Chapter 8

Relations

> *"Mathematicians do not study objects, but the relations between objects." (H.Poincaré)*

The topic of our next chapter is relations, it is about having 2 sets, and connecting related elements from one set to another.

Definition 35. Let A and B be two sets. A binary relation R from A to B is a subset of the cartesian product $A \times B$. Given $x, y \in A \times B$, we say that x is related to y by R, also written $(xRy) \leftrightarrow (x, y) \in R$.

Example 73. Suppose that you have two sets $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, and the relation is given by $(x, y) \in R \leftrightarrow x - y$ is even. Since the relation is a subset of $A \times B$, we start by computing the cartesian product $A \times B$:

$$A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}.$$

Then in this list of pairs, we select those which satisfy the relation R. For example, for $(1, 2)$, we have $x = 1$ and $y = 2$, we compute $x-y = 1-2 = -1$, which is odd, thus it does not belong to R. We try out similarly all the pairs in $A \times B$ to get

$$R = \{(1, 1), (1, 3), (2, 2)\}.$$

This may be visualized using a diagram: draw a circle to represent the set A, and this circle contains two points, one for 1 and one for 2. Similarly, draw a circle to represent B, and points of $1, 2, 3$. Then an arrow from A to B connects x in A with y in B if $x - y$ is even.

Binary Relations between Two Sets

Let A and B be sets. A binary relation R from A to B is a subset of $A \times B$. Given (x,y) in $A \times B$, x is related to y by R ($\boldsymbol{x}\ \mathbf{R}\ \boldsymbol{y}$) $\leftrightarrow$ $(x,y) \in R$.

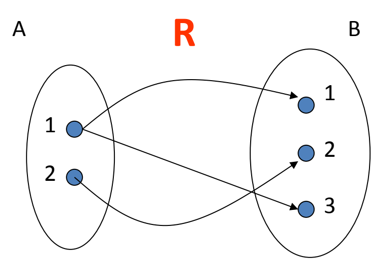
Example. A={1,2}, B={1,2,3}, $(x, y) \in R \leftrightarrow (x - y)$ is even.

- $A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$
- $(1,1) \in R, (1,3) \in R, (2,2) \in R.$

Examples. x > y, x owes y, x divides y

Graphically

- **Example.** A={1,2}, B={1,2,3}, $(x, y) \in R \leftrightarrow (x - y)$ is even.
- $A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$
- $(1,1) \in R, (1,3) \in R, (2,2) \in R.$



Definition 36. Let R be a relation from the set A to the set B.The inverse relation R^{-1} from B to A is defined as

$$R^{-1} = \{(y, x) \in B \times A,\ (x, y) \in R\}.$$

What it says is that for every pair (x, y) in R, you take it, flip the role of x and y to get (y, x), which then belongs to R^{-1}.

Example 74. Consider the sets $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$, with the relation $(x, y) \in R \leftrightarrow x$ divides y. Let us look at it step by step. First we compute the cartesian product $A \times B$:

$$A \times B = \{(2, 2), (2, 6), (2, 8), (3, 2), (3, 6), (3, 8), (4, 2), (4, 6), (4, 8)\}.$$

Then we check for which pair (x, y) it is true that $x \mid y$. For example, if $(x, y) = (2, 2)$, then $2 \mid 2$ and $(2, 2) \in R$, but for $(x, y) = (3, 2)$, 3 does not divide 2, and $(3, 2)$ is not in R. Trying out all the pairs, we get

$$R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}.$$

Now for every pair $(x, y) \in R$, we flip the role of x and y to get

$$R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}.$$

In this case, there is a nice interpretation of what R^{-1} means: $(x, y) \in R \leftrightarrow x|y$, but $x|y \iff y$ is a multiple of x and R^{-1} describes the relation $(y, x) \in R^{-1} \leftrightarrow y$ is a multiple of x. If one draws a diagram, then to go from R to R^{-1}, all is needed is to change the direction of the arrows!

Apart diagrams, another convenient way to represent a relation is to use a matrix representation. Take a binary relation R from the set $A = \{a_1, \ldots, a_m\}$ to the set $B = \{b_1, b_2, \ldots, b_n\}$. Create a matrix whose rows are indexed by the elements of A (thus m rows) and whose columns are indexed by the elements of B (thus n columns). Now the entry (i, j) of the matrix, corresponding to the ith row and jth column, contains $a_i R b_j$, that is, a truth value (True or False), depending on whether it is true or not that $a_i R b_j$ (that is, a_i is related to b_j).

