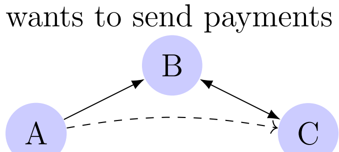
LN fee bounds

sketch

- B - price of 1 on-chain transaction
- r - constant effective interest rate
- c_{ij} - regular payment from node i to node j
- $p_{i_{jk}}$ - the fixed transaction fee i charges for each payment from j to k. It could be changed into a function of c
- λ_{ij} - average transaction frequency between node i and node j, (unit: $tx/minute$)
- m_{ij} - channel size between node i and node j (inbound + outbound capacity)
- k - the number of intermediate transaction a node can transfer
- $\tau(\alpha)$ - an exponential i.i.d. random variable with parameter α, indicating the time period $(0, \tau)$ a channel is open
- T - exponential i.i.d random variable with parameter α, indicating the total transaction fee charged between $(0, \tau)$
- I - exponential i.i.d random variable with parameter α, indicating the interest return (opportunity cost) between $(0, \tau)$
- $3(\frac{2B\lambda}{r})^{1/3}$ - expected cost of a bidirectional channel (opening, reopening, maintaining)
- $\sqrt{\frac{2B\lambda c}{r}}$ - expected cost of an unidirectional channel
- R - intermediate node's revenue through transacting payments

1 3 Nodes

Let there be 1 unidirectional channel between Alice and Bob and 1 bidirectional channel between Bob and Charlie. Let Alice sends payment c_{AB} to Bob at a rate of λ_{AB}, and Alice wants to send payments of c_{AC} to Charlie at a rate of λ_{AC}.



The expected interest return of a channel I and the transaction fee are

$$I(\alpha) = c - c\frac{\alpha}{\alpha + r} \tag{1}$$

$$T(\alpha) = \frac{mp}{c_{AC} + c_{AB} + p} \cdot \frac{\alpha}{\alpha + r} \tag{2}$$

We try to find the bounds of the fixed payment p.

1.0.1 With optimal channel size

The optimal size of the channel is $m = \sqrt{\frac{2B\lambda c}{r}}$. Here, $c = c_{AC} + c_{AB} + p$, such that $m = \sqrt{\frac{2B\lambda(c_{AC}+c_{AB}+p)}{r}}$. Note that both $2B\lambda(c_{AC} + c_{AB} + p) \geqslant 0$ and $r > 0$.
Thus transaction fee with the channel having an optimal channel size is

$$T(\alpha) = \frac{p\alpha}{(c_{AC} + c_{AB} + p)(\alpha + r)} \cdot \sqrt{\frac{2B\lambda(c_{AC} + c_{AB} + p)}{r}} \tag{3}$$

$$= \frac{p\alpha}{(\alpha + r)} \cdot \sqrt{\frac{2B\lambda}{(c_{AC} + c_{AB} + p)r}} \tag{4}$$

1.1 Lower bound

As an intermediate node, Bob is rational to save extra tokens if profit from transaction fees in the node is at least as high as the interest returns if the tokens were saved in a bank. He can also be incentivized to be an intermediate node if their channel capacities becomes unbalanced. With only 2 channels, we don't consider the rebalancing case.
Bob consider the net change in the cost of channels, let C_o be the total cost for old channels, C_n be the total cost for new channels, and sub-subscript i indicates the portion i pays.
His revenue is bounded below by 0, since he will not agree to be an intermediate node otherwise.

$$R(\lambda, \alpha) = T(\alpha) + C_{o_B}(\lambda) - I(\alpha) - C_{n_B}(\lambda) \geqslant 0 \tag{5}$$

$$T(\lambda) \geqslant I(\lambda) + C_{n_B}(\lambda) - C_{o_B}(\lambda) \tag{6}$$

