Commutative Algebra for Algebraic Geometry

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Part I

Algebraic Geometry, Hartshorne

In this part, I will be going through the work of Hartshorne as he laid out in [Har13]. The book is sectioned into 5 distinct chapters - 'Varieties', 'Schemes', 'Cohomology', 'Curves' and 'Surfaces'. We study them in order as they flow nicely.

1 Varieties

Definition 1 ***Affine n-Space over k:*** *Let k be a field and $n \in \mathbb{N}$. Then the affine n-space over k is defined*

$$\mathbb{A}^n_k = \{(k_1, ..., k_n) \in k^n\}$$

It seems that this is a silly definition, but later on we will see that it is useful to have a distinction between the variety $\mathbb{A}^n_k$ and the set of points in k^n. We later view $\mathbb{A}^n_k$ as an affine variety - an object in some arbitrary space rather than the set of k-tuples.

Let $A = k[x_1, ..., x_n]$ be the polynomial ring over k in n variables. Then $f \in A$ is a map $f : k^n \to k$. We define the vanishing locus of this function in the following way:

Definition 2 ***Vanishing Locus of a Polynomial:*** *Let $f \in A = k[x_1, ..., x_n]$, then the vanishing locus is*

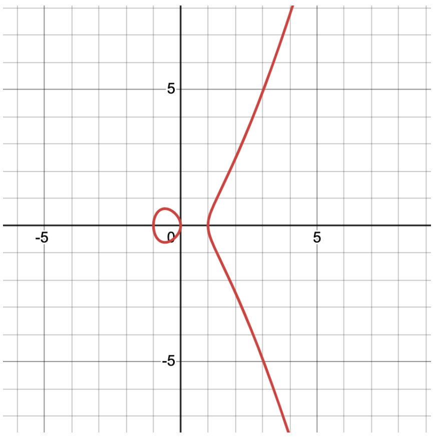
$$\mathbb{V}(f) := \{p \in \mathbb{A}^n_k : f(p) = 0\}$$

We can develop a more advanced analogue of this:

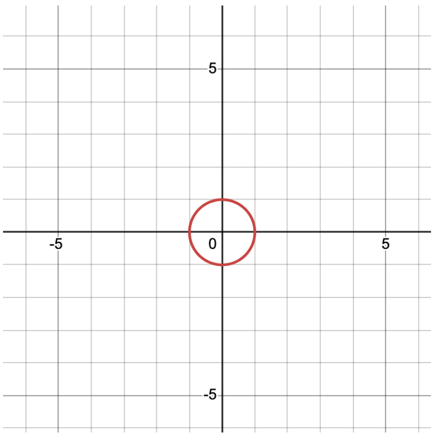
Definition 3 ***Vanishing Locus of a Set of Polynomials:*** *Let $T = \{f_i\}_{i\in I} \subset A = k[x_1, ..., x_n]$, then the vanishing locus of T is*

$$\mathbb{V}(T) := \{p \in \mathbb{A}^n_k : f(p) = 0, \quad \forall f \in T\}$$

Some examples:



(a) $\mathbb{V}(y^2 - x(x+1)(x-1))$



(b) $\mathbb{V}(x^2 + y^2 - 1)$

Figure 1: The vanishing locus of two separate polynomials plotted in $\mathbb{R}^2$

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References

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