

Solutions to the 60th William Lowell Putnam Mathematical Competition

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A-1 Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r, s) = (r + s + |r - s|)/2.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x - 3, 0\} - \max\{5x, 0\} + 3x + 2 \\ &= (-3x - 3 + |3x + 3|)/2 \\ &\quad - (5x + |5x|)/2 + 3x + 2 \\ &= |(3x + 3)/2| - |5x/2| - x + \frac{1}{2}, \end{aligned}$$

so we may set $f(x) = (3x + 3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

A-2 First solution: First factor $p(x) = q(x)r(x)$, where q has all real roots and r has all complex roots. Notice that each root of q has even multiplicity, otherwise p would have a sign change at that root. Thus $q(x)$ has a square root $s(x)$.

Now write $r(x) = \prod_{j=1}^k (x - a_j)(x - \overline{a_j})$ (possible because r has roots in complex conjugate pairs). Write $\prod_{j=1}^k (x - a_j) = t(x) + iu(x)$ with t, u having real coefficients. Then for x real,

$$\begin{aligned} p(x) &= q(x)r(x) \\ &= s(x)^2(t(x) + iu(x))(\overline{t(x) + iu(x)}) \\ &= (s(x)t(x))^2 + (s(x)u(x))^2. \end{aligned}$$

(Alternatively, one can factor $r(x)$ as a product of quadratic polynomials with real coefficients, write each as a sum of squares, then multiply together to get a sum of many squares.)

Second solution: We proceed by induction on the degree of p , with base case where p has degree 0. As in the first solution, we may reduce to a smaller degree in case p has any real roots, so assume it has none. Then $p(x) > 0$ for all real x , and since $p(x) \rightarrow \infty$ for $x \rightarrow \pm\infty$, p has a minimum value c . Now $p(x) - c$ has real roots, so as above, we deduce that $p(x) - c$ is a sum of squares. Now add one more square, namely $(\sqrt{c})^2$, to get $p(x)$ as a sum of squares.

A-3 First solution: Computing the coefficient of x^{n+1} in the identity $(1 - 2x - x^2) \sum_{m=0}^{\infty} a_m x^m = 1$ yields the recurrence $a_{n+1} = 2a_n + a_{n-1}$; the sequence $\{a_n\}$ is then characterized by this recurrence and the initial conditions $a_0 = 1, a_1 = 2$.

Define the sequence $\{b_n\}$ by $b_{2n} = a_{2n-1}^2 +$

a_n^2 , $b_{2n+1} = a_n(a_{n-1} + a_{n+1})$. Then

$$\begin{aligned} 2b_{2n+1} + b_{2n} &= 2a_n a_{n+1} + 2a_{n-1} a_n + a_{n-1}^2 + a_n^2 \\ &= 2a_n a_{n+1} + a_{n-1} a_{n+1} + a_n^2 \\ &= a_{n+1}^2 + a_n^2 = b_{2n+2}, \end{aligned}$$

and similarly $2b_{2n} + b_{2n-1} = b_{2n+1}$, so that $\{b_n\}$ satisfies the same recurrence as $\{a_n\}$. Since further $b_0 = 1, b_1 = 2$ (where we use the recurrence for $\{a_n\}$ to calculate $a_{-1} = 0$), we deduce that $b_n = a_n$ for all n . In particular, $a_n^2 + a_{n+1}^2 = b_{2n+2} = a_{2n+2}$.

Second solution: Note that

$$\begin{aligned} \frac{1}{1 - 2x - x^2} &= \frac{1}{2\sqrt{2}} \left(\frac{\sqrt{2} + 1}{1 - (1 + \sqrt{2})x} + \frac{\sqrt{2} - 1}{1 - (1 - \sqrt{2})x} \right) \end{aligned}$$

and that

$$\frac{1}{1 + (1 \pm \sqrt{2})x} = \sum_{n=0}^{\infty} (1 \pm \sqrt{2})^n x^n,$$

so that

$$a_n = \frac{1}{2\sqrt{2}} \left((\sqrt{2} + 1)^{n+1} - (1 - \sqrt{2})^{n+1} \right).$$

A simple computation (omitted here) now shows that $a_n^2 + a_{n+1}^2 = a_{2n+2}$.

