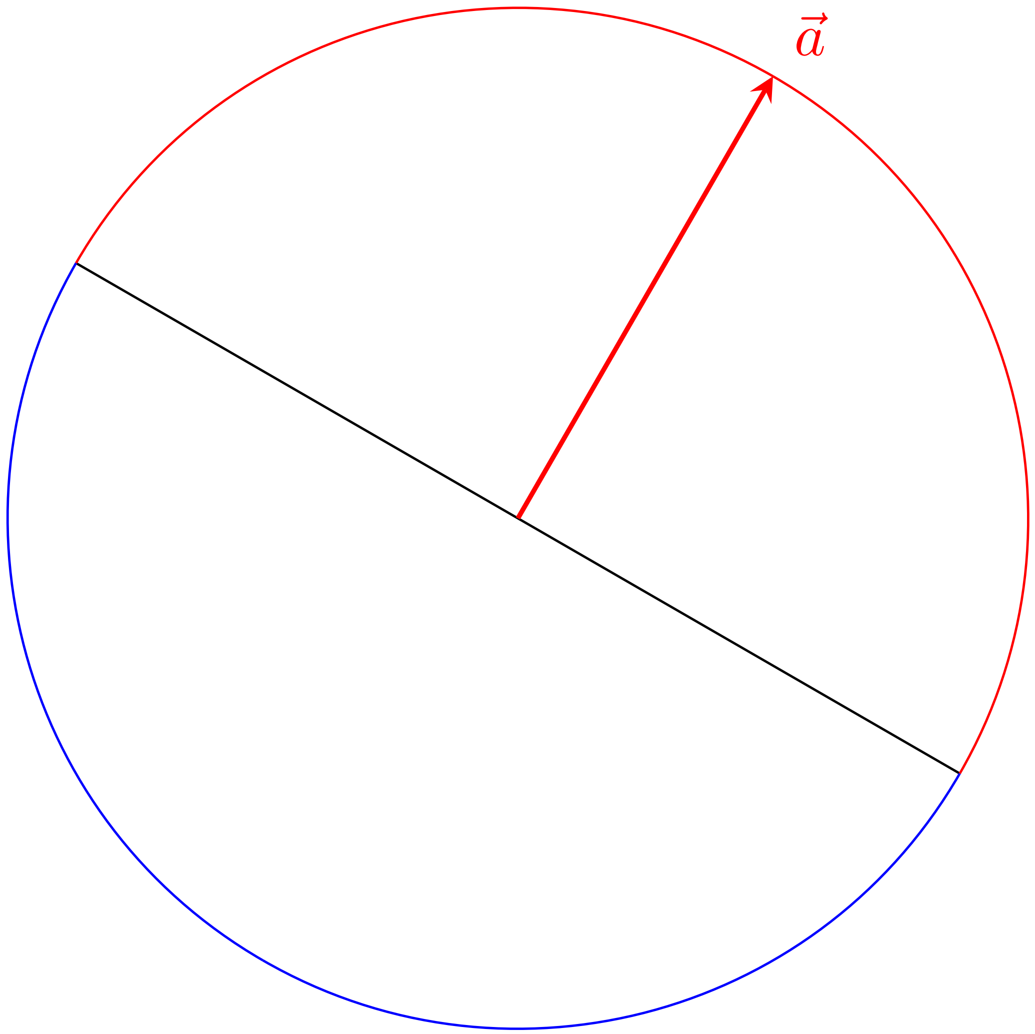
Angle

Thm: If $\vec{a}, \vec{b} \in \mathbb{R}^n$ are two **unit** vectors then $\langle \vec{a}, \vec{b} \rangle = \cos(\theta)$.

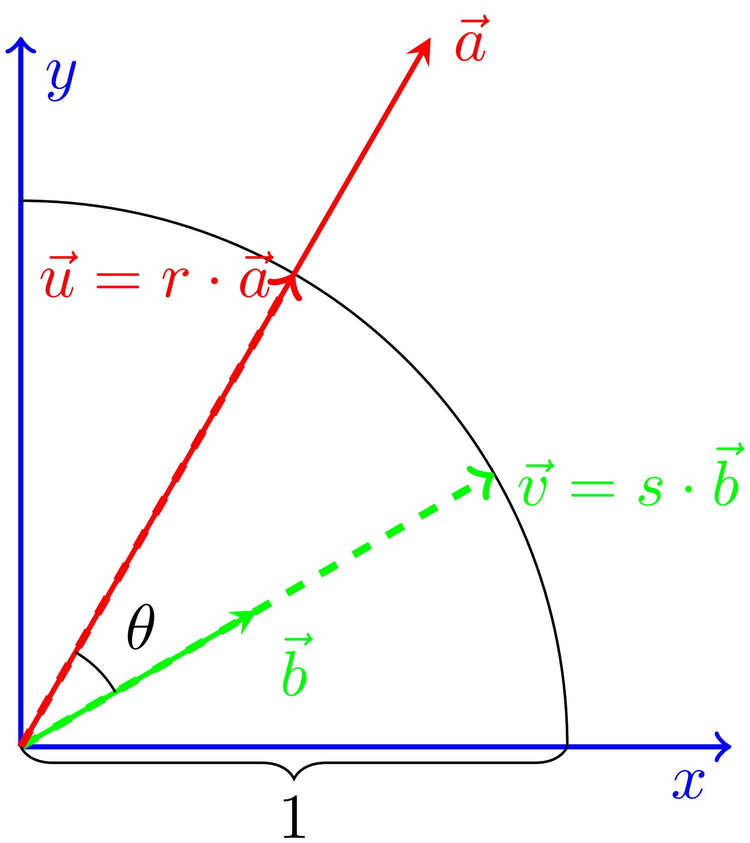
What about non-unit vectors?



Angle

As we said, multiplying a vector by a positive scalar does not change its direction.

If $\vec{a}, \vec{b}$ are nonzero vectors, then the angle between them is the same as the angle between $r \cdot \vec{a}, s \cdot \vec{b}$ for $r, s > 0$.



For what r is $r \cdot \vec{a}$ a unit vector?

(Assume $\vec{a}$ is not the zero vector.)

Scaling to a unit vector

Let's figure out what we should take for r. We want to choose r such that

$$\langle r \cdot \vec{a}, r \cdot \vec{a} \rangle = \|r \cdot \vec{a}\|^2 = 1$$

Let's simplify the expression on the left

$$\begin{aligned} \langle r \cdot \vec{a}, r \cdot \vec{a} \rangle &= r \cdot \langle \vec{a}, r \cdot \vec{a} \rangle \\ &= r^2 \cdot \langle \vec{a}, \vec{a} \rangle \\ &= r^2 \|\vec{a}\|^2 \end{aligned}$$

For this to be 1 and r to be positive we take $r = \dfrac{1}{\|\vec{a}\|}$.

Angle

If $\vec{a}, \vec{b}$ are nonzero vectors, then

$$\cos(\theta) = \frac{\langle \vec{a}, \vec{b} \rangle}{||\vec{a}|| \, ||\vec{b}||}$$

Thm: for any two vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$

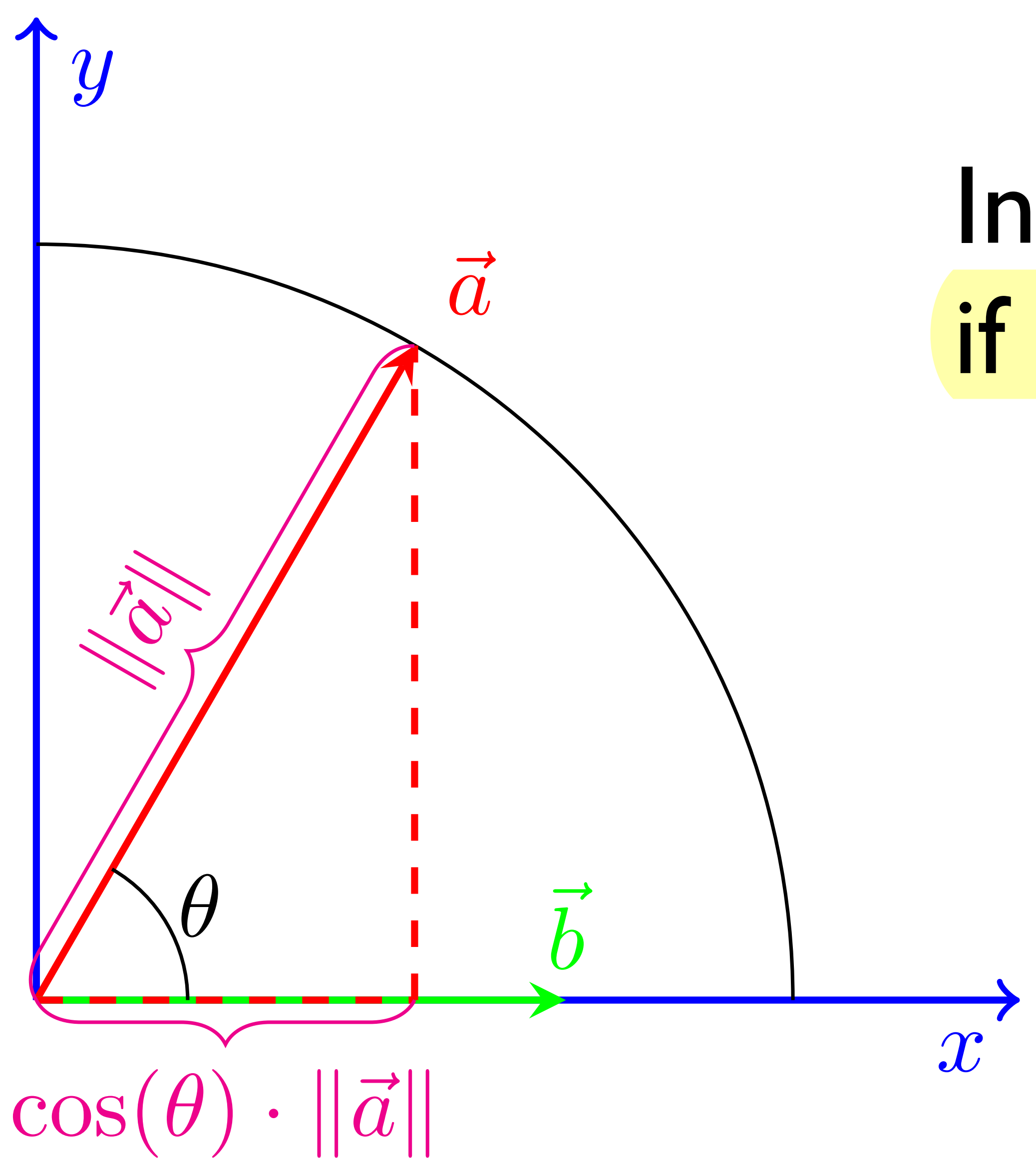
$$\langle \vec{a}, \vec{b} \rangle = \cos(\theta) ||\vec{a}|| \, ||\vec{b}||$$

This holds for any vectors even 0 vector

Cauchy-Schwarz Inequality

From this theorem we can deduce one of the most important inequalities in mathematics.

As $|\cos(\theta)| \leq 1$ we have $|\langle \vec{a}, \vec{b} \rangle| = |\cos(\theta)| \cdot \|a\| \|b\| \leq \|a\| \|b\|$.



