Japan Today's Calculation Of Integral 2006

92 Evaluate $\lim_{n\to\infty} n^2 \int_{-\frac{1}{n}}^{\frac{1}{n}} (2005\sin x + 2006\cos x)|x|dx.$

93 Evaluate

$$\frac{1}{\int_0^{\frac{\pi}{2}} \cos^{2005} x \ \sin 2007x \ dx}.$$

94 Let a be real numbers.Find the following limit value.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (\sin x + \sin ax)^2 dx.$$

95 Evaluate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sin^3 x}{\cos^5 x} dx.$$

96 For $a \geq 0,$ find the minimum value of $\int_{-2}^{1} |x^2 + 2ax| dx.$

97 Answer the following questions.

(1) Evaluate $\int_e^{e^e} \frac{\ln(\ln x)}{x \ln x} dx.$

(2) Let α, β be real numbers.Find the values of α, β for which the following equality holds for any real numbers $p, q.$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (p\cos x + q\sin x)(x^2 + \alpha x + \beta)dx = 0.$$

98 Let

$$I_n = \int_1^{1+\frac{1}{n}} \{[(x+1)\ln x + 1]e^{x(e^x \ln x + 1)} + n\}dx \ (n = 1, 2, \cdots).$$

Evaluate $\lim_{n\to\infty} I_n^n.$

99 Let θ be a constant number such that $0 \leq \theta \leq \pi.$
Evaluate

$$\int_0^{2\pi} \sin 8x |\sin(x-\theta)| \ dx.$$

100 Let a, b, c be positive numbers such that $abc = \frac{1}{16}.$
Prove the following inequality.

$$\int_0^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)(x^2 + c^2)} \ dx \leq \pi.$$

101 Thank you very much, **vidyamanohar.** I will continue to post problems.

For $n > 2$, prove the following inequality.

$$\frac{1}{2} < \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^n}}dx < \frac{\pi}{6}.$$

102 Let a, b be costant numbers such that $a^2 \geq b$. Find the following indefinite integrals.

(1) $I = \int \frac{dx}{x^2+2ax+b}$

(2) $J = \int \frac{dx}{(x^2+2ax+b)^2}$

103 For $0 < a < 1$, let $f(x) = \frac{a-x}{1-ax}$ $(-1 < x < 1)$.

Evaluate $\int_0^a \frac{1-\{f(x)\}^6}{1-x^2}\ dx.$

104 For $0 < x < 1$, let $f(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}\ dt$

(1) Find $\frac{d}{dx}f(\sqrt{1-x^2})$

(2) Find $f\left(\frac{1}{\sqrt{2}}\right)$

(3) Prove that $f(x) + f(\sqrt{1-x^2}) = \frac{\pi}{2}$

105 Let a, b be constant numbers such that $0 < a < b$. If a function $f(x)$ always satisfies $f'(x) > 0$ at $a < x < b$, for $a < t < b$ find the value of t for which the following the integral is minimized.

$$\int_a^b |f(x) - f(t)|x\ dx.$$

106 Evaluate $\int_0^1 \frac{1-x^2}{1+x^2}\frac{dx}{\sqrt{1+x^4}}$

107 Evaluate $\int_{-1}^1 \frac{\sqrt{1-x^2}}{a-x}\ dx\ (a > 1)$

108 For $x \geq 0$, find the minimum value of x for which $\int_0^x 2^t(2^t-3)(x-t)\ dt$ is minimized.

109 Let $I_n = \int_{2006}^{2006+\frac{1}{n}} x\cos^2(x-2006)\ dx\ (n = 1, 2, \cdots)$.

Find $\lim_{n\to\infty} nI_n$.

110 Prove the following inequality.

$$1 \leq \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^3 x}\ dx \leq \frac{1}{2}\{\sqrt{2}+\ln(1+\sqrt{2}\ )\}$$

111 Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-nx}\cos^m x\ dx\ (m, n = 0, 1, 2, \cdots)$.

