

Notes on 'Éléments de Géométrie Algébrique'

Bailey Arm

June 3, 2025

Contents

| | | |
|----------|---|----------|
| I | Éléments de Géométrie Algébrique I: The Language of Schemes | 3 |
| 1 | Affine Schemes | 3 |
| 1.1 | The Prime Spectrum of a Ring | 3 |
| 1.2 | Functorial Properties of Prime Spectra of Rings | 4 |
| 1.2.1 | Application of Functorial Properties | 5 |
| 1.3 | Sheaf Associated to a Module | 7 |
| 1.4 | Quasi-Coherent Sheaves over a Prime Spectrum | 7 |
| 1.5 | Coherent Sheaves over a Prime Spectrum | 7 |
| 1.6 | Functorial Properties of Quasi-Coherent Sheaves over a Prime Spectrum | 7 |
| 1.7 | Characterisation of Morphisms of Affine Schemes | 7 |
| 1.8 | Morphisms from Locally Ringed Spaces to Affine Schemes | 7 |
| 2 | Preschemes and Morphisms of Preschemes | 7 |
| 2.1 | Definition of Preschemes | 7 |
| 2.2 | Morphism of Preschemes | 7 |
| 2.3 | Gluing Preschemes | 7 |
| 2.4 | Preschemes over a Prescheme | 7 |
| 3 | Products of Preschemes | 7 |
| 3.1 | Sums of Preschemes | 7 |
| 3.2 | Products of Preschemes | 7 |
| 3.3 | Formal Properties of the Product, Change of the Base Prescheme | 7 |
| 3.4 | Points of a Prescheme with Values in a Prescheme, Geometric Points | 7 |
| 3.5 | Surjections and Injections | 7 |
| 3.6 | Fibres | 7 |
| 3.7 | Applications: Reduction of a Prescheme Modulo \mathcal{J} | 7 |
| 4 | Subpreschemes and Immersion Morphisms | 7 |
| 4.1 | Subpreschemes | 7 |
| 4.2 | Immersion Morphisms | 7 |
| 4.3 | Products of Immersions | 7 |
| 4.4 | Inverse Images of a Subprescheme | 7 |
| 4.5 | Local Immersions and Local Isomorphisms | 7 |
| 5 | Reduced Preschemes and the Separation Condition | 7 |
| 5.1 | Reduced Preschemes | 7 |
| 5.2 | Existence of a Subprescheme with a Given Underlying Space | 7 |
| 5.3 | Diagonal; Graph of a Morphism | 7 |
| 5.4 | Separated Morphisms and Separated Preschemes | 7 |
| 5.5 | Separation Criteria | 7 |

| | | |
|-----------|--|----------|
| 6 | Finiteness Conditions | 7 |
| 6.1 | Noetherian and Locally Noetherian Preschemes | 7 |
| 6.2 | Artinian Properties | 7 |
| 6.3 | Morphisms of Finite Type | 7 |
| 6.4 | Algebraic Preschemes | 7 |
| 6.5 | Local Determination of a Morphism | 7 |
| 6.6 | Quasi-Compact Morphisms and Morphisms Locally of Finite Type | 7 |
| 7 | Rational Maps | 7 |
| 7.1 | Rational Maps and Rational Functions | 7 |
| 7.2 | Domain of Definition of a Rational Map | 7 |
| 7.3 | Sheaf of Rational Functions | 7 |
| 7.4 | Torsion Sheaves and Torsion-Free Sheaves | 7 |
| 8 | Chevalley Schemes | 7 |
| 8.1 | Allied Local Rings | 7 |
| 8.2 | Local Rings of an Integral Scheme | 7 |
| 8.3 | Chevalley Schemes | 7 |
| 9 | Supplement on Quasi-Coherent Sheaves | 7 |
| 10 | Formal Schemes | 7 |

Part I

Éléments de Géométrie Algébrique I: The Language of Schemes

1 Affine Schemes

1.1 The Prime Spectrum of a Ring

Definition 1 Prime Spectrum of a Ring: Let A be commutative ring and M an A -module. Then

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subset A : \mathfrak{p} \text{ prime ideal in } A\}$$

For points $x \in X = \mathrm{Spec}(A)$ it is usually convenient to write \mathfrak{p}_x . It is also important to note that for $\mathrm{Spec}(A) = \emptyset$ we need A to be the 0 ring.