Example 75. Take $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation R defined by $(x, y) \in R \leftrightarrow x$ divides y. Then the rows of the matrix are indexed by $2, 3, 4$, and the columns by $2, 6, 8$. We thus get

$$\begin{pmatrix} 2R2 & 2R6 & 2R8 \\ 3R2 & 3R6 & 3R8 \\ 4R2 & 4R6 & 4R8 \end{pmatrix} = \begin{pmatrix} T & T & T \\ F & T & F \\ F & F & T \end{pmatrix}.$$

Inverse of a Binary Relation

Let R be a relation from A to B. The inverse relation R^{-1} from B to A is defined as: R^{-1} ={(y,x) $\in B \times A$ |(x,y) $\in R$}.

Example. A={2,3,4}, B={2,6,8}, $(x, y) \in R \leftrightarrow x$ divides y.

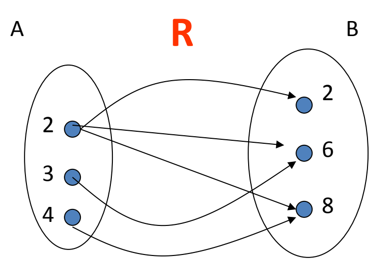
- $A \times B = \{(2,2), (2,6), (2,8), (3,2), (3,6), (3,8), (4,2), (4,6), (4,8)\}$
- $(2,2) \in R, (2,6) \in R, (2,8) \in R, (3,6) \in R, (4,8) \in R$

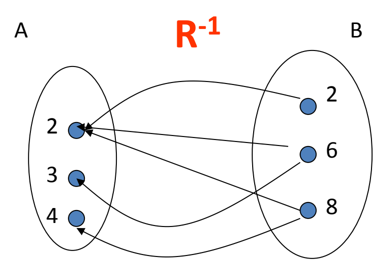
- $(2,2) \in R^{-1}, (6,2) \in R^{-1}, (8,2) \in R^{-1}, (6,3) \in R^{-1}, (8,4) \in R^{-1}$
- $(y, x) \in R^{-1} \leftrightarrow y$ is a multiple of x.

Graphically

Example. A={2,3,4}, B={2,6,8}, $(x, y) \in R \leftrightarrow x$ divides y.

- $(2,2) \in R, (2,6) \in R, (2,8) \in R, (3,6) \in R, (4,8) \in R$
- $(2,2) \in R^{-1}, (6,2) \in R^{-1}, (8,2) \in R^{-1}, (6,3) \in R^{-1}, (8,4) \in R^{-1}$





Matrix Representation (I)

$A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3, b_{4,})$,
$R = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_4)\}$

$a_i R b_j$ is represented by true, false else: $\begin{bmatrix} F & T & F & F \\ T & F & F & F \\ T & F & F & T \end{bmatrix}$

Example. A={2,3,4}, B={2,6,8},
$(x, y) \in R \leftrightarrow x$ divides y.

$$\begin{bmatrix} T & T & T \\ F & T & F \\ F & F & T \end{bmatrix}$$

Matrix Representation (II)

R relation from *A* to *B*: $R^{-1}=\{(y,x) \in B \times A \mid (x,y) \in R\}$.

$A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3, b_{4,})$,
$R = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_4)\}$
$R^{-1} = \{(b_2, a_1), (b_1, a_2), (b_1, a_3), (b_4, a_3)\}$

$a_i R b_j$ =true $\begin{bmatrix} F & T & F & F \\ T & F & F & F \\ T & F & F & T \end{bmatrix}$ $b_i R^{-1} a_j$=true $\begin{bmatrix} F & T & T \\ T & F & F \\ F & F & F \\ F & F & T \end{bmatrix}$

The matrix of R^{-1} is the transpose of the matrix of R.

Composition of Relations

Given R in $A \times B$, and S in $B \times C$, the composition of R and S is a relation on $A \times C$ defined by

$$R \circ S = \{(a,c) \in A \times C \mid \exists\, b \in B, aRb \text{ and } bSc\}.$$

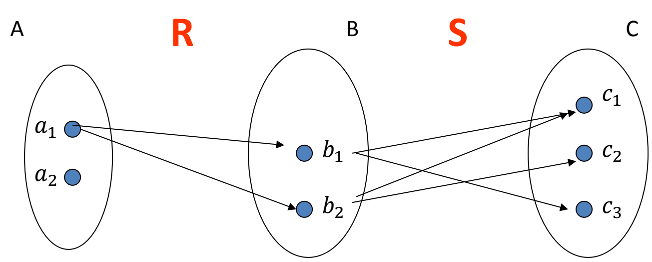
Example. $A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$

- $R = \{(a_1, b_1), (a_1, b_2)\}$
- $S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$
- What is $R \circ S$?

