Notes on 'Algebraic Geometry I: Schemes'

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June 22, 2025

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Part I
Algebraic Geometry I: Schemes, Gortz-Wedhorn

1 Prevarieties

1.1 Affine Algebraic Sets

1.1.1 The Zariski Topology on $\mathbb{A}_k^n$

Definition 1 *Let $M \subseteq k[T_1, ..., T_n] =: k[\underline{T}]$. The set of common zeros of the polynomials in M is defined as*

$$\mathbb{V}(M) := \{p \in k^n : f(p) = 0 \quad \forall f \in M\}$$

Proposition 1 *The sets $\mathbb{V}(\mathfrak{a})$ where $\mathfrak{a}$ is an ideal in $k[\underline{T}]$ form a topoology on $\mathbb{A}_k^n$ called the Zariksi topology.*

This is a very elementary problem in algebraic geometry .

1.1.2 Affine Algebraic Sets

Definition 2 *The closed subspaces of $\mathbb{A}_k^n$ are called affine algebraic sets.*

1.1.3 Hilbert's Nullstellensatz

Theorem 1 ***Hilbert's Nullstellensatz:*** *Let K be field and A a finitely generated K-algebra. Then A is Jacobson, that is for every prime ideal $\mathfrak{p} \subset A$ we have*

$$\mathfrak{p} = \bigcap_{\mathfrak{m} \supseteq \mathfrak{p}, \mathfrak{m} \text{ } maximal} \mathfrak{m}$$

Theorem 2 ***Noether's Normalisation Theorem:*** *Let K be a field and $A \neq 0$ a finitely generated K-algebra. Then there exists $n \in \mathbb{N}$ abd $t_1, ..., t_n$ such that the K-algebra homomorphism $K[T_1, ..., T_n] \to A$, $T_i \to t_i$ is injective and finite*

Lemma 1 *Let A, B be integral domains and $A \to B$ an injective integral ring homomorphism. Then A is a field iff B is a field*

Proof. Let A be a field and $b \in B$ non-zero. Then $A[b]$ is an A-vector space of finite dimension. As B is an integral domain, the multiplication by b, $A[b] \to A[b]$ is injective. As this map is A-linear it is bijective and hence b is a unit. We can extend this type of argument to every $a \neq 0 \in A$ and prove that a must be a unit.

We can further investigate the impact of the Nullstellensatz on algebraically closed fields. Let $K = k$ be an algebraically closed field. Then the Nullstellensatz implies the following:

- Let A be a finitely generated k-algebra and $\mathfrak{m} \subset A$ a maximal ideal. Then $A/\mathfrak{m} = k$
- $\mathfrak{m} \subset k[T_1, ..., T_n]$ maximal implies existence of a point $(x_1, ..., x_n)$ in A such that $\mathfrak{m} = (T_1 - x_1, ..., T_n - x_n)$

These points have nice geometric interpretations. When A is the set of polynomials $A = k[x_1, ..., x_n]$ and $\mathfrak{m}$ is a maximal ideal, the coordinate ring of $X = \mathbb{V}(\mathfrak{m})$ is isomorphic to the underlying field. The second point is also nice and tells us that all maximal ideals in $k[x_1, ..., x_n]$ correspond to points in $\mathbb{A}_k^n$. This is interesting, maximal ideals are large but correspond to very small/finite varities.

1.1.4 The Radical-Affine Correspondence

There is a bijective correspondence between radical ideals and affine varities:

$$\{\mathfrak{p} \subset A : \quad \mathfrak{rad}(\mathfrak{p}) = \mathfrak{p}\} \leftrightarrow \{X \subset \mathbb{A}_k^n : \quad X = \mathbb{V}(f_1, ..., f_n)\}$$
$$\mathfrak{p} \to \mathbb{V}(\mathfrak{p})$$
$$\mathbb{I}(X) \leftarrow X$$

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