

Commutative Algebra for Algebraic Geometry

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Part I

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- 1.7 Intersections in Projective Space

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Commutative Geometry with a View Toward Algebraic Geometry, Eisenbud

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Part III

Algebraic Geometry I: Schemes, Gortz-Wedhorn

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