Inverses

Reading: Strang 2.5

Learning objective: Be fluent with the conditions equivalent to invertibility. Be able to use these to determine if a matrix is invertible, and be able to compute the inverse in this case.

Definition: Invertible

Definition: A square matrix A is invertible if and only if there is a matrix B such that

$$AB = I \qquad \text{and} \qquad BA = I$$

Here I is the identity matrix of the same size as A.

A square matrix that is not invertible is called singular.

Punchline

A big question we are interested in is

When is a square matrix A invertible?

The main punchline is:

An n-by-n matrix A is invertible if and only if Gaussian elimination produces n pivots.

We will (slowly) prove this over today.

Along the way we will find other conditions equivalent to invertibility (there are many).

Left and right inverse

As we go through the proof, it will be convenient to talk about left inverses and right inverses.

A left inverse for A is a matrix B satisfying

$$BA = I\,.$$

A right inverse for A is a matrix C satisfying

$$AC = I\,.$$

Left = Right

If A has a left inverse and a right inverse they must be the same!

Proof: Suppose B is a left inverse for A and C is a right inverse.

So $BA = I$ and $AC = I$.

Then by associativity:

$$C = (BA)C = B(AC) = B$$

Thus $C = B$.

Uniqueness

Recall our definition of invertibility.

Definition: A square matrix A is invertible if and only if there is a matrix B such that

$$AB = I \qquad \text{and} \qquad BA = I$$

if and only if

In our new language this says: A is invertible iff there is a matrix B that is both a left inverse and a right inverse.

Uniqueness

In our new language this says: A is invertible iff there is a matrix B that is both a left inverse and a right inverse.

As left and right inverses must be equal this means:

There can be at most one matrix B witnessing the invertibility of A, that is, satisfying

$$AB = BA = I.$$

We call such a B the inverse of A, denoted $B = A^{-1}$.

Inverse

The notation A^{-1} is meant to recall the case of real numbers (scalars).

if a is invertible, i.e. $a \neq 0$, then $ax = b \implies x = a^{-1}b$.

if A is invertible then $A\vec{x} = \vec{b} \implies \vec{x} = A^{-1}\vec{b}$

This notation makes sense as, if the inverse exists, it is unique.

Invertibility and linear equations

If A is invertible, then for any vector $\vec{b}$ the equation

$$A\vec{x} = \vec{b}$$

has a unique solution, namely $\vec{x} = A^{-1}\vec{b}$.

Contrapositive: If $A\vec{x} = \vec{b}$ does not have a unique solution, then A is not invertible.

Test for singularity

One very useful case of this contrapositive is the following.

Let A be a square matrix. If there is a nonzero vector $\vec{u}$ such that $A\vec{u} = \vec{0}$ then A is singular.

Reason: Then there is not a unique solution to

$$A\vec{x} = \vec{0}$$

as $\vec{u}$ and $\vec{0}$ are two distinct solutions.

Example

Back in Lecture 2b, we said this matrix had no inverse

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Now we can see why:

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Cancellation Law

A similar principle applies in matrix equations.

If A is invertible, then from $AB = AC$ we can conclude

$$A^{-1}AB = A^{-1}AC \implies B = C$$

This is not true in general!

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$$

An n-by-n matrix A is invertible if and only if Gaussian elimination produces n pivots.

Plan:

1) The "easy" direction: if A is invertible then Gaussian elimination produces n pivots.

2) If elimination produces n pivots, then A has a right inverse.

3) If elimination produces n pivots, then A has a left inverse.

(2) and (3) finish the proof as left and right inverses must be equal.

Invertible implies full set of pivots

Proof: if A is invertible then $A\vec{x} = \vec{0}$ has a unique solution.

Do Gaussian elimination on the augmented matrix:

$$\left[\begin{array}{c|c} A & \vec{0} \end{array}\right] \xrightarrow[\text{elimination}]{\text{Gaussian}} \left[\begin{array}{c|c} U & \vec{0} \end{array}\right]$$

where U is upper triangular.

As $U\vec{x} = \vec{0}$ has a unique solution, U cannot have a free column. Every column has a pivot.

Existence of right inverse

Let's think about what having a right inverse means.

