ConstantsOfMotion subpackage of KerrGeodesics

Define usage for public functions

```
BeginPackage["KerrGeodesics`ConstantsOfMotion`"];

KerrGeoEnergy::usage = "KerrGeoEnergy[a, p, e, x] returns the orbital energy."
KerrGeoAngularMomentum::usage = "KerrGeoAngularMomentum[a, p, e, x] returns the o
KerrGeoCarterConstant::usage = "KerrGeoCarterConstant[a, p, e, x] returns the Car
KerrGeoConstantsOfMotion::usage = "KerrGeoConstantsOfMotion[a, p, e, x] returns th

Begin["`Private`"];
```

Schwarzschild (a=0)

Circular (e=0)

```
KerrGeoEnergy[0,p_,0,x_]:=(-2+p)/Sqrt[(-3+p) p]
```

```
KerrGeoAngularMomentum[0,p_,0,x_]:=(p x)/Sqrt[-3+p]
```

```
KerrGeoCarterConstant[0,p_,0,x_]:=-((p^2 (-1+x^2))/(-3+p))
```

Eccentric

```
KerrGeoEnergy[0,p_,e_,x_]:=Sqrt[(-4 e^2+(-2+p)^2)/(p (-3-e^2+p))]
```

```
KerrGeoAngularMomentum[0,p_,e_,x_]:=(p x)/Sqrt[-3-e^2+p]
```

```
KerrGeoCarterConstant[0,p_,e_,x_]:=(p^2 (-1+x^2))/(3+e^2-p)
```

Convenience function to compute all three constants of motion

```
KerrGeoConstantsOfMotion[0,p_,e_,x_]:=
 <|"ℰ" -> KerrGeoEnergy[0,p,e,x],
   "ℒ" -> KerrGeoAngularMomentum[0,p,e,x],
   "𝒬" -> KerrGeoCarterConstant[0,p,e,x] |>
```

Kerr

Equatorial orbits (x^2 = 1)

The Carter constant is zero for all equatorial orbits

```
KerrGeoCarterConstant[a_,p_,e_,x_/;x^2==1]:=0
```

Circular (e=0)

```
KerrGeoEnergy[a_,p_,0,x_/;x^2==1]:=((-2+p) Sqrt[p]+a/x)/Sqrt[2 a/x p^(3/2)+(-3+p)
```

```
KerrGeoAngularMomentum[a_,p_,0,x_/;x^2==1]:=(a^2-2 a/x Sqrt[p]+p^2)/(Sqrt[2 a/x+(
```

Eccentric

Simplified from Glampedakis and Kennefick, Phys. Rev. D66 (2002) 044002, arXiv:gr-qc/0203086, Eq. 7 and appendix A

```
KerrGeoEnergy[a_,p_,e_,x_/;x^2==1]:= Sqrt[1-((1-e^2) (1+((-1+e^2) (a^2 (1+3 e^2+p
```

```
KerrGeoAngularMomentum[a_,p_,e_,x_/;x^2==1]:= p x Sqrt[(a^2 (1+3 e^2+p)+p (-3-e^2
```

Convenience function to compute all three constants of motion

```
KerrGeoConstantsOfMotion[a_,p_,e_,x:(1|-1)]:=
 <|"ℰ" -> KerrGeoEnergy[a,p,e,x],
   "ℒ" -> KerrGeoAngularMomentum[a,p,e,x],
   "𝒬" -> KerrGeoCarterConstant[a,p,e,x] |>
```

Polar orbits (x=0)

The angular momentum is zero for all polar orbits

```
KerrGeoAngularMomentum[a_,p_,e_,(0|0.)]:=0
```

Spherical (e=0)

Simplified formula starting from Stoghianidis & Tsoubelis, Gen. Rel, Grav., vol. 19, No. 12, p. 1235 (1987), Eqs. (17)-(19)

```
KerrGeoEnergy[a_,p_,(0|0.),(0|0.)]:=Sqrt[(p (a^2-2 p+p^2)^2)/((a^2+p^2) (a^2+a^2 
```

```
KerrGeoCarterConstant[a_,p_,(0|0.),(0|0.)]:=(p^2 (a^4+2 a^2 (-2+p) p+p^4))/((a^2+
```

Eccentric

These equations were worked out by N. Warburton starting with Schmidt's formula

```
KerrGeoEnergy[a_,p_,e_,(0|0.)]:=Sqrt[-((p (a^4 (-1+e^2)^2+(-4 e^2+(-2+p)^2) p^2+2
```

```
KerrGeoCarterConstant[a_,p_,e_,(0|0.)]:= -((p^2 (a^4 (-1+e^2)^2+p^4+2 a^2 p (-2+p
```

