Japan Today's Calculation Of Integral 2005

1 Calculate the following indefinite integral.

[1] $\int \frac{e^{2x}}{(e^x+1)^2}dx$

[2] $\int \sin x \cos 3x dx$

[3] $\int \sin 2x \sin 3x dx$

[4] $\int \frac{dx}{4x^2-12x+9}$

[5] $\int \cos^4 x dx$

2 Calculate the following indefinite integrals.

[1] $\int \cos\left(2x-\frac{\pi}{3}\right)dx$

[2] $\int \frac{dx}{\cos^2(3x+4)}$

[3] $\int (x-1)\sqrt[3]{x-2}dx$

[4] $\int x \cdot 3^{x^2+1}dx$

[5] $\int \frac{dx}{\sqrt{1-x}}dx$

3 Calculate the following indefinite integrals.

[1] $\int \sin x \sin 2x dx$

[2] $\int \frac{e^{2x}}{e^x-1}dx$

[3] $\int \frac{\tan^2 x}{\cos^2 x}dx$

[4] $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$

[5] $\int \frac{e^x}{e^x+1}dx$

4 Calculate the following indefinite integrals.

[1] $\int \frac{x}{\sqrt{5-x}}dx$

[2] $\int \frac{\sin x \cos^2 x}{1+\cos x}dx$

[3] $\int (\sin x+\cos x)^2 dx$

[4] $\int \frac{x-\cos^2 x}{x\cos^2 x}dx$

[5] $\int (\sin x+\sin 2x)^2 dx$

5 Calculate the following indefinite integrals.

[1] $\int (4-5\tan x)\cos x dx$

[2] $\int \frac{dx}{\sqrt[3]{(1-3x)^2}}dx$

[3] $\int x^3\sqrt{4-x^2}dx$

[4] $\int e^{-x}\sin\left(x+\frac{\pi}{4}\right)dx$

[5] $\int (3x-4)^2 dx$

6 Calculate the following indefinite integrals.

[1] $\int \sin x\cos^3 x dx$

[2] $\int \frac{dx}{(1+\sqrt{x})\sqrt{x}}dx$

[3] $\int x^2\sqrt{x^3+1}dx$

[4] $\int \frac{e^{2x}-3e^x}{e^x}dx$

[5] $\int (1-x^2)e^x dx$

7 Calculate the following indefinite integrals.

[1] $\int \sqrt{x}(\sqrt{x}+1)^2 dx$

[2] $\int (e^x+2e^{x+1}-3e^{x+2})dx$

[3] $\int (\sin^2 x+\cos x)\sin x dx$

[4] $\int x\sqrt{2-x}dx$

[5] $\int x\ln x dx$

8 Calculate the following indefinite integrals.

[1] $\int x(x^2+3)^2 dx$

[2] $\int \ln(x+2)dx$

[3] $\int x\cos x dx$

[4] $\int \frac{dx}{(x+2)^2}dx$

[5] $\int \frac{x-1}{x^2-2x+3}dx$

9 Calculate the following indefinite integrals.

[1] $\int (x^2+4x-3)^2(x+2)dx$

[2] $\int \frac{\ln x}{x(\ln x+1)}dx$

[3] $\int \frac{\sin(\pi \log_2 x)}{x} dx$

[4] $\int \frac{dx}{\sin x \cos^2 x}$

[5] $\int \sqrt{1-3x}\, dx$

10 Calculate the following indefinite integrals.

[1] $\int (2x+1)\sqrt{x+2}\, dx$

[2] $\int \frac{1+\cos x}{x+\sin x}\, dx$

[3] $\int \sin^5 x \cos^3 x\, dx$

[4] $\int \frac{(x-3)^2}{x^4}\, dx$

[5] $\int \frac{dx}{\tan x}\, dx$

11 Calculate the following indefinite integrals.

[1] $\int \frac{6x+1}{\sqrt{3x^2+x+4}} dx$

[2] $\int \frac{e^x}{e^x+e^{a-x}} dx$

[3] $\int \frac{(\sqrt{x}+1)^3}{\sqrt{x}} dx$

[4] $\int x \ln(x^2-1) dx$

[5] $\int \frac{2(x+2)}{x^2+4x+1} dx$

12 Calculate the following indefinite integrals.

[1] $\int \frac{dx}{1+\cos x}$

[2] $\int x\sqrt{x^2-1} dx$

[3] $\int a^{-\frac{x}{2}} dx \quad (a>0, a \neq 1)$

[4] $\int \frac{\sin^3 x}{1+\cos x} dx$