Without Alice's payments to Charlie, each node is responsible for half of the cost to maintain their channels, and thus their costs of the channels is

$$C_o(\lambda) = \sqrt{\frac{2B\lambda c_{AB}}{r}} + 3(\frac{2B\lambda}{r})^{1/3} \tag{7}$$

$$C_{o_B}(\lambda) = \frac{1}{2}\sqrt{\frac{2B\lambda c_{AB}}{r}} + \frac{3}{2}(\frac{2B\lambda}{r})^{1/3} \tag{8}$$

$$C_{o_A}(\lambda) = \frac{1}{2}\sqrt{\frac{2B\lambda c_{AB}}{r}} \tag{9}$$

$$C_{o_C}(\lambda) = \frac{3}{2}(\frac{2B\lambda}{r})^{1/3} \tag{10}$$

Bob acting as an intermediate node causes him to be responsible for the new frequency $\lambda_{AC} = \lambda$, such that Bob is forced to send transactions as often as necessary.
The new cost of the channels are

$$C_n(\lambda) = \sqrt{\frac{2B\lambda(c_{AB} + c_{AC})}{r}} + 3(\frac{4B\lambda}{r})^{1/3} \tag{11}$$

$$C_{n_A}(\lambda) = C_{o_A}(\lambda) \tag{12}$$

$$C_{n_C}(\lambda) = C_{o_C}(\lambda) \tag{13}$$

$$C_{n_B}(\lambda) = C_n(\lambda) - C_{n_A}(\lambda) - C_{n_C}(\lambda) \tag{14}$$

$$= \sqrt{\frac{2B\lambda(c_{AB} + c_{AC})}{r}} + 3(\frac{4B\lambda}{r})^{1/3} - \frac{1}{2}\sqrt{\frac{2B\lambda c_{AB}}{r}} - \frac{3}{2}(\frac{2B\lambda}{r})^{1/3} \tag{15}$$

From the earlier secions, we have

$$I(\alpha) = c_{AC} - c_{AC}\frac{\alpha}{\alpha + r} \tag{16}$$

$$T(\alpha) = \frac{mp}{c_{AC} + c_{AB} + p} \cdot \frac{\alpha}{\alpha + r} \tag{17}$$

Plugging these equations in

$$T(\alpha) \geqslant I(\alpha) + C_{n_B}(\lambda) - C_{o_B}(\lambda) \tag{18}$$

$$\frac{mp}{c_{AC} + c_{AB} + p} \cdot \frac{\alpha}{\alpha + r} \geqslant \frac{c_{AC} r}{\alpha + r} + \sqrt{\frac{2B\lambda(c_{AB} + c_{AC})}{r}} \tag{19}$$

$$+ 3(\frac{4B\lambda}{r})^{1/3} - \sqrt{\frac{2B\lambda c_{AB}}{r}} - 3(\frac{2B\lambda}{r})^{1/3} \tag{20}$$

We find the lower bound for p

$$\frac{mp}{c_{AC}+c_{AB}+p} \geqslant \frac{c_{AC}r}{\alpha} + \frac{\alpha+r}{\alpha}(((c_{AB}+c_{AC})^{1/2} - c_{AB}^{1/2})\sqrt{\frac{2B\lambda}{r}} \tag{21}$$

$$+ 3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3}) \tag{22}$$

$$p \geqslant \frac{(c_{AC}+c_{AB})(\frac{c_{AC}r}{\alpha} + \frac{\alpha+r}{\alpha}((\sqrt{c_{AB}+c_{AC}} - \sqrt{c_{AB}})\sqrt{\frac{2B\lambda}{r}} + 3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3}))}{m - (\frac{c_{AC}r}{\alpha} + \frac{\alpha+r}{\alpha}(((c_{AB}+c_{AC})^{1/2} - c_{AB}^{1/2})\sqrt{\frac{2B\lambda}{r}} + 3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3}))} \tag{23}$$

