Full Derivation

We start with our basic equation definitions. * $P(X) = \frac{-\beta + \sqrt{\beta^2 - 4\gamma}}{2}$ * $\beta = -\Big(P(C) + P(A)\Big)$ * $\gamma = \frac{P(A)^2 + P(C)^2 - h^2}{2}$

Reformulate

First we want to reformulate in terms of our new δ variable. * $\delta = \frac{P(C) - P(A)}{h}$

We can rewrite this so we have a new definition of $P(C)$ * $P(C) = h\delta + P(A)$

This allows us to derive new values for the other equations. * $\beta = -\Big(P(C) + P(A)\Big)$ * $\beta = -\Big(h\delta + P(A) + P(A)\Big)$ * $\beta = -h\delta - 2P(A)$ * $\gamma = \frac{P(A)^2 + P(C)^2 - h^2}{2}$ * $\gamma = \frac{P(A)^2 + \Big(P(A) + h\delta\Big)^2 - h^2}{2}$ * $\gamma = P(A)^2 + h\delta P(A) + \frac{h^2\delta^2 - h^2}{2}$ * $P(X) = \frac{-\beta + \sqrt{\beta^2 - 4\gamma}}{2}$ *

$$P(X) = \frac{-(-h\delta - 2P(A)) + \sqrt{\Big(-h\delta - 2P(A)\Big)^2 - 4\Big(P(A)^2 + h\delta P(A) + \frac{h^2\delta^2 - h^2}{2}\Big)}}{2}$$

*

$$P(X) = \frac{h\delta + 2P(A) + \sqrt{\Big(-h\delta - 2P(A)\Big)^2 - 4P(A)^2 - 4h\delta P(A) - 2h^2\delta^2 + 2h^2}}{2}$$

*

$$P(X) = \frac{h\delta + 2P(A) + \sqrt{4P(A)^2 + 4h\delta P(A) + h^2\delta^2 - 4P(A)^2 - 4h\delta P(A) - 2h^2\delta^2 + 2h^2}}{2}$$

* $P(X) = \frac{h\delta + 2P(A) + \sqrt{-h^2\delta^2 + 2h^2}}{2}$ * $P(X) = P(A) + \frac{h}{2}\Big(\delta + \sqrt{2 - \delta^2}\Big)$

We're going to call this a new function $f(\delta)$, so now $P(X)$ is in terms of $f(\delta)$ * $f(\delta) = P(A) + \frac{h}{2}\Big(\delta + \sqrt{2 - \delta^2}\Big)$

Taylor Series Expansion

For computational efficiency, we are going to compute the Taylor series expansion. So first, we'll need the derivatives of $f(\delta)$. * $f'(\delta) = \frac{h}{2}(1 - \frac{\delta}{\sqrt{2-\delta^2}})$ * $f''(\delta) = \frac{-h}{(2-\delta^2)^{\frac{3}{2}}}$

0th Order Taylor

- $f_0(\delta, a) = f(a)$
- $f_0(\delta, a) = P(A) + \frac{h}{2}\Big(a + \sqrt{2-a^2}\Big)$

1st Order Taylor

- $f_1(\delta, a) = f(a) + f'(a)(\delta - a)$
- $f_1(\delta, a) = P(A) + \frac{h}{2}\Big(a + \sqrt{2-a^2}\Big) + \frac{h}{2}(1 - \frac{a}{\sqrt{2-a^2}})(\delta - a)$
- $f_1(\delta, a) = P(A) + \frac{h}{2}\Big(a + \sqrt{2-a^2} + (1 - \frac{a}{\sqrt{2-a^2}})(\delta - a)\Big)$
- $f_1(\delta, a) = P(A) + \frac{h}{2}\Big(a + \sqrt{2-a^2} + \delta - a - \frac{a\delta}{\sqrt{2-a^2}} + \frac{a^2}{\sqrt{2-a^2}}\Big)$
- $f_1(\delta, a) = P(A) + \frac{h}{2}\Big(\delta - \frac{a\delta}{\sqrt{2-a^2}} + \sqrt{2-a^2} + \frac{a^2}{\sqrt{2-a^2}}\Big)$
- $f_1(\delta, a) = P(A) + \frac{h}{2}\Big(\delta - \frac{a\delta}{\sqrt{2-a^2}} + \frac{2}{\sqrt{2-a^2}}\Big)$

2nd Order Taylor

- $f_2(\delta, a) = f(a) + f'(a)(\delta - a) + \frac{f''(a)}{2!}(\delta - a)^2$
- $f_2(\delta, a) = f_1(\delta, a) + \frac{1}{2}\frac{-h}{(2-a^2)^{\frac{3}{2}}}(\delta - a)^2$
- $f_2(\delta, a) = f_1(\delta, a) + \frac{h}{2}\frac{-1}{(2-a^2)^{\frac{3}{2}}}(\delta^2 - 2\delta a + a^2)$
- $f_2(\delta, a) = P(A) + \frac{h}{2}\Big(\delta - \frac{a\delta}{\sqrt{2-a^2}} + \frac{2}{\sqrt{2-a^2}}\Big) + \frac{h}{2}\frac{-1}{(2-a^2)^{\frac{3}{2}}}(\delta^2 - 2\delta a + a^2)$
- $f_2(\delta, a) = P(A) + \frac{h}{2}\Big(\delta - \frac{a\delta}{\sqrt{2-a^2}} + \frac{2}{\sqrt{2-a^2}} + \frac{-1}{(2-a^2)^{\frac{3}{2}}}(\delta^2 - 2\delta a + a^2)\Big)$
- $f_2(\delta, a) = P(A) + h\Big(c_0(a) + c_1(a)\delta + c_2(a)\delta^2\Big)$
- $c_0(a) = \frac{1}{2}\Big(\frac{2}{\sqrt{2-a^2}} - \frac{a^2}{(2-a^2)^{\frac{3}{2}}}\Big)$
- $c_0(a) = \frac{4 - 3a^2}{2(2-a^2)^{\frac{3}{2}}}$
- $c_1(a) = \frac{1}{2}\Big(1 - \frac{a}{\sqrt{2-a^2}} + \frac{2a}{(2-a^2)^{\frac{3}{2}}}\Big)$

- $c_2(a) = \frac{-1}{2(2-a^2)^{\frac{3}{2}}}$

Exact coefficients

Now that we have the general equations for the Taylor series, we can evaluate it at different values of a in the range `[0, 1]`.

a	$c_0(a)$	$c_1(a)$	$c_2(a)$
0.0	0.7071	0.5000	-0.1768
0.5	0.7019	0.5270	-0.2160
1.0	0.5000	1.0000	-0.5000

Historically, the values used by `navfn` are

c_0	c_1	c_2
0.7040	0.5307	-0.2301

You can see these values plotted here.

The historical values are pretty close to the values for $\delta = 0.5$, although the exact reason for the difference is unknown, but its close enough to not be overly concerning.