

# Commutative Algebra for Algebraic Geometry

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## Part I

# Algebraic Geometry, Hartshorne

In this part, I will be going through the work of Hartshorne as he laid out in [Har13]. The book is sectioned into 5 distinct chapters - 'Varieties', 'Schemes', 'Cohomology', 'Curves' and 'Surfaces'. We study them in order as they flow nicely.

## 1 Varieties

**Definition 1 Affine  $n$ -Space over  $k$ :** Let  $k$  be a field and  $n \in \mathbb{N}$ . Then the affine  $n$ -space over  $k$  is defined

$$\mathbb{A}_k^n = \{(k_1, \dots, k_n) \in k^n\}$$

It seems that this is a silly definition, but later on we will see that it is useful to have a distinction between the variety  $\mathbb{A}_k^n$  and the set of points in  $k^n$ . We later view  $\mathbb{A}_k^n$  as an affine variety - an object in some arbitrary space rather than the set of  $k$ -tuples.

Let  $A = k[x_1, \dots, x_n]$  be the polynomial ring over  $k$  in  $n$  variables. Then  $f \in A$  is a map  $f : k^n \rightarrow k$ . We define the vanishing locus of this function in the following way:

**Definition 2 Vanishing Locus of a Polynomial:** Let  $f \in A = k[x_1, \dots, x_n]$ , then the vanishing locus is

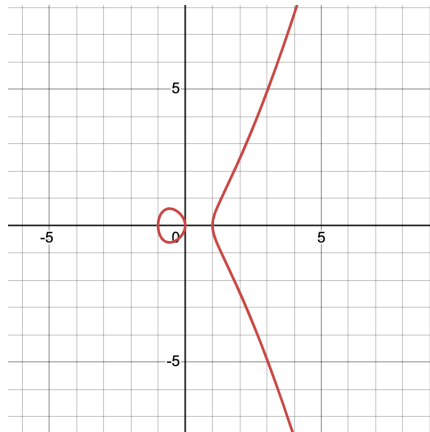
$$\mathbb{V}(f) := \{p \in \mathbb{A}_k^n : f(p) = 0\}$$

We can develop a more advanced analogue of this:

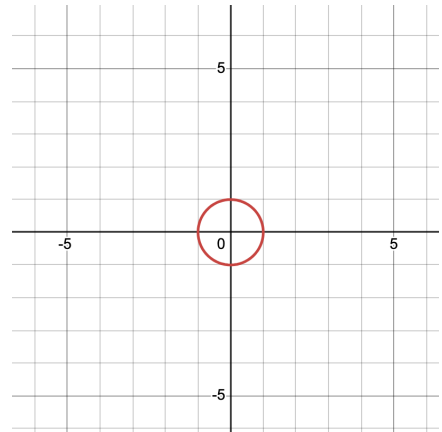
**Definition 3 Vanishing Locus of a Set of Polynomials:** Let  $T = \{f_i\}_{i \in I} \subset A = k[x_1, \dots, x_n]$ , then the vanishing locus of  $T$  is

$$\mathbb{V}(T) := \{p \in \mathbb{A}_k^n : f(p) = 0, \quad \forall f \in T\}$$

Some examples:



(a)  $\mathbb{V}(y^2 - x(x+1)(x-1))$



(b)  $\mathbb{V}(x^2 + y^2 - 1)$

Figure 1: The vanishing locus of two separate polynomials plotted in  $\mathbb{R}^2$



- 1.1 Affine Varieties
- 1.2 Projective Varieties
- 1.3 Morphisms
- 1.4 Rational Maps
- 1.5 Nonsingular Varieties
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  - 4.5 The Canonical Embedding
  - 4.6 Classification of Curves in  $\mathbb{P}^3$

## 5.5 Birational Transformations

## 5.6 Classification of Surfaces

## Part II

# Commutative Geometry with a View Toward Algebraic Geometry, Eisenbud

## 6 Basic Constructions

### 6.1 Roots of Commutative Algebra

#### 6.1.1 Number Theory

#### 6.1.2 Algebraic Curves and Function Theory

#### 6.1.3 Invariant Theory

#### 6.1.4 The Basis Theorem

#### 6.1.5 Graded Rings

#### 6.1.6 Algebra and Rings - The Nullstellensatz

#### 6.1.7 Geometric Invariant Theory

#### 6.1.8 Projective Varieties

#### 6.1.9 Hilbert Functions and Polynomials

#### 6.1.10 Free Resolutions and the Syzygy Theorem

### 6.2 Localisation

A local ring is a ring with a single unique maximal ideal. The technique of localisation reduces many problems in commutative algebra to problems on commutative rings. The idea of localisation is as follows. Given a point  $p$  in an algebraic set  $X \subset \mathbb{A}_k^n$ , we want to investigate what  $X$  looks like near  $p$ , that is we want to investigate arbitrarily small open neighbourhoods of  $p$  in the Zariski topology. The Zariski open neighbourhoods of  $p$  are sets of the form  $X \setminus Y$  for  $p \notin Y \subset X$ .

