Determinants

Reading: Strang 5.2

Learning objective: Recall determinants after recess week and midterm.

Three Properties

The determinant is a function from square matrices to the real numbers that satisfies the following properties:

1) The determinant of the n-by-n identity matrix is one.

2) The determinant changes sign when two rows are exchanged.

3) The determinant is a linear function of each row separately.

Gaussian elimination

These properties tell us how the determinant changes under Gaussian elimination.

1) AMORTA: the determinant does not change.

2) Row swap: the determinant is multiplied by -1.

The determinant of an upper triangular matrix is the product of the diagonal entries.

AMORTAMA?

Add a multiple of one row to a multiple of another?

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1' = 3 \cdot R_1 - 4 \cdot R_2]{} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Does the determinant change in this case?

AMORTAMA?

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1' = 3 \cdot R_1 - 4 \cdot R_2]{} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

AMORTAMA is not one of our elementary operations.

This is a composition of two elementary operations.

$$R_1' = 3 \cdot R_1$$

$$R_1'' = R_1' - 4 \cdot R_2$$

The first operation multiplies the determinant by 3.

Cofactor Formula

We derived the cofactor formula by using applying linearity of the determinant as a function of a row.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$C_{11} =$ cofactor of entry a_{11}

$C_{12} =$ cofactor of entry a_{12}

$C_{13} =$ cofactor of entry a_{13}

Simplified Cofactors

$$\begin{vmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$C_{12} =$ cofactor of entry a_{12}

Let M_{ij} be the matrix A with the i^{th} row and j^{th} column removed. The (i, j)-cofactor is

$$\begin{aligned} C_{ij} &= (-1)^{i-1}(-1)^{j-1}\det(M_{ij}) \\ &= (-1)^{i+j}\det(M_{ij}) \end{aligned}$$

Cofactor Formula

Let M_{ij} be the matrix A with the i^{th} row and j^{th} column removed.

Cofactor expansion along row i

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$$

Remember that $\det(A) = \det(A^T)$

$\longrightarrow$ we can also do cofactor expansion along any column.

Big Formula

Reading: Strang 5.2

Learning objective: Be able to compute the determinant of small matrices by the "big formula".

Big Formula

Now we derive the big formula for the determinant.

With the cofactor formula, we used linearity to expand the determinant along one row.

The big formula doesn't stop here, but keeps expanding along each row successively…

At the end we have many terms, each with only one entry in each row.

Big Formula

We start out like the cofactor formula.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Now let's focus on one term and expand in the second row.

Big Formula

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}_{\text{determinant zero}} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Now we expand each of these terms along the third row.

Big Formula

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}}_{\text{determinant zero}} + \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{vmatrix}}_{\text{determinant zero}} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}}_{\text{determinant zero}} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix}}_{\text{determinant zero}}$$

Big Formula

Only two terms survive.

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix}$$

Summary: expand until have one element from each row.

Observation: those terms that survive have exactly one entry in every column.

If two entries are in the same column, then there will be an all zero column!

Big Formula

Expanding the other terms yields six terms in total.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \quad \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

Next level would have 27 terms!

Only 6 terms survive

Big Formula

Now we can pull out the coefficients using linearity.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$+ a_{12}a_{23}a_{31} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + a_{13}a_{22}a_{31} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

We have an expression in terms of determinants of permutation matrices.

These have determinant in $\{-1, +1\}$.

Using the Big Formula

$$\det(P) = +1 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det(P) = -1 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

There are 6 different 3-by-3 permutation matrices.

$$\det(P) = +1 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det(P) = -1 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Permutation Matrices

How many permutation matrices of size n are there?

For the first row, we can choose any of the n columns

For the second row, can choose any of $n-1$ columns

(avoid the column chosen in first row)

For the third row, can choose any of $n-2$ columns

(avoid the columns chosen in first two rows)

$\vdots$

For the last row, choice of column is fixed.

Permutation Matrices

How many permutation matrices of size n are there?

Overall, we have

$$n(n-1)(n-2)\cdots 2\cdot 1 = n!$$

That is why it is a Big Formula!

Big Formula

For a permutation matrix P let $P(i)$ be the column number of the one in row i.

$$\det(A) = \sum_{P} \det(P) A[1, P(1)] A[2, P(2)] \cdots A[n, P(n)]$$

permutation matrix

Recall that $\det(P) \in \{-1, +1\}$

Viewed this way it is surprising we can compute determinants at all!

Example

What is the determinant of

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Use the Big Formula.

$$\det(A) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 1$$

Example

What is the determinant of

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

Which method to use?

