TỔNG HỢP CÁC BÀI TOÁN TÍCH PHÂN TRÊN BOXMATH

Bài 1. *Tìm nguyên hàm*

$$I = \int \frac{6x^3+8x+1}{(3x^2+4)\sqrt{x^2+1}}\ dx$$

Lời giải

Ta có $\dfrac{6x^3+8x+1}{3x^2+4} = 2x + \dfrac{1}{3x^2+4}$

$$\Rightarrow I = \int \left(2x + \frac{1}{3x^2+4}\right)\frac{1}{\sqrt{x^2+1}}\ \mathrm{d}x = \int \frac{2x}{\sqrt{x^2+1}}\ \mathrm{d}x + \int \frac{1}{(3x^2+4)\sqrt{x^2+1}}\ \mathrm{d}x$$

Tính $I_1 = \displaystyle\int \frac{2x}{\sqrt{x^2+1}}\ \mathrm{d}x$

Đặt $\sqrt{x^2+1} = t,\ x^2+1 = t^2,\ 2\ \mathrm{d}t = 2x\ \mathrm{d}x \Rightarrow I_1 = 2\displaystyle\int \frac{\mathrm{d}t}{t} = 2\ln|t| = 2\ln\sqrt{x^2+1}$

Tính $I_2 = \displaystyle\int \frac{1}{(3x^2+4)\sqrt{x^2+1}}.\ \mathrm{d}x$

Đặt $t = \dfrac{\sqrt{x^2+1}}{x},\ xt = \sqrt{x^2+1},\ x^2t^2 = x^2+1,\ x^2 = \dfrac{1}{t^2-1},\ 3x^2+4 = \dfrac{4t^2-1}{t^2-1}$

$$x\ \mathrm{d}x = -\frac{t\ \mathrm{d}t}{(t^2-1)^2},\ \frac{\mathrm{d}x}{xt} = -\frac{t\ \mathrm{d}t}{(t^2-1)^2x^2t},\ \frac{\mathrm{d}x}{\sqrt{x^2+1}} = \frac{\mathrm{d}t}{1-t^2}$$

$$I_2 = \int \frac{t^2-1}{4t^2-1}\frac{\mathrm{d}t}{1-t^2} = \int \frac{\mathrm{d}t}{1-4t^2} = \frac{1}{2}\int\left(\frac{1}{2t+1} - \frac{1}{2t-1}\right)\mathrm{d}t = \frac{1}{4}\ln\frac{2t+1}{2t-1} = \frac{1}{4}\ln\frac{2\sqrt{x^2+1}+x}{2\sqrt{x^2+1}-x}$$

Vậy
$$I = 2\ln\sqrt{x^2+1} + \frac{1}{4}\ln\frac{2\sqrt{x^2+1}+x}{2\sqrt{x^2+1}-x} + C$$

Bài 2. *Tìm nguyên hàm*

$$I = \int \frac{\cos^2 x}{\sin x + \sqrt{3}\cos x}\ dx$$

Lời giải

Dùng pp hệ số bất định $\cos^2 x = (a\sin x + b\cos x)(\sin x + \sqrt{3}\cos x) + c(\sin^2 x + \cos^2 x)$

$$\cos^2 x = \left(\frac{-1}{4}\sin x + \frac{\sqrt{3}}{4}\cos x\right)(\sin x + \sqrt{3}\cos x) + \frac{1}{4} = \frac{-1}{4}(\sin x - \sqrt{3}\cos x)(\sin x + \sqrt{3}\cos x) + \frac{1}{4}$$

$$I = \int \frac{\frac{-1}{4}(\sin x - \sqrt{3}\cos x)(\sin x + \sqrt{3}\cos x) + \frac{1}{4}}{\sin x + \sqrt{3}\cos x}\ \mathrm{d}x$$

$$I = \frac{-1}{4}\int(\sin x - \sqrt{3}\cos x)\ \mathrm{d}x + \frac{1}{4}\int\frac{1}{\sin x + \sqrt{3}\cos x}\ \mathrm{d}x$$

$$I = \frac{1}{4}(\cos x + \sqrt{3}\sin x) + \frac{1}{4}\int\frac{1}{\sin x + \sqrt{3}\cos x}\ \mathrm{d}x$$

