Japan Today's Calculation Of Integral 2012

770 Find the value of a such that :

$$101a = 6539\int_{-1}^{1}\frac{x^{12}+31}{1+2011^{x}}\,dx.$$

771 (1) Find the range of a for which there exist two common tangent lines of the curve $y=\frac{8}{27}x^3$ and the parabola $y=(x+a)^2$ other than the x axis.

(2) For the range of a found in the previous question, express the area bounded by the two tangent lines and the parabola $y=(x+a)^2$ in terms of a.

772 Given are three points $A(2,\ 0,\ 2),\ B(1,\ 1,\ 0),\ C(0,\ 0,\ 3)$ in the coordinate space. Find the volume of the solid of a triangle ABC generated by a rotation about z-axis.

773 For $x\geq 0$ find the value of x by which $f(x)=\int_0^x 3^t(3^t-4)(x-t)dt$ is minimized.

774 Find the real number a such that $\int_0^a \frac{e^x+e^{-x}}{2}dx=\frac{12}{5}.$

775 Let a be negative constant. Find the value of a and $f(x)$ such that $\int_{\frac{a}{2}}^{\frac{t}{2}} f(x)dx=t^2+3t-4$ holds for any real numbers t.

776 Evaluate $\int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}}(2x^2-1)e^{2x}dx.$

777 Given two points $P,\ Q$ on the parabola $C : y=x^2-x-2$ in the xy plane.
Note that the x coodinate of P is less than that of Q.

(a) If the origin O is the midpoint of the lines□egment PQ, then find the equation of the line PQ.

(b) If the origin O divides internally the line segment PQ by 2:1, then find the equation of PQ.

(c) If the origin O divides internally the line segment PQ by 2:1, find the area of the figure bounded by the parabola C and the line PQ.

778 In the xyz space with the origin O, Let K_1 be the surface and inner part of the sphere centered on the point $(1,\ 0,\ 0)$ with radius 2 and let K_2 be the surface and inner part of the sphere centered on the point $(-1,\ 0,\ 0)$ with radius 2. For three points $P,\ Q,\ R$ in the space, consider points $X,\ Y$ defined by

$$\overrightarrow{OX}=\overrightarrow{OP}+\overrightarrow{OQ},\ \overrightarrow{OY}=\frac{1}{3}(\overrightarrow{OP}+\overrightarrow{OQ}+\overrightarrow{OR}).$$

(1) When $P,\ Q$ move every cranny in $K_1,\ K_2$ respectively, find the volume of the solid generated by the whole points of the point X.

(2) Find the volume of the solid generated by the whole points of the point R for which for any P belonging to K_1 and any Q belonging to K_2, Y belongs to K_1.

(3) Find the volume of the solid generated by the whole points of the point R for which for any P belonging to K_1 and any Q belonging to K_2, Y belongs to $K_1\cup K_2$.

779 Consider parabolas $C_a : y=-2x^2+4ax-2a^2+a+1$ and $C : y=x^2-2x$ in the coordinate plane.
When C_a and C have two intersection points, find the maximum area enclosed by these parabolas.

780 Let $n\geq 3$ be integer. Given a regular n-polygon P with side length 4 on the plane $z=0$ in the xyz-space.Llet G be a circumcenter of P. When the center of the sphere B with radius 1 travels round along the sides of P, denote by K_n the solid swept by B.

Answer the following questions.

(1) Take two adjacent vertices $P_1,\ P_2$ of P. Let Q be the intersection point between the perpendicular dawn from G to P_1P_2,
prove that $GQ>1$.

(2) (i) Express the area of cross section $S(t)$ in terms of $t,\ n$ when K_n is cut by the plane $z=t\ (-1\leq t\leq 1)$.

(ii) Express the volume $V(n)$ of K_n in terms of n.

(3) Denote by l the line which passes through G and perpendicular to the plane $z=0$. Express the volume $W(n)$ of the solid by generated by a rotation of K_n around l in terms of n.

(4) Find $\lim_{n\to\infty}\frac{V(n)}{W(n)}$.

781 Let $l,\ m$ be the tangent lines passing through the point $A(a,\ a-1)$ on the line $y=x-1$ and touch the parabola $y=x^2$. Note that the slope of l is greater than that of m.

