OrbitalFrequencies subpackage of KerrGeodesics

Define usage for public functions

```
BeginPackage["KerrGeodesics`OrbitalFrequencies`",
    {"KerrGeodesics`ConstantsOfMotion`"}];

KerrGeoFrequencies::usage = "KerrGeoFrequencies[a, p, e, x] returns the orbital fr

Begin["`Private`"];
```

Roots of the radial and polar equations

```
(* Returns the roots of the radial equation, as given by Fujita and Hikida *)
KerrGeoRadialRoots[a_, p_, e_, x_, En1_:Null,Q1_:Null] := Module[{M=1,En=En1,Q=Q1
If[En==Null, En = KerrGeoEnergy[a, p, e, x]];
If[Q==Null,  Q = KerrGeoCarterConstant[a, p, e, x]];

r1=p/(1-e);
r2=p/(1+e);
AplusB=(2M)/(1-En^2)-(r1+r2);(*Eq. (11)*)
AB=(a^2 Q)/((1-En^2)r1 r2);(*Eq. (11)*)
r3=(AplusB+Sqrt[(AplusB)^2-4AB])/2;(*Eq. (11)*)
r4=AB/r3;

{r1,r2,r3,r4}

]
```

This code uses the polar equation (z^2-zm^2)(a^2(1-E0^2)z^2-zp^2)==0 as the Polar equation. Hence zp is a*Sqrt[1-E0^2]*zp in other sources.

```
KerrGeoPolarRoots[a_, p_, e_, x_] := Module[{En,L,Q,zm,zp},
  {En,L,Q} = Values[KerrGeoConstantsOfMotion[a, p, e, x]];
  zm = Sqrt[1-x^2];
  zp = (a^2 (1-En^2)+L^2/(1-zm^2))^(1/2);
  {zp,zm}
]
```

Orbital Frequencies

Orbital frequency calculations from Fujita and Hikida, Class. Quantum Grav .26 (2009) 135002, arXiv:0906.1420

Schwarzschild

```
KerrGeoMinoFrequencies[0|0., p_,0,x_] :=
 <| "\!\(\*SubscriptBox[\(ϒ\), \(r\)]\)" -> Sqrt[((-6+p) p)/(-3+p)],
    "\!\(\*SubscriptBox[\(ϒ\), \(θ\)]\)" -> p Sqrt[1/((-3+p) x^2)] x,
    "\!\(\*SubscriptBox[\(ϒ\), \(ϕ\)]\)" -> (p x)/Sqrt[(-3+p) x^2],
    "Γ" -> Sqrt[p^5/(-3+p)] |>;
```

```
KerrGeoMinoFrequencies[0|0., p_,e_,x_] :=
 <| "\!\(\*SubscriptBox[\(ϒ\), \(r\)]\)" -> (Sqrt[-((p (-6+2 e+p))/(3+e^2-p))] π)
    "\!\(\*SubscriptBox[\(ϒ\), \(θ\)]\)" -> p/Sqrt[-3-e^2+p],
    "\!\(\*SubscriptBox[\(ϒ\), \(ϕ\)]\)" -> (p x)/(Sqrt[-3-e^2+p] Abs[x]),
    "Γ" -> 1/2 Sqrt[(-4 e^2+(-2+p)^2)/(p (-3-e^2+p))] (8+1/((-4+p)^2 EllipticK[(4
```

```
KerrGeoBoyerLindquistFrequencies[0|0., p_,0,x_] :=
 <| "\!\(\*SubscriptBox[\(Ω\), \(r\)]\)" -> Sqrt[-6+p]/p^2,
    "\!\(\*SubscriptBox[\(Ω\), \(θ\)]\)" -> (Sqrt[1/x^2] x)/p^(3/2),
    "\!\(\*SubscriptBox[\(Ω\), \(ϕ\)]\)" -> (p x)/Sqrt[p^5 x^2] |>;
```

```
KerrGeoProperFrequencyFactor[0|0., p_,0,x_]:=p^2
```

```
KerrGeoProperFrequencyFactor[0|0. ,p_,e_,x_]:=(p^2 ((1+e) (28+4 e^2+(-12+p) p)-((
```

