \geq Y$, then $\mathrm{E}[X|\mathcal{G}] \geq \mathrm{E}[Y|\mathcal{G}]$.
4. $|\mathrm{E}[X|\mathcal{G}]| \leq \mathrm{E}[|X|\,|\mathcal{G}]$.
5. If ϕ is a convex function on $\mathbb{R}$ and $\phi(X)$ is integrable, then $\phi(\mathrm{E}[X|\mathcal{G}]) \leq \mathrm{E}[\phi(X)|\mathcal{G}]$.
6. If $\lim\limits_{n\to\infty} X_n = X$ and $|X_n| \leq X$, then $\lim\limits_{n\to\infty} \mathrm{E}[X_n|\mathcal{G}] = \mathrm{E}[X|\mathcal{G}]$.

7. $\mathrm{E}[\mathrm{E}[X|\mathcal{G}]] = \mathrm{E}[X]$.
8. $\mathrm{E}[\mathrm{E}[X|\mathcal{H}]|\mathcal{G}] = \mathrm{E}[\mathrm{E}[X|\mathcal{G}]|\mathcal{H}] = \mathrm{E}[X|\mathcal{H}]$
9. If X and $\mathcal{G}$ are independent, that is, whenever $A \in \sigma(X)$ and $B \in \mathcal{G}$, $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$, then $\mathrm{E}[X|\mathcal{G}] = X$.
10. If Z is $\mathcal{G}$-measurable and ZX is integrable, then $\mathrm{E}[ZX|\mathcal{G}] = Z\mathrm{E}[X|\mathcal{G}]$

Appendix

Definition 4.0.2 (associative algebra) Let R be a fixed commutative ring. An associative R-algebra (or more simply, an R-algebra) is an additive abelian group A which has the structure of both a ring and an R-module in such a way that the ring multiplication $* : A \times A \to A$ and the scalar multiplication $\cdot : R \times A \to A$ is compatible in the way that

$$r \cdot (x * y) = (r \cdot x) * y = x * (r \cdot y)$$

for all $r \in R$ and $x, y \in A$.

Definition 4.0.3 (Polish space) a Polish space is a separable completely metrizable topological space; that is, a space homeomorphic to a complete metric space that has a countable dense subset.

Proposition 4.0.3 (convergence of random variable sequences)

$$\begin{tikzcd}
X_n \xrightarrow{L^q} X \arrow[dd, dashed, bend right, "1<p<q"'] & X_n \xrightarrow{a.s.} X \arrow[dr, dashed] & & \\
& & X_n \xrightarrow{p} X \arrow[r, dashed] \arrow[dl, dashed, bend left, "\mathrm{UI}"] & X_n \xrightarrow{d} X \\
X_n \xrightarrow{L^p} X \arrow[r, dashed] & X_n \xrightarrow{L^1} X \arrow[ur, dashed] & &
\end{tikzcd}$$