- $R \circ S = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$

Graphically

- **Example**. $A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$
- $R = \{(a_1, b_1), (a_1, b_2)\}$
- $S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$
- $R \circ S = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$



We may ask next how to interpret the inverse relation R^{-1} on its matrix. First of all, if R goes from $A = \{a_1, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$, then R^{-1} goes from B to A. This means that the rows of the matrix of R^{-1} will be indexed by the set $B = \{b_1, b_2, \dots, b_n\}$, while its columns by the set $A = \{a_1, \dots, a_m\}$. Then, by definition of R^{-1}, whenever there was a T (true) in row i and column j, this meant that $(a_i, b_j) \in R$, thus $(b_j, a_i) \in R^{-1}$, and this becomes a T (true) in row j and column i. If you take the first row of the matrix of R, whenever $(a_1, b_j) \in R$, for the column j, $(b_j, a_1) \in R^{-1}$, and a true in the first row of R becomes a true in the first column of R^{-1}, and the other entries which are false in the first row of R similarly become false in the first column of R^{-1}. This shows that the matrix of R^{-1} is the transpose of R! (recall that the transpose of a matrix is obtained by switching rows and columns).

Example 76. We continue the above example with $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation R defined by $(x, y) \in R \leftrightarrow x$ divides y. We have that $R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$ thus $R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$. Then the matrix of R and R^{-1} are respectively given by

$$\begin{pmatrix} T & T & T \\ F & T & F \\ F & F & T \end{pmatrix}, \quad \begin{pmatrix} T & F & F \\ T & T & F \\ T & F & T \end{pmatrix}.$$

We continue to explore properties of relations.

Definition 37. Given two relations $R \in A \times B$ and $S \in B \times C$, the composition of R and S is a relation on $A \times C$ defined by

$$R \circ S = \{(a, c) \in A \times C,\ \exists b \in B,\ aRb, bSc\}.$$

What it says is that for (a, c) to be part of your relation $R \circ S$, we need to find an element $b \in B$, with the property that a is in relation with b, and b is in relation with c. It is probably best visualize on a diagram: draw 3 circles for A, B, C, and arrows from A to B using the relation R, and arrows from B to C using the relation S. If you can find a path following those arrows from a to c, then (a, c) is in $R \circ S$.

Example 77. Consider the sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$, with relations defined by

$$R = \{(a_1, b_1), (a_1, b_2)\},\ S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}.$$

Reflexivity

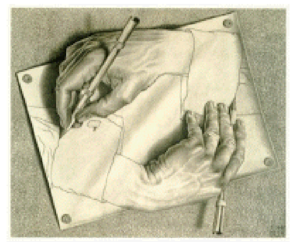
A relation R on a set A is reflexive if every element of A is related to itself: $\forall x \in A, xRx$

Examples.

1. A=Z, xRy ↔ x=y : reflexive

2. A=Z, xRy ↔ x>y : not reflexive

3. Reflexivity on the matrix representing R?

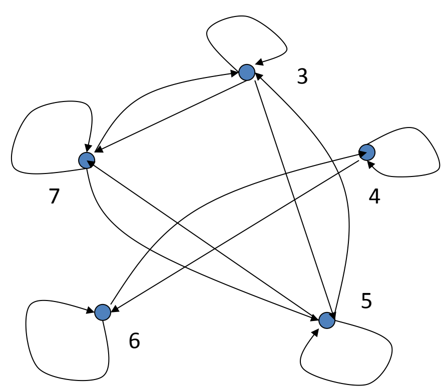


Escher, M C (1948): 'Drawing Hands'

Graphically

A= {3,4,5,6,7}, *xRy* ↔ (*x*-*y*) is even

- *R reflexive*



To compute $R \circ S$, start with (a_1, b_1) and look for pairs starting with b_1 in S: (b_1, c_1) and (b_1, c_3). Therefore, (a_1, b_1) combined with (b_1, c_1) gives the pair (a_1, c_1) and the pair (a_1, b_1) combined with (b_1, c_3) gives (a_1, c_3). We do the same with (a_1, b_2) and pairs starting with b_2 in S to find (a_1, c_2), and

$$R \circ S = \{(a_1, c_1), (a_1, c_2), (a_1, c_3)\}.$$

So far, we were looking at binary relations from A to B. Next we focus on relations where $A = B$, that is we have relations from a set into itself.

Definition 38. A relation R on a set A is reflexive if every element of A is related to itself: $\forall x \in A,\ xRx$.

Example 78. If A is the set $\mathbb{Z}$ of integers, and the relation R is defined by $xRy \leftrightarrow x = y$, then this relation is reflexive, because it is true that x is always in relation with itself ($xRx \leftrightarrow x = x$ is always true).