[5] $\int e^{4x} \sin 2x dx$

13 Calculate the following integarls.

[1] $\int x \cos^2 x dx$

[2] $\int \frac{x-1}{(3x-1)^2} dx$

[3] $\int \frac{x^3}{(2-x^2)^4} dx$

[4] $\int \left(\frac{1}{4\sqrt{x}} + \frac{1}{2x} \right) dx$

[5] $\int (\ln x)^2 dx$

14 Calculate the following indefinite integrals.

[1] $\int \frac{\sin x \cos x}{1+\sin^2 x} dx$

[2] $\int x \log_{10} x dx$

[3] $\int \frac{x}{\sqrt{2x-1}} dx$

[4] $\int (x^2+1) \ln x dx$

[5] $\int e^x \cos x dx$

15 Calculate the following indefinite integrals.

[1] $\int \frac{(x^2-1)^2}{x^4} dx$

[2] $\int \frac{e^{3x}}{\sqrt{e^x+1}} dx$

[3] $\int \sin 2x \cos 3x dx$

[4] $\int x \ln(x+1) dx$

[5] $\int \frac{x}{(x+3)^2} dx$

16 Calculate the following indefinite integrals.

[1] $\int \sin(\ln x) dx$

[2] $\int \frac{x+\sin^2 x}{x \sin^2 x} dx$

[3] $\int \frac{x^3}{x^2+1} dx$

[4] $\int \frac{x^2}{\sqrt{2x-1}} dx$

[5] $\int \frac{x+\cos 2x+1}{x \cos^2 x} dx$

17 Calculate the following indefinite integrals.

[1] $\int \frac{dx}{e^x - e^{-x}}$

[2] $\int e^{-ax} \cos 2x dx \ (a \neq 0)$

[3] $\int (3^x - 2)^2 dx$

[4] $\int \frac{x^4+2x^3+3x^2+1}{(x+2)^5} dx$

[5] $\int \frac{dx}{1-\cos x} dx$

18 Calculate the following indefinite integrals.

[1] $\int (\sin x + \cos x)^4 dx$

[2] $\int \frac{e^{2x}}{e^x + 1} dx$

[3] $\int \sin^4 x dx$

[4] $\int \sin 6x \cos 2x dx$

[5] $\int \frac{x^2}{\sqrt{(x+1)^3}} dx$

19 Calculate the following indefinite integrals.

[1] $\int \tan^3 x dx$

[2] $\int a^{mx+n} dx \ (a > 0, a \neq 1, mn \neq 0)$

[3] $\int \cos^5 x dx$

[4] $\int \sin^2 x \cos^3 x dx$

[5] $\int \frac{dx}{\sin x}$

20 Calculate the following indefinite integrals.

[1] $\int \ln(x^2 - 1) dx$

[2] $\int \frac{1}{e^x + 1} dx$

[3] $\int (ax^2 + bx + c) e^{mx} dx \ (abcm \neq 0)$

[4] $\int \left(\tan x + \frac{1}{\tan x} \right)^2 dx$

[5] $\int \sqrt{1 - \sin x} dx$

21 [1] Tokyo Univ. of Science: $\int \frac{\ln x}{(x+1)^2} dx$

[2] Saitama Univ.: $\int \frac{5}{3\sin x + 4\cos x} dx$

[3] Yokohama City Univ.: $\int_1^{\sqrt{3}} \frac{1}{\sqrt{x^2+1}} dx$

[4] Daido Institute of Technology: $\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x + \cos x} dx$

[5] Gunma Univ.: $\int_0^{\frac{3\pi}{4}} \{(1+x)\sin x + (1-x)\cos x\} dx$

22 Evaluate

$\int_0^1 (1 - x^2)^n dx \ (n = 0, 1, 2, \ldots)$

$\int_0^{\quad}$

23 Evaluate

$$\lim_{a\to\frac{\pi}{2}-0}\int_0^a (\cos x)\ln(\cos x)\,dx\ \left(0\leqq a<\frac{\pi}{2}\right)$$

24 Find the minimum value of $\int_0^{\pi}(x-y)^2(\sin x)|\cos x|dx$.

25 Let $|a|<\frac{\pi}{2}$.

Evaluate

$$\int_0^{\frac{\pi}{2}}\frac{dx}{\{\sin(a+x)+\cos x\}^2}$$

26 Evaluate

$$\int_{e^{e^e}}^{e^{e^{e^e}}}\frac{dx}{x\ln x\cdot\ln(\ln x)\cdot\ln\{\ln(\ln x)\}}$$

