Notes on 'Éléments de Géométrie Algébrique'

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Contents

Part I
Éléments de Géométrie Algébrique I: The Language of Schemes

1 Affine Schemes

1.1 The Prime Spectrum of a Ring

Definition 1 ***Prime Spectrum of a Ring:*** *Let A be commutative ring and M n A-module. Then*

$$Spec(A) = \{\mathfrak{p} \subset A : \mathfrak{p} \text{ prime ideal in } A\}$$

For points $x \in X = Spec(A)$ it is usually convenient to write $\mathfrak{p}_x$. It is also important to note that for $Spec(A) = \phi$ we need A to be the 0 ring.

Definition 2 ***Local Ring of Fractions of a Ring:*** *Let A be a commutative ring. The local ring at $x \in SpecA$ is*

$$A_x = A_{\mathfrak{p}_x} = (A \backslash \mathfrak{p}_x)^{-1} A$$

There is also some more notation

- $\mathfrak{m}_x = \mathfrak{p}_x A_{\mathfrak{p}_x}$ is the (unique) maximal ideal of A_x,
- $k(x) = A_x/\mathfrak{m}_x$ is the residue field of A_x and it canonically isomorphic to the fraction field of the integral ring $A/\mathfrak{p}_x$. It is thus we identify $k(x) = A/\mathfrak{p}_x$,
- $M_x = M \otimes_A A_x$ is the module of fractions with denominators in $A \backslash \mathfrak{p}_x$,
- $V(E) = \{x \in X = \text{Spec}(A) : E \subset j_x\}$
- $\mathfrak{rad}(E) = \bigcap_{x \in V(E)} \mathfrak{p}_x$

Proposition 1
- *$V(0) = X, V(1) = \phi$,*
- *$E \subset E' \implies V(E') \subset V(E)$,*
- *$V(\cup_\lambda E_\lambda) = \cap_\lambda V(E_\lambda)$,*
- *$V(EE') = V(E) \cup V(E')$,*
- *$V(E) = V(\mathfrak{rad}(E))$*

Proof. It is clear that all ideals contain the 0 ideal and so $V(0) = \text{Spec}(A)$. The third and fourth subpropositions are clear via doing the inclusions. The final subproposition is trivial when one notices that

$$V(\mathfrak{rad}(E)) = V(\bigcap_{x \in V(E)} \mathfrak{p}_x) = \bigcup_{x \in V(E)} V(\mathfrak{p}_x) = V(E)$$

For each subset $Y \subset X$ we denoted $\mathfrak{p}(Y)$ to be the set of $f \in A$ such that $f(y) = 0$ for all $y \in Y$. It is hence clear that $\mathfrak{p}(Y) = \bigcap_{y \in Y} \mathfrak{p}_y$. So then more generally

$$\mathfrak{p}(\bigcup_\lambda Y_\lambda) = \bigcap_\lambda \mathfrak{p}(Y_\lambda)$$

And so identifying sets of the form $V(E)$ as closed yields a topology on Spec(A), called either the Zariski or Spectrum topology.

Proposition 2
- *For each subset $E \subset A$ we have $\mathfrak{p}(V(E)) = \mathfrak{rad}(E)$*

- *For each subset $Y \subset X$ we have $V(\mathfrak{p}(Y)) = \bar{Y}$*

Proof. The first part is trivial via the previous propositions. The second part is a result of checking that $V(\mathfrak{p}(Y))$ is the smallest closed subset of X containing Y. Indeed $V(\mathfrak{p}(Y)) \supset Y$ and is closed, and if $Y \subset V(E) \subset V(\mathfrak{p}(Y))$ then $f(y) = 0$ for all $f \in E$ and $y \in Y$. By inclusion reversion, $E \subset \mathfrak{p}(Y) \implies V(E) \supset V(\mathfrak{p}(Y))$ so it is true that $V(\mathfrak{p}(Y))$ is the smallest possible closed set containing Y.