In particular, $|\langle \vec{a}, \vec{b} \rangle| = \|\vec{a}\| \|\vec{b}\|$ if and only if $\vec{a} = c \cdot \vec{b}$.

That is, when $\vec{a}, \vec{b}$ lie on the same line.

Matrices: Quick Preview

Reading: Strang 1.3

Learning objective: Appreciate matrix-vector multiplication as a linear combination of the columns of the matrix.

Matrix

A matrix is a rectangular array of numbers.

They pop up all over the place.

A classy matrix: MH1200

	M+E	Math	Phys	Other
SG	58	126	14	6
MY	8	13	0	0
CN	10	18	1	8
ID	0	5	2	0
VN	1	4	1	3
Other	1	8	0	4

Basic Terminology

An m-by-n matrix has m rows and n columns.

entry

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Diagonal

Anatomy: entries, rows, columns, diagonal.

If $m = n$ then the matrix is called square.

An n-by-n matrix is said to have size n.

Examples

$$\begin{bmatrix} 1 & -5 & 7 \\ 2 & 6 & -4 \end{bmatrix}$$

This is a 2-by-3 matrix.

This is a 3-by-2 matrix.

$$\begin{bmatrix} 1 & 2 \\ -5 & 6 \\ 7 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 0.53 \\ 1.83 \\ -2.25 \\ 0.86 \end{bmatrix}$$

This is a ...?

Also known as a column vector.

Thinking by columns

Let's first think about a matrix by its columns.

We take three column vectors

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and put them together to form a matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Two Questions

The introduction of matrix notation will help us answer questions about $\vec{u}, \vec{v}, \vec{w}$.

There are two main questions we consider today:

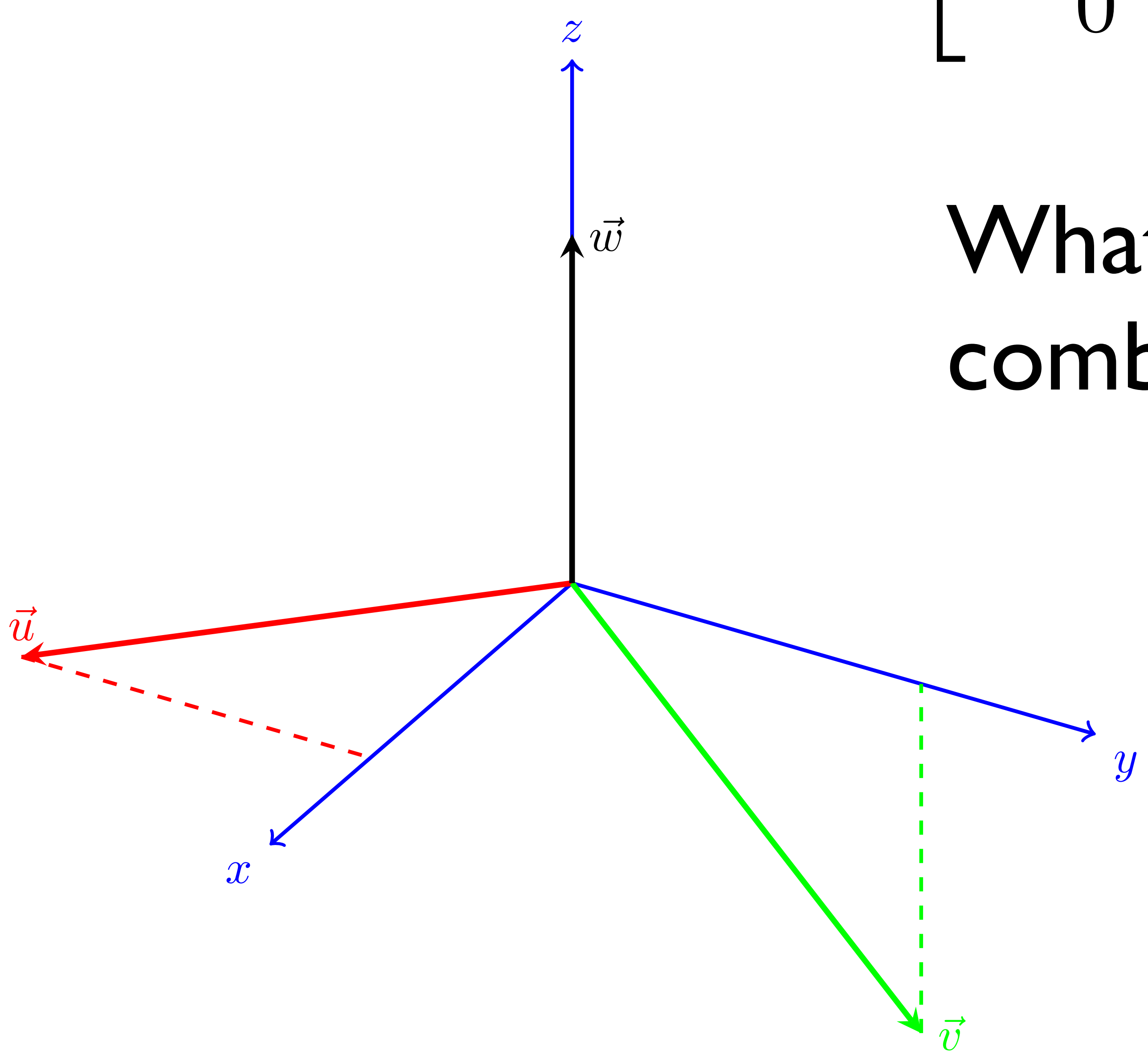
- § What are the possible linear combinations of $\vec{u}, \vec{v}, \vec{w}$?

- § How can $\vec{u}, \vec{v}, \vec{w}$ combine to form the zero vector?

First Question

We take three column vectors

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



What are the possible linear combinations of $\vec{u}, \vec{v}, \vec{w}$?

can take up all the space

$$c \cdot \vec{u} + d \cdot \vec{v} + e \cdot \vec{w}$$

Matrix-vector multiplication

The linear combinations $c \cdot \vec{u} + d \cdot \vec{v} + e \cdot \vec{w}$ motivate our definition of matrix-vector multiplication.

$$\begin{array}{c} \begin{array}{ccc} \vec{u} & \vec{v} & \vec{w} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = c \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= c \cdot \vec{u} + d \cdot \vec{v} + e \cdot \vec{w}$$

A matrix times a vector is a linear combination of the columns of the matrix.

Matrix-vector multiplication

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = c \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= c \cdot \vec{u} + d \cdot \vec{v} + e \cdot \vec{w}$$

A matrix times a vector is a linear combination of the columns of the matrix.

The coefficients in this linear combination are given by the components of the vector.

Matrix-vector multiplication

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = c \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= c \cdot \vec{u} + d \cdot \vec{v} + e \cdot \vec{w}$$

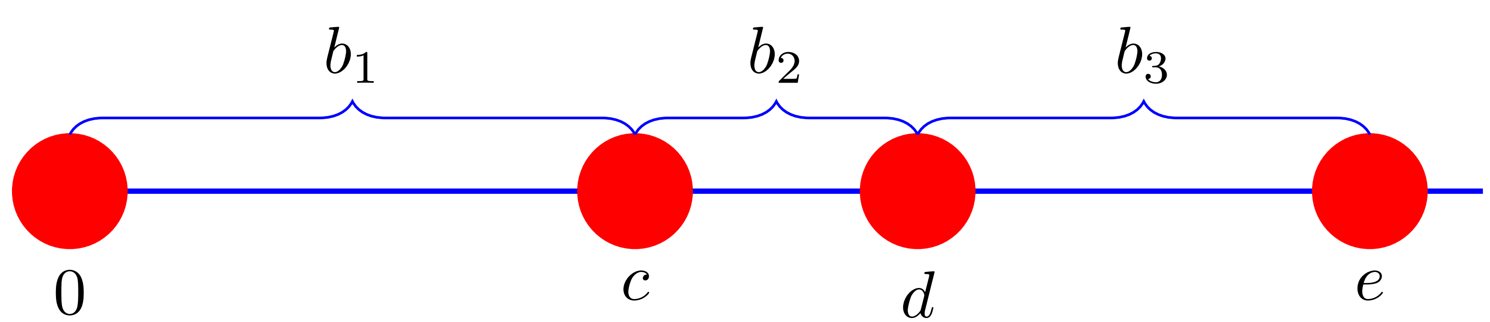
Note that for the matrix-vector product $A\vec{b}$ to make sense, the number of columns of A must equal the number of components of $\vec{b}$.

In this example,

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = c \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c \\ d - c \\ e - d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The matrix-vector multiplication returns the difference between adjacent components of the vector.



Linear Equations

Fundamental question: given a vector $\vec{b}$, can we find a vector $\vec{x} = (x_1, x_2, x_3)$ such that

$$x_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad ?$$

In matrix form, does this equation have a solution $\vec{x}$?

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{?}{=} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Here we fix $\vec{b}$ and solve for $\vec{x}$.

Linear Equations

$$x_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Two vectors are equal if and only if all their components are equal.

This gives us three equations:

$$x_1 = b_1$$

$$x_2 - x_1 = b_2 \implies x_2 = b_1 + b_2$$

$$x_3 - x_2 = b_3 \implies x_3 = b_1 + b_2 + b_3$$

Linear Equations

$$x_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{aligned} x_1 &= b_1 \\ x_2 - x_1 &= b_2 \implies x_2 = b_1 + b_2 \\ x_3 - x_2 &= b_3 \implies x_3 = b_1 + b_2 + b_3 \end{aligned}$$

We can always find a solution!

Every vector $\vec{b} \in \mathbb{R}^3$ can be written as a linear combination of $\vec{u}, \vec{v}, \vec{w}$.

$$x_1 = b_1$$

$$x_2 - x_1 = b_2 \implies x_2 = b_1 + b_2$$

$$x_3 - x_2 = b_3 \implies x_3 = b_1 + b_2 + b_3$$

We can express the solution in matrix form as well:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

call this matrix S.

The solution to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b} = S\vec{b}$.

S is the inverse of A.

Second Question

An important special case is when the right hand side vector $\vec{b} = \vec{0}$ is the zero vector.

$A\vec{x} = \vec{0}$: this is called a homogeneous equation.

How can we combine $\vec{u}, \vec{v}, \vec{w}$ to arrive at the zero vector?

