**Japan Today's Calculation Of Integral 2011**

**673** Let $f(x)=\int_0^x \frac{1}{1+t^2}dt.$ For $-1\leq x<1$, find $\cos \left\{2f\left(\sqrt{\frac{1+x}{1-x}}\right)\right\}.$

*2011 Ritsumeikan University entrance exam/Science and Technology*

**674** Evaluate $\int_0^1 \frac{x^2+5}{(x+1)^2(x-2)}dx.$

*2011 Doshisya University entrance exam/Science and Technology*

**675** In the coordinate plane with the origin $O$, consider points $P(t+2,\ 0),\ Q(0,\ -2t^2-2t+4)\ (t\geq 0)$. If the $y$-coordinate of $Q$ is nonnegative, then find the area of the region swept out by the line segment $PQ$.

*2011 Ritsumeikan University entrance exam/Pharmacy*

**676** Let $f(x)=\cos^4 x+3\sin^4 x.$

Evaluate $\int_0^{\frac{\pi}{2}} |f'(x)|dx.$

*2011 Tokyo University of Science entrance exam/Management*

**677** Let $a,\ b$ be positive real numbers with $a<b$. Define the definite integrals $I_1,\ I_2,\ I_3$ by

$I_1=\int_a^b \sin\ (x^2)\ dx,\ I_2=\int_a^b \frac{\cos\ (x^2)}{x^2}\ dx,\ I_3=\int_a^b \frac{\sin\ (x^2)}{x^4}\ dx.$

(1) Find the value of $I_1+\frac{1}{2}I_2$ in terms of $a,\ b.$

(2) Find the value of $I_2-\frac{3}{2}I_3$ in terms of $a,\ b.$

(3) For a positive integer $n$, define $K_n=\int_{\sqrt{2n\pi}}^{\sqrt{2(n+1)\pi}} \sin\ (x^2)\ dx+\frac{3}{4}\int_{\sqrt{2n\pi}}^{\sqrt{2(n+1)\pi}} \frac{\sin\ (x^2)}{x^4}\ dx.$

Find the value of $\lim_{n\to\infty} 2n\pi\sqrt{2n\pi}K_n.$

*2011 Tokyo University of Science entrance exam/Information Sciences, Applied Chemistry, Mechanical Enginerring, Civil Enginerring*

**678** Evaluate

$$\int_0^{\pi} \left(1+\sum_{k=1}^n k\cos kx\right)^2 dx\ \ (n=1,\ 2,\ \cdots).$$

*2011 Doshisya University entrance exam/Life Medical Sciences*

**679** Find $\sum_{k=1}^{3n} \frac{1}{\int_0^1 x(1-x)^k\ dx}.$

*2011 Hosei University entrance exam/Design and Enginerring*

**680** Let $a>0$. Evaluate $\int_0^a x^2\left(1-\frac{x}{a}\right)^a dx.$

*2011 Keio University entrance exam/Science and Technology*

**681** Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{1-2\sin 2x+3\cos^2 x}\ dx.$

*2011 University of Occupational and Environmental Health/Medicine□entrance exam*

**682** On the $x$-$y$ plane, 3 half-lines $y=0,\ (x\geq 0),\ \ y=x\tan\theta\ (x\geq 0),\ \ y=-\sqrt{3}x\ (x\leq 0)$ intersect with the circle with the center the origin $O$, radius $r\geq 1$ at $A,\ B,\ C$ respectively. Note that $\frac{\pi}{6}\leq \theta \leq \frac{\pi}{3}$.

If the area of quadrilateral $OABC$ is one third of the area of the regular hexagon which inscribed in a circle with radius 1, then

evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r^2d\theta$.

*2011 Waseda University of Education entrance exam/Science*

**683** Evaluate $\int_0^{\frac 12} (x+1)\sqrt{1-2x^2}\ dx$.

*2011 Kyoto University entrance exam/Science, Problem 1B*

**684** On the $xy$ plane, find the area of the figure bounded by the graphs of $y=x$ and $y=\left|\frac{3}{4}x^2-3\right|-2$.

*2011 Kyoto University entrance exam/Science, Problem 3*

**685** Suppose that a cubic function with respect to $x$, $f(x)=ax^3+bx^2+cx+d$ satisfies all of 3 conditions:

$$f(1)=1,\ f(-1)=-1,\ \int_{-1}^1 (bx^2+cx+d)\ dx=1$$

.