Kerr

```
KerrGeoMinoFrequencies[a_,p_,e_,x_]:=Module[{M=1,En,L,Q,r1,r2,r3,r4,ϵ0,zm,a2zp,ϵ0:
{En,L,Q} = Values[KerrGeoConstantsOfMotion[a,p,e,x]];

{r1,r2,r3,r4} = KerrGeoRadialRoots[a,p,e,x,En,Q];
ϵ0=a^2 (1-En^2)/L^2;
zm=1-x^2;
a2zp=(L^2+a^2 (-1+En^2) (-1+zm))/( (-1+En^2) (-1+zm));

ϵ0zp=-((L^2+a^2 (-1+En^2) (-1+zm))/(L^2 (-1+zm)));

(*zmOverZp=If[a==0,0,zm/((L^2+a^2 (-1+En^2) (-1+zm))/(a^2 (-1+En^2) (-1+zm)))];*)
zmOverZp=zm/((L^2+a^2 (-1+En^2) (-1+zm))/(a^2 (-1+En^2) (-1+zm)));



kr=Sqrt[(r1-r2)/(r1-r3) (r3-r4)/(r2-r4)];(*Eq.(13)*)
kθ=Sqrt[zmOverZp];(*Eq.(13)*)
ϒr=(Pi Sqrt[(1-En^2)(r1-r3)(r2-r4)])/(2EllipticK[kr^2]);(*Eq.(15)*)
ϒθ=(Pi L Sqrt[ϵ0zp])/(2EllipticK[kθ^2]);(*Eq.(15)*)

rp=M+Sqrt[M^2-a^2];
rm=M-Sqrt[M^2-a^2];
hr=(r1-r2)/(r1-r3);
hp=((r1-r2)(r3-rp))/((r1-r3)(r2-rp));
hm=((r1-r2)(r3-rm))/((r1-r3)(r2-rm));

(*Eq. (21)*)
ϒϕ=(2ϒθ)/(Pi Sqrt[ϵ0zp]) EllipticPi[zm,kθ^2]+(2a ϒr)/(Pi(rp-rm)Sqrt[(1-En^2)(r1-r:

Γ=4M^2 En + (2a2zp En  ϒθ)/(Pi L Sqrt[ϵ0zp]) (EllipticK[kθ^2]- EllipticE[kθ^2]) +

 <| "\!\(\*SubscriptBox[\(ϒ\), \(r\)]\)" -> ϒr,
    "\!\(\*SubscriptBox[\(ϒ\), \(θ\)]\)" -> Abs[ϒθ],
    "\!\(\*SubscriptBox[\(ϒ\), \(ϕ\)]\)" -> ϒϕ,
    "Γ" -> Γ |>

]
```

```
KerrGeoMinoFrequencies[(1|1.),p_,e_,x_]:=Module[{M=1,a=1,En,L,Q,r1,r2,r3,r4,ϵ0,zm

{En,L,Q} = Values[KerrGeoConstantsOfMotion[a,p,e,x]];

{r1,r2,r3,r4} = KerrGeoRadialRoots[a,p,e,x,En,Q];
ϵ0=a^2 (1-En^2)/L^2;
zm=1-x^2;
a2zp=(L^2+a^2 (-1+En^2) (-1+zm))/( (-1+En^2) (-1+zm));

ϵ0zp=-((L^2+a^2 (-1+En^2) (-1+zm))/(L^2 (-1+zm)));

(*zmOverZp=If[a==0,0,zm/((L^2+a^2 (-1+En^2) (-1+zm))/(a^2 (-1+En^2) (-1+zm)))];*)
zmOverZp=zm/((L^2+a^2 (-1+En^2) (-1+zm))/(a^2 (-1+En^2) (-1+zm)));


kr=Sqrt[(r1-r2)/(r1-r3) (r3-r4)/(r2-r4)];(*Eq.(13)*)
kθ=Sqrt[zmOverZp];(*Eq.(13)*)
ϒr=(Pi Sqrt[(1-En^2)(r1-r3)(r2-r4)])/(2EllipticK[kr^2]);(*Eq.(15)*)
ϒθ=(Pi L Sqrt[ϵ0zp])/(2EllipticK[kθ^2]);(*Eq.(15)*)

hM = ((r1-r2)(r3-M))/((r1-r3)(r2-M));

hr=(r1-r2)/(r1-r3);

(*ϒϕ and Γ from Appendix B for a=M case*)

