Commutative Algebra for Algebraic Geometry

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Part I
Algebraic Geometry, Hartshorne

In this part, I will be going through the work of Hartshorne as he laid out in [Har13]. The book is sectioned into 5 distinct chapters - 'Varieties', 'Schemes', 'Cohomology', 'Curves' and 'Surfaces'. We study them in order as they flow nicely.

1 Varieties

Definition 1 ***Affine n-Space over k:*** *Let k be a field and $n \in \mathbb{N}$. Then the affine n-space over k is defined*

$$\mathbb{A}_k^n = \{(k_1, ..., k_n) \in k^n\}$$

It seems that this is a silly definition, but later on we will see that it is useful to have a distinction between the variety $\mathbb{A}_k^n$ and the set of points in k^n. We later view $\mathbb{A}_k^n$ as an affine variety - an object in some arbitrary space rather than the set of k-tuples.

Let $A = k[x_1, ..., x_n]$ be the polynomial ring over k in n variables. Then $f \in A$ is a map $f : k^n \to k$. We define the vanishing locus of this function in the following way:

Definition 2 ***Vanishing Locus of a Polynomial:*** *Let $f \in A = k[x_1, ..., x_n]$, then the vanishing locus is*

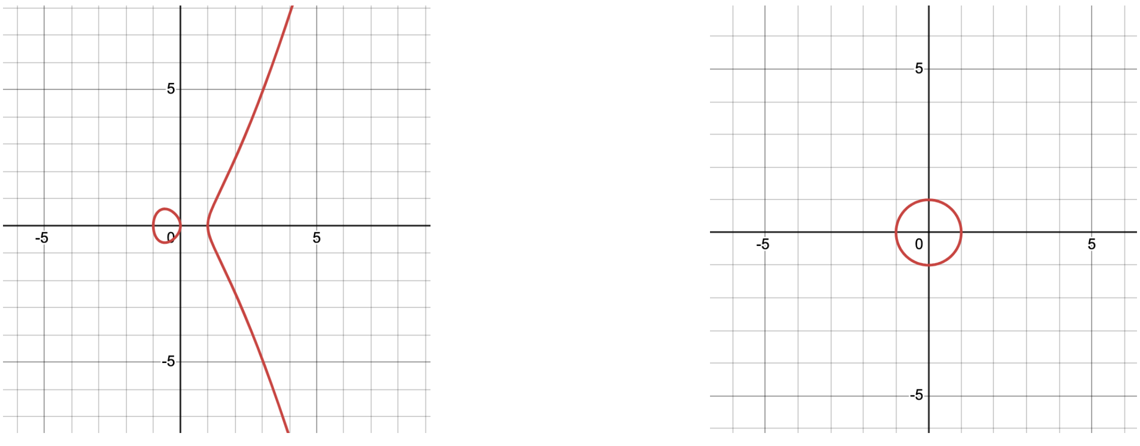
$$\mathbb{V}(f) := \{p \in \mathbb{A}_k^n : f(p) = 0\}$$

We can develop a more advanced analogue of this:

Definition 3 ***Vanishing Locus of a Set of Polynomials:*** *Let $T = \{f_i\}_{i\in I} \subset A = k[x_1, ..., x_n]$, then the vanishing locus of T is*

$$\mathbb{V}(T) := \{p \in \mathbb{A}_k^n : f(p) = 0, \quad \forall f \in T\}$$

Some examples:



(a) $\mathbb{V}(y^2 - x(x+1)(x-1))$ (b) $\mathbb{V}(x^2 + y^2 - 1)$

Figure 1: The vanishing locus of two separate polynomials plotted in $\mathbb{R}^2$

5.5 Birational Transformations

5.6 Classification of Surfaces

Part II

Commutative Geometry with a View Toward Algebraic Geometry, Eisenbud

6 Preliminaries

6.1 Rings and Ideals

Definition 4 ***Ring:*** *A ring is an abelian group R with a multiplication operation $*: R \times R \to R$ as well as an identity element $1 \in R$ such that:*

$$a(bc) = (ab)c \quad \forall a, b, c \in R$$

$$a(b + c) = ab + ac \quad \forall a, b, c \in R$$

$$(b + c) = ba + ca \quad \forall a, b, c \in R$$

$$1a = a1 = a \quad \forall a \in R$$

A ring is commutative if the ring commutes with respect to multiplication, that is $ab = ba \quad \forall a, b \in R$.

