BÀI 5. CÁC PHÉP ĐỔI BIẾN SỐ CƠ BẢN VÀ NÂNG CAO TÍCH PHÂN HÀM LƯỢNG GIÁC

I. CÁC DẠNG TÍCH PHÂN VÀ PHÉP BIẾN ĐỔI CƠ BẢN

• Đặt vấn đề:

Xét tích phân dạng $I = \int R(\sin x, \cos x)\,dx$

1. Đổi biến số tổng quát:

Đặt $t = tg\frac{x}{2} \Rightarrow x = 2\,arctg\,t\,;\, dx = \frac{2\,dt}{1+t^2};\ \sin x = \frac{2t}{1+t^2}\,;\ \cos x = \frac{1-t^2}{1+t^2}$

Khi đó: $I = \int R(\sin x, \cos x)\,dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2\,dt}{1+t^2}$

Ta xét 3 trường hợp đặc biệt thường gặp sau đây mà có thể đổi biến số bằng cách khác để hàm số dưới dấu tích phân nhận được đơn giản hơn.

2. Nếu $R(\sin x, \cos x)$ là hàm lẻ theo sin: $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

thì cần biến đổi hàm số và vi phân để thực hiện phép đổi biến t = cosx.

3. Nếu $R(\sin x, \cos x)$ là hàm lẻ theo cosin: $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

thì cần biến đổi hàm số và vi phân để thực hiện phép đổi biến t = sinx.

4. Nếu $R(\sin x, \cos x)$ thoả mãn điều kiện: $R(-\sin x, -\cos x) = R(\sin x, \cos x)$

thì cần biến đổi hàm số và vi phân để thực hiện phép đổi biến t = tgx.

II. CÁC BÀI TẬP MẪU MINH HỌA

1. Dạng 1: Đổi biến số tổng quát

$$I = \int \frac{3\sin 2x - 2\cos 2x - 1}{3\cos 2x + 4\sin 2x + 5}\,dx$$

Đặt $t = tg\,x \Rightarrow x = arctg\,t\,;\, dx = \frac{dt}{1+t^2};\ \sin 2x = \frac{2t}{1+t^2}\,;\ \cos 2x = \frac{1-t^2}{1+t^2}$

$$\Rightarrow\ I = \int \frac{3.2t - 2(1-t^2) - (1+t^2)}{3(1-t^2) + 4.2t + 5(1+t^2)} \cdot \frac{dt}{1+t^2} = \frac{1}{2}\int \frac{t^2+6t-3}{t^2+4t+4} \cdot \frac{dt}{1+t^2} = \frac{1}{2}\int \frac{(t^2+6t-3)\,dt}{(t+2)^2(1+t^2)}$$

Giả sử $\dfrac{t^2+6t-3}{(t+2)^2(1+t^2)}=\dfrac{A}{t+2}+\dfrac{B}{(t+2)^2}+\dfrac{Ct+D}{1+t^2},\forall t$

$\Leftrightarrow\ t^2+6t-3=A(t+2)(1+t^2)+B(1+t^2)+(Ct+D)(t+2)^2,\forall t\ \ (*)$

$\Leftrightarrow t^2+6t-3=(A+C)t^3+(2A+B+4C+D)t^2+(A+4C+4D)t+(2A+B+4D)$

Thay $t=-2$ vào (*) thì $-11=5B\Rightarrow B=-11/5$

$$(*)\ \Leftrightarrow\begin{cases}A+C=0\\2A+B+4C+D=1\\A+4C+4D=6\\2A+B+4D=-3\end{cases}\Leftrightarrow\begin{cases}A+C=0\\2A+4C+D=16/5\\A+4C+4D=6\\2A+4D=-4/5\end{cases}\Leftrightarrow\begin{cases}A=-34/25\\B=-11/5\\C=34/25\\D=12/25\end{cases}$$

$$I=\frac{1}{2}\int\frac{t^2+6t-3}{(t+2)^2(1+t^2)}dt=-\frac{34}{25}\int\frac{dt}{t+2}-\frac{11}{5}\int\frac{dt}{(t+2)^2}+\frac{1}{25}\int\frac{24t+12}{1+t^2}dt$$

$$=-\frac{34}{25}\int\frac{dt}{t+2}-\frac{11}{5}\int\frac{dt}{(t+2)^2}+\frac{12}{25}\int\frac{d(t^2)}{1+t^2}+\frac{12}{25}\int\frac{dt}{1+t^2}$$

$$=-\frac{34}{25}\ln|t+2|+\frac{11}{5(t+2)}+\frac{12}{25}\ln(1+t^2)+\frac{12}{25}\operatorname{arctg}t+c$$

$$=-\frac{34}{25}\ln|\operatorname{tg}x+2|+\frac{11}{5(\operatorname{tg}x+2)}+\frac{12}{25}\ln(1+\operatorname{tg}^2x)+\frac{12}{25}x+c$$

2. Dạng 2: $R(-\sin x,\cos x)=-R(\sin x,\cos x)$

• $\boldsymbol{J_1=\displaystyle\int\frac{\sin 2x\,dx}{\cos^3x-\sin^2x-1}}=\displaystyle\int\frac{2\sin x\cos x\,dx}{\cos^3x+\cos^2x-2}$

$$R(\sin x,\cos x)=\frac{2\sin x\cos x}{\cos^3x+\cos^2x-2}\Rightarrow R(-\sin x,\cos x)=-R(\sin x,\cos x)$$

Đặt $t=\cos x\Rightarrow\ J_1=\displaystyle\int\frac{-2t\,dt}{t^3+t^2-2}=\int\frac{-2t\,dt}{(t-1)(t^2+2t+2)}=-2\int\left[\frac{A}{t-1}+\frac{Bt+C}{t^2+2t+2}\right]dt$

Ta có: $\dfrac{t}{(t-1)(t^2+2t+2)}=\dfrac{A}{t-1}+\dfrac{Bt+C}{t^2+2t+2}\Leftrightarrow t=A(t^2+2t+2)+(Bt+C)(t-1)$

$$\Leftrightarrow t=(A+B)t^2+(2A-B+C)t+(2A-C)\Leftrightarrow\begin{cases}A+B=0\\2A-B+C=1\\2A-C=0\end{cases}\Leftrightarrow\begin{cases}A=1/5\\B=-1/5\\C=2/5\end{cases}$$

$$J_1 = -\frac{2}{5}\int\left(\frac{1}{t-1} - \frac{t-2}{t^2+2t+2}\right)dt = -\frac{2}{5}\int\frac{dt}{t-1} + \frac{1}{5}\int\frac{2t+2-6}{t^2+2t+2}dt$$

$$= -\frac{2}{5}\int\frac{dt}{t-1} + \frac{1}{5}\int\frac{d\left(t^2+2t+2\right)}{t^2+2t+2} - \frac{6}{5}\int\frac{dt}{(t+1)^2+1}$$

$$= -\frac{2}{5}\ln|t-1| + \frac{1}{5}\ln\left|t^2+2t+2\right| - \frac{6}{5}\operatorname{arctg}(t+1) + c$$

$$= -\frac{2}{5}\ln(1-\cos x) + \frac{1}{5}\ln\left|\cos^2 x + 2\cos x + 2\right| - \frac{6}{5}\operatorname{arctg}(1+\cos x) + c$$

• $\boldsymbol{J_2 = \int\frac{dx}{\sin x\cos^6 x}} = \int\frac{\sin x\,dx}{\sin^2 x\cos^6 x} = \int\frac{-d(\cos x)}{\left(1-\cos^2 x\right)\cos^6 x} = \int\frac{dt}{t^6\left(t^2-1\right)}$

$$= \int\frac{t^6-\left(t^6-1\right)}{t^6\left(t^2-1\right)}dt = \int\left(\frac{1}{t^2-1} - \frac{t^4+t^2+1}{t^6}\right)dt = \ln\left|\frac{t-1}{t+1}\right| + \frac{1}{t} + \frac{1}{3t^3} + \frac{1}{5t^5} + c$$

$$= \ln\left|\frac{1-\cos x}{1+\cos x}\right| + \frac{1}{\cos x} + \frac{1}{3\cos^3 x} + \frac{1}{5\cos^5 x} + c$$

• $\boldsymbol{J_3 = \int\frac{\sin x + \sin 3x}{\cos 2x}dx} = \int\frac{2\sin 2x\cos x}{\cos 2x}dx = \int\frac{4\sin x\cos^2 x}{2\cos^2 x - 1}dx$

$$= \int\frac{4\cos^2 x\,d(\cos x)}{1-2\cos^2 x} = \int\frac{4t^2\,dt}{1-2t^2} = \int\left(\frac{2}{1-2t^2} - 2\right)dt = \int\frac{dt}{\frac{1}{2}-t^2} - 2\int dt$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{1+\sqrt{2}t}{1-\sqrt{2}t}\right| - 2t + c = \frac{1}{\sqrt{2}}\ln\left|\frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x}\right| - 2\cos x + c$$

• $\boldsymbol{J_4 = \int_0^{\pi/2}\frac{4\sin^3 x}{1+\cos x}dx} = \int_0^{\pi/2}\frac{4\sin^2 x}{1+\cos x}\sin x\,dx = -\int_0^{\pi/2}\frac{4\left(1-\cos^2 x\right)}{1+\cos x}d(\cos x)$

$$= -\int_1^0\frac{4\left(1-t^2\right)}{1+t}dt = \int_0^1 4(1-t)\,dt = \left(4t-2t^2\right)\Big|_0^1 = 4-2 = 2$$