Definition 2 Local Ring of Fractions of a Ring: Let A be a commutative ring. The local ring at $x \in \mathrm{Spec} A$ is

$$A_x = A_{\mathfrak{p}_x} = (A \setminus \mathfrak{p}_x)^{-1} A$$

There is also some more notation

- $\mathfrak{m}_x = \mathfrak{p}_x A_{\mathfrak{p}_x}$ is the (unique) maximal ideal of A_x ,
- $k(x) = A_x / \mathfrak{m}_x$ is the residue field of A_x and it canonically isomorphic to the fraction field of the integral ring A / \mathfrak{p}_x . It is thus we identify $k(x) = A / \mathfrak{p}_x$,
- $M_x = M \otimes_A A_x$ is the module of fractions with denominators in $A \setminus \mathfrak{p}_x$,
- $V(E) = \{x \in X = \mathrm{Spec}(A) : E \subset \mathfrak{p}_x\}$
- $\mathrm{rad}(E) = \bigcap_{x \in V(E)} \mathfrak{p}_x$

Proposition 1 • $V(0) = X, V(1) = \emptyset$,

- $E \subset E' \implies V(E') \subset V(E)$,
- $V(\bigcup_{\lambda} E_{\lambda}) = \bigcap_{\lambda} V(E_{\lambda})$,
- $V(EE') = V(E) \cup V(E')$,
- $V(E) = V(\mathrm{rad}(E))$

Proof. It is clear that all ideals contain the 0 ideal and so $V(0) = \mathrm{Spec}(A)$. The third and fourth subpropositions are clear via doing the inclusions. The final subproposition is trivial when one notices that

$$V(\mathrm{rad}(E)) = V\left(\bigcap_{x \in V(E)} \mathfrak{p}_x\right) = \bigcup_{x \in V(E)} V(\mathfrak{p}_x) = V(E)$$

For each subset $Y \subset X$ we denoted $\mathfrak{p}(Y)$ to be the set of $f \in A$ such that $f(y) = 0$ for all $y \in Y$. It is hence clear that $\mathfrak{p}(Y) = \bigcap_{y \in Y} \mathfrak{p}_y$. So then more generally

$$\mathfrak{p}\left(\bigcup_{\lambda} Y_{\lambda}\right) = \bigcap_{\lambda} \mathfrak{p}(Y_{\lambda})$$

And so identifying sets of the form $V(E)$ as closed yields a topology on $\mathrm{Spec}(A)$, called either the Zariski or Spectrum topology.

Proposition 2 • For each subset $E \subset A$ we have $\mathfrak{p}(V(E)) = \mathrm{rad}(E)$

- For each subset $Y \subset X$ we have $V(\mathfrak{p}(Y)) = \bar{Y}$

Proof. The first part is trivial via the previous propositions. The second part is a result of checking that $V(\mathfrak{p}(Y))$ is the smallest closed subset of X containing Y . Indeed $V(\mathfrak{p}(Y)) \supset Y$ and is closed, and if $Y \subset V(E) \subset V(\mathfrak{p}(Y))$ then $f(y) = 0$ for all $f \in E$ and $y \in Y$. By inclusion reversion, $E \subset \mathfrak{p}(Y) \implies V(E) \supset V(\mathfrak{p}(Y))$ so it is true that $V(\mathfrak{p}(Y))$ is the smallest possible closed set containing Y .

The closed subsets of $\text{Spec}(A)$ and the ideals of A that equal their radicals are in bijection

$$\{Y \subseteq \text{Spec}(A) : Y \text{ closed}\} \longleftrightarrow \{\mathfrak{a} \subset A : \mathfrak{a} = \text{rad}(\mathfrak{a})\}$$

$$Y \rightarrow \mathfrak{p}(Y) = \{f : f(y) = 0 \quad \forall y \in Y\}$$

$$\{x \in \text{Spec}(A) : \mathfrak{a} \subset \mathfrak{p}_x\} =: V(\mathfrak{a}) \leftarrow \mathfrak{a}$$

It is interesting to note here that if A is a Noetherian ring then $\text{Spec}(A)$ is a Noetherian space.

Proposition 3 When $f \in A$ the sets $D(f)$ form a basis for the topology of X . Furthermore, for every $f \in A$, $D(f)$ is quasi-compact. In particular, $X = D(1)$ is quasi-compact.

Proof. Let U be an open set in X , then we must have $U = X - V(E)$ (i.e. open sets are the direct complements of closed sets). Therefore we see that $V(E) = \bigcap_{f \in E} V(f)$ and so $U = \bigcup_{f \in E} D(f)$.

Proposition 4 For each ideal \mathfrak{a} of A , $\text{Spec}(A/\mathfrak{a})$ is canonically identified with the closed subspace $V(\mathfrak{a})$ of $\text{Spec}(A)$

Proof. Consider the same set-up. Then $\text{Spec}(A/\mathfrak{a})$ has a bijective correspondence with the (prime) ideals of A/\mathfrak{a} and the (prime) ideals of A containing \mathfrak{a} .