(1) Exress the slope of l in terms of a.

(2) Denote $P,\ Q$ be the points of tangency of the lines $l,\ m$ and the parabola $y=x^2$.
Find the minimum area of the part bounded by the line segment PQ and the parabola $y=x^2$.

(3) Find the minimum distance between the parabola $y=x^2$ and the line $y=x-1$.

782 Let C be the part of the graph $y=\frac{1}{x}\ (x>0)$. Take a point $P\left(t,\ \frac{1}{t}\right)\ (t>0)$ on C.

(i) Find the equation of the tangent l at the point $A(1,\ 1)$ on the curve C.

(ii) Let m be the line passing through the point P and parallel to l. Denote Q be the intersection point of the line m and the curve C other than P. Find the coordinate of Q.

(iii) Express the area S of the part bounded by two line segments $OP,\ OQ$ and the curve C for the origin O in terms of t.

(iv) Express the volume V of the solid generated by a rotation of the part enclosed by two lines passing through the point P and pararell to the y-axis and passing through the point Q and pararell to y-axis, the curve C and the x-axis in terms of t.

(v) $\lim_{t\to 1-0}\frac{S}{V}$.

783 Define a sequence $a_1=0,\ \frac{1}{1-a_{n+1}}-\frac{1}{1-a_n}=2n+1\ (n=1,\ 2,\ 3,\ \cdots)$.

(1) Find a_n.

(2) Let $b_k=\sqrt{\frac{k+1}{k}}\,(1-\sqrt{a_{k+1}})$ for $k=1,\ 2,\ 3,\ \cdots$.

Prove that $\sum_{k=1}^n b_k<\sqrt{2}-1$ for each n.

Last Edited

784 Define for positive integer n, a function $f_n(x)=\frac{\ln x}{x^n}\ (x>0)$. In the coordinate plane, denote by S_n the area of the figure enclosed by $y=f_n(x)\ (x\leq t)$, the x-axis and the line $x=t$ and denote by T_n the area of the rectagle with four vertices $(1,\ 0),\ (t,\ 0),\ (t,\ f_n(t))$ and $(1,\ f_n(t))$.

(1) Find the local maximum $f_n(x)$.

(2) When t moves in the range of $t>1$, find the value of t for which $T_n(t)-S_n(t)$ is maximized.

(3) Find $S_1(t)$ and $S_n(t)\ (n\geq 2)$.

(4) For each $n\geq 2$, prove that there exists the only $t>1$ such that $T_n(t)=S_n(t)$.

Note that you may use $\lim_{x\to\infty}\frac{\ln x}{x}=0$.

785 For a positive real number x, find the minimum value of $f(x)=\int_x^{2x}(t\ln t-t)dt$.

786 For each positive integer n, define $H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$.

(1) Find $H_1(x),\ H_2(x),\ H_3(x)$.

(2) Express $\frac{d}{dx}H_n(x)$ interms of $H_n(x),\ H_{n+1}(x)$. Then prove that $H_n(x)$ is a polynpmial with degree n by induction.

(3) Let a be real number. For $n \geq 3$, express $S_n(a) = \int_0 xH_n(x)e^{-x}\ dx$ in terms of $H_{n-1}(a),\ H_{n-2}(a),\ H_{n-2}(0)$.

(4) Find $\lim_{a\rightarrow\infty} S_6(a)$.
If necessary, you may use $\lim_{x\rightarrow\infty} x^k e^{-x^2} = 0$ for a positive integer k.

787 Take two points $A\ (-1,\ 0),\ B\ (1,\ 0)$ on the xy-plane. Let F be the figure by which the whole points P on the plane satisfies $\frac{\pi}{4} \leq \angle APB \leq \pi$ and the figure formed by $A,\ B$.

Answer the following questions:

(1) Illustrate F.

(2) Find the volume of the solid generated by a rotation of F around the x-axis.

788 For a function $f(x) = \ln(1+\sqrt{1-x^2}) - \sqrt{1-x^2} - \ln x\ (0 < x < 1)$, answer the following questions:

(1) Find $f'(x)$.

(2) Sketch the graph of $y = f(x)$.

