Notes on 'Sheaf Theory' by Bredon

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Part I
Sheaf Theory, Bredon

1 Sheaves and Presheaves

1.1 Definitions

Definition 1 ***Presheaf:*** *A presheaf of abelian groups on a topological space X is a functor*

$$A : \mathfrak{Opn}_X^{op} \to \mathfrak{Ab}$$

i.e. an assignment to each open set $U \in Ob(\mathfrak{Opn}_X^{op})$ an abelian group $A(U) \in Ob(\mathfrak{Ab})$, and on inclusion functions $V \overset{\iota_U^V}{\hookrightarrow} U$ restriction functions $r_V^U : A(U) \to A(V)$ such that:

- $r_U^U = id_U$,
- $U \subset V \subset W \implies r_V^W r_U^V = r_U^W$

One can switch out the target category with other algebraic categories to obtain the definitions of presheaves over R-modules, algebras, etc.

Definition 2 ***Germ:*** *Let $x \in X$ be a point and A a presheaf over X. The set $\mathfrak{M}$ of elements $s \in A(U)$, $x \in U \in Ob\mathfrak{Opn}_X^{op}$. The germs of the presheaf A at x is thus the quotient*

$$\mathfrak{G}_{A,x} := \mathfrak{M}/\sim$$

where $s, t \in \mathfrak{M}$ are equal if there is $W \subset dom(s) \cap dom(t)$ such that $s|_W = t|W$. The equivalence class $[s]$ of germs of $A(U)$ is called the germ of s at $x \in U$.

The set of germs of A at x, $\mathcal{A}_x$ is then the direct limit

$$\mathcal{A}_s = \lim_{\to} A(U)$$

From here we can define a topology on the disjoint union $\sqcup_{x\in X}\mathcal{A}_x$ as follows. Fix an element $s \in A(U)$, for each $x \in U$ we have the germ s_x of s at x. For fixed s the set of all germs $s_x \in \mathcal{A}_x$ is defined to be an open set. With this definition we get the sheaf generated by the presheaf AL

$$\mathcal{A} = \mathfrak{Sheaf}(A) = \mathfrak{Sheaf}(U \to A(U))$$

Definition 3 ***Sheaf:*** *A sheaf on X is a pair $(\mathcal{A}, \pi)$ satisfying:*

- *$\mathcal{A}$ is a (typically non-Hasuforff) topological space,*
- *$\pi : \mathcal{A} \to X$ os a local homeomorphism*
- *Each $\mathcal{A}_x = \pi^{-1}(x)$ for $x \in X$ is an abelian group, and is called the stalk of $\mathcal{A}$ at x*
- *The group operations are continuous*

An example of a sheaf is Ω^0 of germs of $\mathcal{C}^\infty(\mathbb{R})$ functions on a differentiable manifold M^n, this is a sheaf of unital rings, and $\Omega^p(M^n)$ of germs of differential p-forms on M^n which is also a Ω^0-module.

If $\mathcal{A}$ is a sheaf on X with projection $\pi : \mathcal{A} \to X$ and if $Y \subset X$ then the restriction $\mathcal{A}|Y$ of $\mathcal{A}$ is defined:

$$\mathcal{A}|Y = \pi^{-1}(Y)$$

If $\mathcal{A}$ is a sheaf on X, $Y \subset X$ then a section of $\mathcal{A}$ over Y is a map $s : Y \to \mathcal{A}$ such that $\pi \circ s$ is the identity on Y. Note that every point $x \in Y$ admits a section s over some neighborhood U of x. We can then access the 0 section over any open set $U \subset X$ and so we can endow $\mathcal{A}(Y)$ with the structure of an abelian group. Thus the presheaf of sections of A is defined:

$$\Gamma(\mathcal{A}) = \mathcal{A}(X)$$

An example of a non-Hausdorff sheaf is the sheaf on the real line that has 0 stalk everywhere but 0 which has the stalk $\mathbb{Z}_2$. That is $\pi^{-1}(x \neq 0) = 0, \pi^{-1}(0) = \mathbb{Z}_2$