Ta tính $J = \dfrac{1}{4}\displaystyle\int\frac{\mathrm{d}x}{\sin x + \sqrt{3}\cos x} = \frac{1}{8}\int\frac{\mathrm{d}x}{\cos(x-\frac{\pi}{6})} = \frac{1}{8}\int\frac{\cos(x-\frac{\pi}{6})}{1-\sin^2(x-\frac{\pi}{6})}\ \mathrm{d}x$

Đặt $t = \sin(x-\frac{\pi}{6}) \Rightarrow\ \mathrm{d}t = \cos(x-\frac{\pi}{6})\ \mathrm{d}x$

$$\Rightarrow J = \frac{1}{8}\int\frac{\mathrm{d}t}{1-t^2} = \frac{1}{16}\int\left(\frac{1}{t+1} - \frac{1}{t-1}\right)\ \mathrm{d}t = \frac{1}{16}\ln\frac{t+1}{t-1} = \frac{1}{16}\ln\frac{\sin(x-\frac{\pi}{6})+1}{\sin(x-\frac{\pi}{6})-1}$$

Vậy
$$I = \frac{1}{4}(\cos x + \sqrt{3}\sin x) + \frac{1}{16}\ln\frac{\sin(x-\frac{\pi}{6})+1}{\sin(x-\frac{\pi}{6})-1} + C$$

Bài 3. *Tìm nguyên hàm*

$$I = \int \frac{x^3+x^2}{\sqrt[4]{4x+5}}\ dx$$

Lời giải

$$I=\int \frac{x^3+x^2}{\sqrt[4]{4x+5}}\,\mathrm{d}x=\int \frac{x^4+x^3}{\sqrt[4]{4x^5+5x^4}}\,\mathrm{d}x=\frac{1}{20}\int \left(4x^5+5x^4\right)^{-\frac{1}{4}}\,\mathrm{d}(4x^5+5x^4)=\frac{1}{15}\sqrt[4]{(4x^5+5x^4)^3}+C$$

Bài 4. *Tìm nguyên hàm*

$$I=\int \left(\cos 2x+\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)\right)e^{\sin x+\cos x+1}\,dx$$

Lời giải

Ta có $$\cos 2x+\sqrt{2}\cos\left(x+\tfrac{\pi}{4}\right)=(\cos x-\sin x)(\sin x+\cos x+1)$$

$$\begin{aligned}
I&=\int(\cos x-\sin x)(\sin x+\cos x+1)e^{\sin x+\cos x+1}\,\mathrm{d}x\\
I&=\int(\sin x+\cos x+1)e^{\sin x+\cos x+1}\,\mathrm{d}\left(\sin x+\cos x+1\right)\\
I&=\int(\sin x+\cos x+1)\,\mathrm{d}\left(e^{\sin x+\cos x+1}\right)\\
I&=(\sin x+\cos x+1)e^{\sin x+\cos x+1}-\int e^{\sin x+\cos x+1}\,\mathrm{d}\left(\sin x+\cos x+1\right)\\
I&=(\sin x+\cos x+1)e^{\sin x+\cos x+1}-e^{\sin x+\cos x+1}+C\\
I&=(\sin x+\cos x)e^{\sin x+\cos x+1}+C
\end{aligned}$$

Bài 5. *Tìm nguyên hàm*

$$I=\int \sqrt[3]{3x-x^3}\,dx$$

Lời giải

Đặt $t=\dfrac{\sqrt[3]{3x-x^3}}{x}\Rightarrow x^2=\dfrac{3}{t^3+1}\Rightarrow 2x\,\mathrm{d}x=\dfrac{-9t^2\,\mathrm{d}t}{(t^3+1)^2}$

$$I=\frac{1}{2}\int\frac{\sqrt[3]{3x-x^3}}{x}2x\,\mathrm{d}x=\frac{-9}{2}\int\frac{t^3\,\mathrm{d}t}{(t^3+1)^2}=\frac{3}{2}\int t\,\mathrm{d}\left(\frac{1}{t^3+1}\right)=\frac{3t}{2(t^3+1)}-\frac{3}{2}\int\frac{\mathrm{d}t}{t^3+1}$$

Tính $J=\displaystyle\int\frac{\mathrm{d}t}{t^3+1}=\int\frac{\mathrm{d}(t+1)}{(t+1)[(t+1)^2-3(t+1)+3]}=\frac{1}{2}\left(\ln 3(1-t)-2\ln 3t+\ln(1+t)\right)$

