Notes on 'Simplicial and Dendroidal Homotopy Theory'

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Part I

Simplicial and Dendroidal Homotopy Theory

1 Operads

1.1 Operads

Definition 1 ***Operad:*** *An operad P consists of a set of colours C and for each $n \geq 0$ and sequence $c_1, ..., c_n, c$ of colours in C, a set $P(c_1, ..., c_n; c)$ of operations, thought of as taking n inputs of colours $c_1, ..., c_n$ and with output of colour c. Moreover there are the structure maps*

- $\forall c \in C, \exists 1_c \in P(c; c)$,
- *For $\sigma \in \Sigma_n$ a map*
$$\sigma^* : P(c_1, ..., c_n; c) \to P(c_{\sigma(1)}, ..., c_{\sigma(n)}; c)$$
denoted $\sigma^* \circ p = p \circ \sigma$
- *For any sequence $c_1, ..., c_n$ and n-tuple of sequences $d_1^i, ..., d_{k_i}^i$, a composition*
-
$$\gamma : P(c_1, ..., c_n; c) \times \prod_{i=1}^{n} P(d_1^i, ..., d_{k_i}^i; c_i) \to P(d_1^1, ..., d_{k_n}^n; c)$$
which is written as $\gamma(p, q_1, ..., 1_n) \to p \circ (q_1, ..., q_n)$

and there are further requirements on the structure maps:

- $\forall p \in P(c_1, ..., c_n; c), \gamma(1_c, p) = p$,
- $\forall p \in P(c_1, ..., c_n; c), \gamma(p, 1_{c_1}, ..., 1_{c_n}) = p$

There are some classical notation we must be weary of and state here. Let P be an operad with a singleton colour set, i.e. $C = \{*\}$. Then we can write $P(c_1, ..., c_n; c) = P(c, ..., c; c) =: P(n)$. There is an obvious formulation for the compostion in this case:

$$P(n) \times \prod_{i=1}^{n} P(k_i) \to P(k_1 + ... + k_n)$$

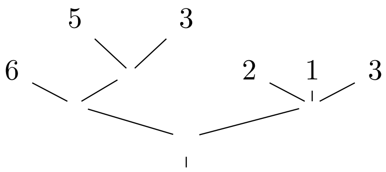
In this case we say P is uncoloured. If $C \notin \{\phi, \{*\}\}$ then we say P is a coloured operad. We can then clearly see that

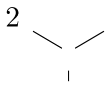
$$\begin{tikzcd} \text{Monoids} \arrow[r, hook] \arrow[d, hook] & \text{Categories} \arrow[d, hook] \\ \text{Uncoloured Operads} \arrow[r, hook] & \text{Operads} \end{tikzcd}$$

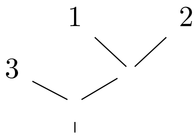
The definition of an operad allows for $n = 0$ in $P(c_1, ..., c_n; c)$, which we define as $P(-; c)$. The elements of $P(-; c)$ are called the constants of colour c, and an operad with $P(-; c) = \{*_c\}$ is called unital. An operad P is open if there are no constants for any colour, i.e. its interior P^o (the set of constants) is empty.

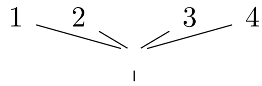
The most fundamental examples of operads are **Com** and **Ass**:

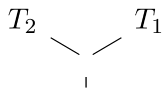
- **Com** is the commutative operad with $\mathbf{Com}(n) = \{*\}$,
- **Ass** is the associative operad with $\mathbf{Ass}(n) = \Sigma_n$
- $\mathbf{Tree}^{pl}$ is the planar tree operad. The n-ary operations here are the set of planar rooted trees with n numbered leaves.

