Notes on 'Éléments de Géométrie Algébrique'

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Part I
Éléments de Géométrie Algébrique I: The Language of Schemes

1 Affine Schemes

1.1 The Prime Spectrum of a Ring

Definition 1 ***Prime Spectrum of a Ring:*** *Let A be commutative ring and M n A-module. Then*

$$Spec(A) = \{\mathfrak{p} \subset A : \mathfrak{p} \text{ prime ideal in } A\}$$

For points $x \in X = Spec(A)$ it is usually convenient to write $\mathfrak{p}_x$. It is also important to note that for $Spec(A) = \phi$ we need A to be the 0 ring.

Definition 2 ***Local Ring of Fractions of a Ring:*** *Let A be a commutative ring. The local ring at $x \in SpecA$ is*

$$A_x = A_{\mathfrak{p}_x} = (A \backslash \mathfrak{p}_x)^{-1} A$$

There is also some more notation

- $\mathfrak{m}_x = \mathfrak{p}_x A_{\mathfrak{p}_x}$ is the (unique) maximal ideal of A_x,
- $k(x) = A_x/\mathfrak{m}_x$ is the residue field of A_x and it canonically isomorphic to the fraction field of the integral ring $A/\mathfrak{p}_x$. It is thus we identify $k(x) = A/\mathfrak{p}_x$,
- $M_x = M \otimes_A A_x$ is the module of fractions with denominators in $A \backslash \mathfrak{p}_x$,
- $V(E) = \{x \in X = \mathrm{Spec}(A) : E \subset j_x\}$
- $\mathfrak{rad}(E) = \bigcap_{x \in V(E)} \mathfrak{p}_x$

Proposition 1
- *$V(0) = X, V(1) = \phi$,*
- *$E \subset E' \implies V(E') \subset V(E)$,*
- *$V(\cup_\lambda E_\lambda) = \cap_\lambda V(E_\lambda)$,*
- *$V(EE') = V(E) \cup V(E')$,*
- *$V(E) = V(\mathfrak{rad}(E))$*

Proof. It is clear that all ideals contain the 0 ideal and so $V(0) = \mathrm{Spec}(A)$. The third and fourth subpropositions are clear via doing the inclusions. The final subproposition is trivial when one notices that

$$V(\mathfrak{rad}(E)) = V(\bigcap_{x \in V(E)} \mathfrak{p}_x) = \bigcup_{x \in V(E)} V(\mathfrak{p}_x) = V(E)$$

For each subset $Y \subset X$ we denoted $\mathfrak{p}(Y)$ to be the set of $f \in A$ such that $f(y) = 0$ for all $y \in Y$. It is hence clear that $\mathfrak{p}(Y) = \bigcap_{y \in Y} \mathfrak{p}_y$. So then more generally

$$\mathfrak{p}(\bigcup_\lambda Y_\lambda) = \bigcap_\lambda \mathfrak{p}(Y_\lambda)$$

And so identifying sets of the form $V(E)$ as closed yields a topology on $\mathrm{Spec}(A)$, called either the Zariski or Spectrum topology.

Proposition 2
- *For each subset $E \subset A$ we have $\mathfrak{p}(V(E)) = \mathfrak{rad}(E)$*

- *For each subset* $Y \subset X$ *we have* $V(\mathfrak{p}(Y)) = \bar{Y}$

Proof. The first part is trivial via the previous propositions. The second part is a result of checking that $V(\mathfrak{p}(Y))$ is the smallest closed subset of X containing Y. Indeed $V(\mathfrak{p}(Y)) \supset Y$ and is closed, and if $Y \subset V(E) \subset V(\mathfrak{p}(Y))$ then $f(y) = 0$ for all $f \in E$ and $y \in Y$. By inclusion reversion, $E \subset \mathfrak{p}(Y) \implies V(E) \supset V(\mathfrak{p}(Y))$ so it is true that $V(\mathfrak{p}(Y))$ is the smallest possible closed set containing Y.