The optimal size of the channel is $m = \sqrt{\frac{2B\lambda c}{r}}$. Here, $c = c_{AC} + c_{AB} + p$, such that $m = \sqrt{\frac{2B\lambda(c_{AC}+c_{AB}+p)}{r}}$. The lower bound with this size of the channel is

$$p \geqslant \frac{(c_{AC}+c_{AB})(\frac{c_{AC}r}{\alpha} + \frac{\alpha+r}{\alpha}((\sqrt{c_{AB}+c_{AC}} - \sqrt{c_{AB}})\sqrt{\frac{2B\lambda}{r}} + 3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3}))}{\sqrt{\frac{2B\lambda(c_{AC}+c_{AB}+p)}{r}} - (\frac{c_{AC}r}{\alpha} + \frac{\alpha+r}{\alpha}(((c_{AB}+c_{AC})^{1/2} - c_{AB}^{1/2})\sqrt{\frac{2B\lambda}{r}} + 3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3}))} \tag{24}$$

$$p \geqslant \frac{(c_{AC}+c_{AB})(c_{AC}r + (\alpha+r)((\sqrt{c_{AB}+c_{AC}} - \sqrt{c_{AB}})\sqrt{\frac{2B\lambda}{r}} + 3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3}))}{\alpha\sqrt{\frac{2B\lambda(c_{AC}+c_{AB}+p)}{r}} - (c_{AC}r + (\alpha+r)(((c_{AB}+c_{AC})^{1/2} - c_{AB}^{1/2})\sqrt{\frac{2B\lambda}{r}} + 3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3}))} \tag{25}$$

1.2 Upper bound

The intermediate nodes must keep their fee at most the cost of their clients to use on-chain transactions (either to make transactions, or open, reopen, or update the channel) or to create a direct channel with the destined node.
Alice analyze when will it be beneficial for her to open a direct channel with Charlie, or pay for Bob's service.
If Alice creates a direct unidirectional channel with Charlie, Bob's cost of channels doesn't change, while Alice and Charlie has a different cost from the previous section.

$$C_{o_B}(\lambda) = C_{n_B}(\lambda) = \frac{1}{2}\sqrt{\frac{2B\lambda c_{AB}}{r}} + \frac{3}{2}(\frac{2B\lambda}{r})^{1/3} \tag{26}$$

$$C_{o_A}(\lambda) = \frac{1}{2}\sqrt{\frac{2B\lambda c_{AB}}{r}} + \frac{1}{2}\sqrt{\frac{2B\lambda c_{AC}}{r}} \tag{27}$$

$$C_{o_C}(\lambda) = \frac{3}{2}(\frac{2B\lambda}{r})^{1/3} + \frac{1}{2}\sqrt{\frac{2B\lambda c_{AC}}{r}} \tag{28}$$

Collectively, Alice and Charlie pays

$$C_{o_{AC}}(\lambda) = \frac{1}{2}\sqrt{\frac{2B\lambda c_{AB}}{r}} + \frac{3}{2}(\frac{2B\lambda}{r})^{1/3} + \sqrt{\frac{2B\lambda c_{AC}}{r}} \tag{29}$$

If Alice and Charlie uses Bob's service, their cost of payment is

$$C_{n_A}(\lambda) = \frac{1}{2}\sqrt{\frac{2B\lambda c_{AB}}{r}} \tag{30}$$

$$c_{n_C}(\lambda) = \frac{3}{2}(\frac{2B\lambda}{r})^{1/3} \tag{31}$$

$$C_{n_{AC}}(\lambda) = \frac{1}{2}\sqrt{\frac{2B\lambda c_{AB}}{r}} + \frac{3}{2}(\frac{2B\lambda}{r})^{1/3} \tag{32}$$

regardless of who actually pays, their maximum willingness to pay is bounded by the change in cost of channels, which is simply the cost of the new channel.

