

The Forty-Sixth Annual William Lowell Putnam Competition

Saturday, December 7, 1985

A-1 Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

- (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and
- (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

A-2 Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T . Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T , and the other two sides of T , with one side along the side of R . For any polygon X , let $A(X)$ denote the area of X . Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

A-3 Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \dots$ by the condition

$$a_m(0) = d/2^m, \\ a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate $\lim_{n \rightarrow \infty} a_n(n)$.

A-4 Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i ?

A-5 Let $I_m = \int_0^{2\pi} \cos(x) \cos(2x) \cdots \cos(mx) dx$. For which integers m , $1 \leq m \leq 10$ is $I_m \neq 0$?

A-6 If $p(x) = a_0 + a_1x + \cdots + a_mx^m$ is a polynomial with real coefficients a_i , then set

$$\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.$$

Let $F(x) = 3x^2 + 7x + 2$. Find, with proof, a polynomial $g(x)$ with real coefficients such that

- (i) $g(0) = 1$, and
- (ii) $\Gamma(f(x)^n) = \Gamma(g(x)^n)$

for every integer $n \geq 1$.

B-1 Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

B-2 Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1)f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

B-3 Let

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n) .

B-4 Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x and y -axes with diagonal pq . What is the probability that no point of R lies outside of C ?

B-5 Evaluate $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})} dt$. You may assume that $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$.

B-6 Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \text{tr}(M_i) = 0$, where $\text{tr}(A)$ denotes the trace of the matrix A . Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix.