Commutative Algebra for Algebraic Geometry

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Part I
Algebraic Geometry, Hartshorne

In this part, I will be going through the work of Hartshorne as he laid out in [Har13]. The book is sectioned into 5 distinct chapters - 'Varieties', 'Schemes', 'Cohomology', 'Curves' and 'Surfaces'. We study them in order as they flow nicely.

1 Varieties

Definition 1 ***Affine n-Space over k:*** *Let k be a field and $n \in \mathbb{N}$. Then the affine n-space over k is defined*

$$\mathbb{A}_k^n = \{(k_1, ..., k_n) \in k^n\}$$

It seems that this is a silly definition, but later on we will see that it is useful to have a distinction between the variety $\mathbb{A}_k^n$ and the set of points in k^n. We later view $\mathbb{A}_k^n$ as an affine variety - an object in some arbitrary space rather than the set of k-tuples.

Let $A = k[x_1, ..., x_n]$ be the polynomial ring over k in n variables. Then $f \in A$ is a map $f : k^n \to k$. We define the vanishing locus of this function in the following way:

Definition 2 ***Vanishing Locus of a Polynomial:*** *Let $f \in A = k[x_1, ..., x_n]$, then the vanishing locus is*

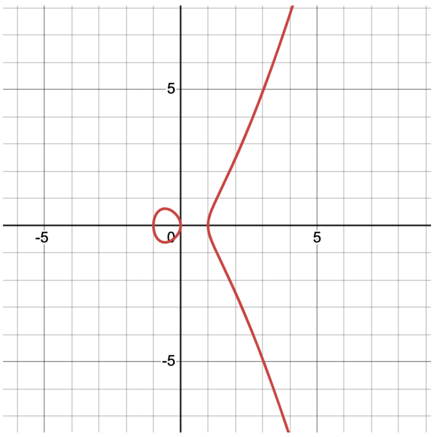
$$\mathbb{V}(f) := \{p \in \mathbb{A}_k^n : f(p) = 0\}$$

We can develop a more advanced analogue of this:

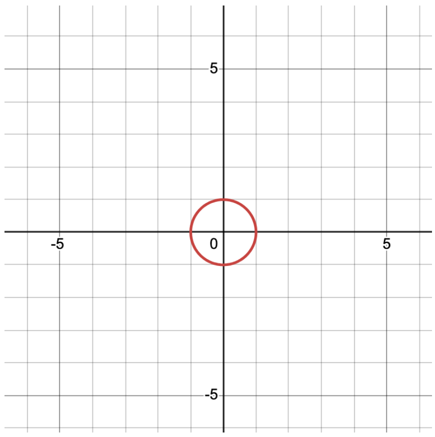
Definition 3 ***Vanishing Locus of a Set of Polynomials:*** *Let $T = \{f_i\}_{i \in I} \subset A = k[x_1, ..., x_n]$, then the vanishing locus of T is*

$$\mathbb{V}(T) := \{p \in \mathbb{A}_k^n : f(p) = 0, \quad \forall f \in T\}$$

Some examples:



(a) $\mathbb{V}(y^2 - x(x+1)(x-1))$



(b) $\mathbb{V}(x^2 + y^2 - 1)$

Figure 1: The vanishing locus of two separate polynomials plotted in $\mathbb{R}^2$

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6.2 Localisation

A local ring is a ring with a single unique maximal ideal. The technique of localisation reduces many problems in commutative algebra to problems on commutative rings. The idea of localisation is as follows. Given a point p in an algebraic set $X \subset \mathbb{A}_k^n$, we want to investigate what X looks like near p, that is we want to investigate arbitrarily small open neighbourhoods of p in the Zariski topology. The Zariski open neighbourhoods of p are sets of the form $X \backslash Y$ for $p \not{i}nY \subset X$.

6.2.1 Fractions

Definition 4 ***Localisation of an R-Module via a Multiplicatively Closed Subset U:*** *Let R be a ring, M an R-module and $U \subset R$ a multiplicatively closed subset. The localisation of M at U, $M[U^{-1}]$ is the set of equivalent classes of pairs $(m, u) \sim (m', u')$ where $m, m' \in Mandu, u' \in U$ are related if there is some $v \in U$ such that $v(mu' - m'u) = 0$.*

Proposition 1 *Let U be multiplicatively closed set of R and let M be an R-module. An element $m \in M$ goes to 0 in $M[U^{-1}]$ under the map $\pi : M \to M[U^{-1}], m \to m/1$ if m is annihilated by an element $u \in U$. In particular, if M is finitely generted then $M[U^{-1}] = 0$ iff M is annihilated by an element of U.*

proof

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64.1 Operads

Definition 5 ***Operad:*** *An operad P consists of a set of colours C and for each $n \geq 0$ and sequence $c_1, ..., c_n, c$ of colours in C, a set $P(c_1, ..., c_n; c)$ of operations, thought of as taking n inputs of colours $c_1, ..., c_n$ and with output of colour c. Moreover there are the structure maps*