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

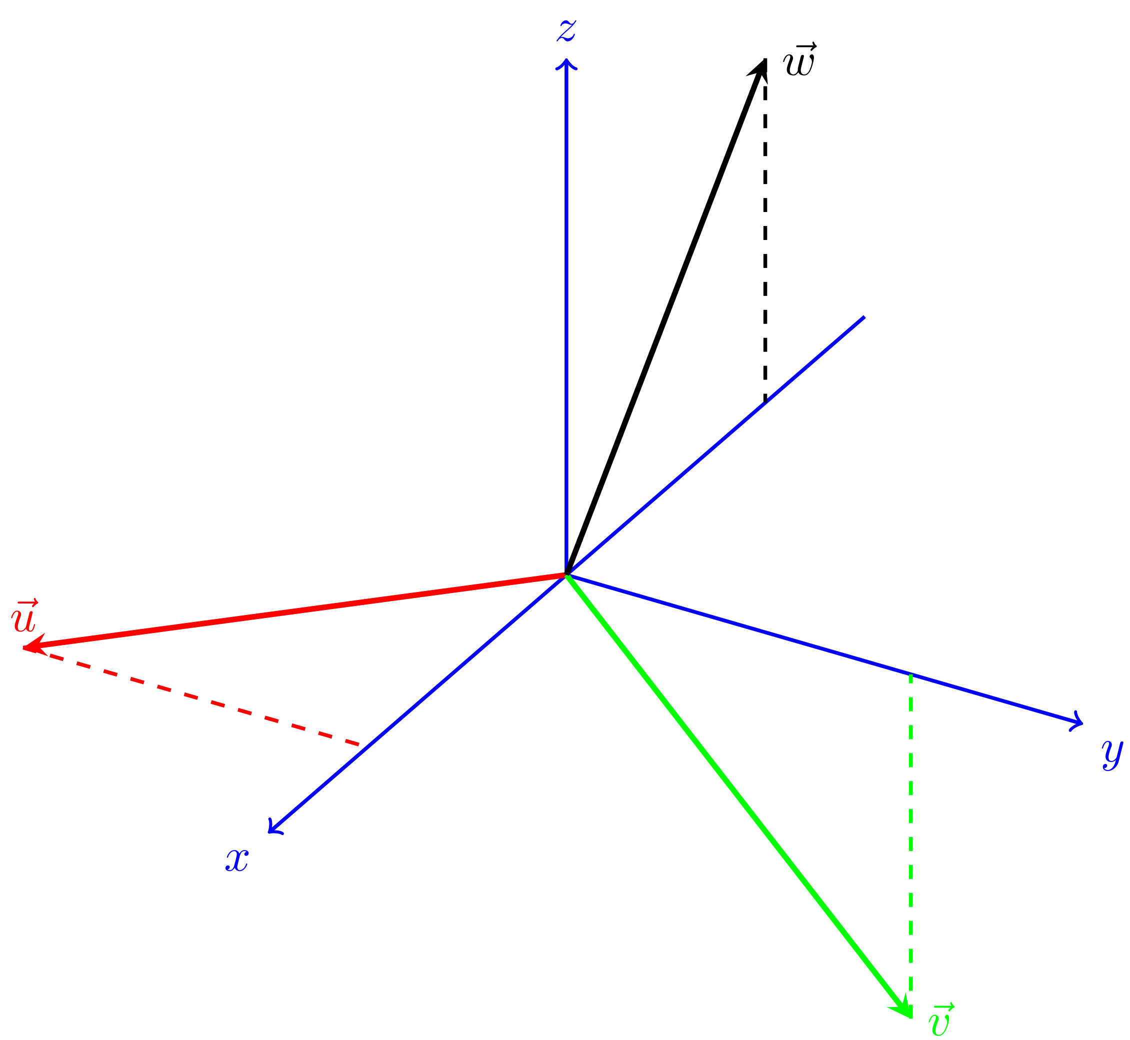
When $\vec{b} = \vec{0}$ the solution is $\vec{x} = \vec{0}$.

There are no other solutions.

Second Example

Now let's make a small change to the set of vectors.

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$



We keep $\vec{u}, \vec{v}$ the same, and change one coordinate of $\vec{w}$.

How can we write $\vec{0}$ as a linear combination of these vectors?

Homogeneous Equation

We again make a matrix whose columns are our vectors.

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we take $x_1 = 0, x_2 = 0, x_3 = 0$ the result will be $\vec{0}$.

Are there any other solutions?

Homogeneous Equation

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If this is $\vec{0}$ then $x_1 = x_3$ and $x_2 = x_1$.

In other words, all coordinates of $\vec{x}$ are equal.

Any vector (c, c, c) is a valid solution!

All vectors possible?

Is every vector $\vec{b} \in \mathbb{R}^3$ a linear combination of $\vec{u}, \vec{v}, \vec{w}^*$?

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If this is the case, then $b_1 + b_2 + b_3 = 0$.

This equation does not always have a solution!

All vectors possible?

Is every vector $\vec{b} \in \mathbb{R}^3$ a linear combination of $\vec{u}, \vec{v}, \vec{w}^*$?

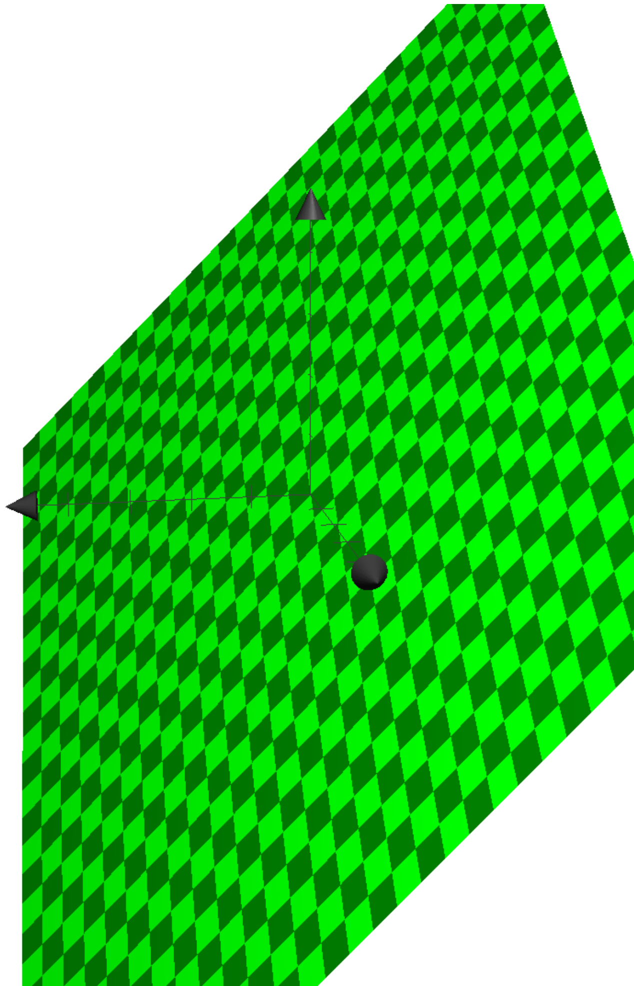
No, all linear combinations of $\vec{u}, \vec{v}, \vec{w}^*$ lie in the plane

$$\{(b_1, b_2, b_3) : b_1 + b_2 + b_3 = 0\}$$

Exercise: Show that

$$= \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has a solution when $b_1 + b_2 + b_3 = 0$



Comparisons

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let A be the matrix with columns $\vec{u}, \vec{v}, \vec{w}$ and B the matrix with columns $\vec{u}, \vec{v}, \vec{w}^*$.

$$A\vec{x} = \vec{0}$$

has the unique solution

$$\vec{x} = \vec{0}$$

$$B\vec{x} = \vec{0}$$

has infinitely many solutions

$$\vec{x} = (c, c, c) \quad c \in \mathbb{R}$$

Comparisons

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let A be the matrix with columns $\vec{u}, \vec{v}, \vec{w}$ and B the matrix with columns $\vec{u}, \vec{v}, \vec{w}^*$.

$$A\vec{x} = \vec{b}$$

has a solution for any

$$\vec{b} \in \mathbb{R}^3$$

$$B\vec{x} = \vec{b}$$

sometimes has no solution

Comparisons

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let A be the matrix with columns $\vec{u}, \vec{v}, \vec{w}$ and B the matrix with columns $\vec{u}, \vec{v}, \vec{w}^*$.

$$A\vec{x} = \vec{b}$$

always has exactly one solution

$$B\vec{x} = \vec{b}$$

sometimes has infinitely many solutions.

Comparisons

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let A be the matrix with columns $\vec{u}, \vec{v}, \vec{w}$ and B the matrix with columns $\vec{u}, \vec{v}, \vec{w}^*$.

A has an inverse	B has no inverse

The phenomena we see here are not special to these two matrices.

We will look at these two questions much more in depth:

§ For what $\vec{b}$ does $A\vec{x} = \vec{b}$ have a solution?

§ How many solutions does $A\vec{x} = \vec{0}$ have?

These two examples already show all the possible answers (for square matrices).

There is a close connection between the answers to these questions!

Linear Independence

Let me introduce one more word: linear independence.

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The vectors $\vec{u}, \vec{v}, \vec{w}^*$ are linearly dependent.

We can write one of them as a linear combination of the others.

$$\vec{u} = -\vec{v} - \vec{w}^*$$

Linear Independence

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

On the other hand, the vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly independent.

We can't write any one of them as a linear combination of the others.

The only way they combine to the zero vector is when all coefficients are zero.

Linear Equations: Geometric View

Reading: Strang 2.1

Learning objective: Visualize the solutions to a system of linear equations in two ways: as intersections of solution sets and as linear combinations of vectors.

Example

Let's start with one linear equation in two variables.

$$2x + y = 1$$

Here x and y are unknowns that we want to "solve" for.

A solution to this equation are values that we can substitute for x and y to make a true statement.

Examples of solutions include $(\frac{1}{2}, 0), (0, 1), (1, -1)$.

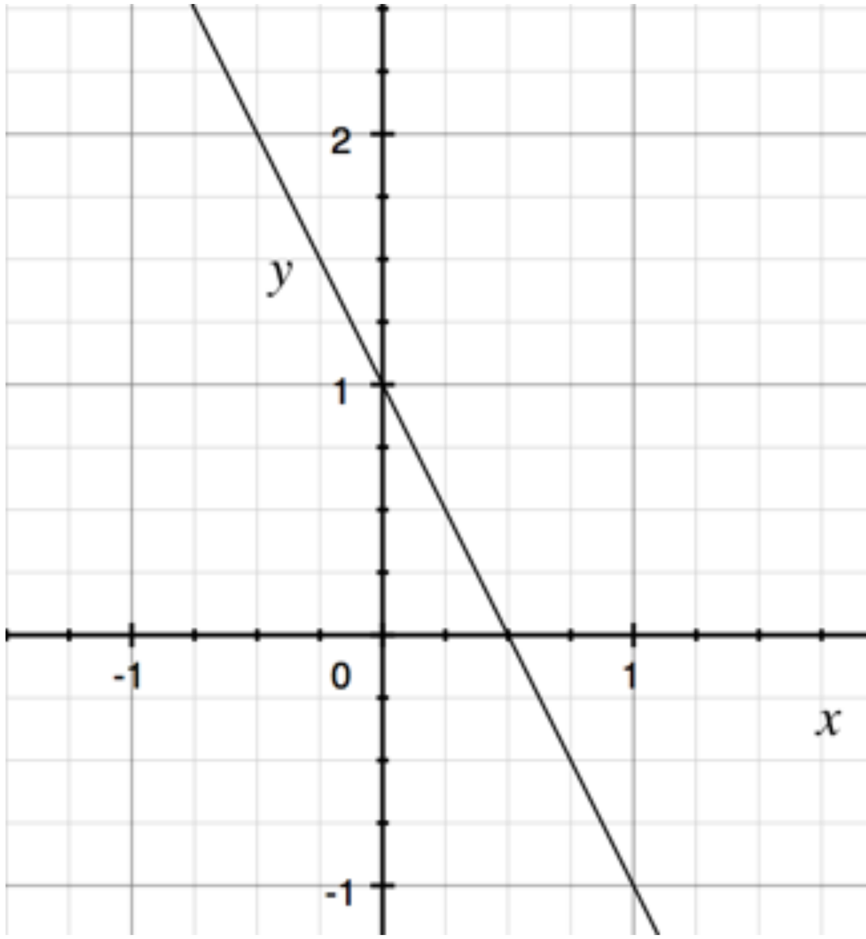
Solution Set

We can ask for all the solutions to this equation

$$2x + y = 1$$

This is called the solution set.