• $\boldsymbol{J_5 = \int_{\pi/6}^{\pi/2}\frac{\sin^2 x}{\sin 3x}dx} = \int_{\pi/6}^{\pi/2}\frac{\sin^2 x\,dx}{3\sin x - 4\sin^3 x} = \int_{\pi/6}^{\pi/2}\frac{\sin x\,dx}{3-4\sin^2 x} = \int_{\pi/6}^{\pi/2}\frac{\sin x\,dx}{4\cos^2 x - 1}$

$$= \int_{\pi/2}^{\pi/6}\frac{d(\cos x)}{4\cos^2 x - 1} = \int_0^{\sqrt{3}/2}\frac{dt}{4t^2-1} = \frac{1}{2}\int_0^{\sqrt{3}/2}\frac{d(2t)}{(2t)^2-1} = \frac{1}{4}\ln\left|\frac{2t-1}{2t+1}\right|\Bigg|_0^{\sqrt{3}/2} = \frac{1}{4}\ln\left(2-\sqrt{3}\right)$$

3. Dạng 3: $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

• $$K_1 = \int \frac{\cos^9 x}{\sin^{20} x} dx = \int \frac{\cos^8 x}{\sin^{20} x} \cos x\, dx = \int \frac{\left(1-\sin^2 x\right)^4}{\sin^{20} x} d(\sin x) = \int \frac{\left(1-t^2\right)^4}{t^{20}} dt$$

$$= \int \frac{1-4t^2+6t^4-4t^6+t^8}{t^{20}} dt = \frac{-1}{19t^{19}} + \frac{4}{17t^{17}} - \frac{6}{15t^{15}} + \frac{4}{13t^{13}} - \frac{1}{11t^{11}} + c$$

$$= \frac{-1}{19(\sin x)^{19}} + \frac{4}{17(\sin x)^{17}} - \frac{6}{15(\sin x)^{15}} + \frac{4}{13(\sin x)^{13}} - \frac{1}{11(\sin x)^{11}} + c$$

• $$K_2 = \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx = \int \frac{\left(\cos^2 x + \cos^4 x\right)}{\sin^2 x + \sin^4 x} \cos x\, dx = \int \frac{\left(\cos^2 x + \cos^4 x\right)}{\sin^2 x + \sin^4 x} d(\sin x)$$

$$= \int \frac{1-t^2+\left(1-t^2\right)^2}{t^2+t^4} dt = \int \frac{t^4-3t^2+2}{t^2\left(1+t^2\right)} dt = \int \left(1 + \frac{2}{t^2} - \frac{6}{1+t^2}\right) dt$$

$$= t - \frac{2}{t} - 6\,\mathrm{arctg}\, t + c = \sin x - \frac{2}{\sin x} - 6\,\mathrm{arctg}(\sin x) + c$$

4. Dạng 4: $R(-\sin x, -\cos x) = R(\sin x, \cos x)$

• $$L_1 = \int_0^{\pi/6} \frac{dx}{\cos x(\sin x - \cos x)} = \int_0^{\pi/6} \frac{dx}{\cos^2 x(tg\, x - 1)} = \int_0^{\pi/6} \frac{d(tg\, x)}{tg\, x - 1} = \ln\left|tg\, x - 1\right|\Big|_0^{\pi/6} = \ln\frac{3-\sqrt{3}}{3}$$

• $$L_2 = \int_{\pi/4}^{\pi/3} \frac{dx}{\sqrt[4]{\sin^3 x \cos^5 x}} = \int_{\pi/4}^{\pi/3} \frac{dx}{\sqrt[4]{tg^3 x \cos^8 x}} = \int_{\pi/4}^{\pi/3} \frac{dx}{\cos^2 x \cdot \sqrt[4]{tg^3 x}} = \int_{\pi/4}^{\pi/3} \frac{d(tg\, x)}{(tg\, x)^{\frac{3}{4}}}$$

$$= \int_{\pi/4}^{\pi/3} (tg\, x)^{\frac{-3}{4}} d(tg\, x) = 4(tg\, x)^{\frac{1}{4}}\Big|_{\pi/4}^{\pi/3} = 4\left[\left(\sqrt{3}\right)^{1/4} - 1\right] = 4\left(\sqrt[8]{3} - 1\right)$$

• $$L_3 = \int_0^{\pi/4} \frac{\sin^2 x\, dx}{\cos x\left(2\sin^3 x + 3\cos^3 x\right)} = \int_0^{\pi/4} \frac{\cos^4 x}{\cos x\left(2\sin^3 x + 3\cos^3 x\right)} \frac{\sin^2 x}{\cos^4 x} dx$$

$$= \int_0^{\pi/4} \frac{tg^2 x}{3+2\,tg^3 x} \cdot \frac{dx}{\cos^2 x} = \int_0^{\pi/4} \frac{tg^2 x}{3+2\,tg^3 x} d(tg\, x) = \frac{1}{6}\int_0^{\pi/4} \frac{d\left(3+2\,tg^3 x\right)}{3+2\,tg^3 x}$$

$$= \frac{1}{6}\ln\left(3+2\,tg^3 x\right)\Big|_0^{\pi/4} = \frac{1}{6}(\ln 5 - \ln 3) = \frac{1}{6}\ln\frac{5}{3}$$

II. BIẾN ĐỔI VÀ ĐỔI BIẾN NÂNG CAO TÍCH PHÂN HÀM SỐ LƯỢNG GIÁC

1. DẠNG 1: MẪU SỐ LÀ BIỂU THỨC THUẦN NHẤT CỦA SIN $\int \frac{dx}{(\sin x)^n}$

• $A_1 = \int \frac{dx}{\sin^3 x} = \int \frac{dx}{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^3} = \int \frac{dx}{8\left(tg\frac{x}{2}\right)^3\left(\cos\frac{x}{2}\right)^6} = \frac{1}{4}\int \frac{\left(1+tg^2\frac{x}{2}\right)^2 d\left(tg\frac{x}{2}\right)}{\left(tg\frac{x}{2}\right)^3}$

$$= \frac{1}{4}\int \frac{1+2\,tg^2\frac{x}{2}+tg^4\frac{x}{2}}{\left(tg\frac{x}{2}\right)^3} d\left(tg\frac{x}{2}\right) = \frac{1}{4}\left[\frac{-1}{2\left(tg\frac{x}{2}\right)^2} + 2\ln\left|tg\frac{x}{2}\right| + \frac{1}{2}\left(tg\frac{x}{2}\right)^2\right] + c$$

Cách 2: $A_1 = \int \frac{dx}{\sin^3 x} = \int \frac{\sin x\, dx}{\sin^4 x} = -\int \frac{d(\cos x)}{(1-\cos^2 x)^2} = -\int \frac{d(\cos x)}{[(1+\cos x)(1-\cos x)]^2}$

$$= \frac{-1}{4}\int \left[\frac{(1+\cos x)+(1-\cos x)}{(1+\cos x)(1-\cos x)}\right]^2 d(\cos x) = \frac{1}{4}\int \left(\frac{1}{1-\cos x} + \frac{1}{1+\cos x}\right)^2 d(\cos x)$$

$$= \frac{-1}{4}\int \left[\frac{1}{(1-\cos x)^2} + \frac{1}{(1+\cos x)^2} + \frac{2}{1-\cos^2 x}\right] d(\cos x) = \frac{-\cos x}{2\sin^2 x} - \frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + c$$

• $A_2 = \int \frac{dx}{\sin^5 x} = \int \frac{dx}{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^5} = \int \frac{dx}{32\left(tg\frac{x}{2}\right)^5\left(\cos\frac{x}{2}\right)^{10}}$

$$= \frac{1}{16}\int \frac{\left(1+tg^2\frac{x}{2}\right)^4 d\left(tg\frac{x}{2}\right)}{\left(tg\frac{x}{2}\right)^5} = \frac{1}{16}\int \frac{1+4\,tg^2\frac{x}{2}+6\,tg^4\frac{x}{2}+4\,tg^6\frac{x}{2}+tg^8\frac{x}{2}}{\left(tg\frac{x}{2}\right)^5} d\left(tg\frac{x}{2}\right)$$

$$= \frac{1}{16}\left[\frac{-1}{4\left(tg\frac{x}{2}\right)^4} - \frac{2}{\left(tg\frac{x}{2}\right)^2} + 6\ln\left|tg\frac{x}{2}\right| + 2\left(tg\frac{x}{2}\right)^2 + \frac{1}{4}\left(tg\frac{x}{2}\right)^4\right] + c$$

Cách 2: $A_2 = \int \frac{dx}{\sin^5 x} = \int \frac{\sin x\, dx}{\sin^6 x} = -\int \frac{d(\cos x)}{(1-\cos^2 x)^3} = -\int \frac{d(\cos x)}{\left[(1+\cos x)(1-\cos x)\right]^3}$

$$= \frac{-1}{8}\int \left[\frac{(1+\cos x)+(1-\cos x)}{(1+\cos x)(1-\cos x)}\right]^3 d(\cos x) = \frac{1}{8}\int \left(\frac{1}{1-\cos x} + \frac{1}{1+\cos x}\right)^3 d(\cos x)$$

$$= \frac{-1}{8}\left(\frac{1}{2(1-\cos x)^2} - \frac{1}{2(1+\cos x)^2} + \frac{3}{2}\int \frac{d(\cos x)}{(1-\cos^2 x)^2}\right) = \frac{-\cos x}{4\sin^4 x} - \frac{3}{4}A_1$$

$$=\frac{-\cos x}{4\sin^4 x}-\frac{3}{4}\left(\frac{-\cos x}{2\sin^2 x}-\frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right|\right)=\frac{-\cos x}{4\sin^4 x}+\frac{3\cos x}{8\sin^2 x}+\frac{3}{8}\ln\left|\frac{1+\cos x}{1-\cos x}\right|+c$$