Third solution (by Richard Stanley): Let A be the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. A simple induction argument shows that

$$A^{n+2} = \begin{pmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{pmatrix}.$$

The desired result now follows from comparing the top left corner entries of the equality $A^{n+2} A^{n+2} = A^{2n+4}$.

A-4 Denote the series by S , and let $a_n = 3^n/n$. Note that

$$\begin{aligned} S &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_m(a_m + a_n)} \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_n(a_m + a_n)}, \end{aligned}$$

where the second equality follows by interchanging m and n . Thus

$$\begin{aligned} 2S &= \sum_m \sum_n \left(\frac{1}{a_m(a_m + a_n)} + \frac{1}{a_n(a_m + a_n)} \right) \\ &= \sum_m \sum_n \frac{1}{a_m a_n} \\ &= \left(\sum_{n=1}^{\infty} \frac{n}{3^n} \right)^2. \end{aligned}$$

But

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}$$

since, e.g., it's $f'(1)$, where

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{3^n} = \frac{3}{3-x},$$

and we conclude that $S = 9/32$.

A-5 First solution: (by Reid Barton) Let r_1, \dots, r_{1999} be the roots of P . Draw a disc of radius ϵ around each r_i , where $\epsilon < 1/3998$; this disc covers a subinterval of $[-1/2, 1/2]$ of length at most 2ϵ , and so of the 2000 (or fewer) uncovered intervals in $[-1/2, 1/2]$, one, which we call I , has length at least $\delta = (1 - 3998\epsilon)/2000 > 0$. We will exhibit an explicit lower bound for the integral of $|P(x)|/P(0)$ over this interval, which will yield such a bound for the entire integral.

Note that

$$\frac{|P(x)|}{|P(0)|} = \prod_{i=1}^{1999} \frac{|x - r_i|}{|r_i|}.$$

Also note that by construction, $|x - r_i| \geq \epsilon$ for each $x \in I$. If $|r_i| \leq 1$, then we have $\frac{|x - r_i|}{|r_i|} \geq \epsilon$. If $|r_i| > 1$, then

$$\frac{|x - r_i|}{|r_i|} = |1 - x/r_i| \geq 1 - |x/r_i| \geq 1/2 > \epsilon.$$

We conclude that $\int_I |P(x)/P(0)| dx \geq \delta\epsilon$, independent of P .

Second solution: It will be a bit more convenient to assume $P(0) = 1$ (which we may achieve by rescaling unless $P(0) = 0$, in which case there is nothing to prove) and to prove that there exists $D > 0$ such that $\int_{-1}^1 |P(x)| dx \geq D$, or even such that $\int_0^1 |P(x)| dx \geq D$.

We first reduce to the case where P has all of its roots in $[0, 1]$. If this is not the case, we can factor $P(x)$ as $Q(x)R(x)$, where Q has all roots in the interval and R has none. Then R is either always positive or always negative on $[0, 1]$; assume the former. Let k be the

largest positive real number such that $R(x) - kx \geq 0$ on $[0, 1]$; then

$$\begin{aligned} \int_{-1}^1 |P(x)| dx &= \int_{-1}^1 |Q(x)R(x)| dx \\ &> \int_{-1}^1 |Q(x)(R(x) - kx)| dx, \end{aligned}$$

and $Q(x)(R(x) - kx)$ has more roots in $[0, 1]$ than does P (and has the same value at 0). Repeating this argument shows that $\int_0^1 |P(x)| dx$ is greater than the corresponding integral for some polynomial with all of its roots in $[0, 1]$.

Under this assumption, we have

$$P(x) = c \prod_{i=1}^{1999} (x - r_i)$$

for some $r_i \in (0, 1]$. Since

$$P(0) = -c \prod r_i = 1,$$

we have

$$|c| \geq \prod |r_i^{-1}| \geq 1.$$

Thus it suffices to prove that if $Q(x)$ is a *monic* polynomial of degree 1999 with all of its roots in $[0, 1]$, then $\int_0^1 |Q(x)| dx \geq D$ for some constant $D > 0$. But the integral of $\int_0^1 \prod_{i=1}^{1999} |x - r_i| dx$ is a continuous function for $r_i \in [0, 1]$. The product of all of these intervals is compact, so the integral achieves a minimum value for some r_i . This minimum is the desired D .