What does the solution set look like?



The solution set is the line

$$y = 1 - 2x$$

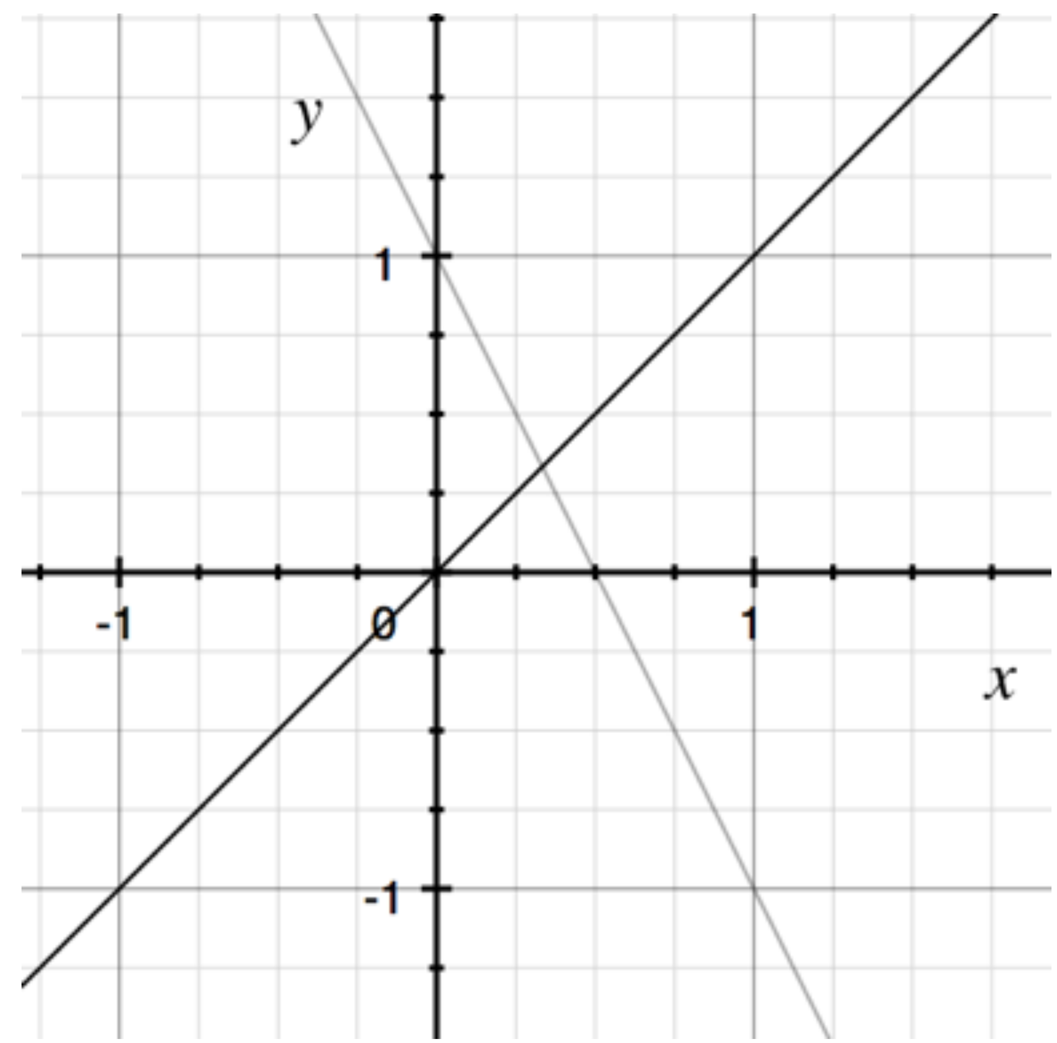
We write the solution set like this $\{(t, 1-2t) : t \in \mathbb{R}\}$.

Two Equations

Now we look at two equations in two unknowns.

$$2x + y = 1$$
$$x - y = 0$$

A solution must satisfy both equations.



Geometrically, this is the intersection of two lines—the intersection of the solution sets of the two equations.

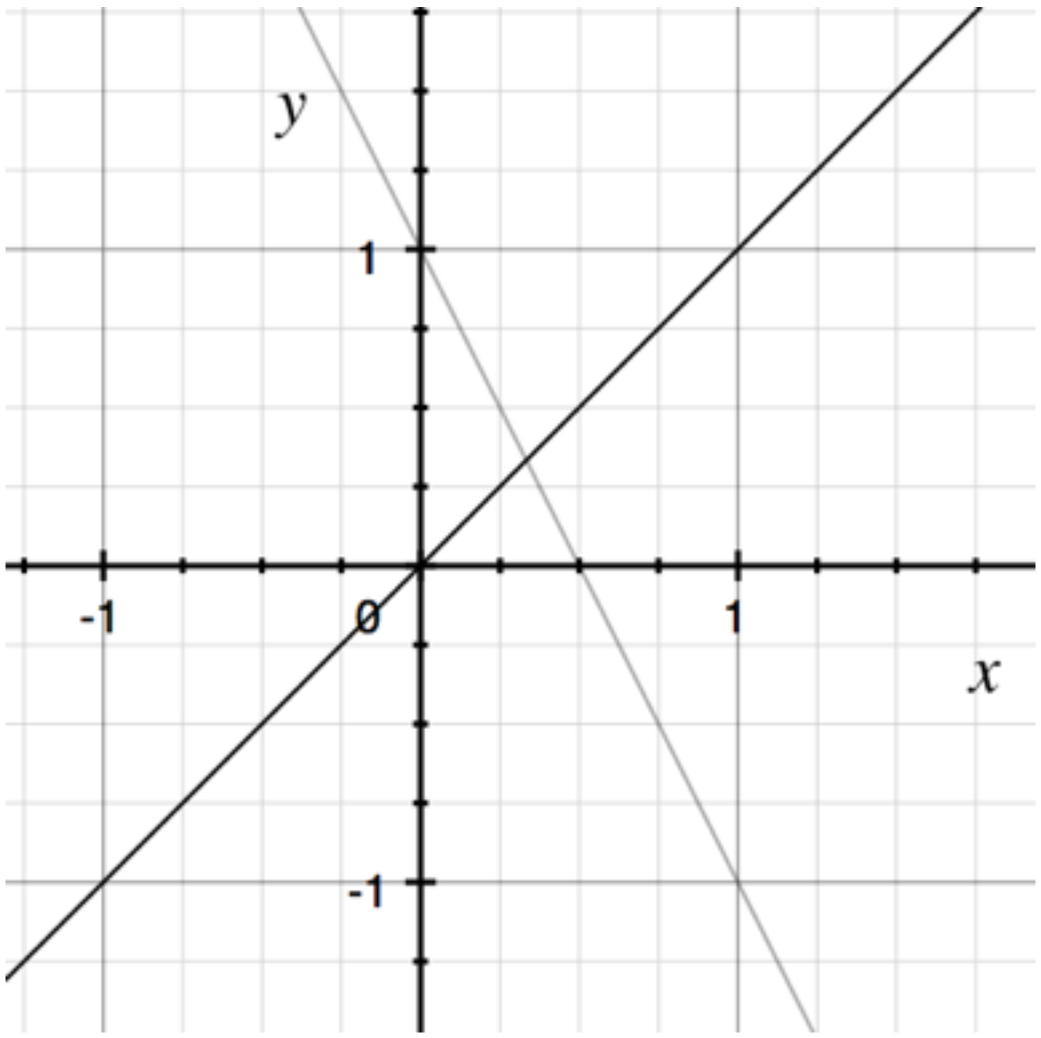
Two Equations

Now we look at two equations in two unknowns.

$$2x + y = 1$$
$$x - y = 0$$

A solution must satisfy both equations.

The solution set has a single point: $\{(\frac{1}{3}, \frac{1}{3})\}$



Column Picture

$$2x + y = 1$$
$$x - y = 0$$

We can think about this system of equations in a different way. Let's consider both equations at the same time.

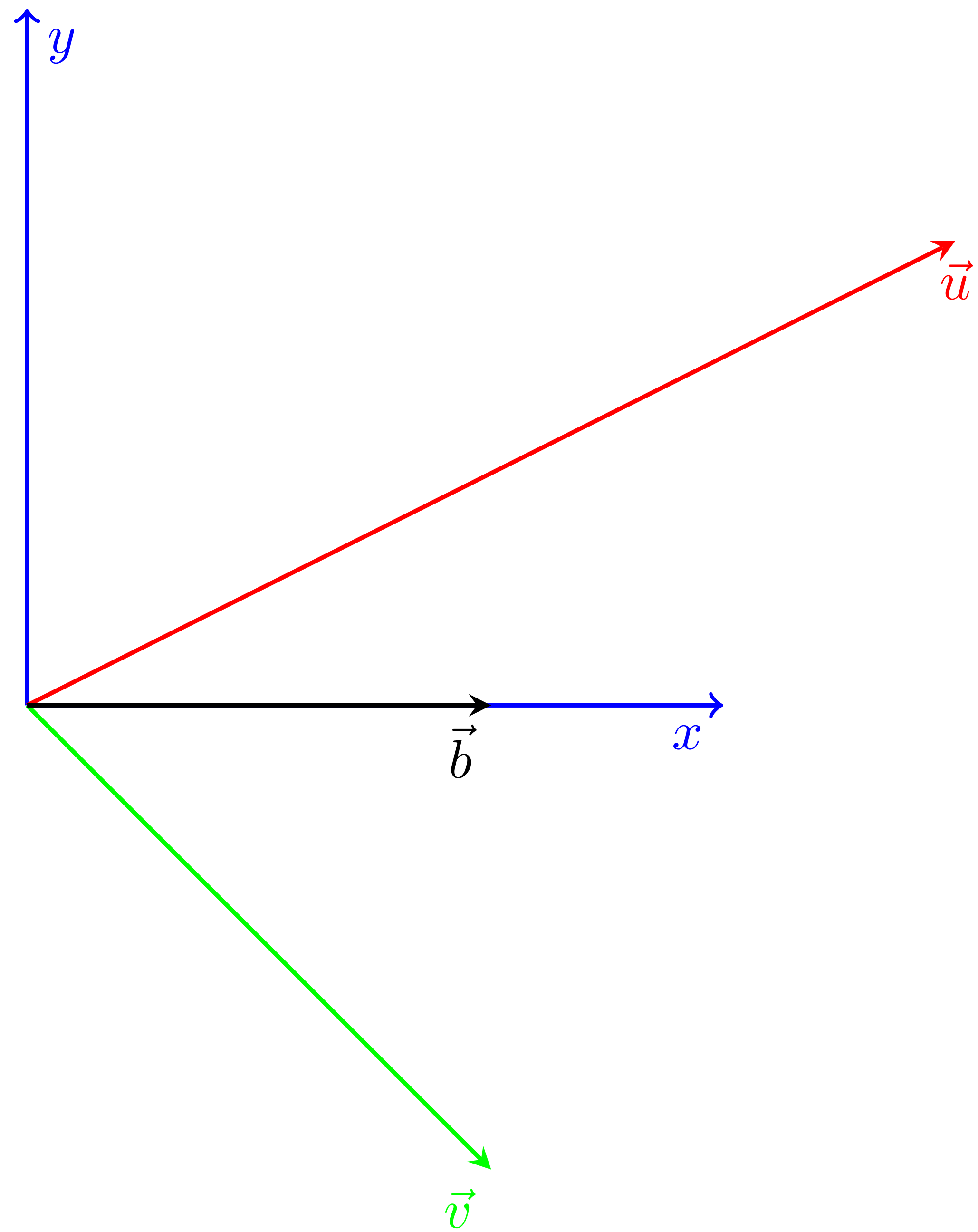
$$x \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We are looking for a linear combination of the vectors $(2,1), (1,-1)$ which is equal to $(1,0)$.

