

# Notes on 'Sheaf Theory' by Bredon

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## Contents

<b>I</b>	<b>Sheaf Theory, Bredon</b>	<b>3</b>
<b>1</b>	<b>Sheaves and Presheaves</b>	<b>3</b>
1.1	Definitions . . . . .	3
1.2	Homomorphisms, Subsheaves and Quotient Sheaves . . . . .	5
1.3	Direct and Inverse Images . . . . .	5
1.4	Cohomomorphisms . . . . .	5
1.5	Algebraic Constructions . . . . .	5
1.6	Supports . . . . .	5
1.7	Classical Cohomology Theories . . . . .	5
<b>2</b>	<b>Sheaf Cohomology</b>	<b>5</b>
2.1	Differential Sheaves and Resolutions . . . . .	5
2.2	The Canonical Resolution and Sheaf Cohomology . . . . .	5
2.3	Injective Sheaves . . . . .	5
2.4	Acyclic Sheaves . . . . .	5
2.5	Flabby Sheaves . . . . .	5
2.6	Connected Sequences of Functors . . . . .	5
2.7	Axioms for Cohomology and the Cup Product . . . . .	5
2.8	Maps of Spaces . . . . .	5
2.9	$\Phi$ -Soft and $\Phi$ -Fine Sheaves . . . . .	5
2.10	Subspaces . . . . .	5
2.11	The Vietoris Mapping Theorem and Homotopy Invariance . . . . .	5
2.12	Relative Cohomology . . . . .	5
2.13	Mayer-Vietoris Theorems . . . . .	5
2.14	Continuity . . . . .	5
2.15	The Künneth and Universal Coefficient Theorems . . . . .	5
2.16	Dimension . . . . .	5
2.17	Local Connectivity . . . . .	5
2.18	Change of Supports and Local Cohomology Groups . . . . .	5
2.19	The Transfer Homomorphism and the Smith Sequences . . . . .	5
2.20	Steenrod's Cyclic Reduced Powers . . . . .	5
2.21	The Steenrod Operations . . . . .	5
<b>3</b>	<b>Comparison with Other Cohomology Theories</b>	<b>5</b>
3.1	Singular Cohomology . . . . .	5
3.2	Alexander-Spanier Cohomology . . . . .	5
3.3	de Rham Cohomology . . . . .	5
3.4	Cech Cohomology . . . . .	5

<b>4</b>	<b>Applications of Spectral Sequences</b>	<b>5</b>
4.1	The Spectral Sequence of a Differential Sheaf . . . . .	5
4.2	The Fundamental Theorems of Sheaves . . . . .	5
4.3	Direct Image Relative to a Support Family . . . . .	5
4.4	The Leray Sheaf . . . . .	5
4.5	Extension of a Support Family by a Family on the Base Space . . . . .	5
4.6	The Leray Spectral Sequence of a Map . . . . .	5
4.7	Fiber Bundles . . . . .	5
4.8	Dimension . . . . .	5
4.9	The Spectral Sequences of Borel and Cartan . . . . .	5
4.10	Characteristic Classes . . . . .	5
4.11	The Spectral Sequence of a Filtered Differential Sheaf . . . . .	5
4.12	The Fary Spectral Sequence . . . . .	5
4.13	Sphere Bundles with Singularities . . . . .	5
4.14	The Oliver Transfer and the Conner Conjecture . . . . .	5
<b>5</b>	<b>Borel-Moore Homology</b>	<b>5</b>
5.1	Cosheaves . . . . .	5
5.2	The Dual of a Differential Cosheaf . . . . .	5
5.3	Homology Theory . . . . .	5
5.4	Maps of Spaces . . . . .	5
5.5	Subspaces and Relative Homology . . . . .	5
5.6	The Viertoris Theorem, Homotopy and Covering Spaces . . . . .	5
5.7	The Homology Sheaf of a Map . . . . .	5
5.8	The Basic Spectral Sequence . . . . .	5
5.9	Poincaré Duality . . . . .	5
5.10	The Cap Product . . . . .	5
5.11	Intersection Theory . . . . .	5
5.12	Uniqueness Theorems . . . . .	5
5.13	Uniqueness Theorems for Maps and Relative Homology . . . . .	5
5.14	The Künneth Formula . . . . .	5
5.15	Change of Rings . . . . .	5
5.16	Generalised Manifolds . . . . .	5
5.17	Locally Homogenous Spaces . . . . .	5
5.18	Homological Fibrations and $p$ -adic Transformation Groups . . . . .	5
5.19	The Transfer Homomorphism on Homology . . . . .	5
5.20	Smith Theory in Homology . . . . .	5
<b>6</b>	<b>Cosheaves and Čech Homology</b>	<b>5</b>
6.1	Theory of Cosheaves . . . . .	5
6.2	Local Triviality . . . . .	5
6.3	Local Isomorphisms . . . . .	5
6.4	Čech Homology . . . . .	5
6.5	The Reflector . . . . .	5
6.6	Spectral Sequences . . . . .	5
6.7	Coresolutions . . . . .	5
6.8	Relative Čech Homology . . . . .	5
6.9	Locally Paracompact Spaces . . . . .	5
6.10	Borel-Moore Homology . . . . .	5
6.11	Modified Borel-Moore Homology . . . . .	5
6.12	Singular Homology . . . . .	5
6.13	Acyclic Coverings . . . . .	5
6.14	Applications to Maps . . . . .	5

## Part I

# Sheaf Theory, Bredon

## 1 Sheaves and Presheaves

### 1.1 Definitions

**Definition 1 Presheaf:** A presheaf of abelian groups on a topological space  $X$  is a functor

$$A : \mathfrak{Opn}_X^{op} \rightarrow \mathfrak{Ab}$$

i.e. an assignment to each open set  $U \in \text{Ob}(\mathfrak{Opn}_X^{op})$  an abelian group  $A(U) \in \text{Ob}(\mathfrak{Ab})$ , and on inclusion functions  $V \xrightarrow{\iota_U^V} U$  restriction functions  $r_V^U : A(U) \rightarrow A(V)$  such that:

- $r_U^U = id_U$ ,
- $U \subset V \subset W \implies r_V^W r_U^V = r_U^W$

One can switch out the target category with other algebraic categories to obtain the definitions of presheaves over  $R$ -modules, algebras, etc.