Third solution (by Abe Kunin): It suffices to prove the stronger inequality

$$\sup_{x \in [-1, 1]} |P(x)| \leq C \int_{-1}^1 |P(x)| dx$$

holds for some C . But this follows immediately from the following standard fact: any two norms on a finite-dimensional vector space (here the polynomials of degree at most 1999) are equivalent. (The proof of this statement is also a compactness argument: C can be taken to be the maximum of the L1-norm divided by the sup norm over the set of polynomials with L1-norm 1.)

Note: combining the first two approaches gives a constructive solution with a constant that is better than that given by the first solution, but is still far from optimal. I don't know offhand whether it is even known what the optimal constant and/or the polynomials achieving that constant are.

A-6 Rearranging the given equation yields the much more tractable equation

$$\frac{a_n}{a_{n-1}} = 6 \frac{a_{n-1}}{a_{n-2}} - 8 \frac{a_{n-2}}{a_{n-3}}.$$

Let $b_n = a_n/a_{n-1}$; with the initial conditions $b_2 = 2, b_3 = 12$, one easily obtains $b_n = 2^{n-1}(2^{n-2} - 1)$, and so

$$a_n = 2^{n(n-1)/2} \prod_{i=1}^{n-1} (2^i - 1).$$

To see that n divides a_n , factor n as $2^k m$, with m odd. Then note that $k \leq n \leq n(n-1)/2$, and that there exists $i \leq m-1$ such that m divides $2^i - 1$, namely $i = \phi(m)$ (Euler's totient function: the number of integers in $\{1, \dots, m\}$ relatively prime to m).

B-1 The answer is $1/3$. Let G be the point obtained by reflecting C about the line AB . Since $\angle ADC = \frac{\pi-\theta}{2}$, we find that $\angle BDE = \pi - \theta - \angle ADC = \frac{\pi-\theta}{2} = \angle ADC = \pi - \angle BDC = \pi - \angle BDG$, so that E, D, G are collinear. Hence

$$|EF| = \frac{|BE|}{|BC|} = \frac{|BE|}{|BG|} = \frac{\sin(\theta/2)}{\sin(3\theta/2)},$$

where we have used the law of sines in $\triangle BDG$. But by l'Hôpital's Rule,

$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta/2)}{\sin(3\theta/2)} = \lim_{\theta \rightarrow 0} \frac{\cos(\theta/2)}{3 \cos(3\theta/2)} = 1/3.$$

B-2 First solution: Suppose that P does not have n distinct roots; then it has a root of multiplicity at least 2, which we may assume is $x = 0$ without loss of generality. Let x^k be the greatest power of x dividing $P(x)$, so that $P(x) = x^k R(x)$ with $R(0) \neq 0$; a simple computation yields

$$P''(x) = (k^2 - k)x^{k-2}R(x) + 2kx^{k-1}R'(x) + x^k R''(x).$$

Since $R(0) \neq 0$ and $k \geq 2$, we conclude that the greatest power of x dividing $P''(x)$ is x^{k-2} . But $P(x) = Q(x)P''(x)$, and so x^2 divides $Q(x)$. We deduce (since Q is quadratic) that $Q(x)$ is a constant C times x^2 ; in fact, $C = 1/(n(n-1))$ by inspection of the leading-degree terms of $P(x)$ and $P''(x)$.

Now if $P(x) = \sum_{j=0}^n a_j x^j$, then the relation $P(x) = Cx^2 P''(x)$ implies that $a_j = Cj(j-1)a_j$ for all j ; hence $a_j = 0$ for $j \leq n-1$, and we conclude that $P(x) = a_n x^n$, which has all identical roots.

Second solution (by Greg Kuperberg): Let $f(x) = P''(x)/P(x) = 1/Q(x)$. By hypothesis, f has at most two poles (counting multiplicity).

