

Notes on 'Sheaf Theory' by Bredon

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June 22, 2025

Contents

I	Sheaf Theory, Bredon	3
1	Sheaves and Presheaves	3
1.1	Definitions	3
1.2	Homomorphisms, Subsheaves and Quotient Sheaves	5
1.3	Direct and Inverse Images	5
1.4	Cohomomorphisms	5
1.5	Algebraic Constructions	5
1.6	Supports	5
1.7	Classical Cohomology Theories	5
2	Sheaf Cohomology	5
2.1	Differential Sheaves and Resolutions	5
2.2	The Canonical Resolution and Sheaf Cohomology	5
2.3	Injective Sheaves	5
2.4	Acyclic Sheaves	5
2.5	Flabby Sheaves	5
2.6	Connected Sequences of Functors	5
2.7	Axioms for Cohomology and the Cup Product	5
2.8	Maps of Spaces	5
2.9	Φ -Soft and Φ -Fine Sheaves	5
2.10	Subspaces	5
2.11	The Vietoris Mapping Theorem and Homotopy Invariance	5
2.12	Relative Cohomology	5
2.13	Mayer-Vietoris Theorems	5
2.14	Continuity	5
2.15	The Künneth and Universal Coefficient Theorems	5
2.16	Dimension	5
2.17	Local Connectivity	5
2.18	Change of Supports and Local Cohomology Groups	5
2.19	The Transfer Homomorphism and the Smith Sequences	5
2.20	Steenrod's Cyclic Reduced Powers	5
2.21	The Steenrod Operations	5
3	Comparison with Other Cohomology Theories	5
3.1	Singular Cohomology	5
3.2	Alexander-Spanier Cohomology	5
3.3	de Rham Cohomology	5
3.4	Cech Cohomology	5

4	Applications of Spectral Sequences	5
4.1	The Spectral Sequence of a Differential Sheaf	5
4.2	The Fundamental Theorems of Sheaves	5
4.3	Direct Image Relative to a Support Family	5
4.4	The Leray Sheaf	5
4.5	Extension of a Support Family by a Family on the Base Space	5
4.6	The Leray Spectral Sequence of a Map	5
4.7	Fiber Bundles	5
4.8	Dimension	5
4.9	The Spectral Sequences of Borel and Cartan	5
4.10	Characteristic Classes	5
4.11	The Spectral Sequence of a Filtered Differential Sheaf	5
4.12	The Fary Spectral Sequence	5
4.13	Sphere Bundles with Singularities	5
4.14	The Oliver Transfer and the Conner Conjecture	5
5	Borel-Moore Homology	5
5.1	Cosheaves	5
5.2	The Dual of a Differential Cosheaf	5
5.3	Homology Theory	5
5.4	Maps of Spaces	5
5.5	Subspaces and Relative Homology	5
5.6	The Viertoris Theorem, Homotopy and Covering Spaces	5
5.7	The Homology Sheaf of a Map	5
5.8	The Basic Spectral Sequence	5
5.9	Poincaré Duality	5
5.10	The Cap Product	5
5.11	Intersection Theory	5
5.12	Uniqueness Theorems	5
5.13	Uniqueness Theorems for Maps and Relative Homology	5
5.14	The Künneth Formula	5
5.15	Change of Rings	5
5.16	Generalised Manifolds	5
5.17	Locally Homogenous Spaces	5
5.18	Homological Fibrations and p -adic Transformation Groups	5
5.19	The Transfer Homomorphism on Homology	5
5.20	Smith Theory in Homology	5
6	Cosheaves and Čech Homology	5
6.1	Theory of Cosheaves	5
6.2	Local Triviality	5
6.3	Local Isomorphisms	5
6.4	Čech Homology	5
6.5	The Reflector	5
6.6	Spectral Sequences	5
6.7	Coresolutions	5
6.8	Relative Čech Homology	5
6.9	Locally Paracompact Spaces	5
6.10	Borel-Moore Homology	5
6.11	Modified Borel-Moore Homology	5
6.12	Singular Homology	5
6.13	Acyclic Coverings	5
6.14	Applications to Maps	5

Part I

Sheaf Theory, Bredon

1 Sheaves and Presheaves

1.1 Definitions

Definition 1 Presheaf: A presheaf of abelian groups on a topological space X is a functor

$$A : \mathfrak{Opn}_X^{op} \rightarrow \mathfrak{Ab}$$

i.e. an assignment to each open set $U \in \text{Ob}(\mathfrak{Opn}_X^{op})$ an abelian group $A(U) \in \text{Ob}(\mathfrak{Ab})$, and on inclusion functions $V \xrightarrow{\iota_U^V} U$ restriction functions $r_V^U : A(U) \rightarrow A(V)$ such that:

- $r_U^U = id_U$,
- $U \subset V \subset W \implies r_V^W r_U^V = r_U^W$

One can switch out the target category with other algebraic categories to obtain the definitions of presheaves over R -modules, algebras, etc.