Definition 5 ***A Unit in a Ring:*** *Let R be a ring. An element $u \in R$ is a unit if it is invertible, that is there exists some $v \in R$ such that $vu = 1 \in R$.*

Proposition 1 ***Uniqueness of the Multiplicative Inverse of an Element in a Ring:*** *Let $u \in R$ where R is a ring. Then $us = ut = 1 \implies s = t$, i.e. inverses are unique in R and we can speak of 'the' inverse of u.*

Proof Consider the same set up. Then we have $su = 1 = ut$ [1]. We then have

$$\begin{aligned} s &= s1 \\ &= s(ut) \\ &= (su)t \\ &= t \end{aligned}$$

So we are done.

Definition 6 ***Field:*** *A field is a non-zero ring such that every non-zero element is invertible.*

Definition 7 ***Zero Divisor of a Ring:*** *Let R be a ring. A zero-divisor in R is a non-zero element u such that there is another non-zero element s with $us = 0$*

Whilst this seems rather abstract, zero divisors crop up more frequently than you would imagine. For example if we consider the hours on a clock, with the multiplication operation between hours being the usual one (that is the hour 3 multiplied by the hour 5 is the hour 15, but on a clock this would be the hour 3), then we have zero-divisors for any integer n such that there is some k with $nk = 12m$ for some $m \in \{0, ..., 11\}$. For example $3, 4$ are a pair of zero-divisors, $2, 6$ is another example.

Definition 8 ***Ideal of a Ring:*** *Let R be a ring. An ideal I in R is an additive subgroup such that if $rinR, s \in I$ then $rs \in I$. An ideal I is said to be generated by the subset $\mathcal{S} \subseteq I$ if any element in I can be expressed as a linear combination (over R) of elements in S. More specifically,*

$$\exists \theta_1, ..., \theta_n \in R, s_1, ..., s_n \in \mathcal{S} : r = \sum_{i=1}^{n} \theta_i s_i$$

[1] It seems that we have used commutativity of R here but we have not. If $us = 1$ then $(su)s = s(us) = s \implies su = (su)ss^{-1} = ss^{-1} = 1$

Some important notes here. A ring is **principal** if it is generated by a single element, in which case we write $I = (s)$. An ideal $I \subset R$ is prime if for any $f, g \in R$, if we have $fg \in I$ then either f or g is in I. A ring R is a domain if (0) is a prime ideal.[2] A maximal ideal of R is a proper ideal $\mathfrak{m}$ that is not contained in any other ideal. Moreover, if $\mathfrak{m}$ is a maximal ideal, then $R/\mathfrak{m}$ is a field.

Proposition 2 *Let R be a ring and $\mathfrak{m}$ a maximal ideal of R. Then $R/\mathfrak{m}$ is a field.*

Definition 9 ***Commutative Algebra over a Ring:*** *Let R be an abelian ring. A commutative algebra over R is a commutative ring S with a ring homomorphism $\alpha : R \to S$.*

Proposition 3 *Any ring is an algebra over the ring over integers $\mathbb{Z}$.*

Definition 10 ***Subalgebras:*** *Let S be an algebra over a commutative ring R. A subring S' is a commutative R-subalgebra of S if $Im(\alpha) = \alpha(R) \subset S'$*

A homomorphism of R-algebras $\phi : S \to T$ is a homomorphism of rings such that $\phi(rs) = r\phi(s) \quad \forall r \in R, s \in S$.

6.1.1 Unique Factorisation

Let R be a ring. An element $r \in R$ is irreducible if it is not a unit and $r = st$ implies that one of s, t is a unit in R. A ring R is a **UFD**if any factorisation is unique up to scaling by units in R.