• $\boldsymbol{A_3=\int\frac{dx}{(\sin x)^{2n+1}}}=\int\frac{dx}{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^{2n+1}}$

$$=\int\frac{dx}{2^{2n+1}\left(\operatorname{tg}\frac{x}{2}\right)^{2n+1}\left(\cos\frac{x}{2}\right)^{4n+2}}=\frac{1}{2^{2n}}\int\frac{\left(1+\operatorname{tg}^2\frac{x}{2}\right)^{2n}d\left(\operatorname{tg}\frac{x}{2}\right)}{\left(\operatorname{tg}\frac{x}{2}\right)^{2n+1}}$$

$$=\frac{1}{2^{2n}}\int\frac{C_{2n}^0+C_{2n}^1\operatorname{tg}^2\frac{x}{2}+\ldots+C_{2n}^n\left(\operatorname{tg}^2\frac{x}{2}\right)^n+\ldots+C_{2n}^{2n}\left(\operatorname{tg}^2\frac{x}{2}\right)^{2n}}{\left(\operatorname{tg}\frac{x}{2}\right)^{2n+1}}d\left(\operatorname{tg}\frac{x}{2}\right)$$

$$=\frac{1}{2^{2n}}\left[\frac{-C_{2n}^0}{2n\left(\operatorname{tg}\frac{x}{2}\right)^{2n}}-\ldots-\frac{C_{2n}^{n-1}}{2\left(\operatorname{tg}\frac{x}{2}\right)^2}+C_{2n}^n\ln\left|\operatorname{tg}\frac{x}{2}\right|+\frac{C_{2n}^{n+1}}{2}\left(\operatorname{tg}\frac{x}{2}\right)^2+\ldots+\frac{C_{2n}^{2n}}{2n}\left(\operatorname{tg}\frac{x}{2}\right)^{2n}\right]+c$$

• $\boldsymbol{A_{10}=\int\frac{dx}{\sin^{2n+2}x}}=-\int\left(1+\operatorname{cotg}^2 x\right)^n d\left(\operatorname{cotg}x\right)=$

$$=-\int\left[C_n^0+C_n^1\operatorname{cotg}^2 x+\ldots+C_n^k\left(\operatorname{cotg}^2 x\right)^k+\ldots+C_n^n\left(\operatorname{cotg}^2 x\right)^n\right]d\left(\operatorname{cotg}x\right)$$

$$=-\left[C_n^0\left(\operatorname{cotg}x\right)+\frac{C_n^1}{3}\operatorname{cotg}^3 x+\ldots+\frac{C_n^k}{2k+1}\left(\operatorname{cotg}x\right)^{2k+1}+\ldots+\frac{C_n^n}{2n+1}\left(\operatorname{cotg}x\right)^{2n+1}\right]+c$$

2. DẠNG 2: MẪU SỐ LÀ BIỂU THỨC THUẦN NHẤT CỦA COSIN $\int\frac{dx}{(\cos x)^n}$

$\boldsymbol{B_1=\int\frac{dx}{\cos^3 x}}=\int\frac{d\left(x+\frac{\pi}{2}\right)}{\sin^3\left(x+\frac{\pi}{2}\right)}=\int\frac{du}{\sin^3 u}=\int\frac{du}{\left(2\sin\frac{u}{2}\cos\frac{u}{2}\right)^3}=\int\frac{du}{8\left(\operatorname{tg}\frac{u}{2}\right)^3\left(\cos\frac{u}{2}\right)^6}$

$$=\frac{1}{4}\int\frac{\left(1+\operatorname{tg}^2\frac{u}{2}\right)^2 d\left(\operatorname{tg}\frac{u}{2}\right)}{\left(\operatorname{tg}\frac{u}{2}\right)^3}=\frac{1}{4}\left[\frac{-1}{2\left(\operatorname{tg}\frac{u}{2}\right)^2}+2\ln\left|\operatorname{tg}\frac{u}{2}\right|+\frac{1}{2}\left(\operatorname{tg}\frac{u}{2}\right)^2\right]+c\,;\left(u=x+\frac{\pi}{2}\right)$$

Cách 2: $\boldsymbol{B_1=\int\frac{dx}{\cos^3 x}}=\int\frac{\cos x\,dx}{\cos^4 x}=\int\frac{d(\sin x)}{(1-\sin^2 x)^2}=\int\frac{d(\sin x)}{[(1+\sin x)(1-\sin x)]^2}$

$$=\frac{1}{4}\int\left[\frac{(1+\sin x)+(1-\sin x)}{(1+\sin x)(1-\sin x)}\right]^2 d(\sin x)=\frac{1}{4}\int\left(\frac{1}{1-\sin x}+\frac{1}{1+\sin x}\right)^2 d(\sin x)$$

$$=\frac{1}{4}\int\left[\frac{1}{(1-\sin x)^2}+\frac{1}{(1+\sin x)^2}+\frac{2}{1-\sin^2 x}\right]d(\sin x)=\frac{\sin x}{2\cos^2 x}+\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+c$$

$$\bullet \boldsymbol{B_2}=\int\frac{dx}{\cos^{2n+1}x}=\int\frac{d\left(x+\frac{\pi}{2}\right)}{\sin^{2n+1}\left(x+\frac{\pi}{2}\right)}=\int\frac{du}{(\sin u)^{2n+1}}=\int\frac{du}{\left(2\sin\frac{u}{2}\cos\frac{u}{2}\right)^{2n+1}}$$

$$=\int\frac{du}{2^{2n+1}\left(\text{tg}\frac{u}{2}\right)^{2n+1}\left(\cos\frac{u}{2}\right)^{4n+2}}=\frac{1}{2^{2n}}\int\frac{\left(1+\text{tg}^2\frac{u}{2}\right)^{2n}d\left(\text{tg}\frac{u}{2}\right)}{\left(\text{tg}\frac{u}{2}\right)^{2n+1}}$$

$$=\frac{1}{2^{2n}}\left[\frac{-C_{2n}^0}{2n\left(\text{tg}\frac{u}{2}\right)^{2n}}-\ldots-\frac{C_{2n}^{n-1}}{2\left(\text{tg}\frac{u}{2}\right)^2}+C_{2n}^n\ln\left|\text{tg}\frac{u}{2}\right|+\frac{C_{2n}^{n+1}}{2}\left(\text{tg}\frac{u}{2}\right)^2+\ldots+\frac{C_{2n}^{2n}}{2n}\left(\text{tg}\frac{u}{2}\right)^{2n}\right]+c$$

$$\bullet \boldsymbol{B_3}=\int\frac{dx}{\cos^{2n+2}x}=\int\left(1+\text{tg}^2 x\right)^n d(\text{tg}\,x)=$$

$$=\int\left[C_n^0+C_n^1\,\text{tg}^2 x+\ldots+C_n^k\left(\text{tg}^2 x\right)^k+\ldots+C_n^n\left(\text{tg}^2 x\right)^n\right]d(\text{tg}\,x)$$

$$=\left[C_n^0(\text{tg}\,x)+\frac{C_n^1}{3}\text{tg}^3 x+\ldots+\frac{C_n^k}{2k+1}(\text{tg}\,x)^{2k+1}+\ldots+\frac{C_n^n}{2n+1}(\text{tg}\,x)^{2n+1}\right]+c$$

3. DẠNG 3: $\mathbf{C}=\int\frac{\mathbf{dx}}{\mathbf{a(sinx)^2+bsinxcosx+c(cosx)^2}}$

$$\bullet \boldsymbol{C}=\int\frac{dx}{(5\sin 3x+2\cos 3x)^2-21}=\int\frac{dx}{\cos^2 3x\left[(5\,\text{tg}\,3x+2)^2-21\left(1+\text{tg}^2 3x\right)\right]}$$

$$=\frac{1}{3}\int\frac{d(\text{tg}\,3x)}{4\,\text{tg}^2 3x+20\,\text{tg}\,3x-17}=\frac{1}{12}\int\frac{d(\text{tg}\,3x)}{\left(\text{tg}\,3x+\frac{5}{2}\right)^2+\frac{42}{4}}=\frac{1}{6\sqrt{42}}\,arc\,\text{tg}\frac{2\,\text{tg}\,3x+5}{\sqrt{42}}+c$$

4. DẠNG 4: $\mathbf{D}=\int\frac{\mathbf{dx}}{\mathbf{a\sin x+b\cos x+c}}$

$$\bullet \boldsymbol{D_1}=\int\frac{dx}{2\sin x+5\cos x+3}=\int\frac{dx}{4\sin\frac{x}{2}\cos\frac{x}{2}+5\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)+3\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)}$$

$$=\int\frac{dx}{\cos^2\frac{x}{2}\left(4\,\text{tg}\frac{x}{2}+8-2\,\text{tg}^2\frac{x}{2}\right)}=-\int\frac{d\left(\text{tg}\frac{x}{2}-1\right)}{\left(\text{tg}\frac{x}{2}-1\right)^2-\left(\sqrt{5}\right)^2}=\frac{-1}{2\sqrt{5}}\ln\left|\frac{\text{tg}\frac{x}{2}-1-\sqrt{5}}{\text{tg}\frac{x}{2}-1+\sqrt{5}}\right|+c$$

5. DẠNG 5: TÍCH PHÂN LIÊN KẾT

• $\boldsymbol{E_1 = \int \frac{cosxdx}{sinx + cosx}}$. Xét tích phân liên kết với E_1 là: $E_1^* = \int \frac{\sin x\,dx}{\sin x + \cos x}$