But $xRy \leftrightarrow x > y$ is not reflexive, because it is never true that xRx (we never have $x > x$).

On the matrix representation of R, reflexivity is shown by having T (true) on the diagonal of the matrix. If one represents a relation on itself with a diagram, reflexivity will be seen by having arrows looping on every element of the diagram!

Definition 39. A relation R on a set A is symmetric if $(x, y) \in R$ implies $(y, x) \in R$: $\forall x \forall y \in A,\ xRy \to yRx$.

On a diagram, this is visualized with having a second arrow between 2 elements of A in the other direction whenever you have one arrow in one direction.

Example 79. If A is the set $\mathbb{Z}$ of integers, and the relation R is defined by $xRy \leftrightarrow x = y$, then this relation is symmetric, because it is true that if x is in relation with y then y is in relation with x ($xRy \leftrightarrow x = y$ implies $y = x \leftrightarrow yRx$).

But $xRy \leftrightarrow x > y$ is not symmetric, because it is never true that xRy implies yRx (we never have $x > y$ that implies $y > x$).

Definition 40. A relation R on a set A is transitive if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$: $\forall x \forall y \forall z \in A,\ xRy \wedge yRz \to xRz$.

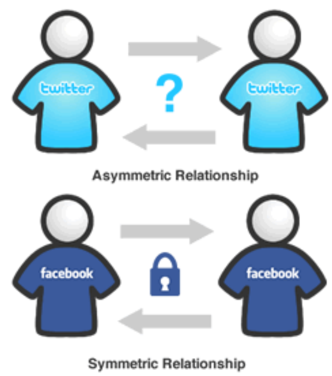
Symmetry

A relation R on a set A is symmetric if $(x, y) \in R$ implies $(y, x) \in R$: $\forall x\, \forall y\, xRy \rightarrow yRx$

Examples.

1. A=Z, xRy ↔ x=y : symmetric

2. A=Z, xRy ↔ x>y : not symmetric

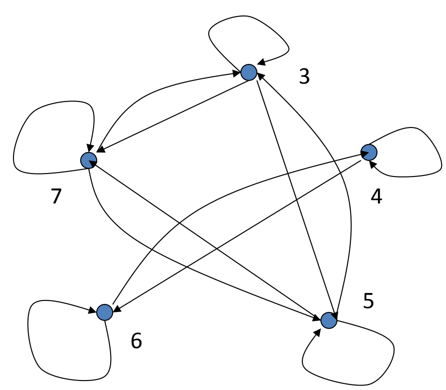


http://bokardo.com/archives/relationship-symmetry-in-social-networks-why-facebook-will-go-fully-asymmetric/

Graphically

A= {3,4,5,6,7}, *xRy* ↔ (*x*-*y*) is even

- *R reflexive*
- *R symmetric*



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Example 80. If A is the set $\mathbb{Z}$ of integers, and the relation R is defined by $xRy \leftrightarrow x = y$, this relation is transitive, because it is true that if x is in relation with y and y is in relation with z then x is in relation with z ($xRy \leftrightarrow x = y$ and $y = z \leftrightarrow yRz$ implies that $x = y = z$ that is $x = z \leftrightarrow xRz$).

Also $xRy \leftrightarrow x > y$ is transitive, because if $xRy \leftrightarrow x > y$ and $yRz \leftrightarrow y > z$, then we have $x > y > z$ that is $x > z \leftrightarrow xRz$.

If a relation R on a set A turns out to satisfy the 3 properties we have just seen: reflexivity, symmetry, and transitivity, then this relation is special, and thus gets a special name:

Definition 41. A relation R on a set A is an equivalence relation if R is reflexive, symmetric and transitive. The equivalence class of a in A is

$$[a] = \{x \in A,\ aRx\}.$$

There is a reason for this name: an equivalence relation is so strong, it so strongly ties together elements that are in relation with each other, that instead of looking at elements one by one, we can just consider all those elements in relation with each other as one entity, called equivalence class.

Example 81. Consider the set $A = \{3, 4, 5, 6, 7\}$ with the relation $xRy \leftrightarrow (x - y)$ is even. Then R is reflexive: indeed, xRx is always true, since $(x - x) = 0$ which is even. Also R is symmetric: indeed, $xRy \leftrightarrow (x - y)$ is even implies that $-(x - y) = y - x$ is also even, and then $(y - x)$ is even $\leftrightarrow yRx$. Finally it is transitive: if $xRy \leftrightarrow (x - y)$ is even, and $yRz \leftrightarrow (y - z)$ is even, then $(x - z) = (x - y) + (y - z)$ which is even (sum of two even numbers is even), thus $(x - z)$ is even $\leftrightarrow xRz$. The equivalence class of $[3]$ is the set of elements in relation with 3, that is $[3] = \{3, 5, 7\}$, similarly $[4] = \{4, 6\}$.