Find $f(x)$ for which $I=\int_{-1}^{\frac 12} \{f''(x)\}^2\ dx$ is minimized, the find the minimum value.

*2011 Tokyo University entrance exam/Humanities, Problem 1*

**686** Let $L$ be a positive constant. For a point $P(t,\ 0)$ on the positive part of the $x$ axis on the coordinate plane, denote $Q(u(t),\ v(t))$ the point at which the point reach starting from $P$ proceeds by□distance $L$ in counter-clockwise on the perimeter of a circle passing the point $P$ with center $O$.

(1) Find $u(t),\ v(t)$.

(2) For real number $a$ with $0<a<1$, find $f(a)=\int_a^1 \sqrt{\{u'(t)\}^2+\{v'(t)\}^2}\ dt$.

(3) Find $\lim_{a\rightarrow +0} \frac{f(a)}{\ln a}$.

*2011 Tokyo University entrance exam/Science, Problem 3*

**687** (1) Let $x>0,\ y$ be real numbers. For variable $t$, find the difference of Maximum and minimum value of the quadratic function $f(t)=xt^2+yt$ in $0\leq t\leq 1$.

(2) Let $S$ be the domain of the points $(x,\ y)$ in the coordinate plane forming the following condition:

For $x>0$ and all real numbers $t$ with $0\leq t\leq 1$ , there exists real number $z$ for which $0\leq xt^2+yt+z\leq 1$ .

Sketch the outline of $S$.

(3) Let $V$ be the domain of the points $(x,\ y,\ z)$ in the coordinate space forming the following condition:

For $0\leq x\leq 1$ and for all real numbers $t$ with $0\leq t\leq 1$, $0\leq xt^2+yt+z\leq 1$ holds.

Find the volume of $V$.

*2011 Tokyo University entrance exam/Science, Problem 6*

**688** For a real number $x$, let $f(x)=\int_0^{\frac{\pi}{2}} |\cos t-x\sin 2t|\ dt$.

(1) Find the minimum value of $f(x)$.

(2) Evaluate $\int_0^1 f(x)\ dx$.

*2011 Tokyo Institute of Technology entrance exam, Problem 2*

**689** Let $C: y=x^2+ax+b$ be a parabola passing through the point $(1,\ -1)$. Find the minimum volume of the figure enclosed by $C$ and the $x$ axis by a rotation about the $x$ axis.

Proposed by kunny

**690** Find the maximum value of $f(x)=\int_0^1 t\sin(x+\pi t)\ dt.$

**691** Let $a$ be a constant. In the $xy$ palne, the curve $C_1 : y=\frac{\ln x}{x}$ touches $C_2 : y=ax^2$.
Find the volume of the solid generated by a rotation of the part enclosed by $C_1,\ C_2$ and the $x$ axis about the $x$ axis.

*2011 Yokohama National Universty entrance exam/Engineering*

**692** Evaluate $\int_0^{\frac{\pi}{12}} \frac{\tan^2 x-3}{3\tan^2 x-1}dx.$

created by kunny

**693** Evaluate $\int_0^{\pi} \sqrt[4]{1+|\cos x|}\ dx.$

created by kunny

**694** Prove the following inequality:

$$\int_1^e \frac{(\ln x)^{2009}}{x^2}dx>\frac{1}{2010\cdot 2011\cdot 2012}$$

created by kunny

**695** For a positive integer $n$, let

$$S_n=\int_0^1 \frac{1-(-x)^n}{1+x}dx,\quad T_n=\sum_{k=1}^n \frac{(-1)^{k-1}}{k(k+1)}$$

Answer the following questions:

(1) Show the following inequality.

$$\left|S_n-\int_0^1 \frac{1}{1+x}dx\right|\leq \frac{1}{n+1}$$

(2) Express $T_n-2S_n$ in terms of $n$.

(3) Find the limit $\lim_{n\to\infty} T_n.$

**696** Let $P(x),\ Q(x)$ be polynomials such that :

$$\int_0^2 \{P(x)\}^2dx=14,\ \int_0^2 P(x)dx=4,\ \int_0^2 \{Q(x)\}^2dx=26,\ \int_0^2 Q(x)dx=2.$$

Find the maximum and the minimum value of $\int_0^2 P(x)Q(x)dx.$

**697** Find the volume of the solid of the domain expressed by the inequality $x^2-x\leq y\leq x$, generated by a rotation about the line $y=x.$

**698** For a positive integer $n$, let denote $C_n$ the figure formed by the inside and perimeter of the circle with center the origin, radius $n$ on the $x$-$y$ plane.