Vậy $I=\dfrac{1}{2}x\sqrt[3]{3x-x^3}-\dfrac{3}{4}\left(\ln 3\left(1-\dfrac{\sqrt[3]{3x-x^3}}{x}\right)-2\ln 3\dfrac{\sqrt[3]{3x-x^3}}{x}+\ln\left(1+\dfrac{\sqrt[3]{3x-x^3}}{x}\right)\right)+C$

Bài 6. *Tìm nguyên hàm*

$$I=\int\frac{1}{x^4+4x^3+6x^2+7x+4}\,dx$$

Lời giải

Tổng các hệ số bậc chẵn bằng tổng các hệ số bậc lẻ nên đa thức ở mẫu nhận $x=-1$ làm nghiệm

$$\begin{aligned}
I&=\int\frac{\mathrm{d}x}{(x+1)[(x+1)^3+3]}\\
I&=\frac{1}{3}\int\frac{(x+1)^3+3-(x+1)^3}{(x+1)[(x+1)^3+3]}\,\mathrm{d}x\\
I&=\frac{1}{3}\left[\int\frac{\mathrm{d}x}{x+1}-\int\frac{(x+1)^2}{(x+1)^3+3}\,\mathrm{d}x\right]\\
I&=\frac{1}{3}\left[\ln|x+1|-\frac{1}{3}\int\frac{\mathrm{d}((x+1)^3)}{(x+1)^3+3}\right]\\
I&=\frac{1}{3}\ln|x+1|-\frac{1}{9}\ln|(x+1)^3+3|+C
\end{aligned}$$

Bài 7. *Tính tích phân*

$$I=\int_0^1\frac{x\ln\left(x+\sqrt{1+x^2}\right)}{x+\sqrt{1+x^2}}\,dx$$

Lời giải

Đặt $u = \ln(x + \sqrt{x^2+1})$, $\quad \mathrm{d}v = \dfrac{x\ \mathrm{d}x}{x + \sqrt{x^2+1}} = x(\sqrt{x^2+1} - x)\ \mathrm{d}x$

Suy ra $\mathrm{d}u = \dfrac{1 + \dfrac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}\ \mathrm{d}x = \dfrac{\mathrm{d}x}{\sqrt{x^2+1}}$, $\quad v = \dfrac{1}{2}\displaystyle\int (1+x^2)^{\frac{1}{2}}\mathrm{d}(1+x^2) - \int x^2 \mathrm{d}x = \dfrac{1}{3}[(1+x^2)^{\frac{3}{2}} - x^3]$

$$I = \frac{1}{3}[(1+x^2)^{\frac{3}{2}} - x^3]\ln(x + \sqrt{1+x^2})\Big|_0^1 - \frac{1}{3}\int_0^1 [(1+x^2)^{\frac{3}{2}} - x^3]\frac{dx}{\sqrt{1+x^2}}$$

Mà $\quad J = \displaystyle\int [(1+x^2)^{\frac{3}{2}} - x^3]\frac{dx}{\sqrt{1+x^2}} = \int \frac{dx}{1+x^2} - \int \frac{x^3 dx}{\sqrt{1+x^2}} = \arctan x - \frac{1}{3}(x^2-2)\sqrt{x^2+1}$

Nên $\quad I = \dfrac{1}{3}[(1+x^2)^{\frac{3}{2}} - x^3]\ln(x+\sqrt{1+x^2})\Big|_0^1 - \dfrac{1}{3}\arctan x\Big|_0^1 + \dfrac{1}{9}(x^2-2)\sqrt{x^2+1}\Big|_0^1$

Vậy $\quad I = \dfrac{1}{3}(\sqrt{8}-1)\ln(1+\sqrt{2}) - \dfrac{\pi}{12} + \dfrac{1}{9}(2+\sqrt{2})$

Bài 8. *Tính tích phân*

$$I = \int_0^{\frac{1}{2}} x \ln\frac{1+x}{1-x}\ dx$$

Lời giải

Với $u = \ln\dfrac{1+x}{1-x}$, $\quad \mathrm{d}v = x\ \mathrm{d}x\quad$ nên $\ \mathrm{d}u = \dfrac{2}{1-x^2}\ \mathrm{d}x$, $\quad v = \dfrac{1}{2}x^2$

$$I = \frac{1}{2}x^2\ln\frac{1+x}{1-x}\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}\frac{x^2}{1-x^2}\ \mathrm{d}x$$

$$I = \frac{1}{8}\ln 3 + \int_0^{\frac{1}{2}}\frac{1-x^2-1}{1-x^2}\ \mathrm{d}x$$

