Full Derivation

We start with our basic equation definitions. * !eq[P(X) = \frac{-\beta + \sqrt{\beta^2 - 4 \gamma}}{2}] * !eq[\beta = -\Big(P(C) + P(A)\Big)] * !eq[\gamma = \frac{P(A)^2 + P(C)^2 - h^2}{2}]

Reformulate

First we want to reformulate in terms of our new !eq[\delta] variable. * !eq[\delta = \frac{P(C) - P(A)}{h}]

We can rewrite this so we have a new definition of !eq[P(C)] * !eq[P(C) = h\delta + P(A)]

This allows us to derive new values for the other equations. * !eq[\beta = -\Big(P(C) + P(A)\Big)] * !eq[\beta = -\Big(h\delta + P(A) + P(A)\Big)] * !eq[\beta = -h\delta - 2 P(A)] * !eq[\gamma = \frac{P(A)^2 + P(C)^2 - h^2}{2}] * !eq[\gamma = \frac{P(A)^2 + \Big(P(A) +h\delta\Big)^2 - h^2}{2}] * !eq[\gamma = P(A)^2 + h\delta P(A) + \frac{h^2\delta^2 - h^2}{2}] * !eq[P(X) = \frac{-\beta + \sqrt{\beta^2 - 4 \gamma}}{2}] * !eq[P(X) = \frac{-(-h\delta - 2 P(A)) + \sqrt{\Big(-h\delta - 2 P(A)\Big)^2 - 4 \Big(P(A)^2 + h\delta P(A) + \frac{h^2\delta^2 - h^2}{2} \Big)}}{2}] * !eq[P(X) = \frac{h\delta + 2 P(A) + \sqrt{\Big(-h\delta - 2 P(A)\Big)^2 - 4 P(A)^2 -4 h\delta P(A) - 2h^2\delta^2 + 2h^2}}{2}] * !eq[P(X) = \frac{h\delta + 2 P(A) + \sqrt{4P(A)^2 + 4h\delta P(A) + h^2\delta^2 - 4 P(A)^2 -4 h\delta P(A) - 2h^2\delta^2 + 2h^2}}{2}] * !eq[P(X) = \frac{h\delta + 2 P(A) + \sqrt{-h^2\delta^2 + 2h^2}}{2}] * !eq[P(X) = P(A) + \frac{h}{2} \Big(\delta + \sqrt{2-\delta^2}\Big)]

We're going to call this a new function !eq[f(\delta)], so now !eq[P(X)] is in terms of !eq[f(\delta)] * !eq[f(\delta) = P(A) + \frac{h}{2} \Big(\delta + \sqrt{2-\delta^2}\Big)]

Taylor Series Expansion

For computational efficiency, we are going to compute the Taylor series expansion. So first, we'll need the derivatives of !eq[f(\delta)]. * !eq[f'(\delta) = \frac{h}{2} (1 - \frac{\delta}{\sqrt{2-\delta^2}})] * !eq[f''(\delta) = \frac{-h}{(2-\delta^2)^\frac{3}{2}}]

0th Order Taylor

- !eq[f_0(\delta, a) = f(a)]
- !eq[f_0(\delta, a) = P(A) + \frac{h}{2} \Big(a + \sqrt{2-a^2}\Big)]

1st Order Taylor

- !eq[f_1(\delta, a) = f(a) + f'(a) (\delta - a)]
- !eq[f_1(\delta, a) = P(A) + \frac{h}{2} \Big(a + \sqrt{2-a^2}\Big) + \frac{h}{2} (1 - \frac{a}{\sqrt{2-a^2}}) (\delta - a)]
- !eq[f_1(\delta, a) = P(A) + \frac{h}{2} \Big(a + \sqrt{2-a^2} + (1 - \frac{a}{\sqrt{2-a^2}}) (\delta - a)\Big)]
- !eq[f_1(\delta, a) = P(A) + \frac{h}{2} \Big(a + \sqrt{2-a^2} + \delta - a - \frac{a\delta}{\sqrt{2-a^2}} + \frac{a^2}{\sqrt{2-a^2}}\Big)]
- !eq[f_1(\delta, a) = P(A) + \frac{h}{2} \Big(\delta - \frac{a\delta}{\sqrt{2-a^2}} + \sqrt{2-a^2} + \frac{a^2}{\sqrt{2-a^2}}\Big)]
- !eq[f_1(\delta, a) = P(A) + \frac{h}{2} \Big(\delta - \frac{a\delta}{\sqrt{2-a^2}} + \frac{2}{\sqrt{2-a^2}}\Big)]

2nd Order Taylor

- !eq[f_2(\delta, a) = f(a) + f'(a) (\delta - a) + \frac{f''(a)}{2!}(\delta - a)^2]

- !eq[f_2(\delta, a) = f_1(\delta, a) + \frac{1}{2} \frac{-h}{(2-a^2)^\frac{3}{2}} (\delta - a)^2]
- !eq[f_2(\delta, a) = f_1(\delta, a) + \frac{h}{2} \frac{-1}{(2-a^2)^\frac{3}{2}} (\delta^2 - 2\delta a + a^2)]
- !eq[f_2(\delta, a) = P(A) + \frac{h}{2} \Big(\delta - \frac{a\delta}{\sqrt{2-a^2}} + \frac{2}{\sqrt{2-a^2}}\Big) + \frac{h}{2} \frac{-1}{(2-a^2)^\frac{3}{2}} (\delta^2 - 2\delta a + a^2)]
- !eq[f_2(\delta, a) = P(A) + \frac{h}{2} \Big(\delta - \frac{a\delta}{\sqrt{2-a^2}} + \frac{2}{\sqrt{2-a^2}} + \frac{-1}{(2-a^2)^\frac{3}{2}} (\delta^2 - 2\delta a + a^2)\Big)]
- !eq[f_2(\delta, a) = P(A) + h\Big(c_0(a) + c_1(a)\delta + c_2(a) \delta^2\Big)]
- !eq[c_0(a) = \frac{1}{2}\Big(\frac{2}{\sqrt{2-a^2}} - \frac{a^2}{(2-a^2)^\frac{3}{2}}\Big)]
- !eq[c_0(a) = \frac{4 - 3a^2}{2(2-a^2)^\frac{3}{2}}]
- !eq[c_1(a) = \frac{1}{2}\Big(1 - \frac{a}{\sqrt{2-a^2}} + \frac{2a}{(2-a^2)^\frac{3}{2}} \Big)]
- !eq[c_2(a) = \frac{-1}{2(2-a^2)^\frac{3}{2}}]

Exact coefficients

Now that we have the general equations for the Taylor series, we can evaluate it at different values of a in the range `[0, 1]`.

!eq[a]	!eq[c_0(a)]	!eq[c_1(a)]	!eq[c_2(a)]
0.0	0.7071	0.5000	-0.1768
0.5	0.7019	0.5270	-0.2160
1.0	0.5000	1.0000	-0.5000

Historically, the values used by `navfn` are

!eq[c_0]	!eq[c_1]	!eq[c_2]
0.7040	0.5307	-0.2301

You can see these values plotted here.

The historical values are pretty close to the values for !eq[\delta=0.5], although the exact reason for the difference is unknown, but its close enough to not be overly concerning.