$\mathbf{Tree}^{pl}(6) \ni \hat{T} =$ 

$\mathbf{Tree}^{pl}(2) \ni T =$ 

, $\mathbf{Tree}^{pl}(3) \ni T_1 =$ 

, $\mathbf{Tree}^{pl}(4) \ni T_2 =$ 

$\mathbf{Tree}^{pl}(7) \ni \tilde{T} =$ 

Then in fact $\gamma(T, T_1, T_2) = \tilde{T}$. The operation of composition on the operad $\mathbf{Tree}^{pl}$ is computed as $\gamma(T \in \mathbf{Tree}^{pl}(n), T_1 \in \mathbf{Tree}^{pl}(k_1), ..., T_n \in \mathbf{Tree}^{pl}(k_n)) = \hat{T}$ where $\hat{T} \in \mathbf{Tree}^{pl}(k_1 + ... + k_n)$ is the original T but with the subtree T_i grafted onto the leaf i for all choices of i.

Definition 2 ***Topological Operad:*** *A topological operad is an operad P where each set of operations $P(c_1, ..., c_n; c)$ is equipped with some topology and all the structure maps are continuous with respect to this topology.*

The most basic form of a topological operad is the little d-cubs operad $\mathbf{E}_d$. The space $\mathbf{E}_d(n)$ is the space of n numbered d-dimensional cubes inside the d-dimensional unit cube $[0,1]^d$. The operadic composition between $p \in \mathbf{E}_d(n)$ with operations $q_1, ..., q_n$ is given by substituting the rescaled q_i into the ith cube of p. Note that this is really just a topological analogue to the planar tree operad $\mathbf{Tree}^{pl}$, where instead of grafting trees onto leaves we are scaling and embedding cubes in some smooth way.

More specifically, a point in $\mathbf{E}_d(n)$ is an n-tuple of embeddings $f_1, ..., f_n : [0,1]^d \to [0,1]^d$ satisfying:

- Each f_i is the composition of d affine embeddings,
- The interiors of the cubes embedded by f_i are mutually disjoint

We can now observe that operads form a category in a very natural fashion. Given two operads P, Q, a morphism $\varphi : P \to Q$ is a function $f : C_p \to C_Q$ on operadic colours and for each sequence $c_1, ..., c_n; c$ of C_P, we have

$$\varphi_{(c_1,...,c_n;c)} : P(c_1, ..., c_n; c) \to Q(f(c_1), ..., f(c_n); f(c))$$

that is compatible in the natural way with Σ_n actions

1.2 Algebras for Operads

Definition 3 ***Operadic Algebras:*** *Let P be an operad. A P-algebra A is a family of sets $\{A_c\}_{c \in C_P}$ together with maps*

$$P(c_1, ..., c_n; c) \times A_{c_1} \times ... \times A_{c_n} \to A_c$$

written $(p, a_1, ..., a_n) \to A(p)(a_1, ..., a_n)$. These maps also satisfy:

- $1_c(a) = a \quad \forall a \in A_c$
- $\sigma \in \Sigma_n, a_i \in A_{c_i}, \sigma^* p(a_{\sigma(1)}, ..., a_{\sigma(n)}) = p(a_1, ..., a_n)$

Definition 4 ***Morphisms of Operadic Algebras:*** *Let A, B be two P-algebras. A morphism $f : A \to B$ is a family of maps*

$$f_c : A_c \to B_c$$

which are compatible:

$$f_c(A(p)(a_1, ..., a_n)) = B(p)(f_{c_1}(a_1), ..., f_{c_n}(a_n))$$

Definition 5 ***Category of P-Algebras:*** *Let P be an operad. Then we have a category of P-algebras Alg_P with:*

- $ObAlg_P = \{P - algebras\ A\}$
- $Hom_{Alg_P}(A, B) = \{f : A \to B : f \text{ is a morphism of algebras}\}$

We can now see some examples of operadic algebras. A **Com** algebra is a set A together with a map $\mu_n : A^{\times n} \to A$ for each $n \geq 0$. We can then verify that the category of algebras over the commutative operad, $\text{Alg}_{\mathbf{Com}}$ is the category of commutative monoids. In a similar way $\text{Alg}_{\mathbf{Ass}}$ is the category of associative monoids.