Ta có: $\begin{cases} E_1 + E_1^* = \int \frac{\cos x + \sin x}{\sin x + \cos x}dx = \int dx = x + (c_1) \\ E_1 - E_1^* = \int \frac{\cos x - \sin x}{\sin x + \cos x}dx = \int \frac{d(\sin x + \cos x)}{\sin x + \cos x} = \ln|\sin x + \cos x| + (c_2) \end{cases}$

Giải hệ phương trình suy ra: $\begin{cases} E_1 = \frac{1}{2}(x + \ln|\sin x + \cos x|) + c \\ E_1^* = \frac{1}{2}(x - \ln|\sin x + \cos x|) + c \end{cases}$

• $\boldsymbol{E_2 = \int \frac{sin3xdx}{2cos3x - 5sin3x}}$. Xét tích phân liên kết là: $E_2^* = \int \frac{\cos 3x\,dx}{2\cos 3x - 5\sin 3x}$

Ta có:

$$\begin{cases} 2E_2^* - 5E_2 = \int \frac{2\cos 3x - 5\sin 3x}{2\cos 3x - 5\sin 3x}dx = \int dx = x + (c_1) \\ 5E_2^* + 2E_2 = \int \frac{5\cos 3x + 2\sin 3x}{2\cos 3x - 5\sin 3x}dx = -\frac{1}{3}\int \frac{d(2\cos 3x - 5\sin 3x)}{2\cos 3x - 5\sin 3x} = -\frac{\ln|2\cos 3x - 5\sin 3x|}{3} + (c_2) \end{cases}$$

Giải hệ phương trình suy ra:

$$\begin{cases} E_2 = \frac{1}{29}\cdot\begin{vmatrix} 2 & x \\ 5 & -\frac{\ln|2\cos 3x - 5\sin 3x|}{3} \end{vmatrix} + c = \frac{-1}{29}\left[\frac{2\ln|2\cos 3x - 5\sin 3x|}{3} + 5x\right] + c \\ E_2^* = \frac{1}{29}\cdot\begin{vmatrix} x & -5 \\ -\frac{\ln|2\cos 3x - 5\sin 3x|}{3} & 2 \end{vmatrix} + c = \frac{1}{29}\left[2x - \frac{5\ln|2\cos 3x - 5\sin 3x|}{3}\right] + c \end{cases}$$

• $\boldsymbol{E_3 = \int \frac{(\sin x)^4}{(\sin x)^4 + (\cos x)^4}dx}$. Xét tích phân liên kết là: $E_3^* = \int \frac{(\cos x)^4}{(\sin x)^4 + (\cos x)^4}dx$

Ta có: $E_3^* + E_3 = \int \frac{(\sin x)^4 + (\cos x)^4}{(\sin x)^4 + (\cos x)^4}dx = \int dx = x + (c_1)$ (1). Mặt khác:

$$E_3^* - E_3 = \int \frac{(\cos x)^4 - (\sin x)^4}{(\sin x)^4 + (\cos x)^4}dx = \int \frac{(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)}{(\cos^2 x + \sin^2 x)^2 - 2\cos^2 x \sin^2 x}dx$$

$$=\int \frac{\cos 2x}{1-\frac{1}{2}\sin^2 2x}dx=\int \frac{d(\sin 2x)}{\left(\sqrt{2}\right)^2-\sin^2 2x}=\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sin 2x}{\sqrt{2}-\sin 2x}\right|+c \quad (2)$$

Từ (1) và (2) suy ra:

$$E_3=\frac{1}{2}\left(x-\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sin 2x}{\sqrt{2}-\sin 2x}\right|\right)+c\,;\,E_3^*=\frac{1}{2}\left(x+\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sin 2x}{\sqrt{2}-\sin 2x}\right|\right)+c$$

• $\boldsymbol{E_4=\int_0^{\pi/2}\frac{(\cos x)^{99}}{(\sin x)^{99}+(\cos x)^{99}}dx}$. Xét tích phân: $E_4^*=\int_0^{\pi/2}\frac{(\sin x)^{99}}{(\sin x)^{99}+(\cos x)^{99}}dx$

Đặt $x=\frac{\pi}{2}-u \Rightarrow dx=-du$. Với $x=\frac{\pi}{2}$ thì $u=0$ và $x=0$ thì $u=\frac{\pi}{2}$. Ta có:

$$E_4^*=\int_0^{\pi/2}\frac{(\sin x)^{99}dx}{(\sin x)^{99}+(\cos x)^{99}}=\int_{\pi/2}^{0}\frac{\left[\sin\left(\frac{\pi}{2}-u\right)\right]^{99}(-du)}{\left[\sin\left(\frac{\pi}{2}-u\right)\right]^{99}+\left[\cos\left(\frac{\pi}{2}-u\right)\right]^{99}}=\int_0^{\pi/2}\frac{(\cos u)^{99}du}{(\cos u)^{99}+(\sin u)^{99}}=E_4$$

Ta có: $E_4^*+E_4=\int_0^{\pi/2}\frac{(\sin x)^{99}+(\cos x)^{99}}{(\sin x)^{99}+(\cos x)^{99}}dx=\int_0^{\pi/2}dx=x\Big|_0^{\pi/2}=\frac{\pi}{2}\Rightarrow E_4=E_4^*=\frac{\pi}{4}$

• $\boldsymbol{E_5=\int_0^{\pi/2}(\cos 3x)^2(\cos 6x)^2dx}$. Xét tích phân: $E_5^*=\int_0^{\pi/2}(\sin 3x)^2(\cos 6x)^2dx$

Ta có: $E_5+E_5^*=\int_0^{\pi/2}\left[(\cos 3x)^2+(\sin 3x)^2\right](\cos 6x)^2dx=\int_0^{\pi/2}(\cos 6x)^2dx$

$$=\frac{1}{2}\int_0^{\pi/2}(1+\cos 12x)dx=\frac{1}{2}\left(x+\frac{\sin 12x}{12}\right)\Bigg|_0^{\pi/2}=\frac{\pi}{4}.$$ Mặt khác:

$$E_5-E_5^*=\int_0^{\pi/2}\left[(\cos 3x)^2-(\sin 3x)^2\right](\cos 6x)^2dx=\int_0^{\pi/2}\cos 6x(\cos 6x)^2dx$$

$$=\frac{1}{6}\int_0^{\pi/2}\left[1-(\sin 6x)^2\right]d(\sin 6x)=\frac{1}{6}\left[\sin 6x-\frac{(\sin 6x)^3}{3}\right]\Bigg|_0^{\pi/2}=0\Rightarrow E_6=E_6^*=\frac{\pi}{8}$$

• $\boldsymbol{E_6=\int_0^{\pi/2}\frac{\sin x\,dx}{(\sin x+\cos x)^3}}$. Xét tích phân: $E_6^*=\int_0^{\pi/2}\frac{\cos x\,dx}{(\sin x+\cos x)^3}$

Ta có: $E_6^* + E_6 = \int_0^{\pi/2} \frac{(\cos x + \sin x)\,dx}{(\sin x + \cos x)^3} = \int_0^{\pi/2} \frac{dx}{(\sin x + \cos x)^2}$

$$= \int_0^{\pi/2} \frac{dx}{\left[\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right]^2} = \frac{1}{2}\int_0^{\pi/2} \frac{dx}{\sin^2\left(x + \frac{\pi}{4}\right)} = \frac{-1}{2}\operatorname{cotg}\left(x + \frac{\pi}{4}\right)\Bigg|_0^{\pi/2} = \frac{1}{2} + \frac{1}{2} = 1$$

Mặt khác: $E_6^* - E_6 = \int_0^{\pi/2} \frac{(\cos x - \sin x)\,dx}{(\sin x + \cos x)^3} = \int_0^{\pi/2} \frac{d(\sin x + \cos x)}{(\sin x + \cos x)^3}$

$$= \frac{-1}{2(\sin x + \cos x)^2}\Bigg|_0^{\pi/2} = 0 \Rightarrow E_6 = E_6^* = \frac{1}{2}$$

6. DẠNG 6: $F = \int \frac{a\sin x + b\cos x}{m\sin x + n\cos x}dx$

a. Phương pháp:

Giả sử: $a\sin x + b\cos x = \alpha(m\sin x + n\cos x) + \beta(m\cos x - n\sin x)$, $\forall x$

$\Leftrightarrow a\sin x + b\cos x = (m\alpha - n\beta)\sin x + (n\alpha + m\beta)\cos x$, $\forall x$

$\Leftrightarrow \begin{cases} m\alpha - n\beta = a \\ n\alpha + m\beta = b \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{am + bn}{m^2 + n^2} \\ \beta = \frac{bm - an}{m^2 + n^2} \end{cases}$. Khi đó ta có:

$$F = \frac{am + bn}{m^2 + n^2}\int \frac{m\sin x + n\cos x}{m\sin x + n\cos x}dx + \frac{bm - an}{m^2 + n^2}\int \frac{m\cos x - n\sin x}{m\sin x + n\cos x}dx$$

$$= \frac{am + bn}{m^2 + n^2}\int dx + \frac{bm - an}{m^2 + n^2}\int \frac{d(m\sin x + n\cos x)}{m\sin x + n\cos x}$$

$$= \frac{am + bn}{m^2 + n^2}x + \frac{bm - an}{m^2 + n^2}\ln|m\sin x + n\cos x| + c$$

b. Các bài tập mẫu minh họa:

• $F_1 = \int \frac{4\sin 2x - 7\cos 2x}{5\sin 2x + 3\cos 2x}dx = \frac{1}{2}\int \frac{4\sin 2x - 7\cos 2x}{5\sin 2x + 3\cos 2x}d(2x) = \frac{1}{2}\int \frac{4\sin u - 7\cos u}{5\sin u + 3\cos u}du$