A has a right inverse if there is a matrix X such that

$$AX = I$$

Example: Does there exist an X to make this true for the following matrix?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's think about this using the column picture of matrix multiplication.

This matrix equation is equivalent to 3 systems of linear equations.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For each of these, we know how to determine if there is a solution.

Let's see if the first column of the inverse exists.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 1 & 2 & 4 & \mathbf{0} \\ 1 & 3 & 9 & \mathbf{0} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 0 & 1 & 3 & -\mathbf{1} \\ 0 & 2 & 8 & -\mathbf{1} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 0 & 1 & 3 & -\mathbf{1} \\ 0 & 0 & 2 & \mathbf{1} \end{bmatrix}$$

$$R_2' = R_2 - R_1$$
$$R_3' = R_3 - R_1$$

$$R_3'' = R_3' - 2R_2'$$

Back substitution:

$$x_{31} = \frac{1}{2} \qquad x_{21} = -\frac{5}{2} \qquad x_{11} = 3$$

First column

We have found the first column of the right inverse!

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ -\frac{5}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

What about the second column, does it exist?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

What about the second column, does it exist?

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{A} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{b}}$$

We already know what Gaussian elimination does to the coefficient matrix A.

We just need to see what happens to the new right hand side vector $\vec{b}$.

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow[\substack{R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1}]{} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow[R_3'' = R_3' - 2R_2']{} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

Second column

The endpoint of Gaussian elimination in solving for the second column will be

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{0} \\ 0 & 1 & 3 & \mathbf{1} \\ 0 & 0 & 2 & -\mathbf{2} \end{bmatrix}$$

Back substitution:

$$x_{32} = -1 \qquad x_{22} = 4 \qquad x_{12} = -3$$

Second column

We have found the second column of the right inverse.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Does the third column of the right inverse exist?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Third column

Does the third column of the right inverse exist?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Again we know how the coefficient matrix transforms under Gaussian elimination.

What happens to this new right hand side?

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow[\substack{R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1}]{} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow[R_3'' = R_3' - 2R_2']{} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Third column

The endpoint of Gaussian elimination in solving for the third column will be

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{0} \\ 0 & 1 & 3 & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{1} \end{bmatrix}$$

Back substitution:

$$x_{33} = \frac{1}{2} \qquad x_{23} = -\frac{3}{2} \qquad x_{13} = 1$$

The Right Inverse

We have now solved for each column of the right inverse. Putting these together:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Existence of right inverse

We have seen that the existence of a right inverse is equivalent to solving n systems of linear equations

$$A\vec{x}_1 = \vec{e}_1, A\vec{x}_2 = \vec{e}_2, \ldots, A\vec{x}_n = \vec{e}_n$$

where $\vec{e}_i$ is the i^{th} column of the identity matrix.

Why did all of these equations have a solution in our example?

Existence of right inverse

$$A\vec{x}_1 = \vec{e}_1, A\vec{x}_2 = \vec{e}_2, \ldots, A\vec{x}_n = \vec{e}_n$$

where $\vec{e}_i$ is the i^{th} column of the identity matrix.

Why did all of these equations have a solution in our example?

Because after Gaussian elimination, the matrix A had n pivots.

Full Set of Pivots

If A is an n-by-n matrix and

$$A\vec{x} = \vec{b} \xrightarrow[\text{elimination}]{\text{Gaussian}} U\vec{x} = \vec{c}$$

Gaussian elimination produces a matrix U with n pivots, then $A\vec{x} = \vec{b}$ always has a solution, for any $\vec{b}$.

The only reason $U\vec{x} = \vec{c}$ will not have a solution is if U has an all zero row, and the corresponding component of $\vec{c}$ is nonzero.

With n pivots, all rows of U are nonzero.

Existence of right inverse

We have now shown the following:

Theorem: If after Gaussian elimination the n-by-n matrix A has n pivots, then A has a right inverse.

Reason: In this case all of the equations

$$A\vec{x}_1 = \vec{e}_1, A\vec{x}_2 = \vec{e}_2, \ldots, A\vec{x}_n = \vec{e}_n$$

where $\vec{e}_i$ is the i^{th} column of the identity matrix, will have a solution.

These solutions form the columns of the right inverse.