$$I = \frac{1}{8}\ln 3 + \frac{1}{2} - \frac{1}{2}\int_0^{\frac{1}{2}}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)\ \mathrm{d}x$$

$$I = \frac{1}{8}\ln 3 + \frac{1}{2} - \frac{1}{2}\ln\frac{1+x}{1-x}\Big|_0^{\frac{1}{2}}$$

$$I = \frac{1}{2} - \frac{3}{8}\ln 3$$

Bài 9. *Tính tích phân*

$$I = \int_0^{\pi} e^{-x}\cos 2x\ dx$$

Lời giải

$$I = \int_0^{\pi} e^{-x}\cos 2x\ \mathrm{d}x$$

$$I = -\int_0^{\pi}\cos 2x\ \mathrm{d}(e^{-x})$$

$$I = -e^{-x}\cos 2x\Big|_0^{\pi} - 2\int_0^{\pi} e^{-x}\sin 2x\ \mathrm{d}x$$

$$I = e^{-\pi} + 1 + 2\int_0^{\pi}\sin 2x\ \mathrm{d}(e^{-x})$$

$$I = e^{-\pi} + 1 + 2e^{-x}\sin 2x\Big|_0^{\pi} - 4\int_0^{\pi} e^{-x}\cos 2x\ \mathrm{d}x$$

$$I = \frac{1}{5}(e^{-\pi}+1)$$

Bài 10. *Tính tích phân*

$$I=\int_0^{\sqrt{3}}\frac{x^5+2x^3}{\sqrt{x^2+1}}\ dx$$

Lời giải

$$I=\int_0^{\sqrt{3}}\frac{x(x^4+2x^2)}{\sqrt{x^2+1}}\ \mathrm{d}x=\int_0^{\sqrt{3}}(x^4+2x^2)\ \mathrm{d}(\sqrt{x^2+1})$$

$$I=(x^4+2x^2)\sqrt{x^2+1}\Big|_0^{\sqrt{3}}-\int_0^{\sqrt{3}}\sqrt{x^2+1}\ \mathrm{d}(x^4+2x^2)$$

Tính $$J=\int\sqrt{x^2+1}\ \mathrm{d}(x^4+2x^2)=\int 4x(x^2+1)\sqrt{x^2+1}\ \mathrm{d}x=4\int\frac{x(x^2+1)^2}{\sqrt{x^2+1}}\ \mathrm{d}x$$

$$=4\int(\sqrt{x^2+1})^4\ \mathrm{d}(\sqrt{x^2+1})=\frac{4}{5}(x^2+1)^2\sqrt{x^2+1}$$

Nên $$I=(x^4+2x^2)\sqrt{x^2+1}\Big|_0^{\sqrt{3}}-\frac{4}{5}(x^2+1)^2\sqrt{x^2+1}\Big|_0^{\sqrt{3}}$$

Bài 11. *Tính tích phân*

$$I=\int_1^e\frac{1+x^2\ln x}{x+x^2\ln x}\ dx$$

Lời giải

$$\begin{aligned}I&=\int_1^e\frac{1+x^2\ln x}{x+x^2\ln x}\ \mathrm{d}x\\&=\int_1^e\frac{\frac{1}{x^2}+\ln x}{\frac{1}{x}+\ln x}\ \mathrm{d}x\\&=\int_1^e\frac{\frac{1}{x}+\ln x}{\frac{1}{x}+\ln x}\ \mathrm{d}x+\int_1^e\frac{\frac{1}{x^2}-\frac{1}{x}}{\frac{1}{x}+\ln x}\ \mathrm{d}x\\&=\int_1^e\mathrm{d}x-\int_1^e\frac{\mathrm{d}\left(\frac{1}{x}+\ln x\right)}{\frac{1}{x}+\ln x}\\&=x\Big|_1^e-\ln\left(\frac{1}{x}+\ln x\right)\Big|_1^e\\&=e-1-\ln\left(\frac{1}{e}+1\right)\end{aligned}$$

Bài 12. *Tính nguyên hàm*

$$I=\int\frac{2(1+\ln x)+x\ln x(1+\ln x)}{1+x\ln x}\ dx$$

Lời giải

Đặt $$u=1+x\ln x\Rightarrow\ \mathrm{d}u=(1+\ln x)\ \mathrm{d}x$$

$$I=\int\frac{(2+x\ln x)(1+\ln x)}{1+x\ln x}\ \mathrm{d}x=\int\frac{u+1}{u}\ \mathrm{d}u=u+\ln|u|+C=1+x\ln x+\ln|1+x\ln x|+C$$