Denote by $N(n)$ the number of a unit square such that all of unit square, whose $x,\ y$ coordinates of 4 vertices are integers, and the vertices are included in $C_n$.

Prove that $\lim \frac{N(n)}{n^2}=\pi.$

$n\to\infty \quad n^2$

**699** Find the volume of the part bounded by $z=x+y,\ z=x^2+y^2$ in the $xyz$ space.

**700** Evaluate

$$\int_0^{\pi} \frac{x^2\cos^2 x - x\sin x - \cos x - 1}{(1+x\sin x)^2}dx$$

**701** Evaluate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{(1+\cos x)\{1-\tan^2\frac{x}{2}\tan(x+\sin x)\tan(x-\sin x)\}}{\tan(x+\sin x)}\,dx$$

**702** $f(x)$ is a continuous function defined in $x>0$. For all $a,\ b\ (a>0,\ b>0)$, if $\int_a^b f(x)\ dx$ is determined by only $\frac{b}{a}$, then prove that $f(x)=\frac{c}{x}\ (c: constant)$.

**703** Given a line segment $PQ$ with endpoints on the parabola $y=x^2$ such that the area bounded by $PQ$ and the parabola always equal to $\frac{4}{3}$. Find the equation of the locus of the midpoint $M$.

**704** A function $f_n(x)\ (n=0,\ 1,\ 2,\ 3,\ \cdots)$ satisfies the following conditions:

(i) $f_0(x)=e^{2x}+1$.

(ii) $f_n(x)=\int_0^x (n+2t)f_{n-1}(t)dt-\frac{2x^{n+1}}{n+1}\ (n=1,\ 2,\ 3,\ \cdots)$.

Find $\sum_{n=1}^{\infty} f_n'\left(\frac{1}{2}\right)$.

**705** The parametric equations of a curve are given by $x=2(1+\cos t)\cos t,\ y=2(1+\cos t)\sin t\ (0\leq t\leq 2\pi)$.

(1) Find the maximum and minimum values of $x$.

(2) Find the volume of the solid enclosed by the figure of revolution about the $x$-axis.

**706** In the $xyz$ space, consider a right circular cylinder with radius of base 2, altitude 4 such that

$$\begin{cases} x^2+y^2\leq 4 \\ 0\leq z\leq 4 \end{cases}$$

Let $V$ be the solid formed by the points $(x,\ y,\ z)$ in the circular cylinder satisfying

$$\begin{cases} z\leq (x-2)^2 \\ z\leq y^2 \end{cases}$$

Find the volume of the solid $V$.

**707** In the $xyz$ space, consider a right circular cylinder with radius of base 2, altitude 4 such that

$$\begin{cases} x^2+y^2\leq 4 \\ 0\leq z\leq 4 \end{cases}$$

Let $V$ be the solid formed by the points $(x,\ y,\ z)$ in the circular cylinder satisfying

$$\begin{cases} z\leq (x-2)^2 \\ z\leq y^2 \end{cases}$$

Find the volume of the solid $V$.

**708** Find $\lim_{n\to\infty}\int_0^1 x^2|\sin n\pi x|\ dx\ (n=1,\ 2,\cdots)$.

**709** Evaluate $\int_0^1 \frac{x}{1+x}\sqrt{1-x^2}\ dx.$

**710** Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin\theta(\sin\theta\cos\theta+2)}{\cos^4\theta}\ d\theta.$

**711** Evaluate $\int_e^{e^2} \frac{4(\ln x)^2+1}{(\ln x)^{\frac{3}{2}}}\ dx.$

**712** Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left\{\frac{1}{\tan x\ (\ln\sin x)}+\frac{\tan x}{\ln\cos x}\right\}\ dx.$

**713** If a positive sequence $\{a_n\}_{n\geq 1}$ satisfies $\int_0^{a_n} x^n\ dx=2$, then find $\lim_{n\to\infty} a_n$.

**714** Find the area enclosed by the graph of $a^2x^4=b^2x^2-y^2\ (a>0,\ b>0)$.

**715** Find the differentiable function $f(x)$ with $f(0)\neq 0$ satisfying $f(x+y)=f(x)f'(y)+f'(x)f(y)$ for all real numbers $x,\ y$.