Computing the right inverse

Now we understand when the right inverse exists and that we can compute it by solving n systems of linear equations.

Now we will see a way to organize the computation of the inverse.

We solve all n systems of linear equations simultaneously using the super augmented matrix.

Super Augmented Matrix

The super augmented matrix lets us handle multiple right hand sides at the same time.

Going back to our inverse example,

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 2 & 4 & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 1 & 3 & 9 & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Now we put all three right hand sides we want to solve for into the augmented matrix.

We can do Gaussian elimination on everything at once!

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 2 & 4 & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 1 & 3 & 9 & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 3 & -\mathbf{1} & \mathbf{1} & \mathbf{0} \\ 0 & 2 & 8 & -\mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 3 & -\mathbf{1} & \mathbf{1} & \mathbf{0} \\ 0 & 2 & 8 & -\mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \xrightarrow[R_3'' = R_3' - 2R_2']{} \begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 3 & -\mathbf{1} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{1} & -\mathbf{2} & \mathbf{1} \end{bmatrix}$$

This is the normal endpoint of Gaussian elimination.

From here we can do back substitution to solve for each right hand side separately.

We can also keep going, with Gauss-Jordan elimination.

Gauss-Jordan Elimination

This is the normal endpoint of Gaussian elimination.

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 3 & -\mathbf{1} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{1} & -\mathbf{2} & \mathbf{1} \end{bmatrix}$$

Gauss-Jordan elimination continues with two more steps.

1) Make all pivots equal to one.

2) Eliminate the nonzero entries above each pivot, starting from the bottom.

Making pivots one

To make the pivots equal to one, we introduce a new elementary row operation:

$$\boxed{\text{Multiply a row by a nonzero constant.}}$$

As with the other elementary row operations, this does not change the solution set:

$$x_1 - 2x_2 + 3x_3 = 3$$

and

$$3 \cdot (x_1 - 2x_2 + 3x_3) = 3 \cdot 3$$

have the same solution set.

Elementary matrix

Multiply a row by a nonzero constant.

There is an elementary matrix that implements this row operation by left multiplication.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3' = \frac{1}{2} R_3]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

We find the elementary matrix in the same way: apply the row operation to the identity matrix.

Undo

Multiply a row by a nonzero constant.

How do we undo this row operation?

We undo multiplying the third row by c by multiplying the third row by $1/c$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary matrices corresponding to multiplying a row by a nonzero constant are invertible.

Gauss-Jordan Example

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 3 & \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{1} & \mathbf{-2} & \mathbf{1} \end{bmatrix}$$

In our example, the third pivot is 2.

We multiply the third row by 1/2.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 3 & \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{1} & \mathbf{-2} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 3 & \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{\frac{1}{2}} & \mathbf{-1} & \mathbf{\frac{1}{2}} \end{bmatrix}$$

elementary matrix that multiplies third row by 1/2.

Gauss-Jordan Example

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 3 & -\mathbf{1} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{\frac{1}{2}} & -\mathbf{1} & \mathbf{\frac{1}{2}} \end{array}\right]$$

Now starting from the bottom, we add a multiple of one row to a row above it to create zeros above the pivots.

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 3 & -\mathbf{1} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{\frac{1}{2}} & -\mathbf{1} & \mathbf{\frac{1}{2}} \end{array}\right] \longrightarrow \left[\begin{array}{cccccc} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & -\mathbf{\frac{5}{2}} & \mathbf{4} & -\mathbf{\frac{3}{2}} \\ 0 & 0 & 1 & \mathbf{\frac{1}{2}} & -\mathbf{1} & \mathbf{\frac{1}{2}} \end{array}\right]$$

$$R_2' = R_2 - 3R_3$$

Gauss-Jordan Example

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & -\frac{\mathbf{5}}{\mathbf{2}} & \mathbf{4} & -\frac{\mathbf{3}}{\mathbf{2}} \\ 0 & 0 & 1 & \frac{\mathbf{1}}{\mathbf{2}} & -\mathbf{1} & \frac{\mathbf{1}}{\mathbf{2}} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{\mathbf{1}}{\mathbf{2}} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{2}} \\ 0 & 1 & 0 & -\frac{\mathbf{5}}{\mathbf{2}} & \mathbf{4} & -\frac{\mathbf{3}}{\mathbf{2}} \\ 0 & 0 & 1 & \frac{\mathbf{1}}{\mathbf{2}} & -\mathbf{1} & \frac{\mathbf{1}}{\mathbf{2}} \end{array}\right]$$

