Notes on 'Algebraic Geometry I: Schemes'

Bailey Arm

June 6, 2025

Contents

I Algebraic Geometry I: Schemes, Gortz-Wedhorn 3

1 Prevarieties 3
- 1.1 Affine Algebraic Sets . . . 3
  - 1.1.1 The Zariski Topology on $\mathbb{A}_k^n$ . . . 3
  - 1.1.2 Affine Algebraic Sets . . . 3
  - 1.1.3 Hilbert's Nullstellensatz . . . 3
- 1.2 Affine Algebraic Sets as Spaces with Functions . . . 5
- 1.3 Prevarieties . . . 5
- 1.4 Projective Varieties . . . 5
- 1.5 Exercises . . . 5

2 Spectrum of a Ring 5

3 Schemes 5

4 Fibre Products 5

5 Schemes over Fields 5

6 Local Porpoerties of Schemes 5

7 Quasi-Coherent Modules 5
- 7.1 Excursion into $\mathcal{O}_X$-Modules . . . 5
- 7.2 Quasi-Coherent Modules on a Scheme . . . 5
- 7.3 Properties of Quasi-Coherent Modules . . . 5
- 7.4 Exercises . . . 5

8 Representable Functors 5
- 8.1 Representable Functors . . . 5
- 8.2 The Grassmannian . . . 5
- 8.3 Brauer-Severi Schemes . . . 5
- 8.4 Exercises . . . 5

9 Separated Morphisms 5
- 9.1 Diagonal of Scheme Morphism s and Separated Morphisms . . . 5
- 9.2 Rational Maps and Function Fields . . . 5
- 9.3 Exercises . . . 5

10 Finiteness Conditions **5**
10.1 . . . 5
10.2 . . . 5
10.3 . . . 5
10.4 . . . 5
10.5 Exercises . . . 5

11 Vector Bundles **5**
11.1 . . . 5
11.2 . . . 5
11.3 . . . 5
11.4 . . . 5
11.5 Exercises . . . 5

12 Affine and Proper Morphisms **5**
12.1 . . . 5
12.2 . . . 5
12.3 . . . 5
12.4 . . . 5
12.5 . . . 5
12.6 . . . 5
12.7 Exercises . . . 5

13 Projective Morphisms **5**
13.1 . . . 5
13.2 . . . 5
13.3 . . . 5
13.4 Exercises . . . 5

14 Flat Morphisms and Dimension **5**
14.1 . . . 5
14.2 . . . 5
14.3 . . . 5
14.4 . . . 5
14.5 . . . 5
14.6 . . . 5
14.7 Exercises . . . 5

15 One-Dimensional Schemes **5**
15.1 . . . 5
15.2 . . . 5
15.3 . . . 5
15.4 . . . 5
15.5 Exercises . . . 5

16 Examples **5**
16.1 Determinal Varieties . . . 5
16.2 Cubic Surfaces and Hilbert Modular Surface . . . 5
16.3 Cyclic Quotient Singularities . . . 5
16.4 Abelian Varieties . . . 5
16.5 Exercises . . . 5

Part I
Algebraic Geometry I: Schemes, Gortz-Wedhorn

1 Prevarieties

1.1 Affine Algebraic Sets

1.1.1 The Zariski Topology on $\mathbb{A}_k^n$

Definition 1 *Let $M \subseteq k[T_1, ..., T_n] =: k[\underline{T}]$. The set of common zeros of the polynomials in M is defined as*

$$\mathbb{V}(M) := \{p \in k^n : f(p) = 0 \quad \forall f \in M\}$$

Proposition 1 *The sets $\mathbb{V}(\mathfrak{a})$ where $\mathfrak{a}$ is an ideal in $k[\underline{T}]$ form a topoology on $\mathbb{A}_k^n$ called the Zariksi topology.*

This is a very elementary problem in algebraic geometry .

1.1.2 Affine Algebraic Sets

Definition 2 *The closed subspaces of $\mathbb{A}_k^n$ are called affine algebraic sets.*

1.1.3 Hilbert's Nullstellensatz

Theorem 1 ***Hilbert's Nullstellensatz:*** *Let K be field and A a finitely generated K-algebra. Then A is Jacobson, that is for every prime ideal $\mathfrak{p} \subset A$ we have*

$$\mathfrak{p} = \bigcap_{\mathfrak{m} \supseteq \mathfrak{p}, \mathfrak{m} \ maximal} \mathfrak{m}$$

1.2 Affine Algebraic Sets as Spaces with Functions

1.3 Prevarieties

1.4 Projective Varieties

1.5 Exercises

2 Spectrum of a Ring

3 Schemes

4 Fibre Products

5 Schemes over Fields

6 Local Porpoerties of Schemes

7 Quasi-Coherent Modules

7.1 Excursion into $\mathcal{O}_X$-Modules

7.2 Quasi-Coherent Modules on a Scheme

7.3 Properties of Quasi-Coherent Modules

7.4 Exercises

8 Representable Functors

8.1 Representable Functors

8.2 The Grassmannian

8.3 Brauer-Severi Schemes

8.4 Exercises

9 Separated Morphisms

9.1 Diagonal of Scheme Morphism s and Separated Morphisms

9.2 Rational Maps and Function Fields

9.3 Exercises

10 Finiteness Conditions

10.1

10.2

10.3

10.4

10.5 Exercises

11 Vector Bundles

11.1

11.2

11.3

11.4