Column Picture

$$x \cdot \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{u}} + y \cdot \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\vec{v}} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{b}}$$

We are looking for a linear combination of $\vec{u}, \vec{v}$ equal to $\vec{b}$.

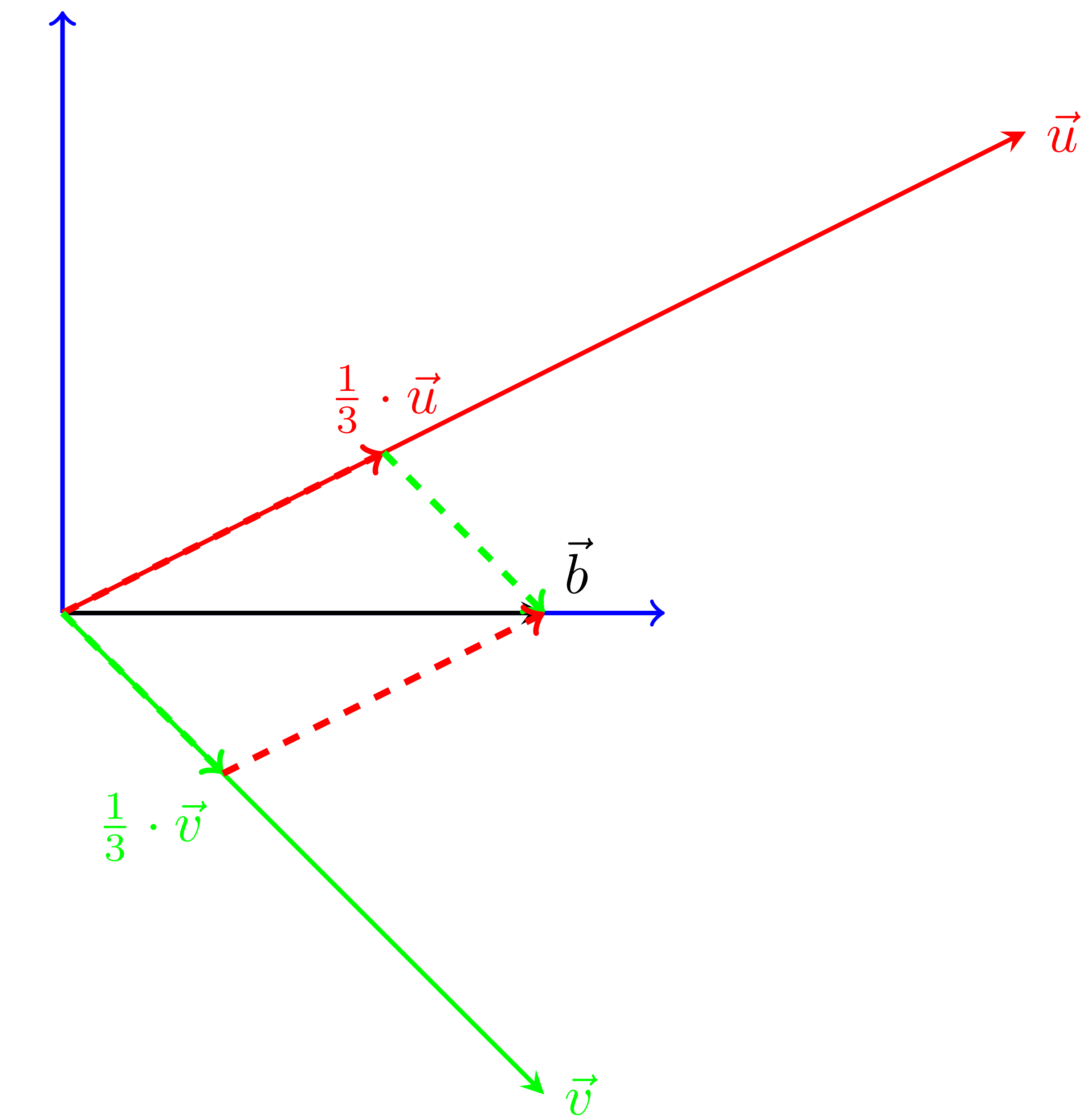


Column Picture

$$x \cdot \underset{\vec{u}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + y \cdot \underset{\vec{v}}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

We find the same solution,

$$\frac{1}{3} \cdot \vec{u} + \frac{1}{3} \cdot \vec{v} = \vec{b}$$



Perspective

Do you find one point of view easier than the other?

To actually solve a system of linear equations, usually the first way is easier.

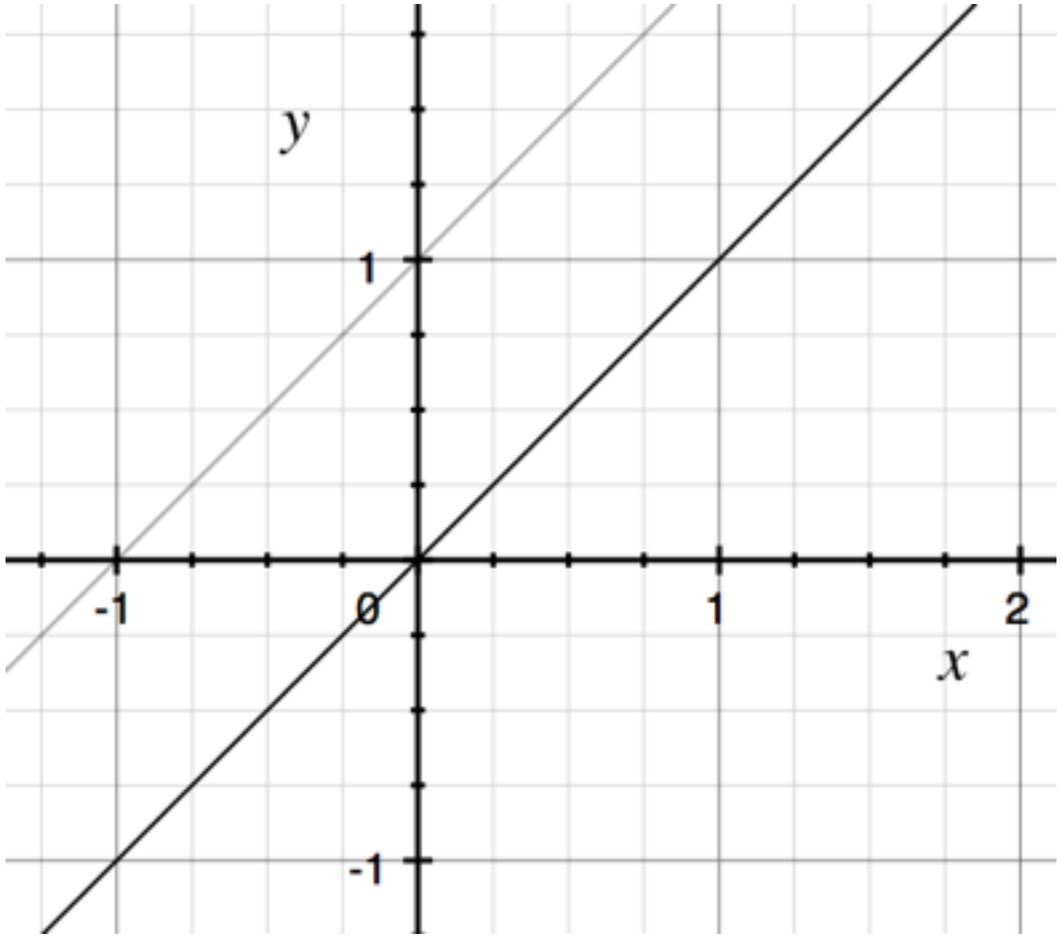
The column picture will be important in the second half of the course when we talk about the column space.

Example 2

Let's look at another example with two equations and two unknowns.

$$x - y = 0$$
$$2x - 2y = 2$$

In this case, the solution sets of the two equations are two parallel lines.



There is no intersection.

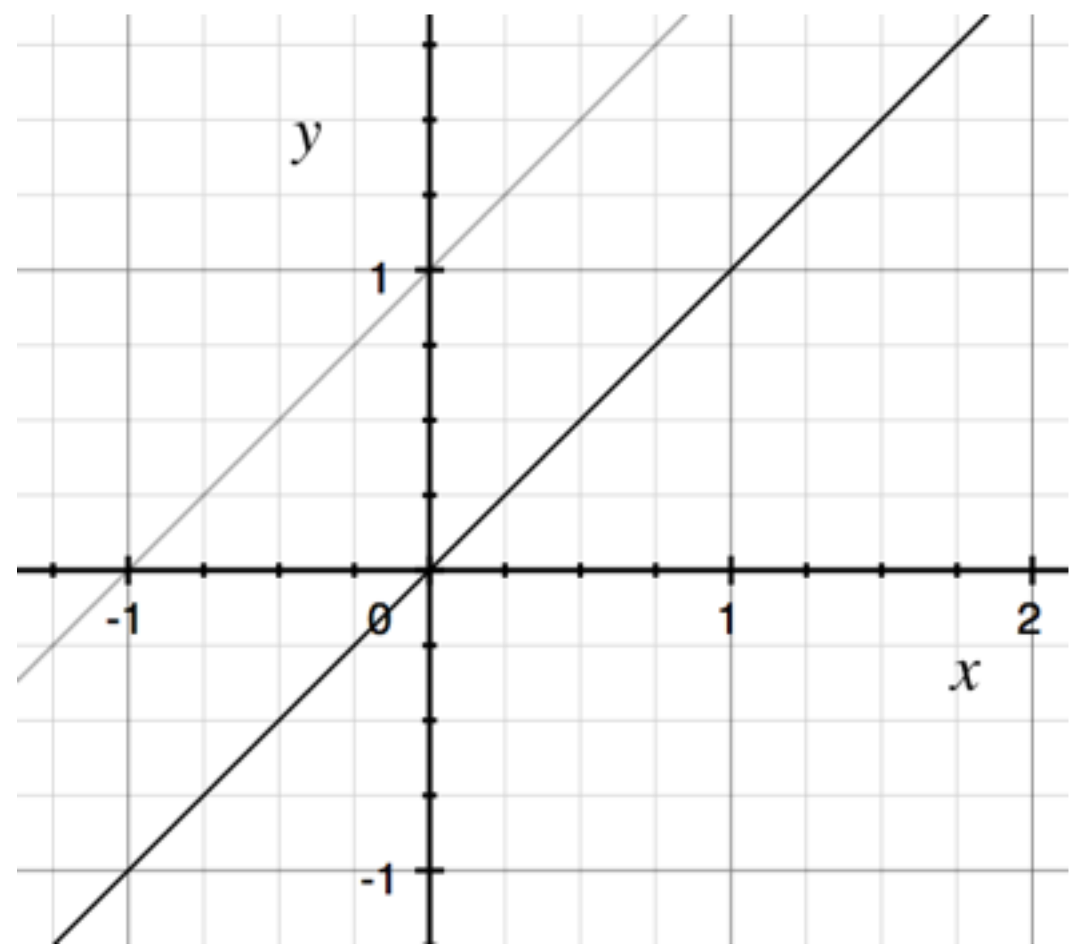
Example 2

Let's look at another example with two equations and two unknowns.