- $\forall c \in C, \exists 1_c \in P(c; c)$,
- *For $\sigma \in \Sigma_n$ a map*
$$\sigma^* : P(c_1, ..., c_n; c) \to P(c_{\sigma(1)}, ..., c_{\sigma(n)}; c)$$
denoted $\sigma^* \circ p = p \circ \sigma$
- *For any sequence $c_1, ..., c_n$ and n-tuple of sequences $d^i_1, ..., d^i_{k_i}$, a composition*
-
$$\gamma : P(c_1, ..., c_n; c) \times \prod_{i=1}^{n} P(d^i_1, ..., d^i_{k_i}; c_i) \to P(d^1_1, ..., d^n_{k_n}; c)$$
which is written as $\gamma(p, q_1, ..., 1_n) \to p \circ (q_1, ..., q_n)$

and there are further requirements on the structure maps:

- $\forall p \in P(c_1, ..., c_n; c), \gamma(1_c, p) = p$,
- $\forall p \in P(c_1, ..., c_n; c), \gamma(p, 1_{c_1}, ..., 1_{c_n}) = p$

There are some classical notation we must be weary of and state here. Let P be an operad with a singleton colour set, i.e. $C = \{*\}$. Then we can write $P(c_1, ..., c_n; c) = P(c, ..., c; c) =: P(n)$. There is an obvious formulation for the compostion in this case:

$$P(n) \times \prod_{i=1}^{n} P(k_i) \to P(k_1 + ... + k_n)$$

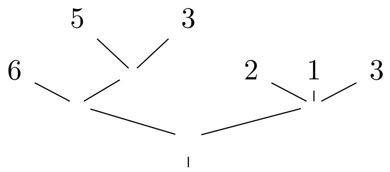
In this case we say P is uncoloured. If $C \notin \{\phi, \{*\}\}$ then we say P is a coloured operad. We can then clearly see that

$$\begin{tikzcd}
\text{Monoids} \arrow[r, hook] \arrow[d, hook] & \text{Categories} \arrow[d, hook] \\
\text{Uncoloured Operads} \arrow[r, hook] & \text{Operads}
\end{tikzcd}$$

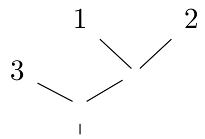
The definition of an operad allows for $n = 0$ in $P(c_1, ..., c_n; c)$, which we define as $P(-; c)$. The elements of $P(-; c)$ are called the constants of colour c, and an operad with $P(-; c) = \{*_c\}$ is called unital. An operad P is open if there are no constants for any colour, i.e. its interior P^o (the set of constants) is empty.

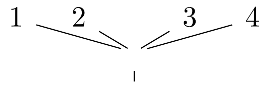
The most fundamental examples of operads are **Com** and **Ass**:

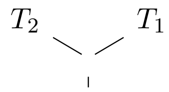
- **Com** is the commutative operad with $\mathbf{Com}(n) = \{*\}$,
- **Ass** is the associative operad with $\mathbf{Ass}(n) = \Sigma_n$
- $\mathbf{Tree}^{pl}$ is the planar tree operad. The n-ary operations here are the set of planar rooted trees with n numbered leaves.

$\mathbf{Tree}^{pl}(6) \ni \hat{T} =$ 

$\mathbf{Tree}^{pl}(2) \ni T =$ 

, $\mathbf{Tree}^{pl}(3) \ni T_1 =$ 

, $\mathbf{Tree}^{pl}(4) \ni T_2 =$ 

$\mathbf{Tree}^{pl}(7) \ni \tilde{T} =$ 

Then in fact $\gamma(T, T_1, T_2) = \tilde{T}$. The operation of composition on the operad $\mathbf{Tree}^{pl}$ is computed as $\gamma(T \in \mathbf{Tree}^{pl}(n), T_1 \in \mathbf{Tree}^{pl}(k_1), ..., T_n \in \mathbf{Tree}^{pl}(k_n)) = \hat{T}$ where $\hat{T} \in \mathbf{Tree}^{pl}(k_1 + ... + k_n)$ is the original T but with the subtree T_i grafted onto the leaf i for all choices of i.

Definition 6 ***Topological Operad:*** *A topological operad is an operad P where each set of operations $P(c_1, ..., c_n; c)$ is equipped with some topology and all the structure maps are continuous with respect to this topology.*

The most basic form of a topological operad is the little d-cubs operad $\mathbf{E}_d$. The space $\mathbf{E}_d(n)$ is the space of n numbered d-dimensional cubes inside the d-dimensional unit cube $[0,1]^d$. The operadic composition between $p \in \mathbf{E}_d(n)$ with operations $q_1, ..., q_n$ is given by substituting the rescaled q_i into the ith cube of p. Note that this is really just a topological analogue to the planar tree operad $\mathbf{Tree}^{pl}$, where instead of grafting trees onto leaves we are scaling and embedding cubes in some smooth way.