Bài 13. *Tính tích phân*

$$I=\int_0^{\frac{\pi}{4}}\frac{x^2(x^2\sin 2x+1)-(x-1)\sin 2x}{\cos x(x^2\sin x+\cos x)}\ dx$$

Lời giải

$$
\begin{aligned}
I &= \int \frac{x^4 \sin 2x + x^2 - (x-1)\sin 2x}{x^2 \sin x \cos x + \cos^2 x}\ \mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \frac{2x^4 \sin 2x + 2x^2 - 2x\sin x + 2\sin 2x}{x^2 \sin 2x + \cos 2x + 1}\ \mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \frac{2x^2(x^2 \sin 2x + \cos 2x + 1) - (x^2 \sin 2x + \cos 2x + 1)'}{x^2 \sin 2x + \cos 2x + 1}\ \mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} 2x^2\ \mathrm{d}x - \int_0^{\frac{\pi}{4}} \frac{\mathrm{d}(x^2 \sin 2x + \cos 2x + 1)}{x^2 \sin 2x + \cos 2x + 1} \\
&= \frac{2}{3}x^3\Big|_0^{\frac{\pi}{4}} - \ln|x^2 \sin 2x + \cos 2x + 1|\Big|_0^{\frac{\pi}{4}} \\
&= \frac{\pi^3}{96} + \ln 2 - \ln\left(\frac{\pi^2}{16} + 1\right)
\end{aligned}
$$

Bài 14. *Tính nguyên hàm*

$$I = \int \frac{(x^2+1) + (x^3 + x\ln x + 2)\ln x}{1 + x\ln x}\ dx$$

Lời giải

$$I = \int \frac{(x^2 + \ln x) + x\ln x(x^2 + \ln x) + (1 + \ln x)}{1 + x\ln x}\ \mathrm{d}x$$

$$I = \int \frac{(x^2 + \ln x)(1 + x\ln x) + (1 + \ln x)}{1 + x\ln x}\ \mathrm{d}x$$

$$I = \int (x^2 + \ln x)\ \mathrm{d}x + \int \frac{\mathrm{d}(1 + x\ln x)}{1 + x\ln x}$$

$$I = \frac{1}{3}.x^3 + x\ln x - x + \ln|1 + x\ln x| + C$$

Bài 15. *Tính nguyên hàm*

$$I = \int \frac{x^2(x^2\sin^2 x + \sin 2x + \cos x) + \sin x(2x - 1 - \sin x) + 1}{x^2 \sin x + \cos x}\ dx$$

Lời giải

Vì $x^2(x^2\sin^2 x + \sin 2x + \cos x) + \sin x(2x - 1 - \sin x) + 1 = (x^2\sin x + \cos x)^2 + (x^2 \sin x + \cos x)'$

$I = \displaystyle\int (x^2 \sin x + \cos x)\ \mathrm{d}x + \int \frac{\mathrm{d}(x^2 \sin x + \cos x)}{x^2 \sin x + \cos x} = \int x^2 \sin x\ \mathrm{d}x + \sin x + \ln|x^2 \sin x + \cos x|$

Tính $J = \displaystyle\int x^2 \sin x\ \mathrm{d}x = -\int x^2\ \mathrm{d}(\cos x) = -x^2\cos x + 2\int x\cos x\ \mathrm{d}x = -x^2 \cos x + 2\int x\ \mathrm{d}(\sin x)$

$$J = -x^2\cos x + 2x\sin x - 2\int \sin x\ \mathrm{d}x = -x^2 \cos x + 2x\sin x + 2\cos x$$

Vậy
$$I = -x^2\cos x + 2x\sin x + 2\cos x + \sin x + \ln|x^2 \sin x + \cos x| + C$$

Bài 16. *Tìm nguyên hàm*

$$I = \int \Big(x(x+2)(3\sin x - 4\sin^3 x) + 2\cos x(\cos x - 2\sin x) + 3x^2\cos 3x - 1\Big)e^x\ dx$$

Lời giải

$$
\begin{aligned}
&\Big(x(x+2)(3\sin x - 4\sin^3 x) + 2\cos x(\cos x - 2\sin x) + 3x^2\cos 3x - 1\Big)e^x \\
&= \Big(x^2 \sin 3x + (x^2\sin 3x)' + \cos 2x + (\cos 2x)'\Big)e^x \\
&\Rightarrow I = (x^2\sin 3x + \cos 2x)e^x
\end{aligned}
$$