It turns out that equivalence classes partition A (for A a set with R a relation which is an equivalence relation). See Exercise 81.

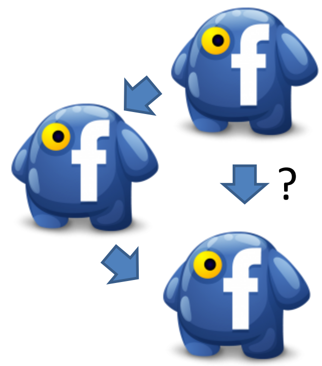
The above example does form an equivalence relation, but it probably does not explain well the concept of equivalence relation, so let us try to get a better feeling using something that we already know (even though we do not know yet that these are equivalence classes!) namely, integers modulo n.

Transitivity

A relation R on a set A is transitive if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$: $\forall x\, \forall y\, \forall z\; xRy \wedge yRz \rightarrow xRz$

Examples.

1. A=Z, xRy ↔ x=y : transitive
2. A=Z, xRy ↔ x>y : transitive



http://www.apkdad.com/tag/atrium-for-facebook-apk/

Equivalence Relation

A relation R on a set A is an equivalence relation if

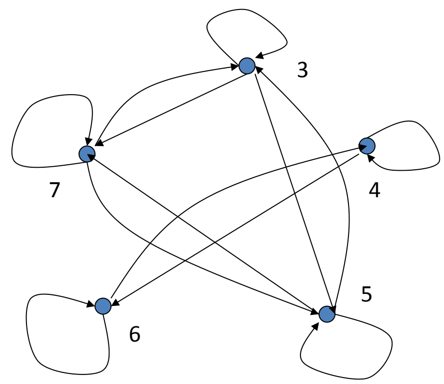
1. R is reflexive: $\forall x \in A, xRx$
2. R is symmetric: $\forall x\, \forall y\; xRy \rightarrow yRx$
3. R is transitive: $\forall x\, \forall y\, \forall z\; xRy \wedge yRz \rightarrow xRz$

Equivalence class of a in A: [a] ={x ∈ A | aRx} for R an equivalence relation.

Example

$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x-y)$ is even

- *R reflexive*
- *R symmetric*
- *R transitive*
- [3]={3,5,7},[4]={4,6}

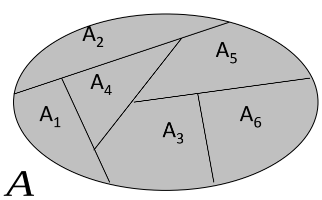


Equivalence Classes

Partition of a set A:

$A_i \cap A_j = \phi$ whenever $i \neq j$

$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 = A$



Equivalence classes of A form a partition of A.

Integers mod n (I)

$a \equiv b \pmod{n} \leftrightarrow a=qn+b$

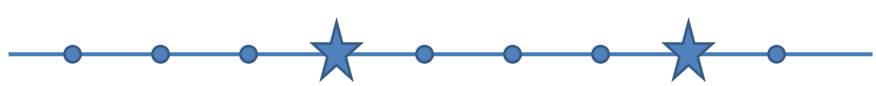
≡ (mod n) is an equivalence relation:

1. ≡ (mod n) is reflexive: $\forall x \in A, x \equiv x \ mod \ n$
2. ≡ (mod n) is symmetric: $\forall x \, \forall y \; x \equiv y \ (mod \ n) \rightarrow y \equiv x \ (mod \ n)$
3. ≡ (mod n) is transitive: $\forall x \, \forall y \, \forall z$ $x \equiv$ y (mod n)∧ y≡z (mod n) → x≡z (mod n).

Integers mod n (II)

Equivalence class of [0]={0,n,2n,3n,…,-n,-2n,-3n…}
Equivalence class of [1]={1,n+1,2n+1,3n+1,…,-n+1,-2n+1…}

Example. Integers mod 4



- Integers mod *n* can be represented as elements between 0 and *n-1*: {0,1,2,…,*n-1*}

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Example 82. The relation $\equiv \pmod n$ is an equivalence relation on $\mathbb{Z}$.

- It is reflexive: $x \equiv x \pmod n$ is always true.
- It is symmetric: $x \equiv y \pmod n$ means that $x = qn + y$ for some integer q, thus $y = -qn + x$ and $y \equiv x \pmod n$.
- It is transitive: if $x \equiv y \pmod n$ and $y \equiv z \pmod n$ then we have $x = qn + y$ and $y = rn + z$ thus $x = qn + y = qn + rn + z = n(q + r) + z$ and $x \equiv z \pmod n$.