Giả sử $4\sin u - 7\cos u = \alpha(5\sin u + 3\cos u) + \beta(5\cos u - 3\sin u)$, $\forall u$

$\Leftrightarrow 4\sin u - 7\cos u = (5\alpha - 3\beta)\sin u + (3\alpha + 5\beta)\cos u$, $\forall u$

$\Leftrightarrow \begin{cases} 5\alpha - 3\beta = 4 \\ 3\alpha + 5\beta = -7 \end{cases} \Leftrightarrow \begin{cases} \alpha = -1/34 \\ \beta = -47/34 \end{cases}$. Khi đó ta có:

$$F_1 = \frac{1}{2}\int \frac{4\sin u - 7\cos u}{5\sin u + 3\cos u}du = \frac{-1}{68}\int \frac{5\sin u + 3\cos u}{5\sin u + 3\cos u}du - \frac{47}{68}\int \frac{5\cos u - 3\sin u}{5\sin u + 3\cos u}du$$

$$= \frac{-1}{68}\int du - \frac{47}{68}\int \frac{d(5\sin u + 3\cos u)}{5\sin u + 3\cos u} = \frac{-1}{68}\left(u + 47\ln|5\sin u + 3\cos u|\right) + c$$

$$= \frac{-1}{68}\left(2x + 47\ln|5\sin 2x + 3\cos 2x|\right) + c$$

c. Các bài tập dành cho bạn đọc tự giải:

$$F_1 = \int \frac{4\sin 3x + 5\cos 3x}{7\cos 3x - 8\sin 3x}dx\ ;\ F_2 = \int \frac{2\sin 5x - 7\cos 5x}{3\sin 5x - 4\cos 5x}dx\ ;\ F_3 = \int \frac{4\sin 9x + 5\cos 9x}{7\cos 9x - 3\sin 9x}dx$$

7. DẠNG 7: $\mathbf{G = \int \frac{a\sin x + b\cos x + c}{m\sin x + n\cos x + p}dx}$

a. Phương pháp:

Giả sử $a\sin x + b\cos x + c = \alpha(m\sin x + n\cos x + p) + \beta(m\cos x - n\sin x) + \gamma, \forall x$

$\Leftrightarrow a\sin x + b\cos x + c = (m\alpha - n\beta)\sin x + (n\alpha + m\beta)\cos x + p\alpha + \gamma, \forall x$

$\Leftrightarrow \begin{cases} m\alpha - n\beta = a \\ n\alpha + m\beta = b \\ p\alpha + \gamma = c \end{cases} \Leftrightarrow \begin{cases} \alpha = (am + bn)/(m^2 + n^2) \\ \beta = (bm - an)/(m^2 + n^2) \\ \gamma = c - \dfrac{am + bn}{m^2 + n^2}p \end{cases}$. Khi đó ta có:

$$G = \frac{am+bn}{m^2+n^2}\int \frac{m\sin x + n\cos x + p}{m\sin x + n\cos x + p}dx + \frac{bm-an}{m^2+n^2}\int \frac{m\cos x - n\sin x}{m\sin x + n\cos x + p}dx +$$
$$+\left(c - \frac{am+bn}{m^2+n^2}p\right)\int \frac{dx}{m\sin x + n\cos x + p}$$

$$= \frac{am+bn}{m^2+n^2}\int dx + \frac{bm-an}{m^2+n^2}\int \frac{d(m\sin x + n\cos x + p)}{m\sin x + n\cos x + p} + \left(c - \frac{am+bn}{m^2+n^2}p\right)\int \frac{dx}{m\sin x + n\cos x + p}$$

$$= \frac{am+bn}{m^2+n^2}x + \frac{bm-an}{m^2+n^2}\ln|m\sin x + n\cos x + p| + \left(c - \frac{am+bn}{m^2+n^2}p\right)\int \frac{dx}{m\sin x + n\cos x + p}$$

b. Các bài tập mẫu minh họa:

• $\boldsymbol{G_1 = \int \frac{sinx + 2cosx - 3}{sinx - 2cosx + 3}dx}$.

Giả sử $\sin x+2\cos x-3=\alpha(\sin x-2\cos x+3)+\beta(\cos x+2\sin x)+\gamma,\forall x$

$\Leftrightarrow \sin x+2\cos x-3=(\alpha+2\beta)\sin x+(-2\alpha+\beta)\cos x+(3\alpha+\gamma),\forall x$

$$\Leftrightarrow\begin{cases}\alpha+2\beta=1\\-2\alpha+\beta=2\\3\alpha+\gamma=-3\end{cases}\Leftrightarrow\begin{cases}\alpha=-3/5\\\beta=4/5\\\gamma=-6/5\end{cases}$$. Khi đó ta có:

$$G_1=\frac{-3}{5}\int\frac{\sin x-2\cos x+3}{\sin x-2\cos x+3}dx+\frac{4}{5}\int\frac{\sin x-2\cos x}{\sin x-2\cos x+3}dx-\frac{6}{5}\int\frac{dx}{\sin x-2\cos x+3}$$

$$=\frac{-3}{5}\int dx+\frac{4}{5}\int\frac{d(\sin x-2\cos x+3)}{\sin x-2\cos x+3}dx-\frac{6}{5}\int\frac{dx}{\sin x-2\cos x+3}$$

$$=\frac{-3}{5}x+\frac{4}{5}\ln|\sin x-2\cos x+3|-\frac{6}{5}J$$

$$J=\int\frac{dx}{\sin x-2\cos x+3}=\int\frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}-2\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)+3\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)}=$$

$$=\int\frac{dx}{\cos^2\frac{x}{2}\left(2\,\mathrm{tg}\frac{x}{2}+1+5\,\mathrm{tg}^2\frac{x}{2}\right)}=\frac{2}{5}\int\frac{d\left(\mathrm{tg}\frac{x}{2}\right)}{\left(\mathrm{tg}\frac{x}{2}\right)^2+\frac{2}{5}\left(\mathrm{tg}\frac{x}{2}\right)+\frac{1}{5}}$$

$$=\frac{2}{5}\int\frac{d\left(\mathrm{tg}\frac{x}{2}\right)}{\left(\mathrm{tg}\frac{x}{2}+\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2}=\frac{2}{5}\cdot\frac{5}{2}\,\mathrm{arctg}\frac{1+5\,\mathrm{tg}\frac{x}{2}}{2}+c=\mathrm{arctg}\frac{1+5\,\mathrm{tg}\frac{x}{2}}{2}+c$$

$$\Rightarrow G_1=\frac{-3}{5}x+\frac{4}{5}\ln|\sin x-2\cos x+3|-\frac{6}{5}\mathrm{arctg}\frac{5\,\mathrm{tg}\frac{x}{2}+1}{2}+c$$

• $\boldsymbol{G_2=\int_0^{\pi/2}\frac{\sin x-\cos x+1}{\sin x+2\cos x+3}dx}$.

Giả sử $\sin x-\cos x+1=\alpha(\sin x+2\cos x+3)+\beta(\cos x-2\sin x)+\gamma,\forall x$

$\Leftrightarrow \sin x-\cos x+1=(\alpha-2\beta)\sin x+(2\alpha+\beta)\cos x+(3\alpha+\gamma),\forall x$

$$\Leftrightarrow\begin{cases}\alpha-2\beta=1\\2\alpha+\beta=-1\\3\alpha+\gamma=1\end{cases}\Leftrightarrow\begin{cases}\alpha=-1/5\\\beta=-3/5\\\gamma=8/5\end{cases}$$. Khi đó ta có:

$$G_2 = -\frac{1}{5}\int_0^{\pi/2}\frac{\sin x+2\cos x+3}{\sin x+2\cos x+3}dx - \frac{3}{5}\int_0^{\pi/2}\frac{\cos x-2\sin x}{\sin x+2\cos x+3}dx + \frac{8}{5}\int_0^{\pi/2}\frac{dx}{\sin x+2\cos x+3}$$

$$= -\frac{1}{5}\int_0^{\pi/2}dx - \frac{3}{5}\int_0^{\pi/2}\frac{d(\sin x+2\cos x+3)}{\sin x+2\cos x+3} + \frac{8}{5}\int_0^{\pi/2}\frac{dx}{\sin x+2\cos x+3}$$

$$= \left(\frac{-1}{5}x - \frac{3}{5}\ln|\sin x+2\cos x+3|\right)\Bigg|_0^{\pi/2} + \frac{8}{5}J = \frac{-\pi}{10} + \frac{3}{5}\ln\frac{5}{4} + \frac{8}{5}J$$

$$J = \int_0^{\pi/2}\frac{dx}{\sin x+2\cos x+3} = \int_0^{\pi/2}\frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2} + 2\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) + 3\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right)}$$

$$= \int_0^{\pi/2}\frac{dx}{\cos^2\frac{x}{2}\left(2\,\mathrm{tg}\frac{x}{2} + 2 - 2\,\mathrm{tg}^2\frac{x}{2} + 3 + 3\,\mathrm{tg}^2\frac{x}{2}\right)} = 2\int_0^{\pi/2}\frac{d\left(\mathrm{tg}\frac{x}{2}\right)}{\mathrm{tg}^2\frac{x}{2} + 2\,\mathrm{tg}\frac{x}{2} + 5}$$

$$= 2\int_0^{\pi/2}\frac{d\left(1+\mathrm{tg}\frac{x}{2}\right)}{\left(1+\mathrm{tg}\frac{x}{2}\right)^2 + 2^2} = \mathrm{arctg}\frac{1+\mathrm{tg}\frac{x}{2}}{2}\Bigg|_0^{\pi/2} = \frac{\pi}{4} - \mathrm{arctg}\frac{1}{2} \Rightarrow G_2 = \frac{3\pi}{10} + \frac{3}{5}\ln\frac{5}{4} - \frac{8}{5}\mathrm{arctg}\frac{1}{2}$$