More specifically, a point in $\mathbf{E}_d(n)$ is an n-tuple of embeddings $f_1, ..., f_n : [0,1]^d \to [0,1]^d$ satisfying:

- Each f_i is the composition of d affine embeddings,
- The interiors of the cubes embedded by f_i are mutually disjoint

We can now observe that operads form a category in a very natural fashion. Given two operads P, Q, a morphism $\varphi : P \to Q$ is a function $f : C_p \to C_Q$ on operadic colours and for each sequence $c_1, ..., c_n; c$ of C_P, we have

$$\varphi_{(c_1, ..., c_n; c)} : P(c_1, ..., c_n; c) \to Q(f(c_1), ..., f(c_n); f(c))$$

that is compatible in the natural way with Σ_n actions

64.2 Algebras for Operads

Definition 7 ***Operadic Algebras:*** *Let P be an operad. A P-algebra A is a family of sets $\{A_c\}_{c \in C_P}$ together with maps*

$$P(c_1, ..., c_n; c) \times A_{c_1} \times ... \times A_{c_n} \to A_c$$

written $(p, a_1, ..., a_n) \to A(p)(a_1, ..., a_n)$. These maps also satisfy:

- $1_c(a) = a \quad \forall a \in A_c$
- $\sigma \in \Sigma_n, a_i \in A_{c_i}, \sigma^* p(a_{\sigma(1)}, ..., a_{\sigma(n)}) = p(a_1, ..., a_n)$

Definition 8 ***Morphisms of Operadic Algebras:*** *Let A, B be two P-algebras. A morphism $f : A \to B$ is a family of maps*

$$f_c : A_c \to B_c$$

which are compatible:

$$f_c(A(p)(a_1, ..., a_n)) = B(p)(f_{c_1}(a_1), ..., f_{c_n}(a_n))$$

Definition 9 ***Category of P-Algebras:*** *Let P be an operad. Then we have a category of P-algebras Alg_P with:*

- $ObAlg_P = \{P - algebras\ A\}$
- $Hom_{Alg_P}(A, B) = \{f : A \to B : f \text{ is a morphism of algebras}\}$

We can now see some examples of operadic algebras. A **Com** algebra is a set A together with a map $\mu_n : A^{\times n} \to A$ for each $n \geq 0$. We can then verify that the category of algebras over the commutative operad, $\mathrm{Alg}_{\mathbf{Com}}$ is the category of commutative monoids. In a similar way $\mathrm{Alg}_{\mathbf{Ass}}$ is the category of associative monoids.

Consider the little-d cubes operad $\mathbf{E}_d$. Let X be a topological space with basepoint x_0. Then the loop space of X is ΩX, the space of basepoint preserving maps $S^1 \to X$, or in otherwords $\{\omega : [0,1] \to X, \omega(\partial[0,1]) = x_0\}$. One can then inductively construct the d-fold loop space $\Omega^d X = \Omega(\Omega^{d-1} X)$. $\Omega^d X$ is very naturally a $\mathbf{E}_d$ algebra.

64.3 Trees

64.4 Alternative Definitions for Operads

64.5 Free Operads

64.6 The Tensor Product of Operads

64.7 The Boardman-Vogt Resolution of an Operad

64.8 Configuration Spaces and the Fulton-MacPherson Operad

64.9 Configuration Spaces and the Operad of Little Cubes

65 Simplicial Sets

65.1 The Simplex Category Δ

Definition 10 ***The Simplex Category Δ:*** *Δ is the category with:*

- $Ob\Delta = \mathbb{N}$,
- $Hom_\Delta([n], [m]) = \{\text{order preserving maps } [n] \to [m]\}$

There are special maps in Δ - the elementary faces $\delta^i : [m-1] \to [m]$ and elementary degeneracies $\sigma^i : [m] \to [m-1]$ $0 \leq i \leq m-1$:

$$\delta^i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}, \quad \sigma^i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

These have some nice relations, called the cosimplicial identities:

- $\sigma_i \sigma_j = \sigma_{j-1} \sigma_i, \quad i < j$
- $\delta_j \delta_i = \delta_i \delta_{j-1}$

65.1.1 Limits and Colimits of The Simplicial Category

$$\begin{tikzcd} k \arrow[r, "f"] \arrow[d, "g"] & n \arrow[d] \\ m \arrow[r] & m+n \end{tikzcd}$$

is a pushout, where $f(i) = i, g(i) = m - k + i$

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