**Definition 2 Germ:** Let  $x \in X$  be a point and  $A$  a presheaf over  $X$ . The set  $\mathfrak{M}$  of elements  $s \in A(U)$ ,  $x \in U \in \text{Ob}\mathfrak{Opn}_X^{op}$ . The germs of the presheaf  $A$  at  $x$  is thus the quotient

$$\mathfrak{G}_{A,x} := \mathfrak{M} / \sim$$

where  $s, t \in \mathfrak{M}$  are equal if there is  $W \subset \text{dom}(s) \cap \text{dom}(t)$  such that  $s|_W = t|_W$ . The equivalence class  $[s]$  of germs of  $A(U)$  is called the germ of  $s$  at  $x \in U$ .

The set of germs of  $A$  at  $x$ ,  $\mathcal{A}_x$  is then the direct limit

$$\mathcal{A}_x = \varinjlim A(U)$$

From here we can define a topology on the disjoint union  $\sqcup_{x \in X} \mathcal{A}_x$  as follows. Fix an element  $s \in A(U)$ , for each  $x \in U$  we have the germ  $s_x$  of  $s$  at  $x$ . For fixed  $s$  the set of all germs  $s_x \in \mathcal{A}_x$  is defined to be an open set. With this definition we get the sheaf generated by the presheaf  $A$

$$\mathcal{A} = \mathfrak{Sh}(\mathcal{A}) = \mathfrak{Sh}(U \rightarrow A(U))$$

**Definition 3 Sheaf:** A sheaf on  $X$  is a pair  $(\mathcal{A}, \pi)$  satisfying:

- $\mathcal{A}$  is a (typically non-Hausdorff) topological space,
- $\pi : \mathcal{A} \rightarrow X$  is a local homeomorphism
- Each  $\mathcal{A}_x = \pi^{-1}(x)$  for  $x \in X$  is an abelian group, and is called the stalk of  $\mathcal{A}$  at  $x$
- The group operations are continuous

An example of a sheaf is  $\Omega^0$  of germs of  $\mathcal{C}^\infty(\mathbb{R})$  functions on a differentiable manifold  $M^n$ , this is a sheaf of unital rings, and  $\Omega^p(M^n)$  of germs of differential  $p$ -forms on  $M^n$  which is also a  $\Omega^0$ -module.

If  $\mathcal{A}$  is a sheaf on  $X$  with projection  $\pi : \mathcal{A} \rightarrow X$  and if  $Y \subset X$  then the restriction  $\mathcal{A}|_Y$  of  $\mathcal{A}$  is defined:

$$\mathcal{A}|_Y = \pi^{-1}(Y)$$

If  $\mathcal{A}$  is a sheaf on  $X$ ,  $Y \subset X$  then a section of  $\mathcal{A}$  over  $Y$  is a map  $s : Y \rightarrow \mathcal{A}$  such that  $\pi \circ s$  is the identity on  $Y$ . Note that every point  $x \in Y$  admits a section  $s$  over some neighborhood  $U$  of  $x$ . We can then access the 0 section over any open set  $U \subset X$  and so we can endow  $\mathcal{A}(Y)$  with the structure of an abelian group. Thus the presheaf of sections of  $\mathcal{A}$  is defined:

$$\Gamma(\mathcal{A}) = \mathcal{A}(X)$$

An example of a non-Hausdorff sheaf is the sheaf on the real line that has 0 stalk everywhere but 0 which has the stalk  $\mathbb{Z}_2$ . That is  $\pi^{-1}(x \neq 0) = 0, \pi^{-1}(0) = \mathbb{Z}_2$



## 1.2 Homomorphisms, Subsheaves and Quotient Sheaves

## 1.3 Direct and Inverse Images

## 1.4 Cohomomorphisms

## 1.5 Algebraic Constructions

## 1.6 Supports

## 1.7 Classical Cohomology Theories

# 2 Sheaf Cohomology

## 2.1 Differential Sheaves and Resolutions

## 2.2 The Canonical Resolution and Sheaf Cohomology

## 2.3 Injective Sheaves

## 2.4 Acyclic Sheaves

## 2.5 Flabby Sheaves

## 2.6 Connected Sequences of Functors

## 2.7 Axioms for Cohomology and the Cup Product

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## 2.21 The Steenrod Operations

# 3 Comparison with Other Cohomology Theories

## 3.1 Singular Cohomology

## 3.2 Alexander-Spanier Cohomology

## 3.3 de Rham Cohomology

## 3.4 Čech Cohomology

# 4 Applications of Spectral Sequences

## 4.1 The Spectral Sequence of a Differential Sheaf

## 4.2 The Fundamental Theorems of Sheaves

## 4.3 Direct Image Relative to a Support Family