Now what is the equivalence class of 0? it is formed by all multiples of n:

$$[0] = \{\ldots, -2n, -n, 0, n, 2n, \ldots\},$$

and similarly the equivalence class of 1 is all multiples of n, plus 1, and we see that there are exactly n equivalence classes, which partition $\mathbb{Z}$:

$$[0], [1], [2], \ldots, [n-1].$$

This is why when we do operations modulo n, we are allowed to pick one element per equivalence class, namely $0, 1, \ldots, n-1$ and work with them!!

We add one more property to those we know: reflexivity, symmetry, and transitivity.

Definition 42. A relation R on a set A is antisymmetric if $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$: $\forall x \forall y, xRy \wedge yRx \to x = y$.

Note that symmetry and antisymmetric are not related, despite their name, see Exercise 78.

Example 83. If A is the set $\mathbb{Z}$ of integers, and the relation R is defined by $xRy \leftrightarrow x = y$, this relation is antisymmetric, because it is true that if x is in relation with y and y is in relation with x then $x = y$ ($xRy \leftrightarrow x = y$ and $y = x \leftrightarrow yRx$ implies that $x = y$).

Also $xRy \leftrightarrow x > y$ is antisymmetric, because we have a statement which is vacuously true!! if $xRy \leftrightarrow x > y$ and $yRx \leftrightarrow y > x$, well, this statement is always false...when we have a $p \to q$ where p is false then $p \to q$ is true (apply here with $p =$ "$xRy \wedge yRx$" and $q =$ "$x = y$").

Consider two sets B and C and the relation B is in relation with $C \leftarrow B \subseteq C$. Then $B \subseteq C$ and $C \subseteq B$ implies that $B = C$! this is what we used to show set equality (double inclusion), and this shows that this relation is antisymmetric!

Antisymmetry

A relation R on a set A is antisymmetric if $(x, y) \in R$ and $(y, x) \in R$ implies x=y: $\forall x\ \forall y\ xRy \wedge yRx \rightarrow x = y$

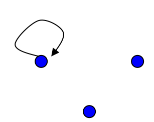
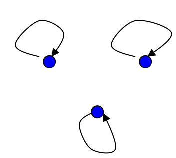
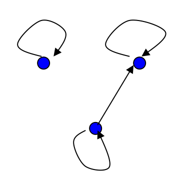
Examples.

1. A=Z, xRy $\leftrightarrow$ x=y : antisymmetric

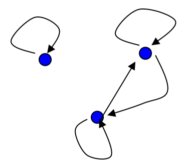
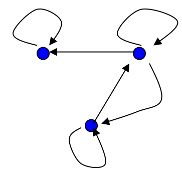
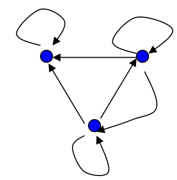
2. A=Z, xRy $\leftrightarrow$ x>y : vacuously true

3. BRC $\leftrightarrow$ $B \subseteq C$: antisymmetric

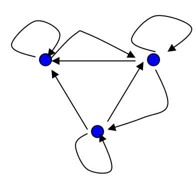
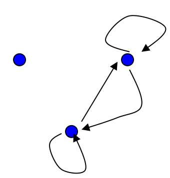
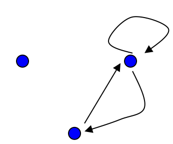
More Examples

			
Reflexive?	N	Y	Y
Symmetric?	Y	Y	N
Antisymmetric?	Y	Y	Y
Transitive?	Y	Y	Y

More Examples

			
Reflexive?	Y	Y	Y
Symmetric?	Y	N	N
Antisymmetric?	N	N	N
Transitive?	Y	N	Y

More Examples

			
Reflexive?	Y	N	N
Symmetric?	N	Y	Y
Antisymmetric?	N	N	N
Transitive?	N	Y	N

Partial Order

A relation R on a set A is a partial order if R is reflexive, antisymmetric and transitive.

Example. A=Z, $xRy \leftrightarrow x \leq y$

Notion of partial order useful for scheduling problems across possibly different domains.

Closure

Let A be a set and R a binary relation on A.

The closure of a relation $R \subseteq A \times A$ with respect to a property P (P being reflexive, symmetric, or transitive) is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P.

Definition 43. A relation R on a set A is a partial order if R is reflexive, antisymmetric and transitive.

The word partial order can be explained by the antisymmetry property. It is not possible to have "a loop" between two elements, namely a relation from one element to another, and back.