#### 6.2.1 Fractions

**Definition 4** *Localisation of an  $R$ -Module via a Multiplicatively Closed Subset  $U$ :* Let  $R$  be a ring,  $M$  an  $R$ -module and  $U \subset R$  a multiplicatively closed subset. The localisation of  $M$  at  $U$ ,  $M[U^{-1}]$  is the set of equivalent classes of pairs  $(m, u) \sim (m', u')$  where  $m, m' \in M$  and  $u, u' \in U$  are related if there is some  $v \in U$  such that  $v(mu' - m'u) = 0$ .

**Proposition 1** *Let  $U$  be multiplicatively closed set of  $R$  and let  $M$  be an  $R$ -module. An element  $m \in M$  goes to 0 in  $M[U^{-1}]$  under the map  $\pi : M \rightarrow M[U^{-1}], m \rightarrow m/1$  if  $m$  is annihilated by an element  $u \in U$ . In particular, if  $M$  is finitely generated then  $M[U^{-1}] = 0$  iff  $M$  is annihilated by an element of  $U$ .*

*proof*



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- 6.2.3 The Construction of Primes
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- 6.3 Associated Primes and Primary Decomposition
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## 64 Operads

### 64.1 Operads

**Definition 5 Operad:** An operad  $P$  consists of a set of colours  $C$  and for each  $n \geq 0$  and sequence  $c_1, \dots, c_n, c$  of colours in  $C$ , a set  $P(c_1, \dots, c_n; c)$  of operations, thought of as taking  $n$  inputs of colours  $c_1, \dots, c_n$  and with output of colour  $c$ . Moreover there are the structure maps

- $\forall c \in C, \exists 1_c \in P(c; c)$ ,
- For  $\sigma \in \Sigma_n$  a map

$$\sigma^* : P(c_1, \dots, c_n; c) \rightarrow P(c_{\sigma(1)}, \dots, c_{\sigma(n)}; c)$$

denoted  $\sigma^* \circ p = p \circ \sigma$

- For any sequence  $c_1, \dots, c_n$  and  $n$ -tuple of sequences  $d_1^i, \dots, d_{k_i}^i$ , a composition

•

$$\gamma : P(c_1, \dots, c_n; c) \times \prod_{i=1}^n P(d_1^i, \dots, d_{k_i}^i; c_i) \rightarrow P(d_1^1, \dots, d_{k_n}^n; c)$$

which is written as  $\gamma(p, q_1, \dots, q_n) \rightarrow p \circ (q_1, \dots, q_n)$

and there are further requirements on the structure maps:

- $\forall p \in P(c_1, \dots, c_n; c), \gamma(1_c, p) = p$ ,
- $\forall p \in P(c_1, \dots, c_n; c), \gamma(p, 1_{c_1}, \dots, 1_{c_n}) = p$

There are some classical notation we must be weary of and state here. Let  $P$  be an operad with a singleton colour set, i.e.  $C = \{*\}$ . Then we can write  $P(c_1, \dots, c_n; c) = P(c, \dots, c; c) =: P(n)$ . There is an obvious formulation for the composition in this case:

$$P(n) \times \prod_{i=1}^n P(k_i) \rightarrow P(k_1 + \dots + k_n)$$

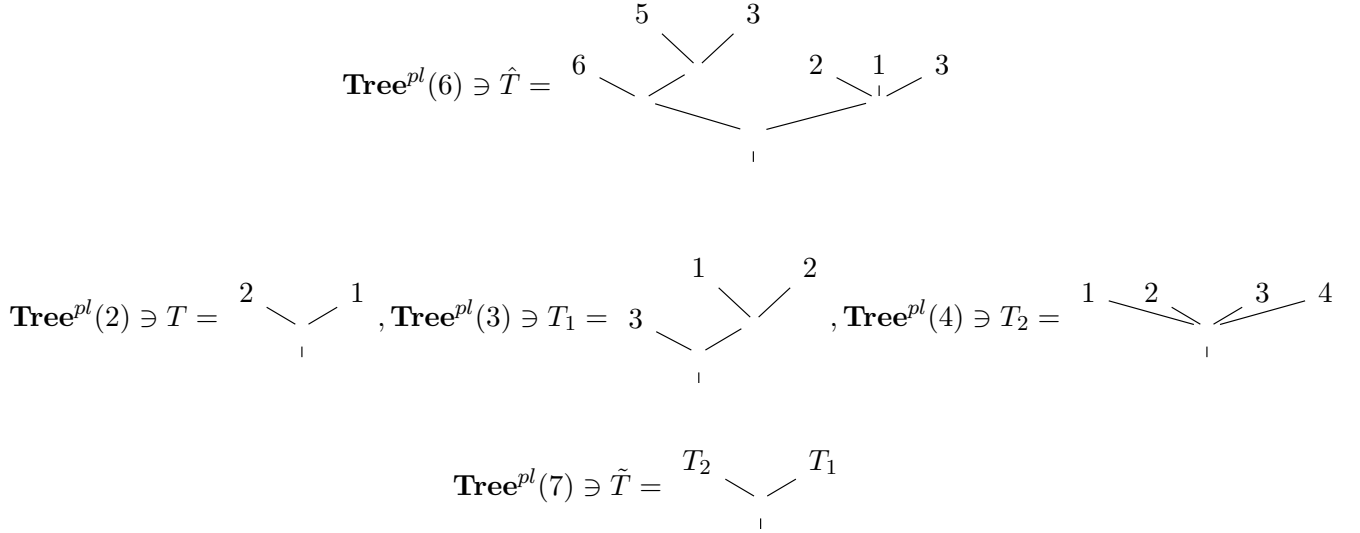
In this case we say  $P$  is uncoloured. If  $C \notin \{\phi, \{*\}\}$  then we say  $P$  is a coloured operad. We can then clearly see that

$$\begin{array}{ccc} \text{Monoids} & \hookrightarrow & \text{Categories} \\ \downarrow & & \downarrow \\ \text{Uncoloured Operads} & \hookrightarrow & \text{Operads} \end{array}$$

The definition of an operad allows for  $n = 0$  in  $P(c_1, \dots, c_n; c)$ , which we define as  $P(-; c)$ . The elements of  $P(-; c)$  are called the constants of colour  $c$ , and an operad with  $P(-; c) = \{*_c\}$  is called unital. An operad  $P$  is open if there are no constants for any colour, i.e. its interior  $P^o$  (the set of constants) is empty.

The most fundamental examples of operads are **Com** and **Ass**:

- **Com** is the commutative operad with  $\mathbf{Com}(n) = \{*\}$ ,
- **Ass** is the associative operad with  $\mathbf{Ass}(n) = \Sigma_n$
- **Tree**<sup>pl</sup> is the planar tree operad. The  $n$ -ary operations here are the set of planar rooted trees with  $n$  numbered leaves.



Then in fact  $\gamma(T, T_1, T_2) = \tilde{T}$ . The operation of composition on the operad  $\mathbf{Tree}^{pl}$  is computed as  $\gamma(T \in \mathbf{Tree}^{pl}(n), T_1 \in \mathbf{Tree}^{pl}(k_1), \dots, T_n \in \mathbf{Tree}^{pl}(k_n)) = \hat{T}$  where  $\hat{T} \in \mathbf{Tree}^{pl}(k_1 + \dots + k_n)$  is the original  $T$  but with the subtree  $T_i$  grafted onto the leaf  $i$  for all choices of  $i$ .

**Definition 6 Topological Operad:** A topological operad is an operad  $P$  where each set of operations  $P(c_1, \dots, c_n; c)$  is equipped with some topology and all the structure maps are continuous with respect to this topology.

The most basic form of a topological operad is the little  $d$ -cubs operad  $\mathbf{E}_d$ . The space  $\mathbf{E}_d(n)$  is the space of  $n$  numbered  $d$ -dimensional cubes inside the  $d$ -dimensional unit cube  $[0, 1]^d$ . The operadic composition between  $p \in \mathbf{E}_d(n)$  with operations  $q_1, \dots, q_n$  is given by substituting the rescaled  $q_i$  into the  $i$ th cube of  $p$ . Note that this is really just a topological analogue to the planar tree operad  $\mathbf{Tree}^{pl}$ , where instead of grafting trees onto leaves we are scaling and embedding cubes in some smooth way.

More specifically, a point in  $\mathbf{E}_d(n)$  is an  $n$ -tuple of embeddings  $f_1, \dots, f_n : [0, 1]^d \rightarrow [0, 1]^d$  satisfying:

- Each  $f_i$  is the composition of  $d$  affine embeddings,
- The interiors of the cubes embedded by  $f_i$  are mutually disjoint