$$R_1' = R_1 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{\mathbf{1}}{\mathbf{2}} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{2}} \\ 0 & 1 & 0 & -\frac{\mathbf{5}}{\mathbf{2}} & \mathbf{4} & -\frac{\mathbf{3}}{\mathbf{2}} \\ 0 & 0 & 1 & \frac{\mathbf{1}}{\mathbf{2}} & -\mathbf{1} & \frac{\mathbf{1}}{\mathbf{2}} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \mathbf{3} & -\mathbf{3} & \mathbf{1} \\ 0 & 1 & 0 & -\frac{\mathbf{5}}{\mathbf{2}} & \mathbf{4} & -\frac{\mathbf{3}}{\mathbf{2}} \\ 0 & 0 & 1 & \frac{\mathbf{1}}{\mathbf{2}} & -\mathbf{1} & \frac{\mathbf{1}}{\mathbf{2}} \end{array}\right]$$

$$R_1' = R_1 - R_2$$

Now the matrix on the right hand side is our right inverse.

Gauss-Jordan Example

Now the matrix on the right hand side is our right inverse.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \mathbf{3} & \mathbf{-3} & \mathbf{1} \\ 0 & 1 & 0 & -\frac{\mathbf{5}}{\mathbf{2}} & \mathbf{4} & -\frac{\mathbf{3}}{\mathbf{2}} \\ 0 & 0 & 1 & \frac{\mathbf{1}}{\mathbf{2}} & \mathbf{-1} & \frac{\mathbf{1}}{\mathbf{2}} \end{array}\right]$$

The first column of the right inverse is the solution to the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \mathbf{3} \\ 0 & 1 & 0 & -\frac{\mathbf{5}}{\mathbf{2}} \\ 0 & 0 & 1 & \frac{\mathbf{1}}{\mathbf{2}} \end{array}\right] \iff \begin{aligned} x_{11} &= 3 \\ x_{21} &= -\frac{5}{2} \\ x_{31} &= \frac{1}{2} \end{aligned}$$

Similarly for the other columns of the right inverse.

Reduced Row Echelon Form

After Gauss-Jordan elimination, the coefficient matrix is in what is called reduced row echelon form.

- § All nonzero rows are above any all-zero row.
- § Every pivot is strictly to the right of any pivot of a row above it.
- § Every pivot is one and is the only nonzero entry in its column.

The first two items are accomplished by Gaussian elimination, the third by Gauss-Jordan.

Reduced Row Echelon Form

§ All nonzero rows are above any all-zero row.

§ Every pivot is strictly to the right of any pivot of a row above it.

§ Every pivot is one and is the only nonzero entry in its column.

A matrix only satisfying the first two items is said to be in row echelon form.

The reduced row echelon form of a matrix is unique!

Example

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Is this matrix in reduced row echelon form?

Theorem: If the n-by-n matrix A has n pivots after Gaussian elimination, then its reduced row echelon form is the identity matrix.

Proof: Say that Gaussian elimination takes A to the upper triangular matrix U.

U has n pivots means all its diagonal entries are nonzero.

Gauss-Jordan elimination will first set all diagonal entries to one.

Then it will eliminate all nonzero entries above each diagonal entry.

Gauss-Jordan terminates with the identity matrix.

Existence of Left Inverse

We have already seen that the existence of n pivots after elimination means A has a right inverse.

Now we will see it means A has a left inverse as well.

We use the last result: when elimination produces n pivots the endpoint of Gauss-Jordan elimination is the identity matrix.

Let's look at Gauss-Jordan elimination from the matrix point of view.