8. DẠNG 8: $H = \int \frac{a\sin x + b\cos x}{(m\sin x + n\cos x)^2}dx$

a. Phương pháp:

Giả sử $a\sin x + b\cos x = \alpha(m\sin x + n\cos x) + \beta(m\cos x - n\sin x),\ \forall x$

$\Leftrightarrow\ a\sin x + b\cos x = (m\alpha - n\beta)\sin x + (n\alpha + m\beta)\cos x,\ \forall x$

$\Leftrightarrow \begin{cases} m\alpha - n\beta = a \\ n\alpha + m\beta = b \end{cases} \Leftrightarrow \begin{cases} \alpha = \dfrac{am+bn}{m^2+n^2} \\ \beta = \dfrac{bm-an}{m^2+n^2} \end{cases}$. Khi đó ta có:

$$H = \frac{am+bn}{m^2+n^2}\int\frac{m\sin x+n\cos x}{(m\sin x+n\cos x)^2}dx + \frac{bm-an}{m^2+n^2}\int\frac{m\cos x-n\sin x}{(m\sin x+n\cos x)^2}dx$$

$$= \frac{am+bn}{m^2+n^2}\int\frac{dx}{m\sin x+n\cos x} + \frac{bm-an}{m^2+n^2}\int\frac{d(m\sin x+n\cos x)}{(m\sin x+n\cos x)^2}$$

$$= \frac{am+bn}{m^2+n^2}\int\frac{dx}{m\sin x+n\cos x} - \frac{bm-an}{m^2+n^2}\cdot\frac{1}{m\sin x+n\cos x} + c$$

2. Các bài tập mẫu minh họa:

• $H_1 = \int \frac{7\sin x - 5\cos x}{(3\sin x + 4\cos x)^2}dx$.

Giả sử $7\sin x - 5\cos x = \alpha(3\sin x + 4\cos x) + \beta(3\cos x - 4\sin x);\forall x$

$\Leftrightarrow$ $7\sin x - 5\cos x = (3\alpha - 4\beta)\sin x + (4\alpha + 3\beta)\cos x;\forall x$

$\Leftrightarrow \begin{cases} 3\alpha - 4\beta = 7 \\ 4\alpha + 3\beta = -5 \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{1}{5} \\ \beta = \frac{-43}{5} \end{cases}$. Khi đó ta có:

$$H_1 = \int \frac{7\sin x - 5\cos x}{(3\sin x + 4\cos x)^2}dx = \frac{1}{5}\int \frac{3\sin x + 4\cos x}{(3\sin x + 4\cos x)^2}dx - \frac{43}{5}\int \frac{3\cos x - 4\sin x}{(3\sin x + 4\cos x)^2}dx$$

$$= \frac{1}{5}\int \frac{dx}{3\sin x + 4\cos x} - \frac{43}{5}\int \frac{d(3\sin x + 4\cos x)}{(3\sin x + 4\cos x)^2} = \frac{1}{5}J + \frac{43}{5(3\sin x + 4\cos x)}$$

$$J = \int \frac{dx}{3\sin x + 4\cos x} = \int \frac{dx}{\cos^2\frac{x}{2}\left(6\,\mathrm{tg}\frac{x}{2} + 4 - 4\,\mathrm{tg}^2\frac{x}{2}\right)} = 2\int \frac{d\left(\mathrm{tg}\frac{x}{2}\right)}{6\,\mathrm{tg}\frac{x}{2} + 4 - 4\,\mathrm{tg}^2\frac{x}{2}}$$

$$= \frac{-2}{5}\ln\left|\frac{2\,\mathrm{tg}\frac{x}{2} - 4}{2\,\mathrm{tg}\frac{x}{2} + 1}\right| + c \Rightarrow H_1 = \frac{-2}{25}\ln\left|\frac{2\,\mathrm{tg}\frac{x}{2} - 4}{2\,\mathrm{tg}\frac{x}{2} + 1}\right| + \frac{43}{5(3\sin x + 4\cos x)} + c$$

3. Các bài tập dành cho bạn đọc tự giải:

$$H_1 = \int \frac{2\sin 5x - 3\cos 5x}{(4\cos 5x + 9\cos 5x)^2}dx\ ; H_2 = \int \frac{5\sin 7x + 4\cos 7x}{(2\sin 7x - 3\cos 7x)^2}dx$$

9. DẠNG 9: $\mathbf{I = \int \frac{a(\sin x)^2 + b\sin x\cos x + c(\cos x)^2}{m\sin x + n\cos x}dx}$

a. Phương pháp:

Giả sử: $a(\sin x)^2 + b\sin x\cos x + c(\cos x)^2 =$

$$= (p\sin x + q\cos x)(m\sin x + n\cos x) + r(\sin^2 x + \cos^2 x), \forall x$$

$\Leftrightarrow$ $a(\sin x)^2 + b\sin x\cos x + c(\cos x)^2 =$

$$= (mp + r)(\sin x)^2 + (np + mq)\sin x\cos x + (nq + r)(\cos x)^2;\forall x$$

$$\Leftrightarrow \begin{cases} mp + r = a \\ np + mq = b \\ nq + r = c \end{cases} \Leftrightarrow \begin{cases} mp + r = a \\ np + mq = b \\ mp - nq = a - c \end{cases} \Leftrightarrow \begin{cases} p = \dfrac{(a-c)m + bn}{m^2 + n^2} \\ q = \dfrac{(a-c)n - bm}{m^2 + n^2} \\ r = \dfrac{an^2 + cm^2 - bmn}{m^2 + n^2} \end{cases}$$. Khi đó ta có:

$$I = \int \left[\frac{(a-c)m + bn}{m^2 + n^2} \sin x + \frac{(a-c)n - bm}{m^2 + n^2} \cos x \right] dx + \frac{an^2 + cm^2 - bmn}{m^2 + n^2} \int \frac{dx}{m \sin x + n \cos x}$$

$$= \frac{(a-c)n - bm}{m^2 + n^2} \sin x - \frac{(a-c)m + bn}{m^2 + n^2} \cos x + \frac{an^2 + cm^2 - bmn}{m^2 + n^2} \int \frac{dx}{m \sin x + n \cos x}$$

b. Các bài tập mẫu minh họa:

• $\boldsymbol{I_1 = \int_0^{\pi/3} \frac{(\cos x)^2\, dx}{\sin x + \sqrt{3}\cos x}}$.

Giả sử $(\cos x)^2 = (a \sin x + b \cos x)(\sin x + \sqrt{3} \cos x) + c(\sin^2 x + \cos^2 x); \forall x$

$$\Leftrightarrow (\cos x)^2 = (a+c)(\sin x)^2 + (a\sqrt{3} + b)\sin x \cos x + (b\sqrt{3} + c)(\cos x)^2; \forall x$$

$$\Leftrightarrow \begin{cases} a + c = 0 \\ a\sqrt{3} + b = 0 \\ b\sqrt{3} + c = 1 \end{cases} \Leftrightarrow \begin{cases} a = -1/4 \\ b = \sqrt{3}/4 \\ c = 1/4 \end{cases} \Rightarrow I = \frac{1}{2}\int_0^{\pi/3} \left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x \right) dx + \frac{1}{4}\int_0^{\pi/3} \frac{dx}{\sin x + \sqrt{3}\cos x}$$

$$= \frac{1}{2}\int_0^{\pi/3} \left(\cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x \right) dx + \frac{1}{8}\int_0^{\pi/3} \frac{dx}{\cos\frac{\pi}{3}\sin x + \sin\frac{\pi}{3}\cos x}$$

$$= \frac{1}{2}\int_0^{\pi/3} \cos\left(x + \frac{\pi}{6} \right) dx + \frac{1}{8}\int_0^{\pi/3} \frac{dx}{\sin\left(x + \frac{\pi}{3} \right)} = \left[\frac{1}{2}\sin\left(x + \frac{\pi}{6} \right) + \frac{1}{8}\ln\left| \operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{6} \right) \right| \right]_0^{\pi/3}$$

$$= \left(\frac{1}{2} + \frac{1}{8}\ln\sqrt{3} \right) - \left(\frac{1}{4} - \frac{1}{8}\ln\sqrt{3} \right) = \frac{1}{4} + \frac{1}{4}\ln\sqrt{3} = \frac{1}{4}\left(1 + \ln\sqrt{3} \right)$$

10. DẠNG 10: $\mathbf{J = \int \dfrac{m\sin x + n\cos x}{a(\sin x)^2 + 2b\sin x\cos x + c(\cos x)^2}dx}$

a. Phương pháp:

•Gọi λ_1, λ_2 là nghiệm của phương trình $\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0$

$$\Leftrightarrow \lambda^2 - (a+c)\lambda + ac - b^2 = 0 \Leftrightarrow \lambda_{1,2} = \frac{a+c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$$

Biến đổi $a(\sin x)^2 + 2b\sin x\cos x + c(\cos x)^2 = \lambda_1 A_1^2 + \lambda_2 A_2^2 =$

$$= \frac{\lambda_1}{1 + \dfrac{b^2}{(a-\lambda_1)^2}}\left(\cos x - \frac{b}{a-\lambda_1}\sin x\right)^2 + \frac{\lambda_2}{1 + \dfrac{b^2}{(a-\lambda_2)^2}}\left(\cos x - \frac{b}{a-\lambda_2}\sin x\right)^2$$