Example 84. If A is the set $\mathbb{Z}$ of integers, and the relation R is defined by $xRy \leftrightarrow x \leq y$, this relation is a partial order:

- It is reflexive: $x \leq x$ always.
- It is antisymmetric: $x \leq y$ and $y \leq x$ implies that $x = y$.
- It is transitive: if $x \leq y$ and $y \leq z$ then $x \leq y \leq z$ and thus $x \leq z$ as needed.

A set with a relation R may not satisfy a given property, may it be reflexivity, symmetry, or transitivity, but then, one may wonder whether it is possible to "complete" the set with more elements to obtain the desired property. This gives rise to the notion of closure:

Definition 44. Consider a relation R on a set A. The closure of R with respect to the property P (where $P \in \{$ reflexivity, symmetry, transitivity $\}$) is the binary relation obtained by adding the minimum number of ordered pairs to R to obtain the property P.

When the property P is transitivity, we speak of transitive closure:

Definition 45. Consider a relation R on a set A. The transitive closure of R is the binary relation R^t, that satisfies the properties:

- R^t is transitive,
- $R \subseteq R^t$,
- If S is any other transitive relation that contains R, then $R^t \subset S$.

The first property says the property of transitivity is satisfied, the second one that R is contained in R^t and the third one says R^t is minimal with this property!

Transitive Closure

Let A be a set and R a binary relation on A.
The transitive closure of R is the binary relation R^t on A that satisfies the following three properties:

1. R^t is Transitive
2. $R \subseteq R^t$
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$

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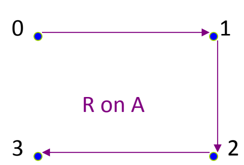
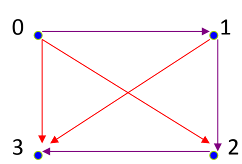
Example

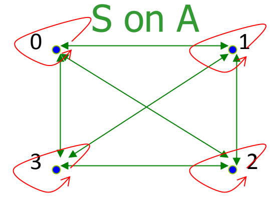
Let A = {0,1,2,3}
Consider a relation R = {(0,1),(1,2),(2,3)} on A.





R^t on A

S on A



S is transitive and $R \subseteq$ **S**
Thus, $R^t \subseteq$ **S**

R^t = {(0,1),(1,2),(2,3),(0,2),(0,3),(1,3)}

Example 85. Take the relation $R = \{(0,1),(1,2),(2,3)\}$ on the set $A = \{0,1,2,3\}$. We have $0R1$ and $1R2$ therefore we need $0R2$, that is $(0,2)$, by transitivity. Then we have $1R2$ and $2R3$, so we need $1R3$, that is $(1,3)$, by transitivity. So our candidate for R^t is now

$$R^t = \{(0,1),(1,2),(2,3),(0,2),(1,3)\}.$$

Then we notice that now, $0R1$, and $1R3$, so we need $0R3$, that is $(0,3)$, which gives the final answer for the transitive closure:

$$R^t = \{(0,1),(1,2),(2,3),(0,2),(1,3),(0,3)\}.$$

We know that we got the final answer by checking every two pairs. Note that the relation S given by

$$S = \{(0,1),(1,2),(2,3),(0,2),(1,3),(0,3),(0,0),(1,1),(2,2),(3,3)\}$$

is not a transitive closure, because it is not minimal! Namely, it contains R^t.

This gives an algorithm to construct a transitive closure. Start with a set A and a binary relation R on A.

1. Start with R, and $\forall x,y,z \in A$, if $(xRy \wedge yRz \wedge x \not R z)$, which means that the transitivity property is not satisfied, then add (x,z).

2. Repeat the above step, until the relation becomes transitive. This can be done in a finite number of steps when the cardinality $|A|$ of A is finite.

We note that the ordering of the edges does not matter. It will give the same result, since we are considering the minimum number of edges to be added to get a transitive relation.

So far, we have considered only "binary" relations, that is, relations defined over two sets. We can extend the definition to non-binary relations, defined over n sets, with $n \geq 3$. For example, say you would like to associate to a set of students their respective grade to their respective courses. This would need three sets, a set of students, a set of courses, and a set of grades.

Definition 46. Let $A_1,\dots,A_n$ be sets. An n-ary relation R is a subset of $A_1 \times \cdots \times A_n$. We say that $a_1,\dots,a_n$ are related if $(a_1,\dots,a_n) \in R$.