Recall that for any complex polynomial P , the roots of P' lie within the convex hull of P . To show this, it suffices to show that if the roots of P lie on one side of a line, say on the positive side of the imaginary axis, then P' has no roots on the other side. That follows because if r_1, \dots, r_n are the roots of P ,

$$\frac{P'(z)}{P(z)} = \sum_{i=1}^n \frac{1}{z - r_i}$$

and if z has negative real part, so does $1/(z - r_i)$ for $i = 1, \dots, n$, so the sum is nonzero.

The above argument also carries through if z lies on the imaginary axis, provided that z is not equal to a root of P . Thus we also have that no roots of P' lie on the sides of the convex hull of P , unless they are also roots of P .

From this we conclude that if r is a root of P which is a vertex of the convex hull of the roots, and which is not also a root of P' , then f has a single pole at r (as r cannot be a root of P''). On the other hand, if r is a root of P which is also a root of P' , it is a multiple root, and then f has a double pole at r .

If P has roots not all equal, the convex hull of its roots has at least two vertices.

B-3 We first note that

$$\sum_{m,n>0} x^m y^n = \frac{xy}{(1-x)(1-y)}.$$

Subtracting S from this gives two sums, one of which is

$$\sum_{m \geq 2n+1} x^m y^n = \sum_n y^n \frac{x^{2n+1}}{1-x} = \frac{x^3 y}{(1-x)(1-x^2 y)}$$

and the other of which sums to $xy^3/[(1-y)(1-xy^2)]$. Therefore

$$\begin{aligned} S(x, y) &= \frac{xy}{(1-x)(1-y)} - \frac{x^3 y}{(1-x)(1-x^2 y)} - \frac{xy^3}{(1-y)(1-xy^2)} \\ &= \frac{xy(1+x+y+xy-x^2 y^2)}{(1-x^2 y)(1-xy^2)} \end{aligned}$$

and the desired limit is $\lim_{(x,y) \rightarrow (1,1)} xy(1+x+y+xy-x^2 y^2) = 3$.

B-4 (based on work by Daniel Stronger) We make repeated use of the following fact: if f is a differentiable function on all of \mathbb{R} , $\lim_{x \rightarrow -\infty} f(x) \geq 0$, and $f'(x) > 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$. (Proof: if $f(y) < 0$ for some x , then $f(x) < f(y)$ for all $x < y$ since $f' > 0$, but then $\lim_{x \rightarrow -\infty} f(x) \leq f(y) < 0$.)

From the inequality $f'''(x) \leq f(x)$ we obtain

$$f'' f'''(x) \leq f''(x)f(x) < f''(x)f(x) + f'(x)^2$$

since $f'(x)$ is positive. Applying the fact to the difference between the right and left sides, we get

$$\frac{1}{2}(f''(x))^2 < f(x)f'(x). \quad (1)$$

On the other hand, since $f(x)$ and $f'''(x)$ are both positive for all x , we have

$$2f'(x)f''(x) < 2f'(x)f''(x) + 2f(x)f'''(x).$$

Applying the fact to the difference between the sides yields

$$f'(x)^2 \leq 2f(x)f''(x). \quad (2)$$

Combining (1) and (2), we obtain

$$\begin{aligned} \frac{1}{2} \left(\frac{f'(x)^2}{2f(x)} \right)^2 &< \frac{1}{2} (f''(x))^2 \\ &< f(x)f'(x), \end{aligned}$$

or $(f'(x))^3 < f(x)^3$. We conclude $f'(x) < 2f(x)$, as desired.

Note: one can actually prove the result with a smaller constant in place of 2, as follows. Adding $\frac{1}{2}f'(x)f'''(x)$ to both sides of (1) and again invoking the original bound $f'''(x) \leq f(x)$, we get

$$\begin{aligned} \frac{1}{2}[f'(x)f'''(x) + (f''(x))^2] &< f(x)f'(x) + \frac{1}{2}f'(x)f'''(x) \\ &\leq \frac{3}{2}f(x)f'(x). \end{aligned}$$

Applying the fact again, we get

$$\frac{1}{2}f'(x)f''(x) < \frac{3}{4}f(x)^2.$$

Multiplying both sides by $f'(x)$ and applying the fact once more, we get

$$\frac{1}{6}(f'(x))^3 < \frac{1}{4}f(x)^3.$$

From this we deduce $f'(x) < (3/2)^{1/3}f(x) < 2f(x)$, as desired.