**716** Prove that :

$$\int_1^{\sqrt{e}} (\ln x)^n\ dx=(-1)^{n-1}n!+\sqrt{e}\sum_{m=0}^{n}(-1)^{n-m}\frac{n!}{m!}\left(\frac{1}{2}\right)^m$$

**717** Let $a_n$ be the area of the part enclosed by the curve $y=x^n\ (n\geq 1)$, the line $x=\frac{1}{2}$ and the $x$ axis.
Prove that :

$$0\leq \ln 2-\frac{1}{2}-(a_1+a_2+\cdots\cdots+a_n)\leq \frac{1}{2^{n+1}}$$

**718** Find $\sum_{n=1}^{\infty}\frac{1}{2^n}\int_{-1}^{1}(1-x)^2(1+x)^n dx\ (n\geq 1).$

**719** Compute $\int_0^x \sin t\cos t\sin(2\pi\cos t)\ dt.$

**720** Evaluate $\int_0^{2\pi} |x^2-\pi^2-\sin^2 x|\ dx.$

**721** For constant $a$, find the differentiable function $f(x)$ satisfying $\int_0^x (e^{-x}-ae^{-t})f(t)dt=0.$

**722** Find the continuous function $f(x)$ such that :

$$\int_0^x f(t)\left(\int_0^t f(t)dt\right)dt=f(x)+\frac{1}{2}$$

**723** Evaluate $\int_1^e \frac{\{1-(x-1)e^x\}\ln x}{(1+e^x)^2}dx.$

**724** Find $\lim_{n\to\infty}\left\{(1+n)^{\frac{1}{n}}\left(1+\frac{n}{2}\right)^{\frac{2}{n}}\left(1+\frac{n}{3}\right)^{\frac{3}{n}}\cdots\cdots 2\right\}^{\frac{1}{n}}.$

**725** For $a>1$, evaluate $\int_{\frac{1}{a}}^{a}\frac{1}{x}(\ln x)\ln(x^2+1)dx.$

**726** Let $P(x,\ y)\ (x>0,\ y>0)$ be a point on the curve $C : x^2-y^2=1$. If $x=\frac{e^u+e^{-u}}{2}\ (u\geq 0)$, then find the area bounded by the line $OP$, the $x$ axis and the curve $C$ in terms of $u$.

**727** For positive constant $a$, let $C: y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$. Denote by $l(t)$ the length of the part $a\leq y\leq t$ for $C$ and denote by $S(t)$ the area of the part bounded by the line $y=t\ (a<t)$ and $C$. Find $\lim_{t\to\infty} \frac{S(t)}{l(t)\ln t}$.

**728** Evaluate

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sin x-\cos x-x(\sin x+\cos x)+1}{x^2-x(\sin x+\cos x)+\sin x\cos x}\ dx.$$

**729** Evaluate $\int_1^e \frac{\ln x-1}{x^2-(\ln x)^2}dx.$

**730** Let $a_n$ be the local maximum of $f_n(x)=\frac{x^ne^{-x+n\pi}}{n!}\ (n=1,\ 2,\ \cdots)$ for $x>0$.

Find $\lim_{n\to\infty} \ln \left(\frac{a_{2n}}{a_n}\right)^{\frac{1}{n}}$.

**731** Let $C$ be the point of intersection of the tangent lines $l,\ m$ at $A(a,\ a^2),\ B(b,b^2)\ (a<b)$ on the parabola $y=x^2$ respectively.
When $C$ moves on the parabola $y=\frac12x^2-x-2$, find the minimum area bounded by 2 lines $l,\ m$ and the parabola $y=x^2$.

**732** Let $a$ be parameter such that $0<a<2\pi$. For $0<x<2\pi$, find the extremum of $F(x)=\int_x^{x+a}\sqrt{1-\cos\theta}\ d\theta$.

**733** Find $\lim_{n\to\infty} \int_0^1 x^2e^{-\left(\frac{x}{n}\right)^2}dx.$

**734** Find the extremum of $f(t)=\int_1^t \frac{\ln x}{x+t}dx\ (t>0)$.

**735** Evaluate the following definite integrals:

(a) $\int_0^{\frac{\sqrt{\pi}}{2}} x\tan(x^2)\ dx$

(b) $\int_0^{\frac13} xe^{3x}\ dx$

(c) $\int_e^{e^e} \frac{1}{x\ln x}\ dx$

(d) $\int_2^3 \frac{x^2+1}{x(x+1)}\ dx$

**736** Evaluate

$$\int_0^1 \frac{(e^x+1)\{e^x+1+(1+x+e^x)\ln(1+x+e^x)\}}{1+x+e^x}\ dx$$

**737** Let $a,\ b$ real numbers such that $a>1,\ b>1$.
Prove the following inequality.

$$\int_{-1}^1 \left(\frac{1+b^{|x|}}{1+a^x}+\frac{1+a^{|x|}}{1+b^x}\right)\ dx<a+b+2$$

**738** Answer the following questions:

(1) Find the value of $a$ for which $S=\int_{-\pi}^{\pi}(x-a\sin 3x)^2\,dx$ is minimized, then find the minimum value.