Consider the little-d cubes operad $\mathbf{E}_d$. Let X be a topological space with basepoint x_0. Then the loop space of X is ΩX, the space of basepoint preserving maps $S^1 \to X$, or in otherwords $\{\omega : [0,1] \to X, \omega(\partial[0,1]) = x_0\}$. One can then inductively construct the d-fold loop space $\Omega^d X = \Omega(\Omega^{d-1} X)$. $\Omega^d X$ is very naturally a $\mathbf{E}_d$ algebra.

1.3 Trees

1.4 Alternative Definitions for Operads

1.5 Free Operads

1.6 The Tensor Product of Operads

1.7 The Boardman-Vogt Resolution of an Operad

1.8 Configuration Spaces and the Fulton-MacPherson Operad

1.9 Configuration Spaces and the Operad of Little Cubes

2 Simplicial Sets

2.1 The Simplex Category Δ

Definition 6 ***The Simplex Category Δ:*** *Δ is the category with:*

- $Ob\Delta = \mathbb{N}$,
- $Hom_\Delta([n], [m]) = \{\text{order preserving maps } [n] \to [m]\}$

There are special maps in Δ - the elementary faces $\delta^i : [m-1] \to [m]$ and elementary degeneracies $\sigma^i : [m] \to [m-1]$ $0 \leq i \leq m-1$:

$$\delta^i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}, \quad \sigma^i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

These have some nice relations, called the cosimplicial identities:

- $\sigma_i \sigma_j = \sigma_{j-1} \sigma_i, \quad i < j$
- $\delta_j \delta_i = \delta_i \delta_{j-1}$

2.1.1 Limits and Colimits of The Simplicial Category

$$\begin{array}{ccc} k & \xrightarrow{f} & n \\ \downarrow g & & \downarrow \\ m & \rightarrow & m+n \end{array}$$

is a pushout, where $f(i) = i, g(i) = m - k + i$

2.2 Simplicial Sets and the Geometric Realisation

Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a functor $X : \Delta^{op} \to \mathcal{C}$. The morphisms between two simplicial objects over $\mathcal{C}$ are the natural transformations of functors.

From this we obtain a category of simpicial objects on $\mathcal{C}$, which has two notations:

$$\mathcal{C}^{\Delta^{op}}, \quad s\mathcal{C}$$

where we use the left notation when we want to be reminded that elements are functors (and thus we treat the category in the standard way one treats a functor category) and the right when we want to be reminded of the underlying structure.

When $\mathcal{C} = Set$ then there is really nice structure here. First off, in this case we say $X \in sSet$ is a simplicial *set*. We get back to this case soon but we expand on the definition of a simplicial object further.

A simplicial object X in $\mathcal{C}$ is given by a sequence of objects $X_n = X([n])$ in $\mathcal{C}$ together with maps $\alpha^* : X_n \to X_m$ where $\alpha \in \mathrm{Hom}_\Delta(m, n)$. These maps should be functorial, i.e.

$$id^* = id : X_n \to X_n$$

$$(\alpha \circ \beta)^* = \beta^* \alpha^* \quad \alpha : [m] \to [n], \beta : [k] \to [m]$$

and a morphism between two simplicial objects is a morphism on objects $f_n : X_n \to Y_n$ such that it is functorially compatible $f_m \alpha^* = \alpha f_n$.

Back to the case of simplicial sets. In this case we refer to X_n as the set of n-simplices of X. We have some special maps

$$d_i = (\delta_i)^* : X_n \to X_{n-1}$$

$$s_i = \sigma_i^* : X_{n-1} \to X_n$$

called the face maps and face degeneracies of the simplicial object X.