I don't know what the best constant is, except that it is not less than 1 (because $f(x) = e^x$ satisfies the given conditions).

B-5 We claim that the eigenvalues of A are 0 with multiplicity $n - 2$, and $n/2$ and $-n/2$, each with multiplicity 1. To prove this claim, define vectors $v^{(m)}$, $0 \leq m \leq n - 1$, componentwise by $(v^{(m)})_k = e^{ikm\theta}$, and note that the $v^{(m)}$ form a basis for \mathbb{C}^n . (If we arrange the $v^{(m)}$ into an $n \times n$ matrix, then the determinant of this matrix is a Vandermonde product which is nonzero.) Now note that

$$\begin{aligned} (Av^{(m)})_j &= \sum_{k=1}^n \cos(j\theta + k\theta) e^{ikm\theta} \\ &= \frac{e^{ij\theta}}{2} \sum_{k=1}^n e^{ik(m+1)\theta} + \frac{e^{-ij\theta}}{2} \sum_{k=1}^n e^{ik(m-1)\theta}. \end{aligned}$$

Since $\sum_{k=1}^n e^{ik\ell\theta} = 0$ for integer ℓ unless $n \mid \ell$, we conclude that $Av^{(m)} = 0$ for $m = 0$ or for $2 \leq m \leq n - 1$. In addition, we find that $(Av^{(1)})_j = \frac{n}{2}e^{-ij\theta} = \frac{n}{2}(v^{(n-1)})_j$ and $(Av^{(n-1)})_j = \frac{n}{2}e^{ij\theta} = \frac{n}{2}(v^{(1)})_j$, so that $A(v^{(1)} \pm v^{(n-1)}) = \pm \frac{n}{2}(v^{(1)} \pm v^{(n-1)})$. Thus $\{v^{(0)}, v^{(2)}, v^{(3)}, \dots, v^{(n-2)}, v^{(1)} + v^{(n-1)}, v^{(1)} - v^{(n-1)}\}$ is a basis for \mathbb{C}^n of eigenvectors of A with the claimed eigenvalues.

Finally, the determinant of $I + A$ is the product of $(1 + \lambda)$ over all eigenvalues λ of A ; in this case, $\det(I + A) = (1 + n/2)(1 - n/2) = 1 - n^2/4$.

B-6 First solution: Choose a sequence p_1, p_2, \dots of primes as follows. Let p_1 be any prime dividing an element of S . To define p_{j+1} given p_1, \dots, p_j , choose an integer $N_j \in S$ relatively prime to $p_1 \cdots p_j$ and let p_{j+1} be a prime divisor of N_j , or stop if no such N_j exists.

Since S is finite, the above algorithm eventually terminates in a finite sequence p_1, \dots, p_k . Let m be the smallest integer such that $p_1 \cdots p_m$ has a divisor in S . (By the assumption on S with $n = p_1 \cdots p_k$, $m = k$ has this property, so m is well-defined.) If $m = 1$, then $p_1 \in S$, and we are done, so assume $m \geq 2$. Any divisor d of $p_1 \cdots p_m$ in S must be a multiple of p_m , or else it would also be a divisor of $p_1 \cdots p_{m-1}$, contradicting the choice of m . But now $\gcd(d, N_{m-1}) = p_m$, as desired.

Second solution (from `sci.math`): Let n be the smallest integer such that $\gcd(s, n) > 1$ for all s in n ; note that n obviously has no repeated prime factors. By the condition on S , there exists $s \in S$ which divides n .

On the other hand, if p is a prime divisor of s , then by the choice of n , n/p is relatively prime to some element t of S . Since n cannot be relatively prime to t , t is divisible by p , but not by any other prime divisor of n (as those primes divide n/p). Thus $\gcd(s, t) = p$, as desired.