27 Let $f(x)=t\sin x+(1-t)\cos x\ (0\leqq t\leqq 1)$.

Find the maximum and minimum value of the following $P(t)$.

$$P(t)=\left\{\int_0^{\frac{\pi}{2}}e^x f(x)dx\right\}\left\{\int_0^{\frac{\pi}{2}}e^{-x}f(x)dx\right\}$$

28 Evaluate

$$\int_0^{\frac{\pi}{4}}\frac{x\cos 5x}{\cos x}dx$$

29 Let a be a real number.
Evaluate

$$\int_{-\pi+a}^{3\pi+a}|x-a-\pi|\sin\left(\frac{x}{2}\right)dx$$

30 A sequence $\{a_n\}$ is defined by $a_n=\int_0^1 x^3(1-x)^n dx\ (n=1,2,3.\cdots)$

Find the constant number c such that $\sum_{n=1}^{\infty}(n+c)(a_n-a_{n+1})=\frac{1}{3}$

31 Evaluate

$$\lim_{n\to\infty}\int_0^{\pi}x^2|\sin nx|dx$$

32 Evaluate

$$\int_0^1 e^{x+e^x+e^{e^x}+e^{e^{e^x}}}dx$$

33 Evaluate

$$\int_{-\ln 2}^0 \frac{dx}{\cos^2 hx \cdot \sqrt{1-2a\tanh x+a^2}} \ (a>0)$$

34 Let p be a constant number such that $0<p<1$.
Evaluate

$$\sum_{k=0}^{2004} \frac{p^k(1-p)^{2004-k}}{\int_0^1 x^k(1-x)^{2004-k}dx}$$

35 Determine the value of a, b for which $\int_0^1 (\sqrt{1-x}-ax-b)^2 dx$ is minimized.

36 A sequence of polynomial $f_n(x)\ (n=0,1,2,\cdots)$ satisfies $f_0(x)=2, f_1(x)=x$,

$$f_n(x)=xf_{n-1}(x)-f_{n-2}(x),\ (n=2,3,4,\cdots)$$

Let $x_n\ (n\geqq 2)$ be the maximum real root of the equation $f_n(x)=0\ (|x|\leqq 2)$

Evaluate

$$\lim_{n\to\infty} n^2\int_{x_n}^2 f_n(x)dx$$

37 Evaluate

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{\sin x\sqrt{1-\cos x}}dx$$

38 Let a be a constant number such that $0<a<1$ and $V(a)$ be the volume formed by the revolution of the figure which is enclosed by the curve $y=\ln(x-a)$, the x-axis and two lines $x=1, x=3$ about the x-axis.

If a varies in the range of $0<a<1$, find the minimum value of $V(a)$.

39 Find the minimum value of the following function $f(x)$ defined at $0<x<\frac{\pi}{2}$.

$$f(x)=\int_0^x \frac{d\theta}{\cos\theta}+\int_x^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}$$

40 Evaluate

$$\int_0^1 x^{2005}e^{-x^2}dx$$

41 Evaluate

$$\int_0^a \sqrt{2ax-x^2}\,dx\ (a>0)$$

42 Let $0<t<\frac{\pi}{2}$.

Evaluate

$$\lim_{t\to\frac{\pi}{2}}\int_0^t \tan\theta\sqrt{\cos\theta}\ln(\cos\theta)d\theta$$

43 Evaluate

$$\int_0^{\frac{\pi}{2}} \cos^{2004}x\cos 2004x\ dx$$

44 Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2005x}{\sin x}dx$$

45 Find the function $f(x)$ which satisfies the following integral equation.

$$f(x)=\int_0^x t(\sin t-\cos t)dt+\int_0^{\frac{\pi}{2}} e^t f(t)dt$$

46 Find the minimum value of $\int_0^1 \frac{|t-x|}{t+1}dt$

47 Find the condition of a,b for which the function $f(x)\ (0\le x\le 2\pi)$ satisfying the following equality can be determined uniquely,then determine $f(x)$, assuming that $f(x)$ is a continuous function at $0\le x\le 2\pi$.

$$f(x)=\frac{a}{2\pi}\int_0^{2\pi}\sin(x+y)f(y)dy+\frac{b}{2\pi}\int_0^{2\pi}\cos(x-y)f(y)dy+\sin x+\cos x$$

48 Evaluate

$$\lim_{n\to\infty}\left(\int_0^{\pi}\frac{\sin^2 nx}{\sin x}dx-\sum_{k=1}^{n}\frac{1}{k}\right)$$

49 For $x\ge 0$, Prove that $\int_0^x (t-t^2)\sin^{2002}t<\frac{1}{2004\cdot 2005}$

50 Let a,b be real numbers such that $a<b$.
Evaluate

$$\lim_{b\to a}\frac{\int_a^b \ln|1+(x-a)(b-x)|dx}{(b-a)^3}$$

.