Convenience function to compute all three constants of motion

```
KerrGeoConstantsOfMotion[a_,p_,e_,(0|0.)] :=
 <|"ℰ" -> KerrGeoEnergy[a,p,e,0],
   "ℒ" -> KerrGeoAngularMomentum[a,p,e,0],
   "𝒬" -> KerrGeoCarterConstant[a,p,e,0] |>
```

Spherical orbits (e=0)

```
KerrGeoEnergy[a_,p_,0,x_]:=√(((-3+p) (-2+p)^2 p^5-2 a^5 x (-1+x^2) Sqrt[p^3+a^2 p
```

```
KerrGeoAngularMomentum[a_,p_,0,x_,En1_:Null]:=Block[{En=En1,g,d,h,f},
If[En==Null,En=KerrGeoEnergy[a,p,0,x]];

g=2 a p;
d=(a^2+(-2+p) p) (p^2-a^2 (-1+x^2));
h=((-2+p) p-a^2 (-1+x^2))/x^2;
f=p^4+a^2 (p (2+p)-(a^2+(-2+p) p) (-1+x^2));

(-En g + x Sqrt[(-d h + En^2 (g^2+ f h))/x^2])/h

]
```

CarterConstant and ConstantsOfMotion calculations are covered by the generic case

Generic orbits

```
KerrGeoEnergy[a_,p_,e_,x_]:= Module[{r1,r2,zm,△,f,g,h,d,κ,ρ,ϵ,σ,η,r},

    r1 = p/(1-e);
    r2 = p/(1+e);

    zm = Sqrt[1-x^2];

    △[r_] = r^2 - 2 r + a^2;

    f[r_] = r^4 + a^2 (r (r + 2) + zm^2 △[r]);
    g[r_] = 2 a r;
    h[r_] = r (r - 2) + zm^2/(1 - zm^2) △[r];
    d[r_] = (r^2 + a^2 zm^2) △[r];

    κ = d[r1]×h[r2] - h[r1]×d[r2];
    ϵ = d[r1]×g[r2] - g[r1]×d[r2];
    ρ = f[r1]×h[r2] - h[r1]×f[r2];
    η = f[r1]×g[r2] - g[r1]×f[r2];
    σ = g[r1]×h[r2] - h[r1]×g[r2];

    Sqrt[(κ ρ + 2 ϵ σ - x 2 Sqrt[σ (σ ϵ^2 + ρ ϵ κ - η κ^2)/x^2])/(ρ^2 + 4 η σ)]
]
```

```
KerrGeoAngularMomentum[a_,p_,e_,x_,En1_:Null]:= Module[{En=En1,r1,zm,△,f,g,h,d,r}
    If[En==Null,En=KerrGeoEnergy[a,p,e,x]];

    r1 = p/(1-e);

    zm = Sqrt[1-x^2];

    △[r_] = r^2 - 2 r + a^2;

    f[r_] = r^4 + a^2 (r (r + 2) + zm^2 △[r]);
    g[r_] = 2 a r;
    h[r_] = r (r - 2) + zm^2/(1 - zm^2) △[r];
    d[r_] = (r^2 + a^2 zm^2) △[r];

    (-En g[r1] + x Sqrt[(-d[r1]×h[r1] + En^2 (g[r1]^2+ f[r1]×h[r1]))/x^2])/h[r1]
]
```

```
KerrGeoCarterConstant[a_,p_,e_,x_,En1_:Null,L1_:Null]:= Module[{En=En1,L=L1,zm},
    If[En==Null,En=KerrGeoEnergy[a,p,e,x]];
    If[L==Null,L= KerrGeoAngularMomentum[a,p,e,x,En]];
    zm = Sqrt[1-x^2];

    zm^2 (a^2 (1 - En^2) + L^2/(1 - zm^2))
]
```

```
KerrGeoConstantsOfMotion[a_,p_,e_,x_] :=Module[{En,L},
  En = KerrGeoEnergy[a,p,e,x];
  L = KerrGeoAngularMomentum[a,p,e,x];

 <|"ℰ" -> En, "ℒ" -> L, "𝒬" -> KerrGeoCarterConstant[a,p,e,x,En,L] |>
 ]
```

Close the package

```
End[];

EndPackage[];
```