(3) Let P be a mobile point on the curve $y = f(x)$ and Q be a point which is on the tangent at P on the curve $y = f(x)$ and such that $PQ = 1$. Note that the x-coordinate of Q is les than that of P. Find the locus of Q.

789 Find the non-constant function $f(x)$ such that $f(x) = x^2 - \int_0^1 (f(t)+x)^2\,dt$.

790 Define a parabola C by $y = x^2 + 1$ on the coordinate plane. Let $s,\ t$ be real numbers with $t < 0$. Denote by $l_1,\ l_2$ the tangent lines drawn from the point $(s,\ t)$ to the parabola C.

(1) Find the equations of the tangents $l_1,\ l_2$.

(2) Let a be positive real number. Find the pairs of $(s,\ t)$ such that the area of the region enclosed by $C,\ l_1,\ l_2$ is a.

791 Let S be the domain in the coordinate plane determined by two inequalities:

$$y \geq \frac{1}{2}x^2,\quad \frac{x^2}{4} + 4y^2 \leq \frac{1}{8}.$$

Denote by V_1 the volume of the solid by a rotation of S about the x-axis and by V_2, by a rotation of S about the y-axis.

(1) Find the values of $V_1,\ V_2$.

(2) Compare the size of the value of $\frac{V_2}{V_1}$ and 1.

792 Answer the following questions:

(1) Let a be positive real number. Find $\lim_{n\rightarrow\infty} (1+a^n)^{\frac{1}{n}}$.

(2) Evaluate $\int_1^{\sqrt{3}} \frac{1}{x^2}\ln\sqrt{1+x^2}\,dx$.

35 points

793 Find the area of the figure bounded by two curves $y = x^4,\ y = x^2 + 2$.

794 Define a function $f(x) = \int_0^{\frac{\pi}{2}} \frac{\cos|t-x|}{1+\sin|t-x|}\,dt$ for $0 \leq x \leq \pi$.

Find the maximum and minimum value of $f(x)$ in $0 \leq x \leq \pi$.

795 Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2+\sin x}{1+\cos x}\ dx$.

796 Answer the following questions:

(1) Let a be non-zero constant. Find $\int x^2\cos(a\ln x)\,dx$

(1) Let a be non-zero constant. Find $\int x^2 \cos(a \ln x)dx.$

(2) Find the volume of the solid generated by a rotation of the figures enclosed by the curve $y = x\cos(\ln x)$, the x-axis and the lines $x = 1,\ x = e^{\frac{\pi}{4}}$ about the x-axis.

797 In the xyz-space take four points $P(0,\ 0,\ 2),\ A(0,\ 2,\ 0),\ B(\sqrt{3}, -1,\ 0),\ C(-\sqrt{3}, -1,\ 0).$
Find the volume of the part satifying $x^2 + y^2 \geq 1$ in the tetrahedron $PABC$.

50 points

798 Denote by $C,\ l$ the graphs of the cubic function $C : y = x^3 - 3x^2 + 2x$, the line $l : y = ax.$

(1) Find the range of a such that C and l have intersection point other than the origin.

(2) Denote $S(a)$ by the area bounded by C and l. If a move in the range found in (1), then find the value of a for which $S(a)$ is minimized.

50 points

799 Let n be positive integer. Define a sequence $\{a_k\}$ by

$$a_1 = \frac{1}{n(n+1)},\ a_{k+1} = -\frac{1}{k+n+1} + \frac{n}{k}\sum_{i=1}^{k} a_i \quad (k = 1,\ 2,\ 3,\ \cdots).$$

(1) Find a_2 and a_3.

(2) Find the general term a_k.

(3) Let $b_n = \sum_{k=1}^{n} \sqrt{a_k}$. Prove that $\lim_{n\to\infty} b_n = \ln 2.$

50 points

800 For a positive constant a, find the minimum value of $f(x) = \int_0^{\frac{\pi}{2}} |\sin t - ax\cos t|dt.$

801 Answer the following questions:

(1) Let $f(x)$ be a function such that $f''(x)$ is continuous and $f'(a) = f'(b) = 0$ for some $a < b$.

Prove that $f(b) - f(a) = \int_a^b \left(\frac{a+b}{2} - x\right) f''(x)dx.$

(2) Consider the running a car on straight road. After a car which is at standstill at a traffic light started at time 0, it stopped again at the next traffic light apart a distance L at time T. During the period, prove that there is an instant□for which the absolute value of the acceleration of the car is more than or equal to $\frac{4L}{T^2}$.