6.1.2 Modules

Definition 11 ***Modules over Rings:*** *Let R be a ring. An R-module M is an abelian group together with an action with R, i.e. a map $\mathbb{R} \times M \to M$ expressed as $(r, m) \to rm$ satisfying $\forall r, s \in R, mn \in M$:*

- $r(sm) = (rs)m$
- $r(m + n) = rm + rn$
- $(r + s)m = rm + sm$
- $1m = m$

The most interesting R-modules are those that take the form of ideals I and their corresponding factor rings R/I. If M is an R-module then the annihilator of M is

$$\mathfrak{ann}_R(M) := \{r \in R : rM = 0\}$$

An example of which is $\mathfrak{ann}_R(R/I) = I$ for any ideal $I \subset R$. We can generalise this notion of quotients. Let I, J be ideals of R, we write $(I : J) = \{f \in R : fJ \subset I\}$. Generalising further we get the notion of submodules. Let M, N be submodules of an R-module P, we write $(M : N) = \{f \in R : fM \subset N\}$.

If M, N are R-modules then the direct sum $M \oplus N$ is the module $M \oplus N = \{(m, n) : m \in M, n \in N\}$. There are the natural inclusion maps $M \hookrightarrow M \oplus N, m \to (m, 0)$ and projection maps $\pi : M \oplus N \to M, (m, n) \to m$. If we have existence of maps $\alpha : M \to P, \sigma : P \to M, \sigma \circ \alpha = id_P, \alpha \circ \sigma = id_M$ then we say M is a direct summand of P. In this case we actually have a nice formula,

$$P \simeq M \oplus \ker \sigma$$

The simplest form of R-modules are just direct sums of the original ring. Modules of this form are called free modules (over R). A small digression is made here. The direct product of R-modules M_i, $\prod_i M_i$ is the set of tuples (m_i) whereas the direct sum is $\oplus_i M_i \subset \prod_i M_i$ where an element $\tilde{m} \in \oplus_i M_i$ is an n-tuple with the additional constraint that all but finitely many are equal to 0.

[2] This seems like a weird definition at first, but it is equivalent to not having any zero-divisors. If $fg \in (0)$ then (0) prime would men either $f = 0$ or $g = 0$, i.e. no zero-divisors

A free R-module is a module that is isomorphic to a direct sum of copies of R. If M is a finitely generated free R-module then $M \cong R^n$ for some $n \in \mathbb{N}$. IF A, B, C are R-modules and $\alpha : A \to B, \beta : B \to C$ are homomorphisms, then a sequence

$$A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C$$

is exact if $Im(\alpha) = \ker(\beta)$. In general a sequence

$$0 \to A_1 \to A_2 \to ... \to A_n$$

is exact if $\ker(\phi_i : A_i \to A_{i+1}) = Im(\phi_{i-1} : A_{i-1} \to A_i)$ A short exact sequence is an exact sequence of the form

$$0 \to A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C \to 0$$

Some nice examples follow. If M_1, M_2 are submodules of M then $M_1 + M_2 \subset M$ is also a submodule. We get the short exact sequence

$$0 \to M_1 \cap M_2 \stackrel{\iota}{\to} M_1 \oplus M_2 \stackrel{(m_1,m_2) \to m_1 - m_2}{\to} M_1 + M_2 \to 0$$

7 Basic Constructions

7.1 Localisation

A local ring is a ring with a single unique maximal ideal. The technique of localisation reduces many problems in commutative algebra to problems on commutative rings. The idea of localisation is as follows. Given a point p in an algebraic set $X \subset \mathbb{A}^n_k$, we want to investigate what X looks like near p, that is we want to investigate arbitrarily small open neighbourhoods of p in the Zariski topology. The Zariski open neighbourhoods of p are sets of the form $X \backslash Y$ for $p \not{i}n Y \subset X$.

7.1.1 Fractions

Definition 12 ***Localisation of an R-Module via a Multiplicatively Closed Subset U:*** *Let R be a ring, M an R-module and $U \subset R$ a multiplicatively closed subset. The localisation of M at U, $M[U^{-1}]$ is the set of equivalent classes of pairs $(m, u) \sim (m', u')$ where $m, m' \in M, u, u' \in U$ are related if there is some $v \in U$ such that $v(mu' - m'u) = 0$.*

Proposition 4 *Let U be multiplicatively closed set of R and let M be an R-module. An element $m \in M$ goes to 0 in $M[U^{-1}]$ under the map $\pi : M \to M[U^{-1}], m \to m/1$ if m is annihilated by an element $u \in U$. In particular, if M is finitely generted then $M[U^{-1}] = 0$ iff M is annihilated by an element of U.*

Proof Let $Ann_R(m) = \{r \in R : rm = 0\}$ be the *annihilator of M in R*. Then $m \to m/1 \in M[U^{-1}]$ maps to 0 if it is equivalent to 0 under the relation , that is to say $m/1 \sim 0 \iff \exists u \in U : u(m-0) = um = 0$. That is the annihilator of m is some subset of U.