Matrix Point of View

First let's look at Gaussian elimination on the coefficient matrix.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}} \xrightarrow[\substack{R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1}]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix} \xrightarrow[R_3' = R_3 - 2R_2]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

In terms of elementary matrices:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{R_3' = R_3 - 2R_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{R_3' = R_3 - R_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_2' = R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Multiplying the matrices from the last slide we have:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Now we begin with the Gauss-Jordan phase. First we make all pivots one.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[R_3' = \frac{1}{2} R_3]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\1 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\1 & 2 & 4\\1 & 3 & 9\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 3\\0 & 0 & 1\end{bmatrix}$$

Finally, we create zeros above the pivots.

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 3\\0 & 0 & 1\end{bmatrix} \xrightarrow[\substack{R_2' = R_2 - 3R_3\\ R_1' = R_1 - R_3}]{} \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \xrightarrow[R_1' = R_1 - R_2]{} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

The cumulative action is given by the matrix

$$\underset{R_1' = R_1 - R_2}{\begin{bmatrix}1 & -1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}\underset{R_1' = R_1 - R_3}{\begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}\underset{R_2' = R_2 - 3R_3}{\begin{bmatrix}1 & 0 & 0\\0 & 1 & -3\\0 & 0 & 1\end{bmatrix}} = \begin{bmatrix}1 & -1 & 2\\0 & 1 & -3\\0 & 0 & 1\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & -3\\ 0 & 0 & 1\end{bmatrix}}_{\text{Create zeros above pivots}}\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}}_{\text{Scale pivots}}\underbrace{\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -2 & 1\end{bmatrix}}_{\text{Gaussian elimination}}\underbrace{\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 3 & 9\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{\text{Identity}}$$

We have found a **left inverse** for A.

$$\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & -3\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -2 & 1\end{bmatrix} = \begin{bmatrix}3 & -3 & 1\\ -\frac{5}{2} & 4 & -\frac{3}{2}\\ \frac{1}{2} & -1 & \frac{1}{2}\end{bmatrix}$$

It is the same as the right inverse we found earlier (as we know it must be).

Existence of Left Inverse

When Gaussian elimination on the n-by-n matrix A produces n pivots, Gauss-Jordan elimination ends with the identity matrix.

All the row operations of Gauss-Jordan elimination can be implemented by left multiplication by elementary matrices.

The product of all these elementary matrices forms a left inverse for A.

An n-by-n matrix A is invertible if and only if Gaussian elimination produces n pivots.

1) The "easy" direction: if A is invertible then Gaussian elimination produces n pivots.

because $A\vec{x} = \vec{0}$ has a unique solution.

2) If elimination produces n pivots, then A has a right inverse.

because $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$.

3) If elimination produces n pivots, then A has a left inverse.

because Gauss-Jordan produces the identity matrix and is done by left multiplication by elem. matrices.

Invertibility and homogeneous equations

Theorem: A square matrix A is invertible if and only if

$$A\vec{x} = \vec{0}$$

has a unique solution.

Proof: If A is invertible then $\vec{x} = A^{-1}\vec{0} = \vec{0}$ is the only solution.

If $A\vec{x} = \vec{0}$ has a unique solution then Gaussian elimination must end in a matrix with no free columns.

Every column has a pivot, thus there is a full set of pivots and A is invertible by the previous theorem.

Example

Is this matrix invertible?

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Example

Is this matrix invertible?

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Example

Is this matrix invertible?

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

No, a triangular matrix is invertible if and only if all diagonal entries are nonzero.

Example

Is this matrix invertible?

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

Question

Suppose that A and B are invertible.

Is AB invertible?

Question

Suppose that B is singular.

Is AB invertible for some A?

Left Inverse Suffices

Normally to show the square matrix A is invertible, we have to find B such that

$$AB = I \qquad \text{and} \qquad BA = I$$

Theorem: If $BA = I$ for square matrices A, B then A is invertible.

Proof: We show the contrapositive. If A is singular then there is a vector $\vec{u} \neq \vec{0}$ with $A\vec{u} = \vec{0}$.

Then $BA\vec{u} = B\vec{0} = \vec{0}$, thus BA is singular and cannot be the identity matrix.

The Big List

Let A be a square matrix. The following are equivalent:

- A is invertible.
- Gaussian elimination produces a full set of pivots.
- $A\vec{x} = \vec{0}$ has a unique solution.
- A has a left inverse.
- A has a right inverse.
- The reduced row echelon form of A is the identity matrix.