Đặt $u_1 = \cos x - \dfrac{b}{a-\lambda_1}\sin x; u_2 = \cos x - \dfrac{b}{a-\lambda_2}\sin x$; $k_1 = \dfrac{1}{a-\lambda_1}; k_2 = \dfrac{1}{a-\lambda_2}$

$$A_1 = \frac{1}{\sqrt{1+b^2k_1^2}}\left(\cos x - bk_1 \sin x\right); A_2 = \frac{1}{\sqrt{1+b^2k_2^2}}\left(\cos x - bk_2 \sin x\right)$$

Để ý rằng $A_1^2 + A_2^2 = 1 \Rightarrow \lambda_1 A_1^2 + \lambda_2 A_2^2 = (\lambda_1 - \lambda_2)A_1^2 + \lambda_2 = (\lambda_2 - \lambda_1)A_2^2 + \lambda_1$

•Giả sử $m\sin x + n\cos x = p\left(\sin x + \dfrac{b}{a-\lambda_1}\cos x\right) + q\left(\sin x + \dfrac{b}{a-\lambda_2}\cos x\right)$, $\forall$x

$$\Leftrightarrow \begin{cases} p+q=m \\ \dfrac{p}{a-\lambda_1} + \dfrac{q}{a-\lambda_2} = \dfrac{n}{b} \end{cases} \Leftrightarrow p = \frac{bm - n(a-\lambda_2)}{b(\lambda_2-\lambda_1)}(a-\lambda_1); q = \frac{bm - n(a-\lambda_1)}{b(\lambda_1-\lambda_2)}(a-\lambda_2)$$

$$J = \int \frac{m\sin x + n\cos x}{a(\sin x)^2 + 2b\sin x\cos x + c(\cos x)^2}dx = \int \frac{-pdu_1}{(\lambda_1-\lambda_2)A_1^2 + \lambda_2} + \int \frac{-qdu_2}{(\lambda_2-\lambda_1)A_2^2 + \lambda_1}$$

$$= -p\sqrt{1+b^2k_1^2}\int \frac{dA_1}{(\lambda_1-\lambda_2)A_1^2 + \lambda_2} - q\sqrt{1+b^2k_2^2}\int \frac{dA_2}{(\lambda_2-\lambda_1)A_2^2 + \lambda_1}$$

b. Các bài tập mẫu minh họa:

• $\boldsymbol{J_1 = \int \frac{(sinx + cosx)dx}{2sin^2 x - 4sinxcosx + 5cos^2 x}}$

λ_1, λ_2 là nghiệm của phương trình $\begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda_1 = 1; \lambda_2 = 6$

$$2\sin^2 x - 4\sin x\cos x + 5\cos^2 x = \frac{1}{5}(\cos x + 2\sin x)^2 + \frac{24}{5}\left(\cos x - \frac{1}{2}\sin x\right)^2$$

$$A_1 = \frac{1}{\sqrt{5}}(\cos x + 2\sin x); A_2 = \frac{2}{\sqrt{5}}\left(\cos x - \frac{1}{2}\sin x\right); A_1^2 + A_2^2 = 1$$

Giả sử $sin\, x + cos\, x = p(sin\, x - 2\,cos\, x) + q\left(sin\, x + \frac{1}{2}cos\, x\right) \Leftrightarrow p = \frac{-1}{5}; q = \frac{6}{5}$

$$\Rightarrow \sin x + \cos x = \frac{-1}{5}(\sin x - 2\cos x) + \frac{6}{5}\left(\sin x + \frac{1}{2}\cos x\right)$$

$$J_1 = \int \frac{(\sin x + \cos x)\,dx}{2\sin^2 x \text{ - } 4\sin x\cos x + 5\cos^2 x} = \frac{3}{5}\int \frac{(2\sin x + \cos x)\,dx}{(2\cos x - \sin x)^2 + 1} - \frac{1}{5}\int \frac{(\sin x - 2\cos x)\,dx}{6 - (\cos x + 2\sin x)^2}$$

$$= \frac{3}{5}\int \frac{d(\sin x - 2\cos x)}{(\sin x - 2\cos x)^2 + 1} + \frac{1}{5}\int \frac{d(\cos x + 2\sin x)}{6 - (\cos x + 2\sin x)^2}$$

$$= \frac{3}{5}\operatorname{arctg}(\sin x - 2\cos x) + \frac{1}{10\sqrt{6}}\ln\left|\frac{\sqrt{6} + \cos x + 2\sin x}{\sqrt{6} - \cos x - 2\sin x}\right| + c$$

11. DẠNG 11: CÁC PHÉP ĐỔI BIẾN SỐ TỔNG HỢP

• $\boldsymbol{K_1 = \int \frac{dx}{sin(x+a)\,sin(x+b)}} = \frac{1}{\sin(a-b)}\int \frac{\sin[(x+a)-(x+b)]}{\sin(x+a)\sin(x+b)}dx \quad (a \neq b)$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)}{\sin(x+a)\sin(x+b)}dx$$

$$= \frac{1}{\sin(a-b)}\int[\operatorname{cotg}(x+b) - \operatorname{cotg}(x+a)]dx = \frac{1}{\sin(a-b)}\ln\left|\frac{\sin(x+b)}{\sin(x+a)}\right| + c$$

• $\boldsymbol{K_2 = \int \frac{dx}{cos(x+a)\,cos(x+b)}} = \frac{1}{\sin(a-b)}\int \frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}dx$

$$=\frac{1}{\sin(a-b)}\int\frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}dx$$

$$=\frac{1}{\sin(a-b)}\int\left[\operatorname{tg}(x+a)-\operatorname{tg}(x+b)\right]dx=\frac{1}{\sin(a-b)}\ln\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+c$$

• $\boldsymbol{K_3=\int\frac{dx}{\sin(x+a)\cos(x+b)}}=\frac{1}{\cos(a-b)}\int\frac{\cos\left[(x+a)-(x+b)\right]}{\sin(x+a)\cos(x+b)}dx$

$$=\frac{1}{\cos(a-b)}\int\frac{\cos(x+a)\cos(x+b)+\sin(x+a)\sin(x+b)}{\sin(x+a)\cos(x+b)}dx$$

$$=\frac{1}{\cos(a-b)}\int\left[\operatorname{cotg}(x+a)+\operatorname{tg}(x+b)\right]dx=\frac{1}{\cos(a-b)}\ln\left|\frac{\sin(x+a)}{\cos(x+b)}\right|+c$$

• $\boldsymbol{K_4=\int\frac{\sqrt{3}+tgx}{\sqrt{3}-tgx}dx}=\int\frac{\left(\sqrt{3}+\operatorname{tg}x\right)\cos x}{\left(\sqrt{3}-\operatorname{tg}x\right)\cos x}dx=\int\frac{\sqrt{3}\cos x+\sin x}{\sqrt{3}\cos x-\sin x}dx$

$$=\int\frac{\frac{1}{2}\left(\sqrt{3}\cos x-\sin x\right)+\frac{\sqrt{3}}{2}\left(\sqrt{3}\sin x+\cos x\right)}{\sqrt{3}\cos x-\sin x}dx=\frac{1}{2}\int dx+\frac{\sqrt{3}}{2}\int\frac{\sqrt{3}\sin x+\cos x}{\sqrt{3}\cos x-\sin x}dx$$

$$=\frac{x}{2}-\frac{\sqrt{3}}{2}\int\frac{d\left(\sqrt{3}\cos x-\sin x\right)}{\sqrt{3}\cos x-\sin x}=\frac{x}{2}-\frac{\sqrt{3}}{2}\ln\left|\sqrt{3}\cos x-\sin x\right|+c$$

• $\boldsymbol{K_5=\int_{\pi/4}^{\pi/3}\sqrt{tgx}\,dx}=\int_{\pi/4}^{\pi/3}\sqrt{\frac{\sin x}{\cos x}}dx=\int_{\pi/4}^{\pi/3}\frac{\sin x}{\sqrt{\sin x\cos x}}dx=\frac{1}{\sqrt{2}}\int_{\pi/4}^{\pi/3}\frac{2\sin x}{\sqrt{2\sin x\cos x}}dx$

$$=\frac{1}{\sqrt{2}}\int_{\pi/4}^{\pi/3}\frac{(\cos x+\sin x)-(\cos x-\sin x)}{\sqrt{2\sin x\cos x}}dx=\frac{1}{\sqrt{2}}\left(\int_{\pi/4}^{\pi/3}\frac{(\cos x+\sin x)dx}{\sqrt{2\sin x\cos x}}-\int_{\pi/4}^{\pi/3}\frac{(\cos x-\sin x)dx}{\sqrt{2\sin x\cos x}}\right)$$

$$=\frac{1}{\sqrt{2}}\left(\int_{\pi/4}^{\pi/3}\frac{d(\sin x-\cos x)}{\sqrt{1-(\sin x-\cos x)^2}}-\int_{\pi/4}^{\pi/3}\frac{d(\sin x+\cos x)}{\sqrt{(\sin x+\cos x)^2-1}}\right)$$

$$=\frac{1}{\sqrt{2}}\left[\arcsin(\sin x-\cos x)-\ln\left|(\sin x+\cos x)+\sqrt{(\sin x+\cos x)^2-1}\right|\right]\Bigg|_{\pi/4}^{\pi/3}$$

$$=\frac{1}{\sqrt{2}}\left(\arcsin\frac{\sqrt{3}-1}{2}-\ln\left|\frac{\left(\sqrt{3}+1\right)\sqrt[4]{3}}{2\sqrt{2}}\right|+\ln\left(1+\sqrt{2}\right)\right)=\frac{1}{\sqrt{2}}\left(\arcsin\frac{\sqrt{3}-1}{2}-\ln\left|\frac{\left(\sqrt{3}+1\right)\sqrt[4]{3}}{4+2\sqrt{2}}\right|\right)$$