Definition 2 Germ: Let $x \in X$ be a point and A a presheaf over X . The set \mathfrak{M} of elements $s \in A(U)$, $x \in U \in \text{Ob}\mathfrak{Opn}_X^{op}$. The germs of the presheaf A at x is thus the quotient

$$\mathfrak{G}_{A,x} := \mathfrak{M} / \sim$$

where $s, t \in \mathfrak{M}$ are equal if there is $W \subset \text{dom}(s) \cap \text{dom}(t)$ such that $s|_W = t|_W$. The equivalence class $[s]$ of germs of $A(U)$ is called the germ of s at $x \in U$.

The set of germs of A at x , \mathcal{A}_x is then the direct limit

$$\mathcal{A}_x = \varinjlim A(U)$$

From here we can define a topology on the disjoint union $\sqcup_{x \in X} \mathcal{A}_x$ as follows. Fix an element $s \in A(U)$, for each $x \in U$ we have the germ s_x of s at x . For fixed s the set of all germs $s_x \in \mathcal{A}_x$ is defined to be an open set. With this definition we get the sheaf generated by the presheaf AL

$$\mathcal{A} = \mathfrak{Shcaf}(A) = \mathfrak{Shcaf}(U \rightarrow A(U))$$

Definition 3 Sheaf: A sheaf on X is a pair (\mathcal{A}, π) satisfying:

- \mathcal{A} is a (typically non-Hasuforff) topological space,
- $\pi : \mathcal{A} \rightarrow X$ is a local homeomorphism
- Each $\mathcal{A}_x = \pi^{-1}(x)$ for $x \in X$ is an abelian group, and is called the stalk of \mathcal{A} at x
- The group operations are continuous

An example of a sheaf is Ω^0 of germs of $\mathcal{C}^\infty(\mathbb{R})$ functions on a differentiable manifold M^n , this is a sheaf of unital rings, and $\Omega^p(M^n)$ of germs of differential p -forms on M^n which is also a Ω^0 -module.

If \mathcal{A} is a sheaf on X with projection $\pi : \mathcal{A} \rightarrow X$ and if $Y \subset X$ then the restriction $\mathcal{A}|_Y$ of \mathcal{A} is defined:

$$\mathcal{A}|_Y = \pi^{-1}(Y)$$

If \mathcal{A} is a sheaf on X , $Y \subset X$ then a section of \mathcal{A} over Y is a map $s : Y \rightarrow \mathcal{A}$ such that $\pi \circ s$ is the identity on Y . Note that every point $x \in Y$ admits a section s over some neighborhood U of x . We can then access the 0 section over any open set $U \subset X$ and so we can endow $\mathcal{A}(Y)$ with the structure of an abelian group. Thus the presheaf of sections of A is defined:

$$\Gamma(\mathcal{A}) = \mathcal{A}(X)$$

1.2 Homomorphisms, Subsheaves and Quotient Sheaves

1.3 Direct and Inverse Images

1.4 Cohomomorphisms

1.5 Algebraic Constructions

1.6 Supports

1.7 Classical Cohomology Theories

2 Sheaf Cohomology

2.1 Differential Sheaves and Resolutions

2.2 The Canonical Resolution and Sheaf Cohomology

2.3 Injective Sheaves

2.4 Acyclic Sheaves

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2.7 Axioms for Cohomology and the Cup Product

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2.21 The Steenrod Operations

3 Comparison with Other Cohomology Theories

3.1 Singular Cohomology

3.2 Alexander-Spanier Cohomology

3.3 de Rham Cohomology

3.4 Čech Cohomology

4 Applications of Spectral Sequences

4.1 The Spectral Sequence of a Differential Sheaf

4.2 The Fundamental Theorems of Sheaves

4.3 Direct Image Relative to a Support Family