112 Evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-e^2\sin^2\theta)^3}$ $(e < 1$ is a constant number$)$.

113 Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}+\sqrt{\cos x}+3(\sqrt{\sin x}-\sqrt{\cos x})\cos 2x}{\sqrt{\sin 2x}}\ dx.$

114 Let a be positive numbers.For $|x|\leq a,$ find the maximum and minimum value of $\int_{x-a}^{x+a} \sqrt{4a^2-t^2}\ dt.$

115 Find the value of a such that $\int_0^{\frac{\pi}{2}} (\sin x+a\cos x)^3\ dx-\frac{4a}{\pi-2}\int_0^{\frac{\pi}{2}} x\cos x dx=2.$

116 Find $\lim_{t\rightarrow 0} \int_0^{2\pi} \frac{|\sin(x+t)-\sin x|\ dx}{|t|}.$

117 Let a be a real constant number. Evaluate $\lim_{n\to\infty} n\int_{-1}^1 e^{-n|a-x|}\ dx.$

118 Let $f(x)$ be the function defined for $x\geq 0$ which satisfies the following conditions.

(a) $f(x)=\begin{cases}x & (0\leq x<1)\\ 2-x & (1\leq x<2)\end{cases}$

(b) $f(x+2n)=f(x)\ (n=1,2,\cdots)$

Find $\lim_{n\to\infty} \int_0^{2n} f(x)e^{-x}\ dx.$

119 Find the continuous function $f(x)$ and constant number such that $\int_0^x f(t)\ dt=e^x-ae^{2x}\int_0^1 f(t)e^{-t}\ dt.$

120 Let k be real constants.How many real roots can the following quadratic equation have?

$$x^2=-2x+k+\int_0^1 |t+k|\ dt.$$

121 Given the parabola $C: y=x^2.$ If the circle centered at y axis with radius 1 has common tangent lines with C at distinct two points, then find the coordinate of the center of the circle K and the area of the figure surrounded by C and $K.$

122 Let $x(t)=\tan t,\ y(t)=-\ln \cos t\ \left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$. Find the area surrounded by the curve : $x=x(t),\ \ y=y(t)$ and x axis and the line $x=1.$

123 Let $f(x)=\pi x^2\sin \pi x^2.$ Prove that the volume V formed by the revolution of the figure surrounded by the part $0\leq x\leq 1$ of the graph of $y=f(x)$ and x axis about y axis is can be given as $2\pi \int_0^1 xf(x)\ dx$ then find the value of $V.$

124 Let $a>1.$ Find the area $S(a)$ of the part surrounded by the curve $y=\frac{a^4}{\sqrt{(a^2-x^2)^3}}\ (0\leq x\leq 1),\ x$ axis , y axis and the line $x=1,$ then when a varies in the range of $a>1,$ then find the extremal value of $S(a).$

125 Prove the following inequality for $x\geq 0.$

$$\int_0^x (t-t^2)\sin^{2004} t\ dt<\frac{1}{2006}$$

126 For $t>0,$ find the minimum value of $\int_0^1 x|e^{-x^2}-t|dx.$

127 Evaluate $\int_{e^{\frac{\pi}{4}}}^{e^{\frac{\pi}{3}}} \frac{1}{\sin(2\ln x)}\ dx.$

This problem is mistaken

128 Prove the following inequality.

$$-\frac{\pi}{3}\ln 2+\frac{\pi^3}{81}<\int_0^{\frac{\pi}{3}}\ln(\cos x)dx<-\frac{\pi^3}{162}.$$

129 The sequence $\{a_n\}$ is defined as follows.

$$a_1=\frac{\pi}{4},\ a_n=\int_0^{\frac{1}{2}}(\cos\pi x+a_{n-1})\cos\pi x\ dx\ \ (n=2,3,\cdots)$$

Find $\lim_{n\rightarrow\infty}a_n$.

130 Find the value of a such that $\int_0^a\frac{1}{e^x+4e^{-x}+5}\ dx=\ln\sqrt[3]{2}$.

131 For $a>0$, find the minimum value of $\int_0^{\frac{1}{a}}(a^3+4x-a^5x^2)e^{ax}\ dx$.

132 Find the area of the figure such that the points $(x,\ y)$ satisfies the inequality $\lim_{n\rightarrow\infty}(x^{2n}+y^{2n})^{\frac{1}{n}}\geq\frac{3}{2}x^2+\frac{3}{2}y^2-1$.