802 Let k and a are positive constants. Denote by V_1 the volume of the solid generated by a rotation of the figure enclosed by the curve $C : y = \frac{x}{x+k}\ (x \geq 0)$, the line $x = a$ and the x-axis around the x-axis, and denote by V_2 that of the solid by a rotation of the figure enclosed by the curve C, the line $y = \frac{a}{a+k}$ and the y-axis around the y-axis.
Find the ratio $\frac{V_2}{V_1}$.

803 Answer the following questions:

(1) Evaluate $\int_{-1}^{1} (1 - x^2)e^{-2x}dx.$

(2) Find $\lim_{n\to\infty} \left\{\frac{(2n)!}{n!n^n}\right\}^{\frac{1}{n}}.$

804 For $a > 0$, find the minimum value of $I(a) = \int_1^e |\ln ax|\ dx.$

805 Prove the following inequalities:

805 Prove the following inequalities:

(1) For $0 \leq x \leq 1$,

$$1-\frac{1}{3}x \leq \frac{1}{\sqrt{1+x^2}} \leq 1.$$

(2) $\frac{\pi}{3}-\frac{1}{6} \leq \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^4}}dx \leq \frac{\pi}{3}.$

806 Let n be positive integers and t be a positive real number.
Evaluate $\int_0^{\frac{2n}{t}\pi} |x\sin\ tx|\ dx.$

807 Define a sequence a_n satisfying :

$$a_1=1,\quad a_{n+1}=\frac{na_n}{2+n(a_n+1)}\ (n=1,\ 2,\ 3,\ \cdots).$$

Find $\lim_{m\to\infty} m \sum_{n=m+1}^{2m} a_n.$

808 For a constant c, a sequence a_n is defined by $a_n=\int_c^1 nx^{n-1}\left(\ln\left(\frac{1}{x}\right)\right)^n dx\ (n=1,\ 2,\ 3,\ \cdots).$

Find $\lim_{n\to\infty} a_n.$

809 For $a>0$, denote by $S(a)$ the area of the part bounded by the parabolas $y=\frac{1}{2}x^2-3a$ and $y=-\frac{1}{2}x^2+2ax-a^3-a^2$.
Find the maximum area of $S(a)$.

810 Given the functions $f(x)=xe^x+2x\int_0^2 |g(t)|dt-1,\ g(x)=x^2-x\int_0^1 f(t)dt$, evaluate $\int_0^2 |g(t)|dt.$

811 Let a be real number. Evaluate $\int_a^{a+\pi} |x|\cos x\ dx.$

812 Let $f(x)=\frac{\cos 2x-(a+2)\cos x+a+1}{\sin x}$. For constant a such that $\lim_{x\to 0}\frac{f(x)}{x}=\frac{1}{2}$, evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{f(x)}dx.$

813 Let a be a real number. Find the minimum value of $\int_0^1 |ax-x^3|dx.$

How many solutions (including University Mathematics)are there for the problem?

Any advice would be appreciated. :)

814 Find the area of the region bounded by $C: y=-x^4+8x^3-18x^2+11$ and the tangent line which touches C at distinct two points.

815 Prove that : $\left|\sum_{i=0}^n\left(1-\pi\sin\frac{i\pi}{4n}\cos\frac{i\pi}{4n}\right)\right|<1.$

816 Find the volume of the solid of a circle $x^2+(y-1)^2=4$ generated by a rotation about the x-axis.

817 Define two functions $f(t)=\frac{1}{2}\left(t+\frac{1}{t}\right),\ g(t)=t^2-2\ln t$. When real number t moves in the range of $t>0$, denote by C the curve by which the point $(f(t),\ g(t))$ draws on the xy-plane.
Let $a>1$, find the area of the part bounded by the line $x=\frac{1}{2}\left(a+\frac{1}{a}\right)$ and the curve C.