For now let's consider the common topological n-simplex:

$$\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} : \sum_i t_i = 1, t_i \geq 0\}$$

These can be thought of as generalisations of triangles in higher dimensions. Δ^1 is just the line $y = -x + 1$, Δ^2 is the set $\{(x, y, 1 - (x + y) : 0 \leq x, y, x + y \leq 1)\}$ and so on. A function $f : \{0, ..., m\} \to \{0, ..., n\}$ defines an affine map $f_* : \Delta^m \to \Delta^n, v_i \to v_{f(i)}$ where v_i is the ith vertex of the simplex Δ^m.

This construction makes the family of standard simplicies into a functor $\Delta^\bullet : \Delta \to \mathbf{Top}$.

$$\Delta^\alpha(t_0, ..., t_m) = (s_0, ..., s_n), s_j = \sum_{i \in \alpha^{-1}(j)} t_i$$

for $\alpha \in \mathrm{Hom}_\Delta([m], [n])$. Specifically the elementary face map $\delta_i : [n-1] \to [n]$ is lifted to

$$\Delta^{\delta_i} : \Delta^{n-1} \to \Delta^n$$

and embeds Δ^{n-1} as the face opposite the vertex i

Let's see this in low dimension. Let $n = 1$ and $i = 0$. We must keep track of what's going on. We have just claimed that Δ^{δ_0} embeds Δ^0 as the face opposite the vertex v_0 in Δ^1.

Recall that $\Delta^0 = \{1 \in \mathbb{R}^1\}$ is a point and Δ^1 corresponds to the line $y = 1 - x$. Let's call the point $(0,1) = v_0$ the first vertex. We have $\Delta^{\delta_0}(t) = (\sum_{i\in\delta_0^{-1}(0)} t_i, \sum_{i\in\delta_0^{-1}(1)} t_i)$ but $\delta_0^{-1}(1) = \{j \in [1] = \{0,1\} : j+1 = 1 \text{ and } j \geq 0\} = 0$. Thus we get that $\Delta^{\delta_0}(1) = (1,0)$. So now we have gained the intuition for the following proposition

Proposition 1 *For maps $\delta_i : [n-1] \to [n]$, the map $(\delta_i)_* = \Delta^{\delta_i} : \Delta^{n-1} \to \Delta^n$ embeds Δ^{n-1} as the face opposite the vertex labelled i*

Proof

There is also a nice geometric interpretation for Δ^{σ_j} for $\sigma_j : [n] \to [n-1]$. This is the collapsing of Δ^n onto Δ^{n-1} by a projection parallel to the line connecting v_j and v_{j+1}.

We start with a large and rather unintuitive space and try to quotient out by these relations in order to get a nice geometric space. Let $\hat{X} = \bigsqcup_{n\geq 0}(X_n \times \Delta^n)$. Let $(x,t) \in \hat{X}$, that is $x \in X_n, t \in \Delta^n$. We have the relation $(x, \Delta^\alpha t) \sim (\alpha^* x, t)$.

Definition 7 ***The Geometric Realisation of a Simplicial Set:*** *Let $X : \Delta^{op} \to Set$ be a simplicial set. The geometric realisation of X is*

$$|X| := (\bigsqcup_{n\geq 0}(X_n \times \Delta^n))/\sim$$

where we denote $x \otimes t$ the equivalence class of (x,t) in $|X|$.

A map $f : X \to Y$ between simplicial sets induces a continuous map

$$|f| : |X| \to |Y|, x \otimes t \to f(x) \otimes t$$

Whence it is clear we have a functor $|.| : \mathbf{sSet} \to \mathbf{Top}$

If $C^\bullet : \Delta \to \mathcal{C}$ is a functor, each object $c \in \mathcal{C}$ defines a simplicial set $\mathbf{Sing}_{C^\bullet}(c) = \mathrm{Hom}_{\mathcal{C}}(C^n, c)$

2.3 The Geometric Realisation as a Cell Complex

2.4 Simplicial Sets as a Category of Presheaves

3 Dendroidal Sets

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6 Kan Conditions for Dendroidal Sets

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Over the category of cones over a category