The closed subsets of $\mathrm{Spec}(A)$ and the ideals of A that equal their radicals are in bijection

$$\{Y \subseteq \mathrm{Spec}(A) : Y \text{ closed }\} \longleftrightarrow \{\mathfrak{a} \subset A : \mathfrak{a} = \mathfrak{rad}(\mathfrak{a})\}$$

$$Y \to \mathfrak{p}(Y) = \{f : f(y) = 0 \quad \forall y \in Y\}$$

$$\{x \in \mathrm{Spec}(A) : \mathfrak{a} \subset \mathfrak{p}_x\} =: V(\mathfrak{a}) \leftarrow \mathfrak{a}$$

It is interesting to note here that if A is a Noetherian ring then $\mathrm{Spec}(A)$ is a Noetherian space.

Proposition 3 *When $f \in A$ the sets $D(f)$ form a basis for the topology of X. Furthermore, for every $f \in A$, $D(f)$ is quasi-compact. In particular, $X = D(1)$ is quasi-compact.*

Proof. Let U be an open set in X, then we must have $U = X - V(E)$ (i.e. open sets are the direct complements of closed sets). Therefore we see that $V(E) = \cap_{f \in E} V(f)$ and so $U = \cup_{f \in E} D(f)$.

Proposition 4 *For each ideal $\mathfrak{a}$ of A, Spec$(A/\mathfrak{a})$ is canonically identified with the closed subspace $V(\mathfrak{a})$ of Spec(A)*

Proof. Consider the same set-up. Then $\mathrm{Spec}(A/\mathfrak{a})$ has a bijective correspondence with the (prime) ideals of $A/\mathfrak{a}$ and the (prime) ideals of A containing $\mathfrak{a}$.

Let $\mathfrak{N}$ be the set of nilpotent elements of A, and recall that it is equal to the radical of the 0 ideal, that is $\mathfrak{N} = \mathfrak{rad}((0))$ and is clearly then the intersection of all prime ideals in A and hence all elements in $\mathrm{Spec}(A)$

Proposition 5 *If $\mathfrak{J}$ is an ideal in A containing the nilradical of A, $\mathfrak{N}_A$, the only neighbourhood of $V(\mathfrak{J})$ in Spec(A) is the entire space.*

This is trivial under the identification that $\mathrm{Spec}(A) \subset \mathfrak{J}$ for such a space.

1.2 Functorial Properties of Prime Spectra of Rings

Let A, A' be two rings and $\Phi : A' \to A$ be a ring homomorphism. For each prime ideal $x = \mathfrak{p}_x \in \mathrm{Spec}(A)$, the ring $A'/\Phi^{-1}(\mathfrak{p}_x)$ is canonically isomorphic to a subring of $A/\mathfrak{p}_x$ and so it is integral. Whence $\Phi^{-1}(\mathfrak{p}_x)$ is a prime ideal in $\mathrm{Spec}(A')$. We denote this map as ${}^A\Phi$:

$${}^A\Phi : \mathrm{Spec}(A) \to \mathrm{Spec}(A')$$

Proposition 6 *For each subset $E' \subset A'$ we have*

$${}^A\Phi^{-1}(V(E')) = V(\Phi(E'))$$

$${}^A\Phi^{-1}(D(f')) = D(\Phi(f')) \quad \forall f' \in A'$$

A nice result of this is that if S is a mulitplicatively closed subset of A, then $\mathrm{Spec}(S^{-1}A)$ is canonically bijective to the elements of $\mathrm{Spec}(A)$ not containing X

1.2.1 Application of Functorial Properties

Let's look at the variety $X = V(x^2 + y^2 - 1)$. The localisation of this at the point $p = (1, 0)$ is the local ring $\mathcal{O}_{(1,0),X} = (K[x, y]/(x^2 + y^2 - 1))_{(x-1,y)}$. We can then state that

$$\mathrm{Spec}((K[X]\backslash\mathfrak{m}_p)^{-1}K[X]) = \mathrm{Spec}(K[X]_{\mathfrak{m}_p}) = \mathrm{Spec}(\mathcal{O}_{p,X}) \cong \{\mathfrak{p} \in \mathrm{Spec}(K[X]) : \mathfrak{p} \cap K[X]\backslash\mathfrak{m}_p = \phi\}$$

So $\mathrm{Spec}((K[X]\backslash\mathfrak{m}_p)^{-1}K[X])$ can be identified with the vanishing of the prime ideal $\mathfrak{m}_p$, $V(\mathfrak{m}_p)$.

In this case, $V(\mathfrak{m}_p) =$