818 For a function $f(x)=x^3-x^2+x$, find the limit $\lim_{n\to\infty}\int_n^{2n}\frac{1}{f^{-1}(x)^3+|f^{-1}(x)|}\ dx.$

819 For real numbers $a,\ b$ with $0\leq a\leq \pi,\ a<b$, let $I(a,\ b)=\int_a^b e^{-x}\sin x\ dx.$

Determine the value of a such that $\lim_{b\rightarrow\infty} I(a,\ b)=0.$

820 Let P_k be a point whose x-coordinate is $1+\frac{k}{n}\ (k=1,\ 2,\ \cdots ,\ n)$ on the curve $y=\ln x$. For $A(1,\ 0)$, find the limit $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \overline{AP_k}^2.$

821 Prove that : $\ln \frac{11}{27}<\int_{\frac 14}^{\frac 34} \frac{1}{\ln (1-x)}\ dx<\ln \frac{7}{15}.$

822 For $n=0,\ 1,\ 2,\ \cdots$, let

$$a_n=\int_n^{n+1}\{xe^{-x}-(n+1)e^{-n-1}(x-n)\}\ dx,$$

$$b_n=\int_n^{n+1}\{xe^{-x}-(n+1)e^{-n-1}\}\ dx.$$

Find $\lim_{n\to\infty}\sum_{k=0}^n (a_k-b_k).$

823 Let C be the curve expressed by $x=\sin t,\ y=\sin 2t\ \left(0\leq t\leq \frac{\pi}{2}\right).$

(1) Express y in terms of x.

(2) Find the area of the figure D enclosed by the x-axis and C.

(3) Find the volume of the solid generated by a rotation of D about the y-axis.

824 In the xy-plane, for $a>1$ denote by $S(a)$ the area of the figure bounded by the curve $y=(a-x)\ln x$ and the x-axis.

Find the value of integer n for which $\lim_{a\to\infty}\frac{S(a)}{a^n\ln a}$ is non-zero real number.

825 Answer the following questions.

(1) For $x\geq 0$, show that $x-\frac{x^3}{6}\leq \sin x\leq x.$

(2) For $x\geq 0$, show that $\frac{x^3}{3}-\frac{x^5}{30}\leq \int_0^x t\sin t\ dt\leq \frac{x^3}{3}.$

(3) Find the limit

$$\lim_{x\to 0}\frac{\sin x-x\cos x}{x^3}.$$

826 Let G be a hyper elementary abelian $p-$group and let $f: G\to G$ be a homomorphism. Then prove that $\ker f$ is isomorphic to $\mathrm{coker} f$.

827 Find $\lim_{n\to\infty}\sum_{k=0}^{\infty}\int_{2k\pi}^{(2k+1)\pi} xe^{-x}\sin x\ dx.$

828 Find a function $f(x)$, which is differentiable and $f'(x)$ is continuous, such that $\int_0^x f(t)\cos (x-t)\ dt=xe^{2x}.$

829 Let a be a positive constant. Find the value of $\ln a$ such that

$$\frac{\int_1^e \ln(ax)\ dx}{\int_1^e x\ dx} = \int_1^e \frac{\ln(ax)}{x}\ dx.$$

830 Find $\lim_{n\to\infty} \frac{1}{(\ln n)^2} \sum_{k=3}^{n} \frac{\ln k}{k}$.

831 Let n be a positive integer. Answer the following questions.

(1) Find the maximum value of $f_n(x) = x^n e^{-x}$ for $x \geq 0$.

(2) Show that $\lim_{x\to\infty} f_n(x) = 0$.

(3) Let $I_n = \int_0^x f_n(t)\ dt$. Find $\lim_{x\to\infty} I_n(x)$.

832 Find the limit

$$\lim_{n\to\infty} \frac{1}{n\ln n} \int_{\pi}^{(n+1)\pi} (\sin^2 t)(\ln t)\ dt.$$

833 Let $f(x) = \int_0^x e^t(\cos t + \sin t)\ dt,\ g(x) = \int_0^x e^t(\cos t - \sin t)\ dt.$

For a real number a, find $\sum_{n=1}^{\infty} \frac{e^{2a}}{\{f^{(n)}(a)\}^2 + \{g^{(n)}(a)\}^2}$.

834 Find the maximum and minimum areas of the region enclosed by the curve $y = |x|e^{|x|}$ and the line $y = a\ (0 \leq a \leq e)$ at $[-1,\ 1]$.