51 A function $f(x)$ satisfies $f(x)=f\left(\frac{c}{x}\right)$ for some real number $c(>1)$ and all positive number x.

If $\int^{\sqrt{c}}\frac{f(x)}{x}dx=3$, evaluate $\int^{c}\frac{f(x)}{x}dx$

$\int_1^{\quad}$ $\int_1^{\quad}$

52 Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k\sqrt{-1}}$$

53 Find the maximum value of the following integral.

$$\int_0^{\infty} e^{-x}\sin tx\,dx$$

54 evaluate

$$\int_{-1}^{0}\sqrt{\frac{1+x}{1-x}}dx$$

55 Evaluate

$$\lim_{n\to\infty} n\int_0^1 (1+x)^{-n-1}e^{x^2}\,dx\ \ (n=1,2,\cdots)$$

56 Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{[\sqrt{2n^2-k^2}\,]}{n^2}$$

$[x]$ is the greatest integer $\leq x$.

57 Find the value of $n\in\mathbb{N}$ satisfying the following inequality.

$$\left|\int_0^{\pi} x^2\sin nx\,dx\right| < \frac{99\pi^2}{100n}$$

58 Let $f(x)=\dfrac{e^x}{e^x+1}$

Prove the following equation.

$$\int_a^b f(x)dx+\int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b)-af(a)$$

59 Evaluate

$$\int_{-\pi}^{\pi}(\cos 2x)(\cos 2^2x)\cdots(\cos 2^{2006}x)dx$$

60 Let $a_n=\int_0^{\frac{\pi}{2}}\sin 2t\,(1-\sin t)^{\frac{n-1}{2}}dt\ (n=1,2,\cdots)$

Evaluate

$$\sum_{n=1}^{\infty}(n+1)(a_n-a_{n+1})$$

61 Evaluate

$$\sum_{k=0}^{2004}\int_{-1}^{1}\frac{\sqrt{1-x^2}}{\sqrt{k+1}-x}dx$$

62 For $a>1$, let $f(a)=\frac{1}{2}\int_0^1 |ax^n-1|dx+\frac{1}{2}\ (n=1,2,\cdots)$ and let b_n be the minimum value of $f(a)$ at $a>1$.

Evaluate

$$\lim_{m\to\infty} b_m\cdot b_{m+1}\cdot\cdots\cdot b_{2m}\ (m=1,2,3,\cdots)$$

63 For a positive number x, let $f(x)=\lim_{n\to\infty}\sum_{k=1}^{n}\left|\cos\left(\frac{2k+1}{2n}x\right)-\cos\left(\frac{2k-1}{2n}x\right)\right|$

Evaluate

$$\lim_{x\to\infty}\frac{f(x)}{x}$$

64 Let $f(t)$ be the cubic polynomial for t such that $\cos 3x=f(\cos x)$ holds for all real number x.
Evaluate

$$\int_0^1\{f(t)\}^2\sqrt{1-t^2}dt$$

65 Let $a>0$. Find the minimum value of $\int_{-1}^{a}\left(1-\frac{x}{a}\right)\sqrt{1+x}\ dx$

66 Find the minimum value of $\int_0^{\frac{\pi}{2}}|\cos x-a|\sin x\ dx$

67 Evaluate

$$\frac{2005\int_0^{1002}\frac{dx}{\sqrt{1002^2-x^2}+\sqrt{1003^2-x^2}}+\int_{1002}^{1003}\sqrt{1003^2-x^2}dx}{\int_0^1\sqrt{1-x^2}dx}$$

68 Find the minimum value of $\int_1^e\left|\ln x-\frac{a}{x}\right|dx\ (0\le a\le e)$

69 Let $f_1(x)=x$, $f_n(x)=x+\frac{1}{14}\int_0^{\pi}xf_{n-1}(t)\cos^3 t\ dt\ (n\ge 2)$.

Find $\lim_{n\to\infty}f_n(x)$

70 Find the number of root for $\int_0^{\frac{\pi}{2}}e^x\cos(x+a)\ dx=0$ at $0\le a<2\pi$

71 Find the minimum value of $\int_{-1}^{1}\sqrt{|t-x|}\ dt$

72 Let $f(x)$ be a continuous function satisfying $f(x)=1+k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f(t)\sin(x-t)dt\ (k:constant\ number)$

Find the value of k for which $\int_0^{\pi}f(x)dx$ is maximized.