$$C_{o_{AC}}(\lambda) \geqslant C_{n_{AC}}(\lambda) + T(\lambda) \tag{33}$$

$$T(\lambda) \leqslant C_{o_{AC}}(\lambda) - C_{n_{AC}}(\lambda) \tag{34}$$

$$\frac{mp}{c_{AC} + c_{AB} + p} \cdot \frac{\alpha}{\alpha + r} \leqslant \sqrt{\frac{2B\lambda c_{AC}}{r}} \tag{35}$$

Thus the bound of p is

$$p \leqslant \frac{(c_{AC} + c_{AB})\sqrt{\frac{2B\lambda c_{AC}}{r}}\frac{\alpha+r}{\alpha}}{m - \sqrt{\frac{2B\lambda c_{AC}}{r}}\frac{\alpha+r}{\alpha}} \tag{36}$$

With the optimial channel size $m = \sqrt{\frac{2B\lambda(c_{AC}+c_{AB}+p)}{r}}$, the upper bound becomes

$$p \leqslant \frac{(c_{AC} + c_{AB})\sqrt{\frac{2B\lambda c_{AC}}{r}}\frac{\alpha+r}{\alpha}}{\sqrt{\frac{2B\lambda(c_{AC}+c_{AB}+p)}{r}} - \sqrt{\frac{2B\lambda c_{AC}}{r}}\frac{\alpha+r}{\alpha}} \tag{37}$$

$$p \leqslant \frac{(c_{AC} + c_{AB})\sqrt{c_{AC}}(\alpha + r)}{\alpha\sqrt{(c_{AC} + c_{AB} + p)} - \sqrt{c_{AC}}(\alpha + r)} \tag{38}$$

2 Simplifying the bounds

With optimal channel size, and assume that all the c's are 1. The bounds of p are

$$\frac{(c_{AC}+c_{AB})(c_{AC}r+(\alpha+r)((\sqrt{c_{AB}+c_{AC}}-\sqrt{c_{AB}})\sqrt{\frac{2B\lambda}{r}}+3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3}))}{\alpha\sqrt{\frac{2B\lambda(c_{AC}+c_{AB}+p)}{r}}-(c_{AC}r+(\alpha+r)(((c_{AB}+c_{AC})^{1/2}-c_{AB}^{1/2})\sqrt{\frac{2B\lambda}{r}}+3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3}))} \tag{39}$$

$$\leqslant p \leqslant \frac{(c_{AC}+c_{AB})\sqrt{c_{AC}}(\alpha+r)}{\alpha\sqrt{(c_{AC}+c_{AB}+p)}-\sqrt{c_{AC}}(\alpha+r)} \tag{40}$$

$$\Rightarrow \frac{2(r+(\alpha+r)((\sqrt{2}-1)\sqrt{\frac{2B\lambda}{r}}+3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3}))}{\alpha\sqrt{\frac{2B\lambda(2+p)}{r}}-(r+(\alpha+r)((\sqrt{2}-1)\sqrt{\frac{2B\lambda}{r}}+3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3}))} \tag{41}$$

$$\leqslant p \leqslant \frac{2\alpha+2r}{\alpha\sqrt{2+p}-\alpha-r} \tag{42}$$

Simplifying the lower bound

$$2r+2(\alpha+r)(\sqrt{2}-1)\sqrt{\frac{2B\lambda}{r}}+2(\alpha+r)3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3} \leqslant \tag{43}$$

$$p\alpha\sqrt{\frac{2B\lambda(2+p)}{r}}-pr-p(\alpha+r)(\sqrt{2}-1)\sqrt{\frac{2B\lambda}{r}}-p(\alpha+r)3(4^{1/3}-2^{1/3})(\frac{B\lambda}{r})^{1/3} \tag{44}$$

$$\Rightarrow 2r+1.17157(\alpha+r)\sqrt{\frac{B\lambda}{r}}+1.96488(\alpha+r)(\frac{B\lambda}{r})^{1/3} \leqslant \tag{45}$$