$$\begin{aligned} x - y &= 0 \\ 2x - 2y &= 2 \end{aligned}$$

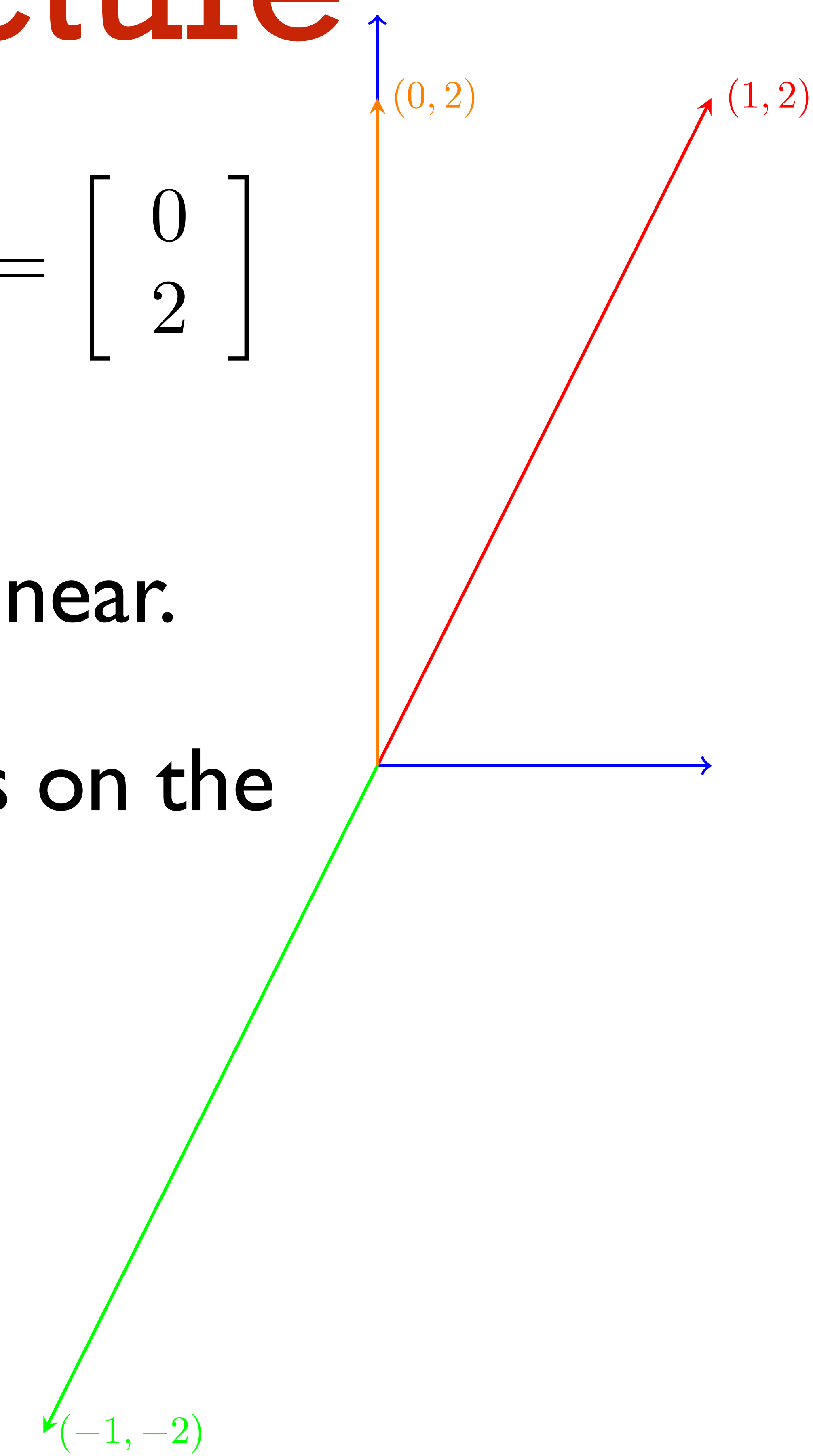
This system of linear equations has no solution.

We say such a system is inconsistent.



Column Picture

$$x \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \cdot \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



The vectors $(1, 2), (-1, -2)$ are collinear.

Any linear combination of them lies on the same line.

But $(0, 2)$ does not lie on this line.

There is no solution.

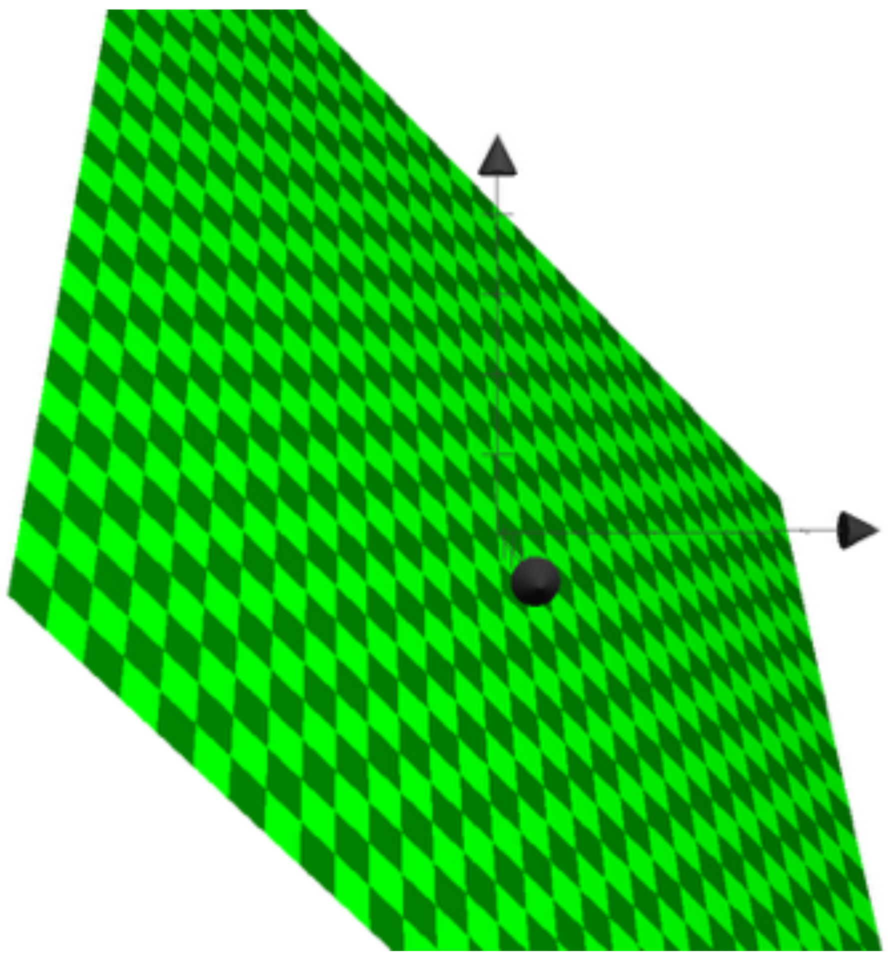
Three Variables

Here is a linear equation with three variables.

$$x + y + z = 0$$

Solutions include $(0, 0, 0), (1, -1, 0), (0, 1, -1), (-2, 1, 1)$.

Geometrically, the solution set is a plane.



Three Variables

Let's look at two equations in three variables.

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + z &= 1 \end{aligned}$$

A solution must satisfy both equations.

Examples of solutions include:

$$(\tfrac{1}{3}, -\tfrac{1}{3}, 0), (1, 0, -1), (-\tfrac{1}{3}, -\tfrac{2}{3}, 1)$$

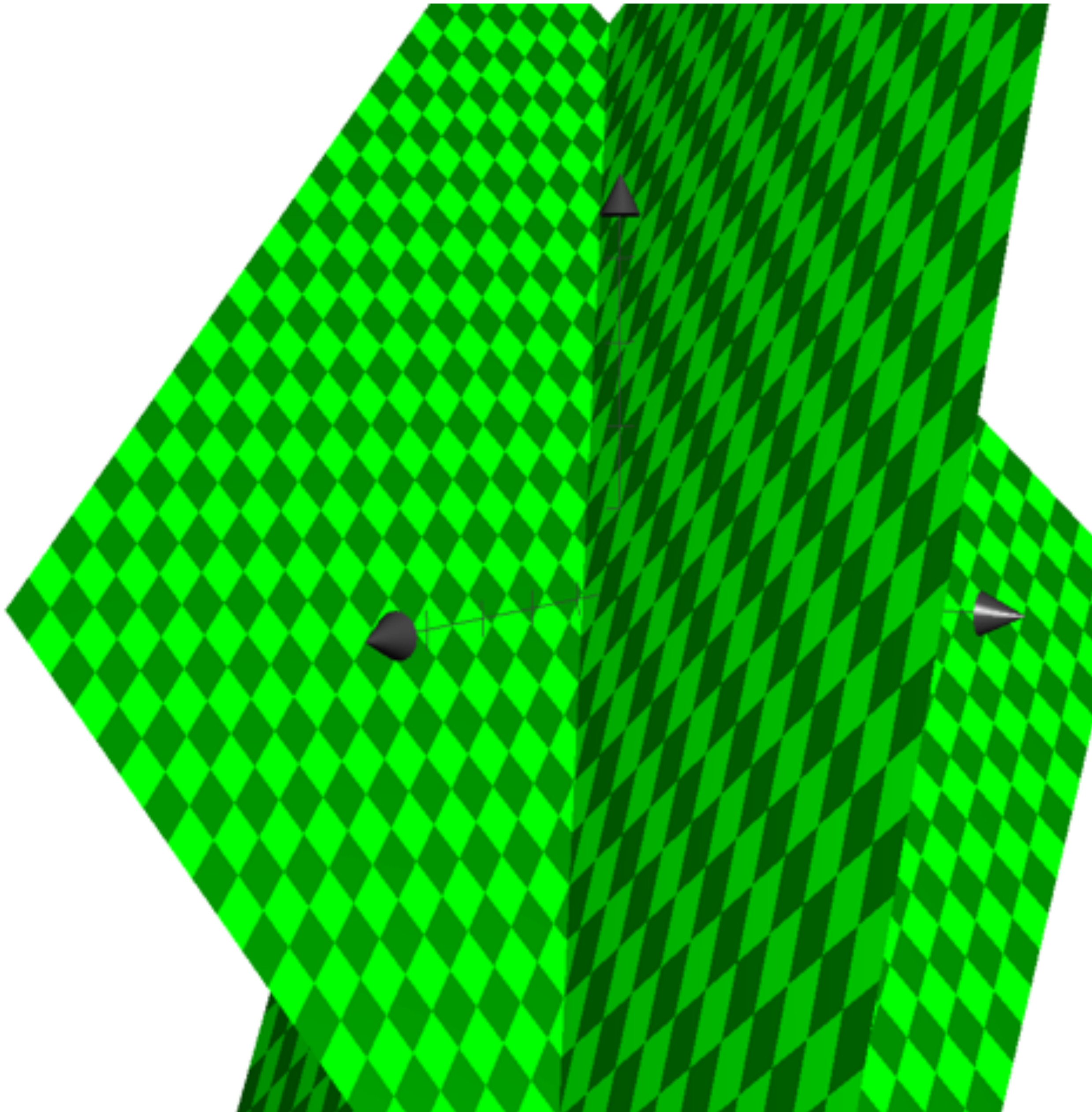
Three Variables

Let's look at two equations in three variables.