The first example of localisation is **quotient field of an integral domain**. Let R be an integral domain, and take the localisation of R with respect to $U = R\backslash\{0\}$. This localisation, $R[U^{-1}] =: K(R)$ is the **total quotient ring of** R.

If P is a prime ideal of R and $U = R\backslash P$, then we have another localisation $R[U^{-1}]$. Let R be the coordinate ring of a variety X then the local ring of X at a point $x \in X$ is then the local ring found via inverting any elements that don't vanish at x. Recall that a point $x \in X$ corresponds to a prime ideal $\mathfrak{m}_x$ of functions that vanish at x. Then the local ring (which I learnt denoted as $\mathcal{O}_{x,X}$) is $R[(R\backslash P)^{-1}]$.

We can compute some examples. Let $X = \mathbb{V}(x^2 + y^2 - 1) \subset \mathbb{A}^2$. The local ring at $(1, 0)$ is then:

$$\mathcal{O}_{(1,0),X} := (K[x, y]/(x^2 + y^2 - 1))_{(x-1,y)}$$

This seems rather abstract but we can directly compute the unique maximal ideal. The maximal ideal $m_{(1,0)} = (x-1, y)$ has $m_{(1,0)}^2 = ((x-1)^2, (x-1)y, y^2)$. Under the relation generated in the coordinate ring we know that $x^2 + y^2 = 1$:

$$\begin{aligned} x^2 + y^2 - 1 &= (x-1)(x+1) + y^2 \\ \implies x - 1 &= y^2/(x+1) \end{aligned}$$

i.e. that $x - 1 \in m_{(1,0)}^2$, so $m_(1,0)^2 = (y)$ and $\mathcal{O}_{(1,0),X}$ is completely generated by y. There are a couple of things to note here. In this case, when $t \in m_p \backslash m_p^2$ is a generator we say t is a uniformiser for the maximal local ring. Also, have have just shown that the local ring is 1 dimensional as viewed as a vector space over $K[X]/\mathfrak{m}_p$. This is an algebraic criterion for non-singularity, that is the point $(1, 0) \in X$ is a non-singular point.

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65.1 Operads

Definition 13 ***Operad:*** *An operad P consists of a set of colours C and for each $n \geq 0$ and sequence $c_1, ..., c_n, c$ of colours in C, a set $P(c_1, ..., c_n; c)$ of operations, thought of as taking n inputs of colours $c_1, ..., c_n$ and with output of colour c. Moreover there are the structure maps*

- $\forall c \in C, \exists 1_c \in P(c; c)$,
- *For $\sigma \in \Sigma_n$ a map*
$$\sigma^* : P(c_1, ..., c_n; c) \to P(c_{\sigma(1)}, ..., c_{\sigma(n)}; c)$$
denoted $\sigma^* \circ p = p \circ \sigma$
- *For any sequence $c_1, ..., c_n$ and n-tuple of sequences $d_1^i, ..., d_{k_i}^i$, a composition*
-
$$\gamma : P(c_1, ..., c_n; c) \times \prod_{i=1}^{n} P(d_1^i, ..., d_{k_i}^i; c_i) \to P(d_1^1, ..., d_{k_n}^n; c)$$
which is written as $\gamma(p, q_1, ..., 1_n) \to p \circ (q_1, ..., q_n)$

and there are further requirements on the structure maps:

- $\forall p \in P(c_1, ..., c_n; c), \gamma(1_c, p) = p$,
- $\forall p \in P(c_1, ..., c_n; c), \gamma(p, 1_{c_1}, ..., 1_{c_n}) = p$