$$p\alpha\sqrt{\frac{4B\lambda+2B\lambda p}{r}}-pr-.585786p(\alpha+r)\sqrt{\frac{B\lambda}{r}}-.98244p(\alpha+r)(\frac{B\lambda}{r})^{1/3} \tag{46}$$

$$\Rightarrow 2r+1.17157(\alpha+r)\sqrt{\frac{B\lambda}{r}}+1.96488(\alpha+r)(\frac{B\lambda}{r})^{1/3} \leqslant \tag{47}$$

$$p\alpha\sqrt{\frac{4B\lambda+2B\lambda p}{r}}-pr-.585786p(\alpha+r)\sqrt{\frac{B\lambda}{r}}-.98244p(\alpha+r)(\frac{B\lambda}{r})^{1/3} \tag{48}$$

$$\Rightarrow (2+p)r^{3/2}+.585786(2+p)(\alpha+r)\sqrt{B\lambda}+.98244(2+p)(\alpha+r)(B\lambda)^{1/3}r^{1/6} \leqslant \tag{49}$$

$$p\alpha\sqrt{2B\lambda(2+p)} \tag{50}$$

$$\Rightarrow r^{3/2}+.585786(\alpha+r)\sqrt{B\lambda}+.98244(\alpha+r)(B\lambda)^{1/3}r^{1/6} \leqslant \frac{p\alpha\sqrt{2B\lambda}}{\sqrt{2+p}} \tag{51}$$

Simplifying the upper bound

$$p\alpha\sqrt{2+p} - p\alpha - pr \leqslant 2\alpha + 2r \tag{52}$$
$$p\alpha\sqrt{2+p} - p\alpha - 2\alpha \leqslant 2r + pr \tag{53}$$
$$\frac{p\sqrt{2+p} - p - 2}{2+p} \leqslant \frac{r}{\alpha} \tag{54}$$
$$\frac{p}{\sqrt{2+p}} - 1 \leqslant \frac{r}{\alpha} \tag{55}$$
$$\tag{56}$$

So in general, without assumption of c, the lower bound

$$(c_{AC} + c_{AB} + p)c_{AC}r + (c_{AC} + c_{AB} + p)(\alpha + r)(\sqrt{c_{AB} + c_{AC}} - \sqrt{c_{AB}})\sqrt{\frac{2B\lambda}{r}} \tag{57}$$
$$+ (c_{AC} + c_{AB} + p)(\alpha + r)3(4^{1/3} - 2^{1/3})(\frac{B\lambda}{r})^{1/3} \leqslant p\alpha\sqrt{\frac{2B\lambda(c_{AC} + c_{AB} + p)}{r}} \tag{58}$$
$$\Rightarrow c_{AC}r + (\alpha + r)(\sqrt{c_{AB} + c_{AC}} - \sqrt{c_{AB}})\sqrt{\frac{2B\lambda}{r}} + 3(\alpha + r)(4^{1/3} - 2^{1/3})(\frac{B\lambda}{r})^{1/3} \tag{59}$$
$$\leqslant \frac{p\alpha\sqrt{2B\lambda}}{\sqrt{r(c_{AC} + c_{AB} + p)}} \tag{60}$$
$$\Rightarrow \frac{c_{AC}r^{3/2}}{\alpha\sqrt{2B\lambda}} + (1 + \frac{r}{\alpha})(\sqrt{c_{AB} + c_{AC}} - \sqrt{c_{AB}}) + 3(1 + \frac{r}{\alpha})(2^{1/6} - 2^{-1/6})(\frac{r}{B\lambda})^{1/6} \tag{61}$$
$$\leqslant \frac{p}{\sqrt{c_{AC} + c_{AB} + p}} \tag{62}$$

and the upper bound

$$p \leqslant \frac{(c_{AC} + c_{AB})\sqrt{c_{AC}}(\alpha + r)}{\alpha\sqrt{(c_{AC} + c_{AB} + p)} - \sqrt{c_{AC}}(\alpha + r)} \tag{63}$$
$$\frac{p}{\sqrt{c_{AC} + c_{AB} + p}} \leqslant \sqrt{c_{AC}} + \frac{\sqrt{c_{AC}}r}{\alpha} \tag{64}$$