Bài 17. *Tìm nguyên hàm*

$$I = \int \frac{2x^4\ln^2 x + x\ln x(x^3+1) + x - \frac{1}{x^2}}{1 + x^3\ln x}\ dx$$

Lời giải

$$\frac{2x^6\ln^2 x + x^6\ln x + x^3\ln x + x^3 - 1}{x^2 + x^5\ln x}$$

$$= \frac{2[(x^3\ln x)^2 - 1] + x^3(x^3\ln x + 1) + (x^3\ln x + 1)}{x^2(1 + x^3\ln x)}$$

$$= \frac{(x^3\ln x + 1)(2x^3\ln x + x^3 - 1)}{x^2(1 + x^3\ln x)} = 2x\ln x + x - \frac{1}{x^2}$$

Nên $$I = \int\left(2x\ln x + x - \frac{1}{x^2}\right)\,\mathrm{d}x = \frac{1}{2}x^2 + \frac{1}{x} + \int 2x\ln x\,\mathrm{d}x = \frac{1}{2}x^2 + \frac{1}{x} + \int \ln x\,\mathrm{d}(x^2)$$

$$I = \frac{1}{2}x^2 + \frac{1}{x} + x^2\ln x - \int x\,\mathrm{d}x = \frac{1}{x} + x^2\ln x + C$$

Bài 18. *Tìm nguyên hàm*

$$I = \int x^2\sin(\ln x)\;dx$$

Lời giải

Đặt $x = e^t, \quad \ln x = t, \quad \mathrm{d}x = e^t\mathrm{d}t$

$$\Rightarrow I = \int e^{3t}\sin t\,\mathrm{d}t = -e^{3t}\cos t + \int 3e^{3t}\cos t\,\mathrm{d}t$$

$$= -e^{3t}\cos t + 3e^{3t}\sin t - \int 9e^{3t}\sin t\,\mathrm{d}t$$

$$\Rightarrow 10I = 3e^{3t}\sin t - e^{3t}\cos t$$

$$\Rightarrow I = \frac{1}{10}\left(3.e^{3\ln x}\sin(\ln x) - e^{3\ln x}\cos(\ln x)\right) + C$$

Bài 19. *Tìm nguyên hàm*

$$I = \int \frac{e^x(x-1) + 2x^3 + x^3(e^x + x(x^2+1))}{e^x.x + x^2(x^2+1)}\;dx$$

Lời giải

$$\frac{e^x(x-1) + 2x^3 + x^3(e^x + x(x^2+1))}{e^x.x + x^2(x^2+1)} = \frac{x^3 - 1}{x} + \frac{3x^2 + e^x + 1}{x^3 + x + e^x} = x^2 - \frac{1}{x} + \frac{(x^3 + x + e^x)'}{x^3 + x + e^x}$$

Do đó

$$I = \frac{x^3}{3} - \ln|x| + \ln|x^3 + x + e^x| + C$$

Bài 20. *Tính tích phân*

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln(\tan x)\;dx$$

Lời giải

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln(\tan x)\,\mathrm{d}x =^{\text{đổi biến } (x = \frac{\pi}{2} - x)} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln(\cot x)\,\mathrm{d}x$$

$$\Rightarrow 2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \ln(\tan x.\cot x)\,\mathrm{d}x = 0 \Rightarrow I = 0$$

Bài 21. *Tìm nguyên hàm*

$$I = \int \frac{dx}{\sin^3 x + \cos^3 x}$$

Lời giải

Ta có $$\frac{1}{\sin^3 x + \cos^3 x} = \frac{(\sin x + \cos x)}{(\sin x + \cos x)^2(1 - \sin x\cos x)} = \frac{(\sin x + \cos x)}{(1 + \sin 2x)(1 - \sin x\cos x)}$$

Đặt $$t = \sin x - \cos x, \quad \sin x\cos x = \frac{1 - t^2}{2}, \mathrm{d}t = (\cos x + \sin x)\;\mathrm{d}x$$

$$I=\int \frac{\mathrm{d}t}{(2-t^2)\left(1-\dfrac{1-t^2}{2}\right)}=2\int \frac{\mathrm{d}t}{(2-t^2)(1+t^2)}=\frac{2}{3}\int \left(\frac{1}{2-t^2}+\frac{1}{1+t^2}\right)\,\mathrm{d}t$$

$$I=\frac{2}{3}\int \frac{\mathrm{d}t}{2-t^2}+\frac{2}{3}\int \frac{\mathrm{d}t}{1+t^2}$$