(2) Find the vlues of $p,\ q$ for which $T=\int_{-\pi}^{\pi}(\sin 3x-px-qx^2)^2dx$ is minimized, then find the minimum value.

**739** Find the function $f(x)$ such that :

$$f(x)=\cos x+\int_0^{2\pi} f(y)\sin(x-y)\ dy$$

**740** Let $r$ be a positive constant. If 2 curves $C_1: y=\frac{2x^2}{x^2+1},\ C_2: y=\sqrt{r^2-x^2}$ have each tangent line at their point of intersection and at which their tangent lines are perpendicular each other, then find the area of the figure bounded by $C_1,\ C_2$.

**741** Evaluate

$$\int_0^1 \frac{(x-1)^2(\cos x+1)-(2x-1)\sin x}{(x-1+\sqrt{\sin x})^2}\ dx$$

**742** Evaluate

$$\int_0^1 \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}}\ dx$$

**743** Evaluate $\int_0^{\frac{\pi}{2}} \ln(1+\sqrt[3]{\sin\theta})\cos\theta\ d\theta.$

**744** Let $a,\ b$ be real numbers. If $\int_0^3 (ax-b)^2dx\leq 3$ holds, then find the values of $a,\ b$ such that $\int_0^3 (x-3)(ax-b)dx$ is minimized.

**745** When real numbers $a,\ b$ move satisfying $\int_0^{\pi}(a\cos x+b\sin x)^2dx=1$, find the maximum value of $\int_0^{\pi}(e^x-a\cos x-b\sin x)^2dx.$

**746** Prove the following inequality.

$$n^ne^{-n+1}\leq n!\leq \frac{1}{4}(n+1)^{n+1}e^{-n+1}.$$

**747** Prove that $\int_0^4 \left(1-\cos\frac{x}{2}\right)e^{\sqrt{x}}dx\leq -2e^2+30.$

**748** Evaluate the following integrals.

(1) $\int_0^{\pi} \cos mx\cos nx\ dx\ (m,\ n=1,\ 2,\ \cdots).$

(2) $\int_1^3 \left(x-\frac{1}{x}\right)(\ln x)^2\,dx.$

**749** Let $m$ be a positive integer. A tangent line at the point $P$ on the parabola $C_1: y=x^2+m^2$ intersects with the parabola $C_2: y=x^2$ at the points $A,\ B$. For the point $Q$ between $A$ and $B$ on $C_2$, denote by $S$ the sum of the areas of the region bounded by the line $AQ$,$C_2$ and the region bounded by the line $QB$, $C_2$. When $Q$ move between $A$ and $B$ on $C_2$, prove that the minimum value of $S$ doesn't depend on how we would take $P$, then find the value in terms of $m$.

**750** Let $a_n\ (n\geq 1)$ be the value for which $\int_x^{2x} e^{-t^n}dt\ (x\geq 0)$ is maximal. Find $\lim_{n\to\infty}\ln a_n.$

**751** Find $\lim_{n\to\infty}\left(\frac{1}{n}\int_0^n (\sin^2 \pi x)\ln(x+n)dx-\frac{1}{2}\ln n\right)$.

**752** Find $f_n(x)$ such that $f_1(x)=x,\ f_n(x)=\int_0^x tf_{n-1}(x-t)dt\ (n=2,\ 3,\ \cdots)$.

**753** Find $\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{n}{2n^2+3nk+k^2}$.

**754** Let $S_n$ be the area of the figure enclosed by a curve $y=x^2(1-x)^n\ (0\leq x\leq 1)$ and the $x$-axis.
Find $\lim_{n\to\infty}\sum_{k=1}^{n} S_k$.