There are some classical notation we must be weary of and state here. Let P be an operad with a singleton colour set, i.e. $C = \{*\}$. Then we can write $P(c_1, ..., c_n; c) = P(c, ..., c; c) =: P(n)$. There is an obvious formulation for the compostion in this case:

$$P(n) \times \prod_{i=1}^{n} P(k_i) \to P(k_1 + ... + k_n)$$

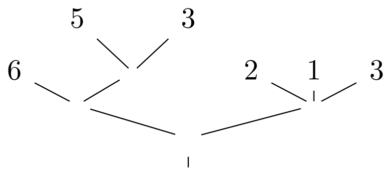
In this case we say P is uncoloured. If $C \notin \{\phi, \{*\}\}$ then we say P is a coloured operad. We can then clearly see that

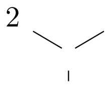
$$\begin{tikzcd}
\text{Monoids} \arrow[r, hook] \arrow[d, hook] & \text{Categories} \arrow[d, hook] \\
\text{Uncoloured Operads} \arrow[r, hook] & \text{Operads}
\end{tikzcd}$$

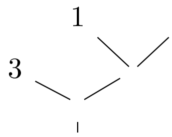
The definition of an operad allows for $n = 0$ in $P(c_1, ..., c_n; c)$, which we define as $P(-; c)$. The elements of $P(-; c)$ are called the constants of colour c, and an operad with $P(-; c) = \{*_c\}$ is called unital. An operad P is open if there are no constants for any colour, i.e. its interior P^o (the set of constants) is empty.

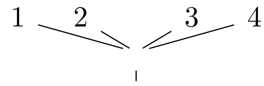
The most fundamental examples of operads are **Com** and **Ass**:

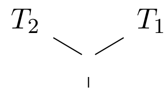
- **Com** is the commutative operad with $\mathbf{Com}(n) = \{*\}$,
- **Ass** is the associative operad with $\mathbf{Ass}(n) = \Sigma_n$
- $\mathbf{Tree}^{pl}$ is the planar tree operad. The n-ary operations here are the set of planar rooted trees with n numbered leaves.

$\mathbf{Tree}^{pl}(6) \ni \hat{T} =$ 

$\mathbf{Tree}^{pl}(2) \ni T =$ 

, $\mathbf{Tree}^{pl}(3) \ni T_1 =$ 

, $\mathbf{Tree}^{pl}(4) \ni T_2 =$ 

$\mathbf{Tree}^{pl}(7) \ni \tilde{T} =$ 

Then in fact $\gamma(T, T_1, T_2) = \tilde{T}$. The operation of composition on the operad $\mathbf{Tree}^{pl}$ is computed as $\gamma(T \in \mathbf{Tree}^{pl}(n), T_1 \in \mathbf{Tree}^{pl}(k_1), ..., T_n \in \mathbf{Tree}^{pl}(k_n)) = \hat{T}$ where $\hat{T} \in \mathbf{Tree}^{pl}(k_1 + ... + k_n)$ is the original T but with the subtree T_i grafted onto the leaf i for all choices of i.

Definition 14 ***Topological Operad:*** *A topological operad is an operad P where each set of operations $P(c_1, ..., c_n; c)$ is equipped with some topology and all the structure maps are continuous with respect to this topology.*

The most basic form of a topological operad is the little d-cubs operad $\mathbf{E}_d$. The space $\mathbf{E}_d(n)$ is the space of n numbered d-dimensional cubes inside the d-dimensional unit cube $[0,1]^d$. The operadic composition between $p \in \mathbf{E}_d(n)$ with operations $q_1, ..., q_n$ is given by substituting the rescaled q_i into the ith cube of p. Note that this is really just a topological analogue to the planar tree operad $\mathbf{Tree}^{pl}$, where instead of grafting trees onto leaves we are scaling and embedding cubes in some smooth way.