ϒϕ= (2ϒθ)/(π Sqrt[ϵ0zp]) EllipticPi[zm, kθ^2]+ (2 a ϒr)/(π Sqrt[(1-En^2)(r1-r3)(r

Γ=4M^2 En+(2a^2 En a2zp ϒθ)/(π L Sqrt[ϵ0zp]) (EllipticK[kθ^2]-EllipticE[kθ^2]) +

<| "\!\(\*SubscriptBox[\(ϒ\), \(r\)]\)" -> ϒr,
   "\!\(\*SubscriptBox[\(ϒ\), \(θ\)]\)" -> Abs[ϒθ],
   "\!\(\*SubscriptBox[\(ϒ\), \(ϕ\)]\)" -> ϒϕ,
   "Γ" -> Γ |>
]
```

```
KerrGeoBoyerLindquistFrequencies[a_,p_,e_,x_]:=Module[{ϒr,ϒθ,ϒϕ,Γ},

{ϒr,ϒθ,ϒϕ,Γ} = Values[KerrGeoMinoFrequencies[a,p,e,x]];

  <| "\!\(\*SubscriptBox[\(Ω\), \(r\)]\)" -> ϒr,
     "\!\(\*SubscriptBox[\(Ω\), \(θ\)]\)" -> ϒθ,
     "\!\(\*SubscriptBox[\(Ω\), \(ϕ\)]\)" -> ϒϕ
   |> / Γ
]
```

```
KerrGeoProperFrequencyFactor[a_,p_,e_,x_]:=
Module[{ρ1,ρ2,ρ3,ρ4,zm,zp,T},
    {ρ1,ρ2,ρ3,ρ4}=KerrGeoRadialRoots[a,p,e,x];
    {zp,zm}=KerrGeoPolarRoots[a,p,e,x];
    T=KerrGeoEnergy[a,p,e,x];
    With[{kr= (ρ1-ρ2)/(ρ1-ρ3) (ρ3-ρ4)/(ρ2-ρ4), kθ=a^2 (1-T^2)(zm/zp)^2, hr=(ρ1-ρ2

        1/2 (-((2 zp^2)/(-1+T^2))+ρ1 (-ρ2+ρ3)+ρ3 (ρ2+ρ3))
        +((ρ1-ρ3) (ρ2-ρ4) EllipticE[kr])/(2 EllipticK[kr])
        +(zp^2 EllipticE[kθ])/((-1+T^2) EllipticK[kθ])+((ρ2-ρ3) (ρ1+ρ2+ρ3+ρ4) Ell
    ]
]
```

```
KerrGeoProperFrequencies[a_,p_,e_,x_]:=Module[{MinoFreqs,P},
    MinoFreqs = KerrGeoMinoFrequencies[a,p,e,x];
    P=KerrGeoProperFrequencyFactor[a,p,e,x];
    <|"\!\(\*SubscriptBox[\(ω\), \(r\)]\)"-> MinoFreqs["\!\(\*SubscriptBox[\(ϒ\),
]
```

Generic function for choosing between frequencies w.r.t different time coordinates

```
Options[KerrGeoFrequencies] = {"Time" -> "BoyerLindquist"}
SyntaxInformation[KerrGeoFrequencies] = {"ArgumentsPattern"->{_,_,_,_,OptionsPatt
KerrGeoFrequencies[a_,p_,e_,x_,OptionsPattern[]] := Module[{M=1,En,L,Q,r1,r2,r3,r4


If[OptionValue["Time"]=="Mino",Return[KerrGeoMinoFrequencies[a,p,e,x][[1;;3]]]];

If[OptionValue["Time"]=="BoyerLindquist", Return[KerrGeoBoyerLindquistFrequencies

If[OptionValue["Time"]=="Proper",Return[KerrGeoProperFrequencies[a,p,e,x][[1;;3]]

]
```

Close the package

```
End[];

EndPackage[];
```