Let \mathfrak{N} be the set of nilpotent elements of A , and recall that it is equal to the radical of the 0 ideal, that is $\mathfrak{N} = \text{rad}((0))$ and is clearly then the intersection of all prime ideals in A and hence all elements in $\text{Spec}(A)$

Proposition 5 If \mathfrak{J} is an ideal in A containing the nilradical of A , \mathfrak{N}_A , the only neighbourhood of $V(\mathfrak{J})$ in $\text{Spec}(A)$ is the entire space.

This is trivial under the identification that $\text{Spec}(A) \subset \mathfrak{J}$ for such a space.

1.2 Functorial Properties of Prime Spectra of Rings

Let A, A' be two rings and $\Phi : A' \rightarrow A$ be a ring homomorphism. For each prime ideal $x = \mathfrak{p}_x \in \text{Spec}(A)$, the ring $A'/\Phi^{-1}(\mathfrak{p}_x)$ is canonically isomorphic to a subring of A/\mathfrak{p}_x and so it is integral. Whence $\Phi^{-1}(\mathfrak{p}_x)$ is a prime ideal in $\text{Spec}(A')$. We denote this map as ${}^A\Phi$:

$${}^A\Phi : \text{Spec}(A) \rightarrow \text{Spec}(A')$$

Proposition 6 For each subset $E' \subset A'$ we have

$${}^A\Phi^{-1}(V(E')) = V(\Phi(E'))$$

$${}^A\Phi^{-1}(D(f')) = D(\Phi(f')) \quad \forall f' \in A'$$

A nice result of this is that if S is a multiplicatively closed subset of A , then $\text{Spec}(S^{-1}A)$ is canonically bijective to the elements of $\text{Spec}(A)$ not containing X

1.2.1 Application of Functorial Properties

Let's look at the variety $X = V(x^2 + y^2 - 1)$. The localisation of this at the point $p = (1, 0)$ is the local ring $\mathcal{O}_{(1,0),X} = (K[x, y]/(x^2 + y^2 - 1))_{(x-1, y)}$. We can then state that

$$\mathrm{Spec}((K[X] \setminus \mathfrak{m}_p)^{-1} K[X]) = \mathrm{Spec}(K[X]_{\mathfrak{m}_p}) = \mathrm{Spec}(\mathcal{O}_{p,X}) \cong \{\mathfrak{p} \in \mathrm{Spec}(K[X]) : \mathfrak{p} \cap K[X] \setminus \mathfrak{m}_p = \emptyset\}$$

So $\mathrm{Spec}((K[X] \setminus \mathfrak{m}_p)^{-1} K[X])$ can be identified with the vanishing of the prime ideal \mathfrak{m}_p , $V(\mathfrak{m}_p)$.

In this case, $V(\mathfrak{m}_p) =$

- 1.3 Sheaf Associated to a Module
- 1.4 Quasi-Coherent Sheaves over a Prime Spectrum
- 1.5 Coherent Sheaves over a Prime Spectrum
- 1.6 Functorial Properties of Quasi-Coherent Sheaves over a Prime Spectrum
- 1.7 Characterisation of Morphisms of Affine Schemes
- 1.8 Morphisms from Locally Ringed Spaces to Affine Schemes
- 2 Preschemes and Morphisms of Preschemes
 - 2.1 Definition of Preschemes
 - 2.2 Morphism of Preschemes
 - 2.3 Gluing Preschemes
 - 2.4 Preschemes over a Prescheme
- 3 Products of Preschemes
 - 3.1 Sums of Preschemes
 - 3.2 Products of Preschemes
 - 3.3 Formal Properties of the Product, Change of the Base Prescheme
 - 3.4 Points of a Prescheme with Values in a Prescheme, Geometric Points
 - 3.5 Surjections and Injections
 - 3.6 Fibres
 - 3.7 Applications: Reduction of a Prescheme Modulo \mathcal{J}
- 4 Subpreschemes and Immersion Morphisms
 - 4.1 Subpreschemes
 - 4.2 Immersion Morphisms
 - 4.3 Products of Immersions
 - 4.4 Inverse Images of a Subprescheme
 - 4.5 Local Immersions and Local Isomorphisms
- 5 Reduced Preschemes and the Separation Condition
 - 5.1 Reduced Preschemes
 - 5.2 Existence of a Subprescheme with a Given Underlying Space
 - 5.3 Diagonal; Graph of a Morphism
 - 5.4 Separated Morphisms and Separated Preschemes
 - 5.5 Separation Criteria
- 6 Finiteness Conditions
 - 6.1 Noetherian and Locally Noetherian Preschemes
 - 6.2 Artinian Properties
 - 6.3 Morphisms of Finite Type
 - 6.4 Algebraic Preschemes