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + z &= 1 \end{aligned}$$

A solution must satisfy both equations.

Geometrically, the solution set is the intersection of two planes.



The solution set is a line.

Column Picture

Let's look at two equations in three variables.

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + z &= 1 \end{aligned}$$

In the column picture, we look for a linear combination of the column vectors equal to $(0, 1)$.

$$x \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} + z \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

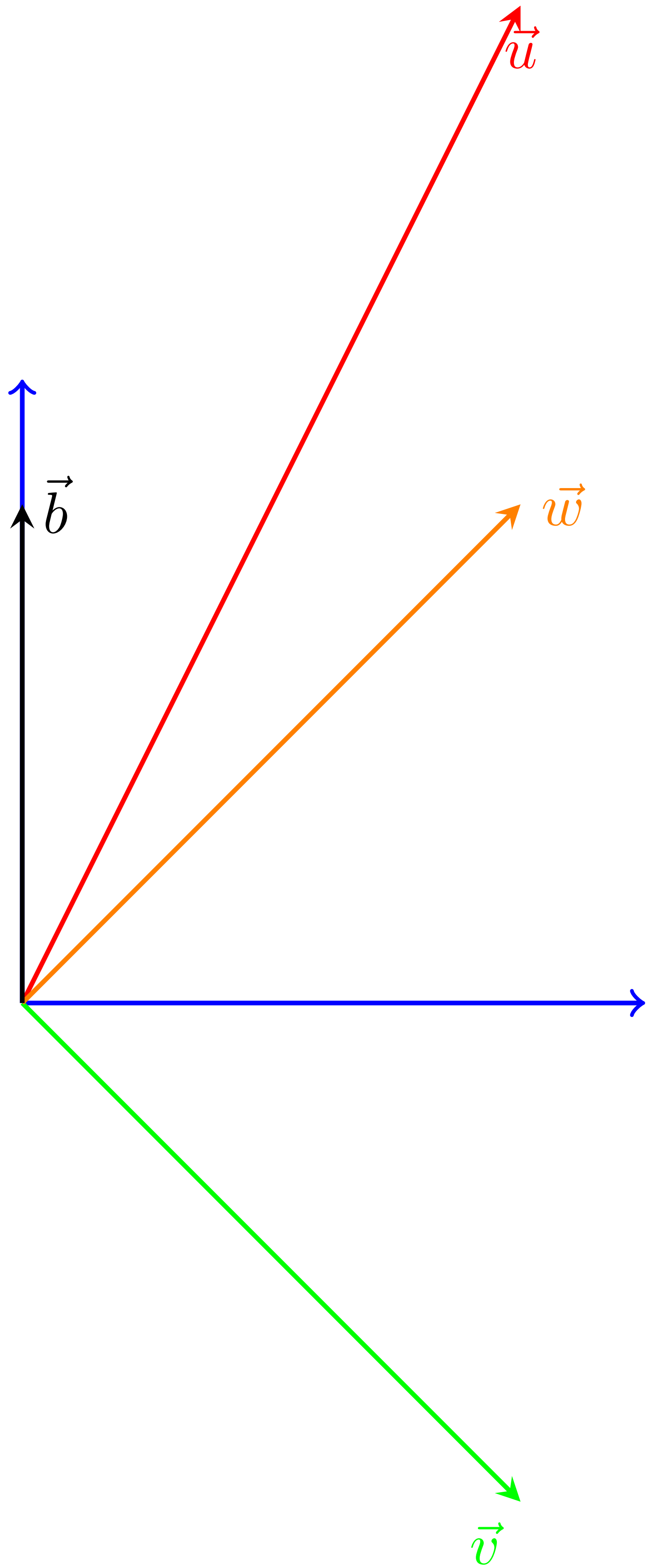
Column Picture

Let's look at two equations in three variables.

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + z &= 1 \end{aligned}$$

In the column picture, we look for a linear combination of the column vectors equal to $(0, 1)$.

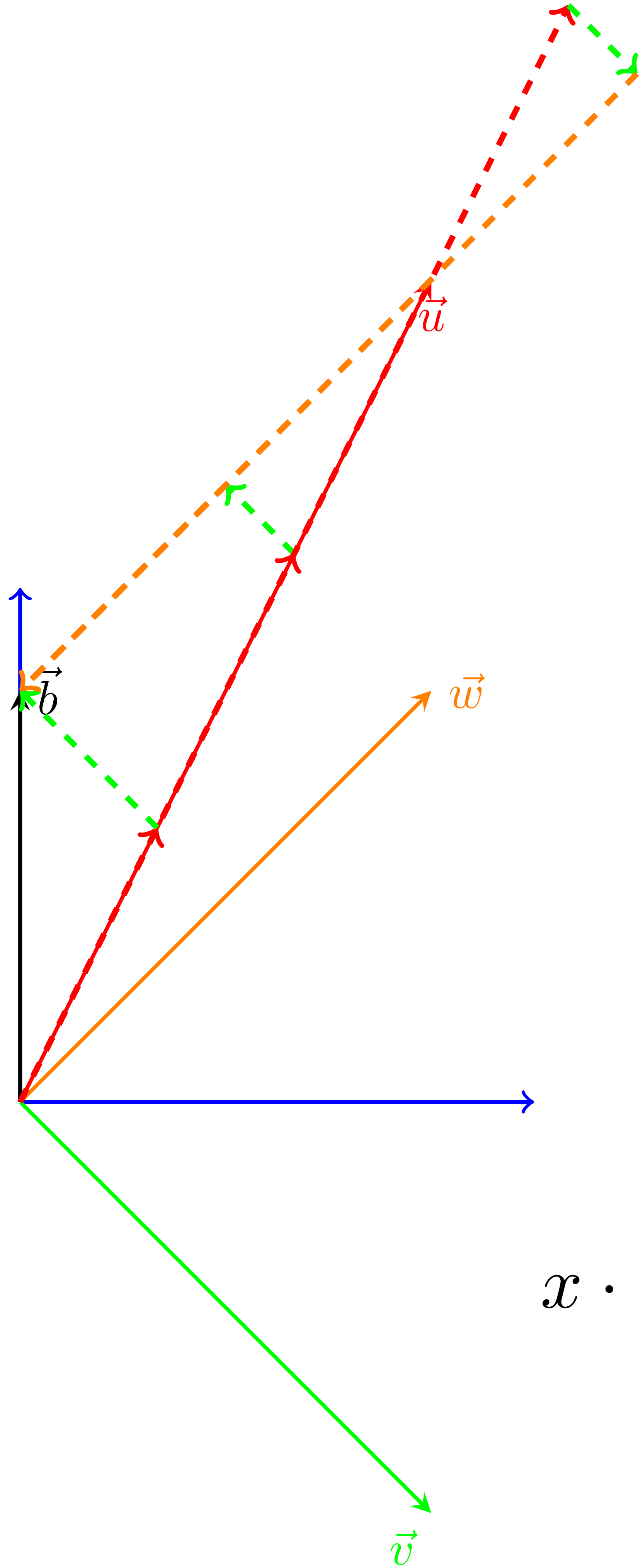
$$\underset{\vec{u}}{x \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \underset{\vec{v}}{y \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}} + \underset{\vec{w}}{z \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$



$$x + y + z = 0$$
$$2x - y + z = 1$$

In the column picture, we look for a linear combination of the column vectors equal to $(0, 1)$.

$$\underset{\vec{u}}{x \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \underset{\vec{v}}{y \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}} + \underset{\vec{w}}{z \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$



We can also see that there are infinitely many solutions in the column picture.

$$x \cdot \underset{\vec{u}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + y \cdot \underset{\vec{v}}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} + z \cdot \underset{\vec{w}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

Three Variables Three Equations

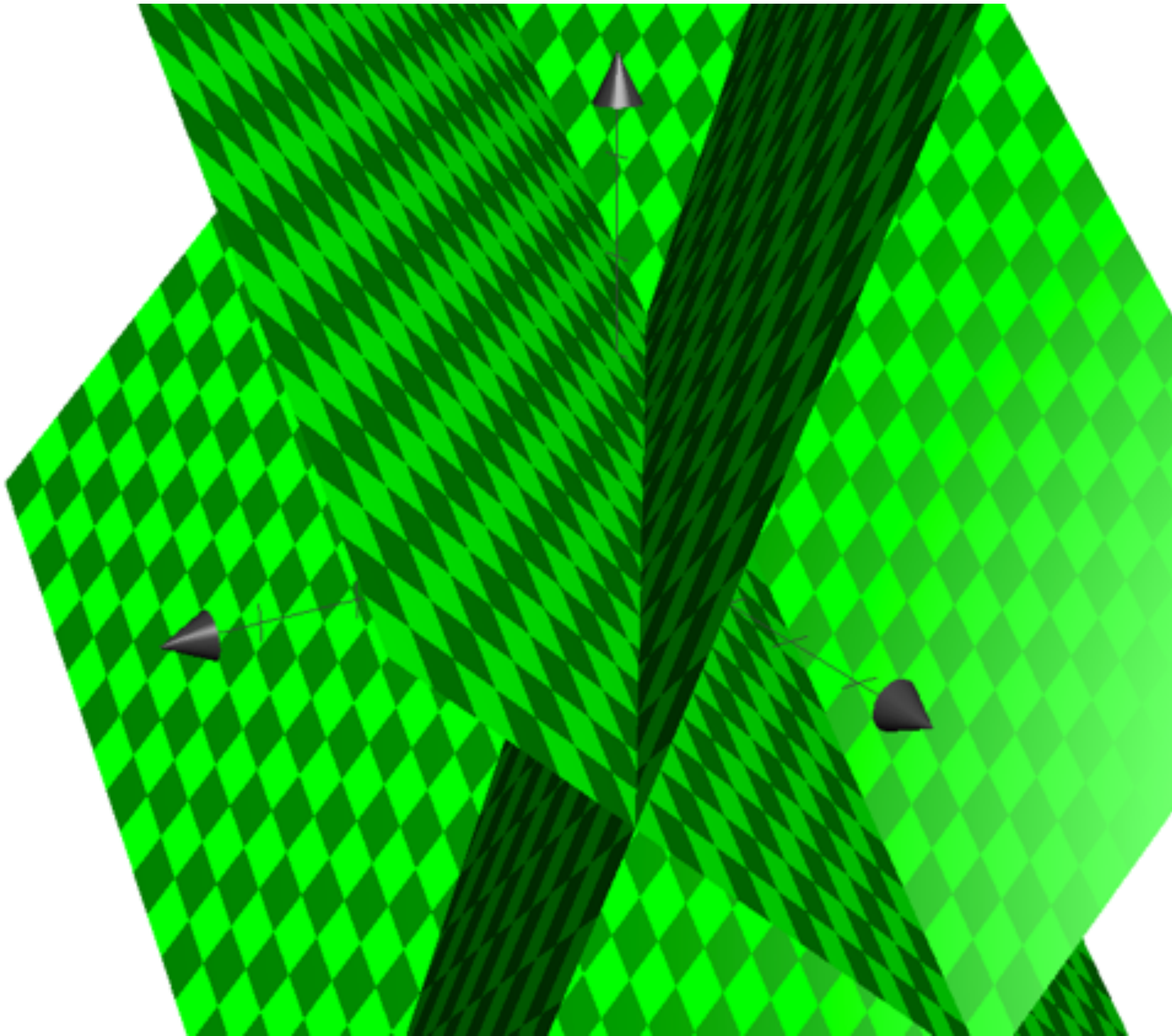
Let's look at three equations in three variables.