Example

What is the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \end{vmatrix}$$

Example

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \end{vmatrix}$$

Now do cofactor expansion in the third row.

$$\begin{aligned} \det(A) &= 1 \cdot (-3) - 2 \cdot 2 \cdot (-3) \\ &= 9 \end{aligned}$$

Cramer’s Rule

Reading: Strang 5.3

Learning objective: Be able to solve systems of linear equations and compute inverses by Cramer’s Rule.

Cramer's Rule

§ Cramer's rule is a method to solve linear equations using determinants.

§ The number of equations has to be the same as the number of unknowns.

§ To solve a system with n equations and n unknowns, you compute $n+1$ many determinants.

§ The determinant of the coefficient matrix must be nonzero.

§ For integer matrices, only integer computations.

Important Example

This case will be important for Cramer's Rule.

$$\begin{bmatrix} 1 & 0 & x_1 & 0 \\ 0 & 1 & x_2 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & x_4 & 1 \end{bmatrix}$$

Look at the identity matrix with one column replaced by a vector.

We denote by $I_{3 \leftarrow \vec{x}}$ the identity matrix with the third column replaced by $\vec{x}$.

What is the determinant of this matrix?

$$\begin{bmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 \\ 0 & 1 & \cdots & x_2 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \cdots & x_j & \cdots & 0 \\ \vdots & & \cdots & & \ddots & \vdots \\ 0 & & \cdots & x_n & \cdots & 1 \end{bmatrix}$$

In general, let $I_{j \leftarrow \vec{x}}$ be the identity matrix with column j replaced by $\vec{x}$.

Do cofactor expansion along row j

$$\det(A) = \sum_{k=1}^{n} A(j,k) C_{jk}$$

Only the j^{th} term in this sum is nonzero.

$$C_{jj} = (-1)^{j+j} \det(M_{jj}) = \det(I) = 1$$

Thus the determinant is x_j

Cramer's Rule

Let A be a square matrix. Consider the equation $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If this holds, then also

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

The i^{th} column of AY is A times the i^{th} column of Y

First Component

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

Now we take determinants of both sides.

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{bmatrix} \right) \det\left(\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} \right)$$

$$= x_1 \det(A)$$

First Component

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

Now we take determinants of both sides:

$$x_1 \det(A) = \det(A_{1 \leftarrow \vec{b}})$$

where $A_{1 \leftarrow \vec{b}}$ is the matrix A with $\vec{b}$ placed in the first column.

If $\det(A) \neq 0$ then

$$x_1 = \frac{\det(A_{1 \leftarrow \vec{b}})}{\det(A)}$$

First Component

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

If $\det(A) \neq 0$ then

$$x_1 = \frac{\det(A_{1 \leftarrow b})}{\det(A)}$$

We have found the first component of the solution!

We can continue in this way to find the other components.

Second Component

Finding the second component of the solution:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix}}_{I_{2\leftarrow\vec{x}}} = \underbrace{\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{12} & b_2 & a_{23} \\ a_{13} & b_3 & a_{33} \end{bmatrix}}_{A_{2\leftarrow\vec{b}}}$$

Again take determinants.

Note

$$\det\left(\begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix}\right) = x_2$$

Second Component

Finding the second component of the solution:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{12} & b_2 & a_{23} \\ a_{13} & b_3 & a_{33} \end{bmatrix}$$

$A_{2\leftarrow\vec{b}}$

Again take determinants.

$$x_2 \det(A) = \det(A_{2\leftarrow\vec{b}})$$

If $\det(A) \neq 0$ then

$$x_2 = \frac{\det(A_{2\leftarrow\vec{b}})}{\det(A)}$$

Cramer's Rule

We can keep going in this fashion to find all the components of the solution.

Theorem: Let A be a square matrix with $\det(A) \neq 0$. The j^{th} component of the solution to $A\vec{x} = \vec{b}$ is given by

$$x_j = \frac{\det(A_{j \leftarrow \vec{b}})}{\det(A)}$$

The key fact we used to derive this is that

$$\det(I_{j \leftarrow \vec{x}}) = x_j$$

Example

Solve this system using Cramer's Rule:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

First we compute the determinant of the coefficient matrix.

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 1 \cdot 3 - (-1)(-2) = 1$$

It is nonzero, so we can use Cramer's rule.

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}}{1} \qquad x_2 = \frac{\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 2 \end{vmatrix}}{1} \qquad x_3 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 0 \end{vmatrix}}{1}$$

$$= 3 \qquad\qquad = 2 \qquad\qquad = 1$$

Solution: $x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Inverses

We can further use Cramer's rule to find inverses.

Recall that if A is invertible, then the i^{th} column of the inverse is the solution to $A\vec{x} = I(:, i)$.

Let's revisit the last example with this in mind.