133 Let $f(x)$ be the polynomial with respect to x, and $g_n(x)=n-2n^2\left|x-\frac{1}{2}\right|+\left|n-2n^2\left|x-\frac{1}{2}\right|\right|$.

Find $\lim_{n\rightarrow\infty}\int_0^1 f(x)g_n(x)\ dx$.

134 For positive integers n, let $A_n=\frac{1}{n}\{(n+1)+(n+2)+\cdots+(n+n)\},\ B_n=\{(n+1)(n+2)\cdots(n+n)\}^{\frac{1}{n}}$.

Find $\lim_{n\rightarrow\infty}\frac{A_n}{B_n}$.

135 Find the value of a for which $\int_0^{\frac{\pi}{2}}|a\sin x-\cos x|\ dx\ (a>0)$ is minimized.

136 Let c be the constant number such that $c>1$.Find the least area of the figure surrounded by the line passing through the point $(1,\ c)$ and the palabola $y=x^2$ on $x-y$ plane.

137 Find the value of a for which $\int_0^1|xe^x-a|\ dx$ is minimized.

138 Let $f(x)$ be the product of functions made by taking four functions from three functions $x,\ \sin x,\ \cos x$ repeatedly. Find the minimum value of $\int_0^{\frac{\pi}{2}}f(x)\ dx$.

139 Let $a,\ b$ be real numbers.
Evaluate

$$\int_0^{2\pi}(a\cos x+b\sin x)^{2n}\ dx\ (n=1,\ 2,\ \cdots).$$

140 Evaluate

$$\int_0^{\frac{\pi}{4}}\left(\frac{\cos x}{\sin x+\cos x}\right)^2\ dx,\quad \int_0^{\frac{\pi}{4}}\left(\frac{\sin x+\cos x}{\cos x}\right)^2\ dx.$$

141 Evaluate $\int_0^{\pi}\frac{\cos 4x-\cos 4\alpha}{\cos x-\cos\alpha}\ dx$.

142 Evaluate $\int_0^{\pi}\frac{\sin x}{\sqrt{1-2a\cos x+a^2}}\ dx\ (a>0)$.

143 Evaluate $\int_0^{\frac{\pi}{2}} \frac{1-\sin 2x}{(1+\sin 2x)^2}\ dx.$

144 Evaluate $\lim_{n\to\infty} \int_0^{\pi} \left|\left(x+\frac{\pi}{n}\right)\sin nx\right|\ dx\ (n=1,\ 2,\ \cdots).$

145 Find the minimum value of $\int_x^{x+l} \left(t+\frac{1}{t}\right) dt\ (x>0,\ l>0).$

146 Find the maximum value of $\int_{-1}^1 |x-a|e^x dx$ for $|a|\leq 1.$

147 Find the area of the figure surrounded by the curve $2(x^2+1)y^2+8x^2y+x^4+4x^2-1=0.$

148 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^2 n\theta}{\sin^2\theta}\ d\theta\ (n=1,\ 2,\ \cdots).$

149 Let $f(x)=(1-x^2)^{\frac{3}{2}}$. Denote M the maximum value of $|f'(x)|$ in $(-1,\ 1)$. Prove that $\int_{-1}^1 f(x)\ dx \leq M.$

1981 Musashi Institute of Technology entrance exam/Electro Telecommunication, Civil Enginerring

150 Find the value of a such that $\lim_{n\to\infty} \frac{3}{2}\int_{-\sqrt[3]{a}}^{\sqrt[3]{a}} \left(1-\frac{t^3}{n}\right)^n t^2\ dt=\sqrt{2}\ \ (n=1,2,\cdots).$

151 Let $a,\ b$ be positive constant numbers.Find the volume of the revolution of the region surrounded by the parabola $y=ax^2$ and the line $y=bx$ about the line $y=bx$ as the axis on xy plane.

152 Let $f(x)$ the function such that $f(0)=0,\ |f'(x)|\leq \frac{1}{1+x}\ (x\geq 0)$. Prove that $\int_0^{e-1} \{f(x)\}^2 dx\leq e-2.$

153 Draw the perpendicular to the tangent line of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>0,\ b>0)$ from the origin $O(0,\ 0).$

Let θ be the angle between the perpendicular and the positive direction of x axis. Denote the length of the perpendicular by $r(\theta)$.