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + z &= 1 \\ -x + y + z &= 2 \end{aligned}$$

A solution must satisfy all three equations.

The only solution is

$$(-1, -1, 2)$$



Column Picture

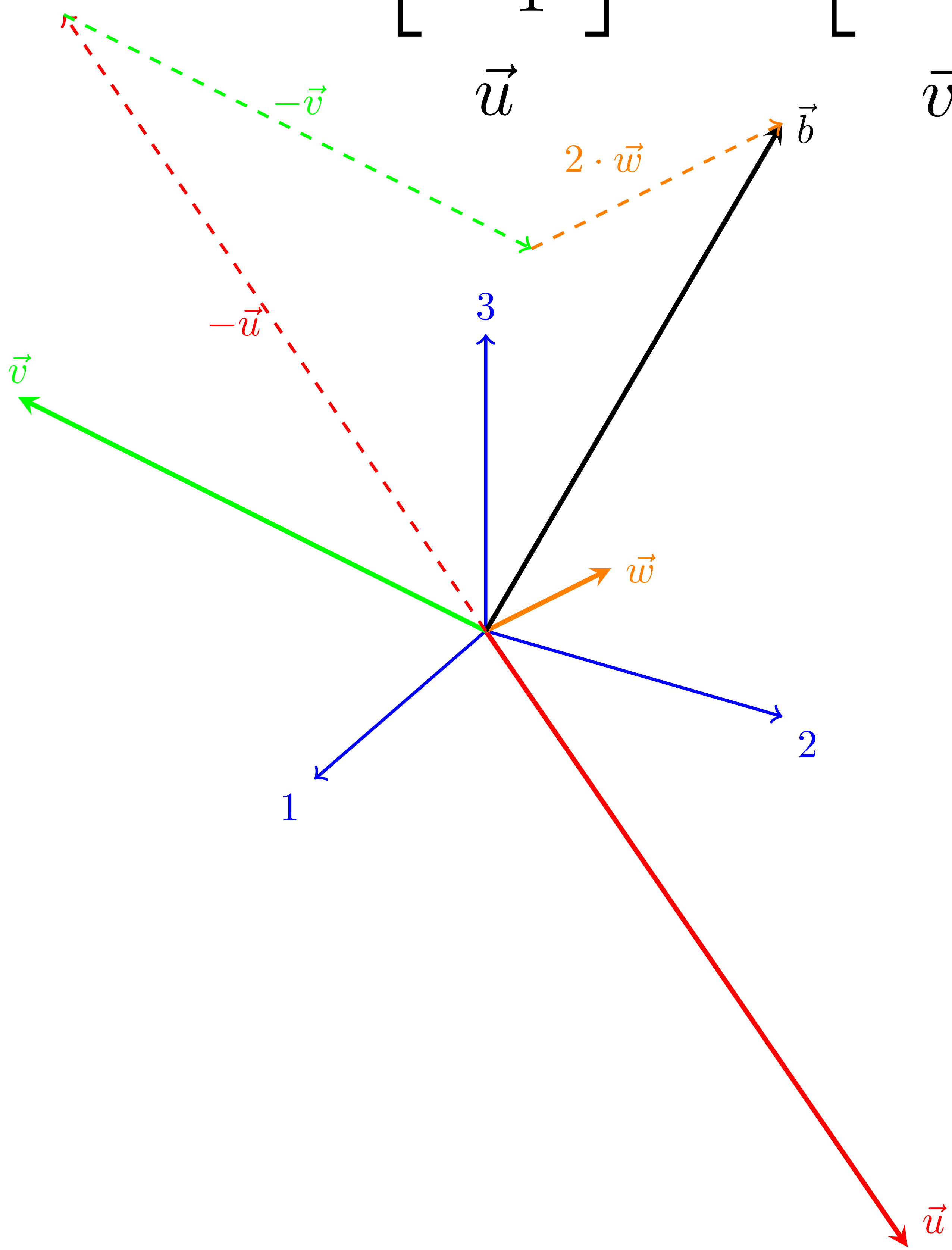
$$
\begin{aligned}
x + y + z &= 0 \\
2x - y + z &= 1 \\
-x + y + z &= 2
\end{aligned}
$$

In the column picture, we look for a linear combination of the column vectors equal to $(0, 1, 2)$.

$$
x \cdot \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{\vec{u}} + y \cdot \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\vec{v}} + z \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{w}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\vec{b}}
$$

$$x \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + z \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{u} \qquad \vec{v} \qquad \vec{w} \qquad \vec{b}$$



We have the same solution as in the row picture:

$$\vec{b} = -\vec{u} - \vec{v} + 2 \cdot \vec{w}$$

$$x = -1, y = -1, z = 2$$

Summary

We have looked at the geometrical picture of linear equations with 2 and 3 variables.

For two equations with two variables, either:

- § The solution sets define the same line

  → infinitely many solutions

- § The solution sets define parallel lines

  → no solution

- § The solution sets define lines that intersect in a single point

  → unique solution

Summary

For three equations with three variables, we saw the most "typical" case:

§ The solution sets define three planes that intersect in a single point.

→ unique solution

In the problem set, you will explore other geometric possibilities for three equations with three variables.

Linear Equation

Now that we have seen some examples, let me give you the formal definition.

Definition: A linear equation in the variables $x_1, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

for real constants $a_1, \ldots, a_n, b$

We say that $t_1, \ldots, t_n$ is a solution to this equation if

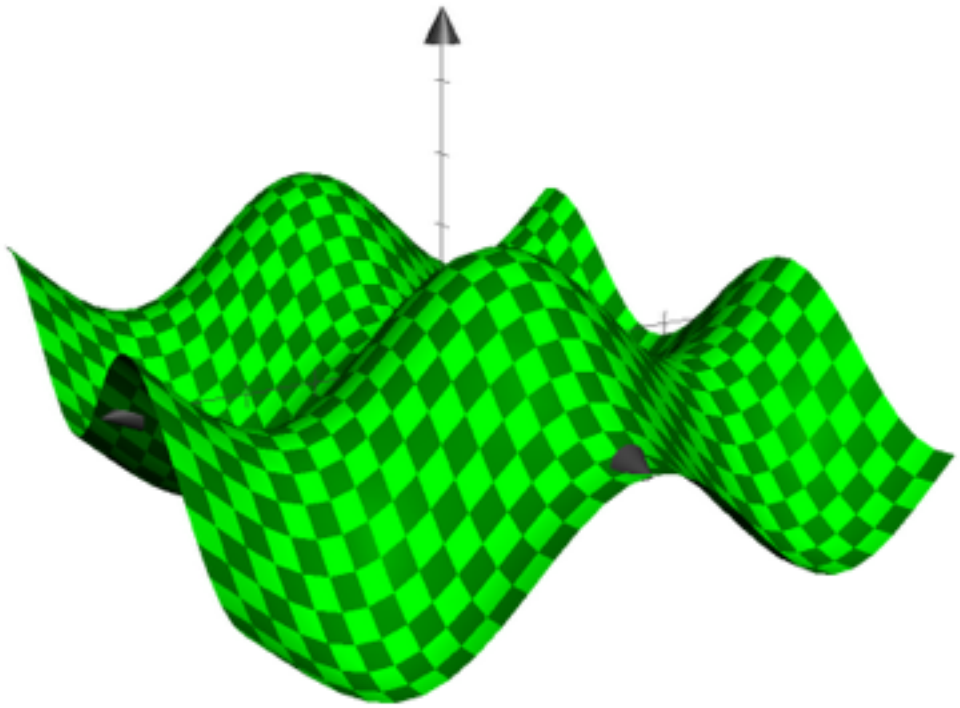
$$a_1t_1 + a_2t_2 + \cdots a_nt_n = b$$

The set of all solutions is the solution set.

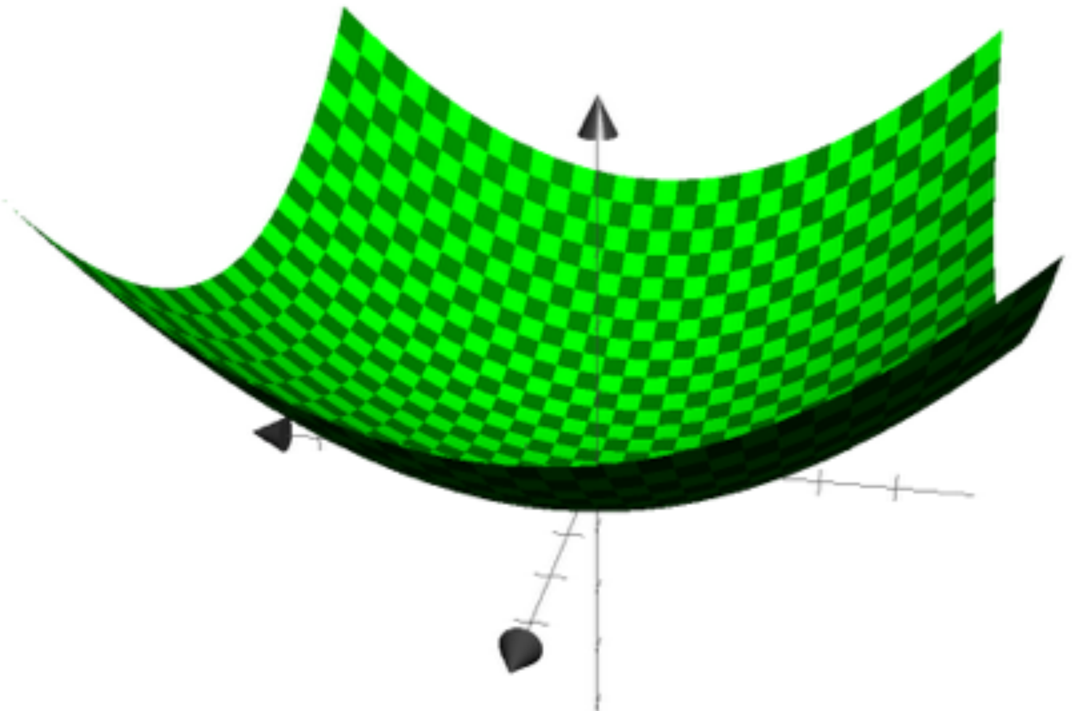
Non-examples

The following equations are not linear equations.

$$-\cos(x) - \cos(y) + z = 0$$



$$-\frac{x^2}{10} - \frac{y^2}{10} + z = 0$$



Systems of linear equations

Definition: A system of linear equations is a finite set of linear equations. A system of m linear equations in variables $x_1, \ldots, x_n$ has the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Systems of linear equations

Numbers $t_1, \ldots, t_n$ are a solution to a system of linear equation if they are a solution to each equation in the system.

The solution set of a system of linear equations is the set of all solutions.