• $\boldsymbol{K_6 = \int_{\pi/8}^{\pi/4} \frac{dx}{\sin^6 x + \cos^6 x}} = \int_{\pi/8}^{\pi/4} \frac{dx}{(\sin^2 x+\cos^2 x)^3 - 3\sin^2 x\cos^2 x} = \int_{\pi/8}^{\pi/4} \frac{dx}{1-3\sin^2 x\cos^2 x}$

$$= \int_{\pi/8}^{\pi/4} \frac{dx}{\cos^4 x\left[(1+\mathrm{tg}^2 x)^2 - 3\,\mathrm{tg}^2 x\right]} = \int_{\pi/8}^{\pi/4} \frac{(1+\mathrm{tg}^2 x)\,d(\mathrm{tg}\,x)}{\mathrm{tg}^4 x - \mathrm{tg}^2 x + 1} = \int_{\sqrt{2}-1}^{1} \frac{(1+u^2)\,du}{u^4 - u^2 + 1}$$

$$= \int_{\sqrt{2}-1}^{1} \frac{\left(\frac{1}{u^2}+1\right)du}{u^2 + \frac{1}{u^2} - 1} = \int_{\sqrt{2}-1}^{1} \frac{d\left(u-\frac{1}{u}\right)}{\left(u-\frac{1}{u}\right)^2+1} = \mathrm{arctg}\frac{u^2-1}{u}\Bigg|_{\sqrt{2}-1}^{1} = \mathrm{arctg}\frac{2\sqrt{2}-1}{\sqrt{2}-1} = \mathrm{arctg}\left(3+\sqrt{2}\right)$$

• $\boldsymbol{K_7 = \int_{\pi/16}^{\pi/12} \frac{\cos 2x\cos 6x}{\mathrm{tg}4x + \mathrm{cotg}8x}dx} = \int_{\pi/16}^{\pi/12} \frac{\cos 2x\cos 6x\cos 4x\sin 8x}{\sin 4x\sin 8x + \cos 4x\cos 8x}dx$

$$= \int_{\pi/16}^{\pi/12} \frac{\cos 2x\cos 6x\cos 4x\sin 8x}{\cos(8x-4x)}dx = \frac{1}{2}\int_{\pi/16}^{\pi/12} (\cos 8x + \cos 4x)\sin 8x\,dx$$

$$= \frac{1}{4}\int_{\pi/16}^{\pi/12} (\sin 16x + \sin 12x + \sin 4x)\,dx = \frac{-1}{4}\left(\frac{1}{16}\cos 16x + \frac{1}{12}\cos 12x + \frac{1}{4}\cos 4x\right)\Bigg|_{\pi/16}^{\pi/12} = \frac{8\sqrt{2}-7}{384}$$

• $\boldsymbol{K_8 = \int_{0}^{\pi/2} \frac{\sin 2x + \sin x}{\sqrt{1+3\cos x}}dx} = -\int_{0}^{\pi/2} \frac{1+2\cos x}{\sqrt{1+3\cos x}}\sin x\,dx = -\int_{0}^{\pi/2} \frac{1+2\cos x}{\sqrt{1+3\cos x}}d(\cos x)$

$$= \frac{-2}{3}\int_{0}^{\pi/2} \frac{(1+3\cos x)+\frac{1}{2}}{\sqrt{1+3\cos x}}d(\cos x) = \frac{-2}{9}\int_{0}^{\pi/2} \sqrt{1+3\cos x}\,d(1+3\cos x) - \frac{1}{9}\int_{0}^{\pi/2} \frac{d(1+3\cos x)}{\sqrt{1+3\cos x}}$$

$$= \frac{-1}{9}\left[\frac{4}{3}(1+3\cos x)^{3/2} + 2\sqrt{1+3\cos x}\right]\Bigg|_{0}^{\pi/2} = \frac{34}{27}$$ ***(Đề thi TSĐH khối A 2005)***

• $\boldsymbol{K_9 = 2\int_{0}^{\pi/2} \frac{\sin x\cos^2 x}{1+\cos x}dx} = 2\int_{0}^{\pi/2} \frac{(1-\cos^2 x)-1}{1+\cos x}d(\cos x) = 2\int_{0}^{\pi/2}\left(1-\cos x - \frac{1}{1+\cos x}\right)d(\cos x)$

$$= 2\left[\cos x - \frac{\cos^2 x}{2} - \ln(1+\cos x)\right]\Bigg|_{0}^{\pi/2} = 2\ln 2 - 1$$ ***(Đề thi TSĐH khối D 2005)***

$$\bullet\ \boldsymbol{K_{10}} = \int_0^{\pi/6} \frac{dx}{\cos x\cos\left(x+\frac{\pi}{4}\right)} = \sqrt{2}\int_0^{\pi/6} \frac{dx}{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)\cos x} = \sqrt{2}\int_0^{\pi/6} \frac{dx}{(\cos x-\sin x)\cos x}$$

$$= \sqrt{2}\int_0^{\pi/6} \frac{dx}{(1-\operatorname{tg}x)\cos^2 x} = \sqrt{2}\int_0^{\pi/6} \frac{d(\operatorname{tg}x)}{1-\operatorname{tg}x} = -\sqrt{2}\ln|1-\operatorname{tg}x|\Big|_0^{\pi/6} = \sqrt{2}\ln\frac{3+\sqrt{3}}{2}$$

$$\bullet\ \boldsymbol{K_{11}} = \int_0^{\pi/4} \frac{dx}{\sqrt{2}+\sin x-\cos x} = \int_0^{\pi/4} \frac{dx}{\sqrt{2}\left[1-\cos\left(x+\frac{\pi}{4}\right)\right]} = \frac{1}{2\sqrt{2}}\int_0^{\pi/4} \frac{dx}{\sin^2\left(\frac{x}{2}+\frac{\pi}{8}\right)}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{d\left(\frac{x}{2}+\frac{\pi}{8}\right)}{\sin^2\left(\frac{x}{2}+\frac{\pi}{8}\right)} = \frac{-1}{\sqrt{2}}\operatorname{cotg}\left(\frac{x}{2}+\frac{\pi}{8}\right)\Big|_0^{\pi/4} = \frac{-1}{\sqrt{2}}\left[1-\left(\sqrt{2}+1\right)\right] = 1$$

$$\bullet\ \boldsymbol{K_{12}} = \int_0^{\pi/4} \frac{\sin x\,dx}{1+\sin 2x} = \int_0^{\pi/4} \frac{\sin x\,dx}{(\sin x+\cos x)^2} = \frac{1}{2}\int_0^{\pi/4} \frac{(\cos x+\sin x)-(\cos x-\sin x)}{(\sin x+\cos x)^2}dx$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{dx}{\sin x+\cos x} - \frac{1}{2}\int_0^{\pi/4}\frac{d(\sin x+\cos x)}{(\sin x+\cos x)^2} = \frac{1}{2\sqrt{2}}\int_0^{\pi/4}\frac{dx}{\sin\left(x+\frac{\pi}{4}\right)} - \frac{1}{2}\int_0^{\pi/4}\frac{d(\sin x+\cos x)}{(\sin x+\cos x)^2}$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/4}\frac{d\left[\cos\left(x+\frac{\pi}{4}\right)\right]}{\cos^2\left(x+\frac{\pi}{4}\right)-1} - \frac{1}{2}\int_0^{\pi/4}\frac{d(\sin x+\cos x)}{(\sin x+\cos x)^2} = \left[\frac{2}{\sqrt{2}}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{8}\right)\right| + \frac{1}{2(\sin x+\cos x)}\right]_0^{\pi/4}$$

$$= \frac{1}{2\sqrt{2}} - \sqrt{2}\ln\left(\sqrt{2}-1\right) - \frac{1}{2} = \sqrt{2}\ln\left(1+\sqrt{2}\right) - \frac{2-\sqrt{2}}{4}$$

$$\bullet\ \boldsymbol{K_{13}} = \int_{\pi/3}^{\pi/2} \frac{dx}{\sin 2x-2\sin x} = \int_{\pi/3}^{\pi/2} \frac{dx}{2\sin x(\cos x-1)} = \frac{1}{2}\int_{\pi/3}^{\pi/2} \frac{\sin x\,dx}{\sin^2 x(\cos x-1)}$$

$$= \frac{1}{2}\int_{\pi/3}^{\pi/2}\frac{d(\cos x)}{(1-\cos^2 x)(1-\cos x)} = \frac{1}{4}\int_{\sqrt{3}/2}^{0}\frac{[(1+u)+(1-u)]}{(1+u)(1-u)^2}du = \frac{1}{4}\left(\int_{\sqrt{3}/2}^{0}\frac{du}{(1-u)^2} + \int_{\sqrt{3}/2}^{0}\frac{du}{1-u^2}\right)$$

$$= \left[\frac{1}{4(1-u)} + \frac{1}{8}\ln\frac{1+u}{1-u}\right]_{\sqrt{3}/2}^{0} = \frac{1}{4} - \frac{2+\sqrt{3}}{2} + \frac{1}{4}\ln\left(2-\sqrt{3}\right) = \frac{1}{4}\ln\left(2-\sqrt{3}\right) - \frac{3+2\sqrt{3}}{4}$$

$$K_1 = \int_0^{\pi/2}\frac{\sin 2x\,dx}{\left(a^2\sin^2 x+b^2\cos^2 x\right)}, (ab\neq 0)\ ;\ K_2 = \int_0^{\pi/6}\frac{(\sin x)^3\,dx}{3\sin 4x-\sin 6x-3\sin 2x}$$