

Notes on 'Éléments de Géométrie Algébrique'

Bailey Arm

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Part I

Éléments de Géométrie Algébrique I: The Language of Schemes

1 Affine Schemes

1.1 The Prime Spectrum of a Ring

Definition 1 Prime Spectrum of a Ring: Let A be commutative ring and M an A -module. Then

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subset A : \mathfrak{p} \text{ prime ideal in } A\}$$

For points $x \in X = \mathrm{Spec}(A)$ it is usually convenient to write \mathfrak{p}_x . It is also important to note that for $\mathrm{Spec}(A) = \emptyset$ we need A to be the 0 ring.

Definition 2 Local Ring of Fractions of a Ring: Let A be a commutative ring. The local ring at $x \in \mathrm{Spec} A$ is

$$A_x = A_{\mathfrak{p}_x} = (A \setminus \mathfrak{p}_x)^{-1} A$$

There is also some more notation

- $\mathfrak{m}_x = \mathfrak{p}_x A_{\mathfrak{p}_x}$ is the (unique) maximal ideal of A_x ,
- $k(x) = A_x / \mathfrak{m}_x$ is the residue field of A_x and it canonically isomorphic to the fraction field of the integral ring A / \mathfrak{p}_x . It is thus we identify $k(x) = A / \mathfrak{p}_x$,
- $M_x = M \otimes_A A_x$ is the module of fractions with denominators in $A \setminus \mathfrak{p}_x$,
- $V(E) = \{x \in X = \mathrm{Spec}(A) : E \subset \mathfrak{p}_x\}$
- $\mathrm{rad}(E) = \bigcap_{x \in V(E)} \mathfrak{p}_x$

Proposition 1 • $V(0) = X, V(1) = \emptyset$,

- $E \subset E' \implies V(E') \subset V(E)$,
- $V(\bigcup_{\lambda} E_{\lambda}) = \bigcap_{\lambda} V(E_{\lambda})$,
- $V(EE') = V(E) \cup V(E')$,
- $V(E) = V(\mathrm{rad}(E))$

Proof. It is clear that all ideals contain the 0 ideal and so $V(0) = \mathrm{Spec}(A)$. The third and fourth subpropositions are clear via doing the inclusions. The final subproposition is trivial when one notices that

$$V(\mathrm{rad}(E)) = V\left(\bigcap_{x \in V(E)} \mathfrak{p}_x\right) = \bigcup_{x \in V(E)} V(\mathfrak{p}_x) = V(E)$$

For each subset $Y \subset X$ we denoted $\mathfrak{p}(Y)$ to be the set of $f \in A$ such that $f(y) = 0$ for all $y \in Y$. It is hence clear that $\mathfrak{p}(Y) = \bigcap_{y \in Y} \mathfrak{p}_y$. So then more generally

$$\mathfrak{p}\left(\bigcup_{\lambda} Y_{\lambda}\right) = \bigcap_{\lambda} \mathfrak{p}(Y_{\lambda})$$

And so identifying sets of the form $V(E)$ as closed yields a topology on $\mathrm{Spec}(A)$, called either the Zariski or Spectrum topology.

Proposition 2 • For each subset $E \subset A$ we have $\mathfrak{p}(V(E)) = \mathrm{rad}(E)$

- For each subset $Y \subset X$ we have $V(\mathfrak{p}(Y)) = \bar{Y}$

Proof. The first part is trivial via the previous propositions. The second part is a result of checking that $V(\mathfrak{p}(Y))$ is the smallest closed subset of X containing Y . Indeed $V(\mathfrak{p}(Y)) \supset Y$ and is closed, and if $Y \subset V(E) \subset V(\mathfrak{p}(Y))$ then $f(y) = 0$ for all $f \in E$ and $y \in Y$. By inclusion reversion, $E \subset \mathfrak{p}(Y) \implies V(E) \supset V(\mathfrak{p}(Y))$ so it is true that $V(\mathfrak{p}(Y))$ is the smallest possible closed set containing Y .

- 1.2 Functorial Properties of Prime Spectra of Rings
- 1.3 Sheaf Associated to a Module
- 1.4 Quasi-Coherent Sheaves over a Prime Spectrum
- 1.5 Coherent Sheaves over a Prime Spectrum
- 1.6 Functorial Properties of Quasi-Coherent Sheaves over a Prime Spectrum
- 1.7 Characterisation of Morphisms of Affine Schemes
- 1.8 Morphisms from Locally Ringed Spaces to Affine Schemes

2 Preschemes and Morphisms of Preschemes

- 2.1 Definition of Preschemes
- 2.2 Morphism of Preschemes
- 2.3 Gluing Preschemes
- 2.4 Preschemes over a Prescheme

3 Products of Preschemes

- 3.1 Sums of Preschemes
- 3.2 Products of Preschemes
- 3.3 Formal Properties of the Product, Change of the Base Prescheme
- 3.4 Points of a Prescheme with Values in a Prescheme, Geometric Points
- 3.5 Surjections and Injections
- 3.6 Fibres
- 3.7 Applications: Reduction of a Prescheme Modulo \mathcal{J}

4 Subpreschemes and Immersion Morphisms

- 4.1 Subpreschemes
- 4.2 Immersion Morphisms
- 4.3 Products of Immersions
- 4.4 Inverse Images of a Subprescheme
- 4.5 Local Immersions and Local Isomorphisms

5 Reduced Preschemes and the Separation Condition

- 5.1 Reduced Preschemes
- 5.2 Existence of a Subprescheme with a Given Underlying Space
- 5.3 Diagonal; Graph of a Morphism
- 5.4 Separated Morphisms and Separated Preschemes
- 5.5 Separation Criteria

6 Finiteness Conditions

- 6.1 Noetherian and Locally Noetherian Preschemes
- 6.2 Artinian Properties
- 6.3 Morphisms of Finite Type