73 Find the minimum value of $\int_0^{\pi}(a\sin x+b\sin 2x+c\sin 3x-x)^2\ dx$

73 Find the minimum value of $\int_0^{\pi} (a\sin x + b\sin 2x + c\sin 3x - x)^2\,dx$

74 p, q satisfies $px + q \geq \ln x$ at $a \leq x \leq b$ $(0 < a < b)$.

Find the value of p, q for which the following definite integral is minimized and then the minimum value.

$$\int_a^b (px + q - \ln x)dx$$

75 A function $f(\theta)$ satisfies the following conditions $(a), (b)$.

(a) $f(\theta) \geq 0$

(b) $\int_0^{\pi} f(\theta)\sin\theta d\theta = 1$

Prove the following inequality.

$$\int_0^{\pi} f(\theta)\sin n\theta\ d\theta \leq n\ (n = 1, 2, \cdots)$$

76 The function $f_n(x)$ $(n = 1, 2, \cdots)$ is defined as follows.

$$f_1(x) = x,\ f_{n+1}(x) = 2x^{n+1} - x^n + \frac{1}{2}\int_0^1 f_n(t)\ dt\ \ (n = 1, 2, \cdots)$$

Evaluate

$$\lim_{n\to\infty} f_n\left(1 + \frac{1}{2n}\right)$$

77 Find the area of the part enclosed by the following curve.

$$x^2 + 2axy + y^2 = 1\ (-1 < a < 1)$$

78 Let α, β be the distinct positive roots of the equation of $2x = \tan x$.
Evaluate

$$\int_0^1 \sin\alpha x \sin\beta x\ dx$$

79 Find the area of the domain expressed by the following system inequalities.

$$x \geq 0,\ y \geq 0,\ x^{\frac{1}{p}} + y^{\frac{1}{p}} \leq 1\ (p = 1, 2, \cdots)$$

80 Let S be the domain surrounded by the two curves $C_1 : y = ax^2,\ C_2 : y = -ax^2 + 2abx$ for constant positive numbers a, b.
Let V_x be the volume of the solid formed by the revolution of S about the axis of x, V_y be the volume of the solid formed by the revolution of S

about the axis of y. Find the ratio of $\frac{V_x}{V_y}$.

81 Prove the following inequality.

$$\frac{1}{12}(\pi - 6 + 2\sqrt{3}) \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \ln(1 + \cos 2x)dx \leq \frac{1}{4}(2 - \sqrt{3})$$

82 Let $0 < a < b$. Prove the following inequaliy.

$$\frac{1}{b-a}\int_a^b \left(\ln\frac{b}{x}\right)^2 dx < 2$$

83 Evaluate

$$\sum_{n=1}^{\infty}\int_{2n\pi}^{2(n+1)\pi} \frac{x\sin x+\cos x}{x^2}\,dx\ (n=1,2,\cdots)$$

84 Evaluate

$$\lim_{n\to\infty} n\int_0^{\pi} e^{-nx}\sin^2 nx\,dx$$

85 Evaluate

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}} \frac{[n\sin x]}{n}\,dx$$

where $[x]$ is the integer equal to x or less than x.

86 Prove

$$\left[\int_{\pi}^{\infty}\frac{\cos x}{x}\,dx\right]^2 < \frac{1}{\pi^2}$$

87 Find the minimum value of $a\ (0<a<1)$ for which the following definite integral is minimized.

$$\int_0^{\pi} |\sin x - ax|\,dx$$

88 A function $f(x)$ satisfies $\begin{cases} f(x) = -f''(x) - (4x-2)f'(x) \\ f(0) = a,\ f(1) = b \end{cases}$

Evaluate $\int_0^1 f(x)(x^2-x)\,dx$.

89 For $f(x) = x^4 + |x|$, let $I_1 = \int_0^{\pi} f(\cos x)\,dx,\ I_2 = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$.

Find the value of $\frac{I_1}{I_2}$.

90 Find $\lim_{n\to\infty}\left(\frac{{}_{3n}C_n}{{}_{2n}C_n}\right)^{\frac{1}{n}}$

where ${}_iC_j$ is a binominal coefficient which means $\frac{i\cdot(i-1)\cdots(i-j+1)}{j\cdot(j-1)\cdots 2\cdot 1}$.

91 Prove the following inequality.