More specifically, a point in $\mathbf{E}_d(n)$ is an n-tuple of embeddings $f_1, ..., f_n : [0,1]^d \to [0,1]^d$ satisfying:

- Each f_i is the composition of d affine embeddings,
- The interiors of the cubes embedded by f_i are mutually disjoint

We can now observe that operads form a category in a very natural fashion. Given two operads P, Q, a morphism $\varphi : P \to Q$ is a function $f : C_p \to C_Q$ on operadic colours and for each sequence $c_1, ..., c_n; c$ of C_P, we have

$$\varphi_{(c_1,...,c_n;c)} : P(c_1, ..., c_n; c) \to Q(f(c_1), ..., f(c_n); f(c))$$

that is compatible in the natural way with Σ_n actions

65.2 Algebras for Operads

Definition 15 ***Operadic Algebras:*** *Let P be an operad. A P-algebra A is a family of sets $\{A_c\}_{c \in C_P}$ together with maps*

$$P(c_1, ..., c_n; c) \times A_{c_1} \times ... \times A_{c_n} \to A_c$$

written $(p, a_1, ..., a_n) \to A(p)(a_1, ..., a_n)$. These maps also satisfy:

- $1_c(a) = a \quad \forall a \in A_c$
- $\sigma \in \Sigma_n, a_i \in A_{c_i}, \sigma^* p(a_{\sigma(1)}, ..., a_{\sigma(n)}) = p(a_1, ..., a_n)$

Definition 16 ***Morphisms of Operadic Algebras:*** *Let A, B be two P-algebras. A morphism $f : A \to B$ is a family of maps*

$$f_c : A_c \to B_c$$

which are compatible:

$$f_c(A(p)(a_1, ..., a_n)) = B(p)(f_{c_1}(a_1), ..., f_{c_n}(a_n))$$

Definition 17 ***Category of P-Algebras:*** *Let P be an operad. Then we have a category of P-algebras Alg_P with:*

- $ObAlg_P = \{P - algebras\ A\}$
- $Hom_{Alg_P}(A, B) = \{f : A \to B : f \text{ is a morphism of algebras}\}$

We can now see some examples of operadic algebras. A **Com** algebra is a set A together with a map $\mu_n : A^{\times n} \to A$ for each $n \geq 0$. We can then verify that the category of algebras over the commutative operad, $\mathrm{Alg}_{\mathbf{Com}}$ is the category of commutative monoids. In a similar way $\mathrm{Alg}_{\mathbf{Ass}}$ is the category of associative monoids.

Consider the little-d cubes operad $\mathbf{E}_d$. Let X be a topological space with basepoint x_0. Then the loop space of X is ΩX, the space of basepoint preserving maps $S^1 \to X$, or in otherwords $\{\omega : [0,1] \to X, \omega(\partial[0,1]) = x_0\}$. One can then inductively construct the d-fold loop space $\Omega^d X = \Omega(\Omega^{d-1} X)$. $\Omega^d X$ is very naturally a $\mathbf{E}_d$ algebra.

65.3 Trees

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66 Simplicial Sets

66.1 The Simplex Category Δ

Definition 18 ***The Simplex Category Δ:*** *Δ is the category with:*

- $Ob\Delta = \mathbb{N}$,
- $Hom_\Delta([n], [m]) = \{\text{order preserving maps } [n] \to [m]\}$

There are special maps in Δ - the elementary faces $\delta^i : [m-1] \to [m]$ and elementary degeneracies $\sigma^i : [m] \to [m-1]$ $0 \leq i \leq m-1$:

$$\delta^i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}, \quad \sigma^i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

These have some nice relations, called the cosimplicial identities:

- $\sigma_i \sigma_j = \sigma_{j-1} \sigma_i, \quad i < j$
- $\delta_j \delta_i = \delta_i \delta_{j-1}$

66.1.1 Limits and Colimits of The Simplicial Category

$$\begin{array}{ccc} k & \xrightarrow{f} & n \\ \downarrow^{g} & & \downarrow \\ m & \to & m+n \end{array}$$

is a pushout, where $f(i) = i, g(i) = m - k + i$

66.2 Simplicial Sets and the Geometric Realisation

Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a functor $X : \Delta^{op} \to \mathcal{C}$. The morphisms between two simplicial objects over $\mathcal{C}$ are the natural transformations of functors.

From this we obtain a category of simpicial objects on $\mathcal{C}$, which has two notations:

$$\mathcal{C}^{\Delta^{op}}, \quad s\mathcal{C}$$

where we use the left notation when we want to be reminded that elements are functors (and thus we treat the category in the standard way one treats a functor category) and the right when we want to be reminded of the underlying structure.