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}}{1} \qquad x_2 = \frac{\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 2 \end{vmatrix}}{1} \qquad x_3 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 0 \end{vmatrix}}{1}$$

We can interpret this in terms of cofactors:

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = C_{11}$$

AMOCTA

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}}{1} \qquad x_2 = \frac{\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 2 \end{vmatrix}}{1} \qquad x_3 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 0 \end{vmatrix}}{1}$$

We can interpret this in terms of cofactors:

$$\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 2 \end{vmatrix} \underset{\substack{\uparrow \\ \text{AMOCTA}}}{=} \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 2 \end{vmatrix} = C_{12}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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We can interpret this in terms of cofactors:

$$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 0 \end{vmatrix} = C_{13}$$

AMOCTA

First column of inverse

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

We have found the first column of the inverse of A.

$$\frac{1}{\det(A)} \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$$

The first column of the inverse is given by the cofactors of the first row of A.

Second column of inverse

Let's move on to the second column of the inverse

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We again apply Cramer's rule.

$$y_j = \frac{\det(A_{j \leftarrow \vec{b}})}{\det(A)}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We again apply Cramer's rule.

$$y_j = \frac{\det(A_{j \leftarrow \vec{b}})}{\det(A)}$$

$$y_1 = \frac{\begin{vmatrix} 0 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}}{1} \qquad y_2 = \frac{\begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{vmatrix}}{1} \qquad y_3 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix}}{1}$$

$$y_1 = \frac{\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 2 \end{vmatrix}}{1} \qquad y_2 = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix}}{1} \qquad y_3 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}}{1}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We again apply Cramer's rule.

$$y_j = \frac{\det(A_{j \leftarrow \vec{b}})}{\det(A)}$$

$$y_1 = \frac{\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 2 \end{vmatrix}}{1} \qquad y_2 = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix}}{1} \qquad y_3 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}}{1}$$

$$y_1 = C_{21} \qquad y_2 = C_{22} \qquad y_3 = C_{23}$$

Conclusion: The second column of the inverse is given by second row of cofactors (divided by $\det(A)$).

Third column of inverse

Let's move on to the third column of the inverse

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We again apply Cramer's rule.

$$z_j = \frac{\det(A_{j \leftarrow \vec{b}})}{\det(A)}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We again apply Cramer's rule $\quad z_j = \dfrac{\det(A_{j\leftarrow b})}{\det(A)}$

$$z_1 = \frac{\begin{vmatrix} 0 & -1 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{vmatrix}}{1} \qquad z_2 = \frac{\begin{vmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}}{1} \qquad z_3 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}{1}$$

$$z_1 = C_{31} \qquad z_2 = C_{32} \qquad z_3 = C_{33}$$

Conclusion: The third column of the inverse is given by the third row of cofactors (divided by $\det(A)$).

Cofactor Matrix

The pattern we have observed holds in general.

Define a matrix C of cofactors. $C_{ij} = (-1)^{i+j} \det(M_{ij})$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

If $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{\det(A)} C^T$$

Another proof

Let's look at another way to view this result.

$$\underset{\underset{A}{\uparrow}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \underset{\underset{C^T}{\uparrow}}{\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}} = \underset{\underset{Z}{\uparrow}}{\begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}}$$

Why is this true?

Let's look at the diagonals of the product first.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}$$

$$\underset{A}{\uparrow} \qquad\qquad \underset{C^T}{\uparrow} \qquad\qquad \underset{Z}{\uparrow}$$

Let's look at the diagonals first.

$$Z_{11} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

This is cofactor expansion in the first row!

$$Z_{11} = \det(A)$$

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \underset{C^T}{\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}} = \underset{Z}{\begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}}$$

The other diagonals are similar.

$$Z_{22} = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

cofactor expansion along the second row.

$$= \det(A)$$

$$Z_{33} = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

cofactor expansion along the third row.

$$= \det(A)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}$$

$$Z$$

Now let's look at an off diagonal entry, say Z_{12}

$$Z_{12} = a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$$

We can also think of this as a determinant,
but it's not of the matrix A.

Off diagonal entries

Now let's look at an off diagonal entry, say Z_{12}

$$Z_{12} = a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$$

Claim:
$$Z_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Do cofactor expansion in the second row.

$$\longrightarrow \quad Z_{12} = 0$$

Off diagonal entries

A similar thing happens for all other off diagonal entries:

$$Z_{ij} = a_{i1}C_{j1} + a_{i2}C_{j2} + a_{i3}C_{j3}$$

This is the determinant of the matrix with i^{th} row of A repeated in row j

This will be zero if $i \neq j$.

Inverse by cofactors

Theorem: If A is invertible, then

$$A^{-1} = \frac{C^T}{\det(A)}$$

where C is the cofactor matrix $C[i,j] = C_{ij}$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}$$