Bài 22. *Tính Tích Phân*

$$I=\int_{\frac{-\pi}{4}}^{0} \frac{\sin 4x}{(1+\sin x)(1+\cos x)}\ dx$$

Lời giải

$$2(1+\sin x)(1+\cos x)=(\sin x+\cos x+1)^2=\frac{4\sin 2x(\cos x+\sin x)(\cos x-\sin x)}{(\sin x+\cos x+1)^2}$$

Đặt $t=\cos x+\sin x,\quad \sin 2x=t^2-1,\ \mathrm{d}t=(\cos x-\sin x)\ \mathrm{d}x,\quad x=\dfrac{-\pi}{4}, t=0,\quad x=0, t=1$

$$I=\int_0^1 \frac{4(t^2-1)t}{(t+1)^2}\ \mathrm{d}t=4\int_0^1 \frac{t^2-t}{t+1}\ \mathrm{d}t=4\int_0^1 \left(t-2+\frac{2}{t+1}\right)\ \mathrm{d}t$$

$$I=\left(2t^2-8t+8\ln(t+1)\right)\Big|_0^1=2(4\ln 2-3)$$

Bài 23. *Tính Tích Phân*

$$I=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2+x^{98}+x^{100}}$$

Lời giải

$$I=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\mathrm{d}x}{(1+x^2)(1+x^{98})}\overset{x=\frac{1}{x}}{=}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\mathrm{d}x}{x^2\left(1+\dfrac{1}{x^2}\right)\left(1+\dfrac{1}{x^{98}}\right)}=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{x^{98}\ \mathrm{d}x}{(x^2+1)(x^{98}+1)}$$

$$\Rightarrow I=\frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\mathrm{d}x}{1+x^2}$$

Bài 24. *Tìm nguyên hàm*

$$I=\int \frac{x^2-3x+\dfrac{5}{4}}{\sqrt[7]{(2x+1)^4}}\ dx$$

Lời giải

$$I=\frac{1}{4}\int \frac{4x^2-12x+5}{(2x+1)^{\frac{4}{7}}}\ \mathrm{d}x$$

$$I=\frac{1}{8}\int \left[(2x+1)^2-8(2x+1)+12\right](2x+1)^{\frac{-4}{7}}\mathrm{d}(2x+1)$$

$$I=\frac{1}{8}\int \left[(2x+1)^{\frac{10}{7}}-8(2x+1)^{\frac{3}{7}}+12(2x+1)^{\frac{-4}{7}}\right]\mathrm{d}(2x+1)$$

$$I=\frac{7}{136}(2x+1)^{\frac{17}{7}}-\frac{7}{10}(2x+1)^{\frac{10}{7}}+\frac{9}{14}(2x+1)^{\frac{3}{7}}+C$$

Bài 25. *Tìm nguyên hàm*

$$I=\int \frac{2x^3+5x^2-11x+4}{(x+1)^{30}}\ dx$$

Lời giải

$$
\begin{aligned}
I &= \int \frac{2x^3+5x^2-11x+4}{(x+1)^{30}}\ \mathrm{d}x \\
&= \int \frac{2(x+1)^3-(x+1)^2-15(x+1)+18}{(x+1)^{30}}\ \mathrm{d}x \\
&= \int \left[2(x+1)^{-27}-(x+1)^{-28}-15(x+1)^{-29}+18(x+1)^{-30}\right]\ \mathrm{d}x \\
&= -\frac{1}{13(x+1)^{26}}+\frac{1}{27(x+1)^{27}}+\frac{15}{28(x+1)^{28}}-\frac{18}{29(x+1)^{29}}+C
\end{aligned}
$$

Bài 26. *Tìm nguyên hàm*

$$I = \int \frac{x^3-3x^2+4x-9}{(x-2)^{15}}\ dx$$

Lời giải

$$
\begin{aligned}
I &= \int \frac{x^3-3x^2+4x-9}{(x-2)^{15}}dx \\
&= \int \frac{(x-2)^3+3(x-2)^2+4(x-2)+3}{(x-2)^{15}}\ \mathrm{d}x \\
&= \int \left[(x-2)^{-12}+3(x-2)^{-13}+4(x-2)^{-14}+3(x-2)^{-15}\right]\ \mathrm{d}x \\
&= -\frac{1}{11(x-2)^{11}}-\frac{1}{4(x-2)^{12}}-\frac{4}{13(x-2)^{13}}-\frac{3}{14(x+1)^{14}}+C
\end{aligned}
$$