**755** Given mobile points $P(0,\ \sin\theta),\ Q(8\cos\theta,\ 0)\ \left(0\leq\theta\leq\frac{\pi}{2}\right)$ on the $x$-$y$ plane.
Denote by $D$ the part in which line segment $PQ$ sweeps. Find the volume $V$ generated by a rotation of $D$ around the $x$-axis.

**756** Let $a$ be real number. A circle $C$ touches the line $y=-x$ at the point $(a,-a)$ and passes through the point $(0,\ 1)$.
Denote by $P$ the center of $C$. When $a$ moves, find the area of the figure enclosed by the locus of $P$ and the line $y=1$.

**757** Evaluate

$$\int_0^1 \frac{(x^2+x+1)^3\{\ln(x^2+x+1)+2\}}{(x^2+x+1)^3}(2x+1)e^{x^2+x+1}dx.$$

**758** Find the slope of a line passing through the point $(0,\ 1)$ with which the area of the part bounded by the line and the parabola $y=x^2$ is $\frac{5\sqrt{5}}{6}$.

**759** Given a regular tetrahedron $PQRS$ with side length $d$. Find the volume of the solid generated by a rotation around the line passing through $P$ and the midpoint $M$ of $QR$.

**760** Prove that there exists a positive integer $n$ such that $\int_0^1 x\sin(x^2-x+1)dx\geq\frac{n}{n+1}\sin\frac{n+2}{n+3}$.

**761** Find $\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{(4n)!}{(3n)!}}$.

**762** Define a function $f_n(x)\ (n=0,\ 1,\ 2,\ \cdots)$ by

$$f_0(x)=\sin x,\ f_{n+1}(x)=\int_0^{\frac{\pi}{2}} f_n\prime(t)\sin(x+t)dt.$$

(1) Let $f_n(x)=a_n\sin x+b_n\cos x$. Express $a_{n+1},\ b_{n+1}$ in terms of $a_n,\ b_n$.

(2) Find $\sum_{n=0}^{\infty} f_n\left(\frac{\pi}{4}\right)$.

**763** Evaluate $\int_1^4 \frac{x-2}{(x^2+4)\sqrt{x}}dx$.

**764** Let $f(x)$ be a continuous function defined on $0\leq x\leq\pi$ and satisfies $f(0)=1$ and

$$\left\{\int_0^{\pi}(\sin x+\cos x)f(x)dx\right\}^2=\pi\int_0^{\pi}\{f(x)\}^2dx.$$

$\int^{\pi}$

Evaluate $\int_0 \{f(x)\}^3 dx.$

**765** Define two functions $g(x),\ f(x)\ (x \geq 0)$ by $g(x) = \int_0^x e^{-t^2} dt,\ f(x) = \int_0^1 \frac{e^{-(1+s^2)x}}{1+s^2} ds.$

Now we know that $f'(x) = -\int_0^1 e^{-(1+s^2)x} ds.$

(1) Find $f(0)$.

(2) Show that $f(x) \leq \frac{\pi}{4} e^{-x}\ (x \geq 0)$.

(3) Let $h(x) = \{g(\sqrt{x})\}^2$. Show that $f'(x) = -h'(x)$.

(4) Find $\lim_{x \to +\infty} g(x)$

Please solve the problem without using Double Integral or Jacobian for those Japanese High School Students who don't study them.

**766** Let $f(x)$ be a continuous function defined on $0 \leq x \leq \pi$ and satisfies $f(0) = 1$ and

$$\left\{\int_0^{\pi} (\sin x + \cos x) f(x) dx\right\}^2 = \pi \int_0^{\pi} \{f(x)\}^2 dx.$$

Evaluate $\int_0^{\pi} \{f(x)\}^3 dx.$

**767** For $0 \leq t \leq 1$, define $f(t) = \int_0^{2\pi} |\sin x - t| dx.$

Evaluate $\int_0^1 f(t) dt.$

**768** Let $r$ be a real such that $0 < r \leq 1$. Denote by $V(r)$ the volume of the solid formed by all points of $(x,\ y,\ z)$ satisfying

$$x^2 + y^2 + z^2 \leq 1,\ x^2 + y^2 \leq r^2$$

in $xyz$-space.

(1) Find $V(r)$.

(2) Find $\lim_{r \to 1-0} \frac{V(1) - V(r)}{(1-r)^{\frac{3}{2}}}.$

(3) Find $\lim_{r \to +0} \frac{V(r)}{r^2}.$

**769** In $xyz$ space, find the volume of the solid expressed by $x^2 + y^2 \leq z \leq \sqrt{3}y + 1.$