Thus the bounds are simplified to

$$\frac{c_{AC}r^{3/2}}{\alpha\sqrt{2B\lambda}} + (1 + \frac{r}{\alpha})(\sqrt{c_{AB} + c_{AC}} - \sqrt{c_{AB}}) + 3(1 + \frac{r}{\alpha})(2^{1/6} - 2^{-1/6})(\frac{r}{B\lambda})^{1/6} \tag{65}$$
$$\leqslant \frac{p}{\sqrt{c_{AC} + c_{AB} + p}} \leqslant \sqrt{c_{AC}} + \frac{\sqrt{c_{AC}}r}{\alpha} \tag{66}$$

2.1 relationship between channel lifetime and frequency

Currently we have λ as the frequency of transactions, and α as the lifetime of a channel. Both of these paramaters are of exponential distributions. We recognize that they are related to each other, as λ affects α (the slower the frequency, the longer lifetime.) Note that an unidirectional channel should make $\frac{m}{c}$ transactions, and the frequency between each transaction builds α.

Intuitively, $\alpha \approx \frac{m}{c\lambda}$

- Let Y_i be i.i.d. random variables of exponential distribution with parameter λ, time between two transactions
- There are $\frac{m}{c}$ transactions
- Let $X = \sum_{i=0}^{\frac{m}{c}} Y_i$

$$\alpha = E[X] = \sum_{i=0}^{\frac{m}{c}} E[Y_i] = \sum_{i=0}^{\frac{m}{c}} \frac{1}{\lambda} = \frac{m}{c\lambda} \tag{67}$$

With optimal channel size $m = \sqrt{\frac{2B\lambda c}{r}}$, with all the parameters being nonnegative, and $c = c_{AC}$

$$\alpha = \frac{\sqrt{\frac{2B\lambda c}{r}}}{c\lambda} = \sqrt{\frac{2B}{r\lambda c_{AC}}} \tag{68}$$

Plugging into the bounds,

$$\frac{c_{AC} r^{3/2}}{\sqrt{\frac{2B}{r\lambda c_{AC}}}\sqrt{2B\lambda}} + (1 + \frac{r}{\sqrt{\frac{2B}{r\lambda c_{AC}}}})(\sqrt{c_{AB} + c_{AC}} - \sqrt{c_{AB}}) + 3(1 + \frac{r}{\sqrt{\frac{2B}{r\lambda c_{AC}}}})(2^{1/6} - 2^{-1/6})(\frac{r}{B\lambda})^{1/6} \tag{69}$$

$$\leqslant \frac{p}{\sqrt{c_{AC} + c_{AB} + p}} \leqslant \sqrt{c_{AC}} + \frac{\sqrt{c_{AC}}r}{\sqrt{\frac{2B}{r\lambda c_{AC}}}} \tag{70}$$

$$\Rightarrow \frac{c_{AC}^{3/2} r^2}{2B} + (1 + \sqrt{\frac{r^3 \lambda c_{AC}}{2B}})(\sqrt{c_{AB} + c_{AC}} - \sqrt{c_{AB}}) + 3(1 + \sqrt{\frac{r^3 \lambda c_{AC}}{2B}})(2^{1/6} - 2^{-1/6})(\frac{r}{B\lambda})^{1/6} \tag{71}$$

$$\leqslant \frac{p}{\sqrt{c_{AC} + c_{AB} + p}} \leqslant \sqrt{c_{AC}} + c_{AC} r^{3/2} \sqrt{\frac{\lambda}{2B}} \tag{72}$$