We can now observe that operads form a category in a very natural fashion. Given two operads  $P, Q$ , a morphism  $\varphi : P \rightarrow Q$  is a function  $f : C_P \rightarrow C_Q$  on operadic colours and for each sequence  $c_1, \dots, c_n; c$  of  $C_P$ , we have

$$\varphi_{(c_1, \dots, c_n; c)} : P(c_1, \dots, c_n; c) \rightarrow Q(f(c_1), \dots, f(c_n); f(c))$$

that is compatible in the natural way with  $\Sigma_n$  actions

## 64.2 Algebras for Operads

**Definition 7 Operadic Algebras:** Let  $P$  be an operad. A  $P$ -algebra  $A$  is a family of sets  $\{A_c\}_{c \in C_P}$  together with maps

$$P(c_1, \dots, c_n; c) \times A_{c_1} \times \dots \times A_{c_n} \rightarrow A_c$$

written  $(p, a_1, \dots, a_n) \rightarrow A(p)(a_1, \dots, a_n)$ . These maps also satisfy:

- $1_c(a) = a \quad \forall a \in A_c$
- $\sigma \in \Sigma_n, a_i \in A_{c_i}, \sigma^* p(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = p(a_1, \dots, a_n)$

**Definition 8 Morphisms of Operadic Algebras:** Let  $A, B$  be two  $P$ -algebras. A morphism  $f : A \rightarrow B$  is a family of maps

$$f_c : A_c \rightarrow B_c$$

which are compatible:

$$f_c(A(p)(a_1, \dots, a_n)) = B(p)(f_{c_1}(a_1), \dots, f_{c_n}(a_n))$$

**Definition 9 Category of  $P$ -Algebras:** Let  $P$  be an operad. Then we have a category of  $P$ -algebras  $\text{Alg}_P$  with:

- $\text{ObAlg}_P = \{P\text{-algebras } A\}$
- $\text{Hom}_{\text{Alg}_P}(A, B) = \{f : A \rightarrow B : f \text{ is a morphism of algebras}\}$

We can now see some examples of operadic algebras. A **Com** algebra is a set  $A$  together with a map  $\mu_n : A^{\times n} \rightarrow A$  for each  $n \geq 0$ . We can then verify that the category of algebras over the commutative operad,  $\text{Alg}_{\text{Com}}$  is the category of commutative monoids. In a similar way  $\text{Alg}_{\text{Ass}}$  is the category of associative monoids.

Consider the little- $d$  cubes operad  $\mathbf{E}_d$ . Let  $X$  be a topological space with basepoint  $x_0$ . Then the loop space of  $X$  is  $\Omega X$ , the space of basepoint preserving maps  $S^1 \rightarrow X$ , or in other words  $\{\omega : [0, 1] \rightarrow X, \omega(\partial[0, 1]) = x_0\}$ . One can then inductively construct the  $d$ -fold loop space  $\Omega^d X = \Omega(\Omega^{d-1} X)$ .  $\Omega^d X$  is very naturally a  $\mathbf{E}_d$  algebra.

### 64.3 Trees

### 64.4 Alternative Definitions for Operads

### 64.5 Free Operads

### 64.6 The Tensor Product of Operads

### 64.7 The Boardman-Vogt Resolution of an Operad

### 64.8 Configuration Spaces and the Fulton-MacPherson Operad

### 64.9 Configuration Spaces and the Operad of Little Cubes

## 65 Simplicial Sets

### 65.1 The Simplex Category $\Delta$

**Definition 10 The Simplex Category  $\Delta$ :**  $\Delta$  is the category with:

- $\text{Ob}\Delta = \mathbb{N}$ ,
- $\text{Hom}_\Delta([n], [m]) = \{\text{order preserving maps } [n] \rightarrow [m]\}$

There are special maps in  $\Delta$  - the elementary faces  $\delta^i : [m-1] \rightarrow [m]$  and elementary degeneracies  $\sigma^i : [m] \rightarrow [m-1]$   $0 \leq i \leq m-1$ :

$$\delta^i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}, \quad \sigma^i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

These have some nice relations, called the cosimplicial identities:

- $\sigma_i \sigma_j = \sigma_{j-1} \sigma_i, \quad i < j$
- $\delta_j \delta_i = \delta_i \delta_{j-1}$

### 65.1.1 Limits and Colimits of The Simplicial Category

$$\begin{array}{ccc} k & \xrightarrow{f} & n \\ \downarrow g & & \downarrow \\ m & \rightarrow & m+n \end{array}$$

is a pushout, where  $f(i) = i, g(i) = m - k + i$

## 66 Dendroidal Sets

## 67 Tensor Products of Dendroidal Sets

## 68 Kan Conditions for Simplicial Sets

## 69 Kan Conditions for Dendroidal Sets

## 70 Model Categories

## 71 Model Structures on the Category of Simplicial Sets

## 72 Three Model Structures on the Category of Dendroidal Sets

## 73 Reedy Categories and Diagrams of Spaces

## 74 Mapping Spaces and Bousfield Localisations

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## 76 Left Fibrations and the Covariant Model Structure

## 77 Simplicial Operads and $\infty$ -Operads

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