Bài 27. *Tìm nguyên hàm*

$$I = \int (x-1)^2(5x+2)^{15}\ dx$$

Lời giải

Ta có

$$25(x-1)^2 = 25x^2-50x+25 = 25x^2+20x+4-70x-28+49 = (5x+2)^2-14(5x+2)+49$$

Nên

$$I = \frac{1}{25}\int (5x+2)^{17}-14(5x+2)^{16}+49(5x+2)^{15}\ \mathrm{d}x$$

$$I = \frac{1}{25}\left[\frac{(5x+2)^{18}}{90}-\frac{14(5x+2)^{17}}{85}+\frac{49(5x+2)^{16}}{80}\right]+C$$

Bài 28. *Tính tích phân*

$$I = \int_4^8 \frac{\sqrt{x^2-16}}{x}\ dx$$

Lời giải

Đặt $x = \dfrac{4}{\sin t}$, $\mathrm{d}x = \dfrac{-4\cos t}{\sin^2 t}\ \mathrm{d}t$, $\quad \sqrt{\left(\dfrac{4}{\sin t}\right)^2-16} = 4\cot t \quad x=4, t=\frac{\pi}{2}; \quad x=8, t=\frac{\pi}{6}$

Ta được

$$
\begin{aligned}
I &= \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{4\cot t}{\dfrac{4}{\sin t}}\frac{-4\cos t}{\sin^2 t}\ \mathrm{d}t = 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2 t\ \mathrm{d}t = 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1+\cot^2 t-1)\ \mathrm{d}t \\
&= 4(-\cot t - t)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 4\sqrt{3}+\frac{4\pi}{3}
\end{aligned}
$$

Bài 29. *Tính tích phân*

$$I = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{\sqrt{(1+x^2)^5}}{x^8}\ dx$$

Lời giải

Đặt $x = \tan t,\ \mathrm{d}x = \dfrac{\mathrm{d}t}{\cos^2 t},\quad \sqrt{(1+x^2)^5} = \sqrt{\dfrac{1}{\cos^{10} t}},\quad x = \dfrac{1}{\sqrt{3}}, t = \dfrac{\pi}{6},\quad x = 1, t = \dfrac{\pi}{4}$
Ta được

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\dfrac{1}{\cos^{10} t}}}{\tan^8 t} \frac{\mathrm{d}t}{\cos^2 t} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\mathrm{d}(\sin t)}{sin^8 t}\ \mathrm{d}t = \frac{1}{7}\sin^7 t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{128 - 8\sqrt{2}}{7}$$

Bài 30. *Tính tích phân*

$$I = \int_1^2 \frac{x - \sqrt{x^2 - 2x + 2}}{x + \sqrt{x^2 - 2x + 2}} \frac{dx}{x^2 - 2x + 2}$$

Lời giải

Đặt $x = u + 1,\ \mathrm{d}x = \ \mathrm{d}u,\quad x = 1, u = 0,\quad x = 2, u = 1$
Ta được

$$I = \int_0^1 \frac{u + 1 - \sqrt{u^2 + 1}}{x + 1\sqrt{x^2 + 1}} \frac{\mathrm{d}u}{u^2 + 1} = \int_0^1 \frac{\mathrm{d}u}{u^2 + 1} - \int_0^1 \frac{2\ \mathrm{d}u}{\sqrt{u^2 + 1}(u + \sqrt{u^2 + 1} + 1)}$$

$$= \int_0^1 \frac{\mathrm{d}u}{u^2 + 1} - \int_1^{1+\sqrt{2}} \frac{2\ \mathrm{d}t}{t(t + 1)} \quad (\text{ với } t = u + \sqrt{u^2 + 1},\ \mathrm{d}t = \frac{\sqrt{u^2 + 1} + u}{\sqrt{u^2 + 1}}\ \mathrm{d}u)$$

$$= \arctan u\Big|_0^1 - 2\ln\frac{t}{t + 1}\Big|_1^{1+\sqrt{2}} = \frac{\pi}{4} - \ln 2$$