When $\mathcal{C} = Set$ then there is really nice structure here. First off, in this case we say $X \in sSet$ is a simplicial *set*. We get back to this case soon but we expand on the definition of a simplicial object further.

A simplicial object X in $\mathcal{C}$ is given by a sequence of objects $X_n = X([n])$ in $\mathcal{C}$ together with maps $\alpha^* : X_n \to X_m$ where $\alpha \in \mathrm{Hom}_\Delta(m, n)$. These maps should be functorial, i.e.

$$id^* = id : X_n \to X_n$$

$$(\alpha \circ \beta)^* = \beta^* \alpha^* \quad \alpha : [m] \to [n], \beta : [k] \to [m]$$

and a morphism between two simplicial objects is a morphism on objects $f_n : X_n \to Y_n$ such that it is functorially compatible $f_m \alpha^* = \alpha f_n$.

Back to the case of simplicial sets. In this case we refer to X_n as the set of n-simplices of X. We have some special maps

$$d_i = (\delta_i)^* : X_n \to X_{n-1}$$

$$s_i = \sigma_i^* : X_{n-1} \to X_n$$

called the face maps and face degeneracies of the simplicial object X.

For now let's consider the common topological n-simplex:

$$\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} : \sum_i t_i = 1, t_i \geq 0\}$$

These can be thought of as generalisations of triangles in higher dimensions. Δ^1 is just the line $y = -x + 1$, Δ^2 is the set $\{(x, y, 1 - (x + y) : 0 \leq x, y, x + y \leq 1)\}$ and so on. A function $f : \{0, ..., m\} \to \{0, ..., n\}$ defines an affine map $f_* : \Delta^m \to \Delta^n, v_i \to v_{f(i)}$ where v_i is the ith vertex of the simplex Δ^m.

This construction makes the family of standard simplicies into a functor $\Delta^\bullet : \Delta \to \mathbf{Top}$.

$$\Delta^\alpha(t_0, ..., t_m) = (s_0, ..., s_n), s_j = \sum_{i \in \alpha^{-1}(j)} t_i$$

for $\alpha \in \mathrm{Hom}_\Delta([m], [n])$. Specifically the elementary face map $\delta_i : [n-1] \to [n]$ is lifted to

$$\Delta^{\delta_i} : \Delta^{n-1} \to \Delta^n$$

and embeds Δ^{n-1} as the face opposite the vertex i

Let's see this in low dimension. Let $n = 1$ and $i = 0$. We must keep track of what's going on. We have just claimed that Δ^{δ_0} embeds Δ^0 as the face opposite the vertex v_0 in Δ^1.

Recall that $\Delta^0 = \{1 \in \mathbb{R}^1\}$ is a point and Δ^1 corresponds to the line $y = 1 - x$. Let's call the point $(0,1) = v_0$ the first vertex. We have $\Delta^{\delta_0}(t) = (\sum_{i \in \delta_0^{-1}(0)} t_i, \sum_{i \in \delta_0^{-1}(1)} t_i)$ but $\delta_0^{-1}(1) = \{j \in [1] = \{0,1\} : j+1 = 1 \text{ and } j \geq 0\} = 0$. Thus we get that $\Delta^{\delta_0}(1) = (1,0)$. So now we have gained the intuition for the following proposition

Proposition 5 *For maps $\delta_i : [n-1] \to [n]$, the map $(\delta_i)_* = \Delta^{\delta_i} : \Delta^{n-1} \to \Delta^n$ embeds Δ^{n-1} as the face opposite the vertex labelled i*

Proof

67 Dendroidal Sets

68 Tensor Products of Dendroidal Sets

69 Kan Conditions for Simplicial Sets

70 Kan Conditions for Dendroidal Sets

71 Model Categories

72 Model Structures on the Category of Simplicial Sets

73 Three Model Structures on the Category of Dendroidal Sets

74 Reedy Categories and Diagrams of Spaces

75 Mapping Spaces and Bousfield Localisations

76 Dendroidal Spaces and ∞-Operads

77 Left Fibrations and the Covariant Model Structure

78 Simplical Operads and ∞-Operads

79 Some Research

79.1 Yubiwal Sets

References

[Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.