Calculate $\int_0^{2\pi} r(\theta)^2\ d\theta.$

154 Find the function $f(x)$ which is defined for $\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$ such that $f(x)+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x-y)\cdot f(y)\ dy=x+1\ \left(-\frac{\pi}{2}\leq x\leq \frac{\pi}{2}\right).$

155 The sequence $\{c_n\}$ is determined by the following equation.

$$c_n=(n+1)\int_0^1 x^n\cos \pi x\ dx\ (n=1,\ 2,\ \cdots).$$

Let λ be the limit value $\lim_{n\to\infty} c_n$. Find $\lim_{n\to\infty} \frac{c_{n+1}-\lambda}{c_n-\lambda}.$

156 For arbiterary integers $n,$ find the continuous function $f(x)$ which satisfies the following equation.

$$\lim_{h\to 0} \frac{1}{h}\int_{x-nh}^{x+nh} f(t)dt=2f(nx).$$

Note that x can range all real numbers and $f(1)=1.$

157 Find the volume of the solid expressed by the following six inequaities in xyz space.

$$x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 3,\ x+2z \leq 4,\ y-z \leq 1.$$

158 (1) Evaluate the definite integral $\int_0^{\pi} e^{-x}\sin x dx.$

(2) Find the limit $\lim_{n\to\infty}\int_0^{n\pi} e^{-x}|\sin x| dx.$

159 A function is defined by $f(x) = \int_0^x \frac{1}{1+t^2}dt.$

(1) Find the equation of normal line at $x = 1$ of $y = f(x)$.

(2) Find the area of the figure surrounded by the normal line found in (1), the x axis and the graph of $y = f(x)$.

Note that you may not use the formula $\int \frac{1}{1+x^2}dx = \tan^{-1}x + Const.$

160 Find the value of m $(0 < m < 1)$ for which $\int_0^{\pi} |\sin x - mx|\ dx$ is minimized.

161 Find the differentiable function $f(x)$ such that $f(x) = -\int_0^x f(t)\tan t\ dt + \int_0^x \tan(t-x)\ dt\ \left(|x| < \frac{\pi}{2}\right).$

162 Let $f(x)$ be the function such that $f(x) > 0$ at $x \geq 0$ and $\{f(x)\}^{2006} = \int_0^x f(t)dt + 1.$

Find the value of $\{f(2006)\}^{2005}$.

163 Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\ dx\ (n = 0,\ 1,\ 2,\ \cdots).$

Find $\sum_{n=0}^{\infty}\{{I_{n+2}}^2 + (I_{n+1} + I_{n+3})I_{n+2} + I_{n+1}I_{n+3}\}.$

164 For positive integers n, let

$$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots\cdots + \frac{1}{\sqrt{n}},\ T_n = \frac{1}{\sqrt{1+\frac{1}{2}}} + \frac{1}{\sqrt{2+\frac{1}{2}}} + \cdots\cdots + \frac{1}{\sqrt{n+\frac{1}{2}}}.$$

Find $\lim_{n\to\infty}\frac{T_n}{S_n}$.

165 On $x-y$ plane, let $C: y = 2006x^3 - 12070102x^2 + \cdots$.
Find the area of the region surrounded by the tangent line of C at $x = 2006$ and the curve C.

166 Express the following the limit values in terms of a definite integral and find them.

(1) $I = \lim_{n\to\infty}\frac{1}{n}\ln\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\cdots\left(1+\frac{n}{n}\right).$

(2) $J = \lim_{n\to\infty}\frac{1}{n^2}(\sqrt{n^2-1} + \sqrt{n^2-2^2} + \cdots\cdots + \sqrt{n^2-n^2}).$

(3) $K = \lim_{n\to\infty}\frac{1}{n^3}(\sqrt{n^2+1} + 2\sqrt{n^2+2^2} + \cdots + n\sqrt{n^2+n^2}).$

167 In xyz plane find the volume of the solid formed by the points $(x,\ y,\ z)$ satisfying the following system of inequalities.

$$0 \leq z \leq 1 + x + y - 3(x-y)y,\ 0 \leq y \leq 1,\ y \leq x \leq y+1.$$