Construction of Transitive Closure

- Let A be a set and R a binary relation on A.
- Start with R, and do the following:

  $\forall x, y, z \in A$, if $(xRy \land yRz \land x \not R z)$ then add (x, z)

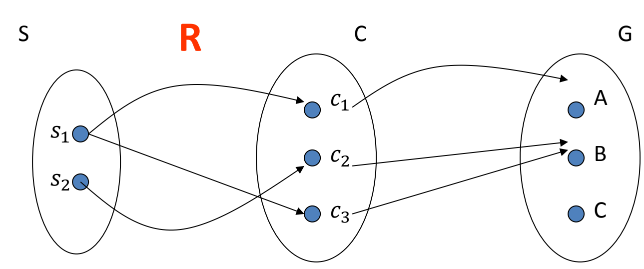
- Repeat until the obtained relation is transitive (will stop when $|A|$ is finite)
- The ordering in which the edges are added does not matter.

Non-binary Relations: Example

Example.

$S = \{s_1, s_2\}$ students, $C = \{c_1, c_2, c_3\}$ courses

$G = \{A, B, C\}$ grades, $(s_1, c_1, A), (s_1, c_3, B), (s_2, c_2, B)$



We next describe possible operations that can be performed on n-ary relations.

Definition 47. Let $R, S \subset A_1 \times \cdots \times A_n$ be two relations. Then

$$\bar{R} = (A_1 \times \cdots \times A_n - R)$$

is the relational complement of R. Namely

$$(a_1, \ldots, a_n) \in \bar{R} \iff (a_1, \ldots, a_n) \notin R.$$

Example 86. Take the sets $A = \{1, 2\}$, $B = \{3, 5\}$, and the relation $R = \{(1, 3), (2, 5)\}$. Then $A \times B = \{(1, 3), (1, 5), (2, 3), (2, 5)\}$. So $\bar{R}$ are the elements which are in $A \times B$, but not in R, namely

$$\bar{R} = \{(1, 5), (2, 3)\}.$$

Definition 48. Let $R, S \subset A_1 \times \cdots \times A_n$ be two relations. Then $R \cup S$ is the union of relations R and S, that is, it is the relation such that

$$(a_1, \ldots, a_n) \in R \cup S \iff (a_1, \ldots, a_n) \in R \vee (a_1, \ldots, a_n) \in S.$$

Example 87. Take the sets $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, and the relations $R = \{(1, 1), (2, 2), (3, 3)\}$ and $S = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$. Then

$$R \cup S = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (1, 4)\}.$$

Since R and S are subsets of $A \times B$, the union of the relations is really the union of the subsets R and S.

Definition 49. Let $R, S \subset A_1 \times \cdots \times A_n$ be two relations. Then $R \cap S$ is the intersection of relations R and S, that is, it is the relation such that

$$(a_1, \ldots, a_n) \in R \cup S \iff (a_1, \ldots, a_n) \in R \wedge (a_1, \ldots, a_n) \in S.$$

Example 88. Take the sets $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, and the relations $R = \{(1, 1), (2, 2), (3, 3)\}$ and $S = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ as in the previous example. Then

$$R \cap S = \{(1, 1)\}.$$

Since R and S are subsets of $A \times B$, the intersection of the relations is really the intersection of the subsets R and S.

Non-binary Relations

Let $A_1, \ldots, A_n$ be sets. A n-ary relation R is a subset of $A_1 \times \cdots \times A_n$. $a_1, \ldots, a_n$ are related if $(a_1, \ldots, a_n) \in R$.

Example.

$S = \{s_1, s_2\}$ students, $C = \{c_1, c_2, c_3\}$ courses

$G = \{A, B, C\}$ grades, $(s_1, c_1, A), (s_1, c_3, B), (s_2, c_2, B)$

Complement of a Relation

Let R, S ⊆ $A_1 \times \cdots \times A_n$ be two relations.

$$\bar{R} = (A_1 \times \cdots \times A_n - R)$$

is the relational complement of R, i.e.

$(a_1, a_2, a_3, \ldots, a_n) \in \bar{R} \Leftrightarrow (a_1, a_2, a_3, \ldots, a_n) \notin R$

Example. A = {1, 2}, B = {3, 5} and R = {(1, 3), (2, 5)} then

$\bar{R}$ = A × B − R = {(1, 5), (2, 3)}

Union of Relations

Let R, S ⊆ $A_1 \times \cdots \times A_n$ be two relations.

R ∪ S is the relation such that

$$(a_1, a_2, a_3, \ldots, a_n) \in R \cup S \Leftrightarrow$$
$$(a_1, a_2, a_3, \ldots, a_n) \in R \vee (a_1, a_2, a_3, \ldots, a_n) \in S$$

Example. A = {1, 2,3}, B = {1,2,3, 4} and R = {(1, 1), (2, 2),(3,3)}, S={(1,1),(1,2),(1,3),(1,4)} then

R ∪ S = {(1, 1), (2, 2),(3,3),(1,2),(1,3),(1,4)}

Intersection of Relations

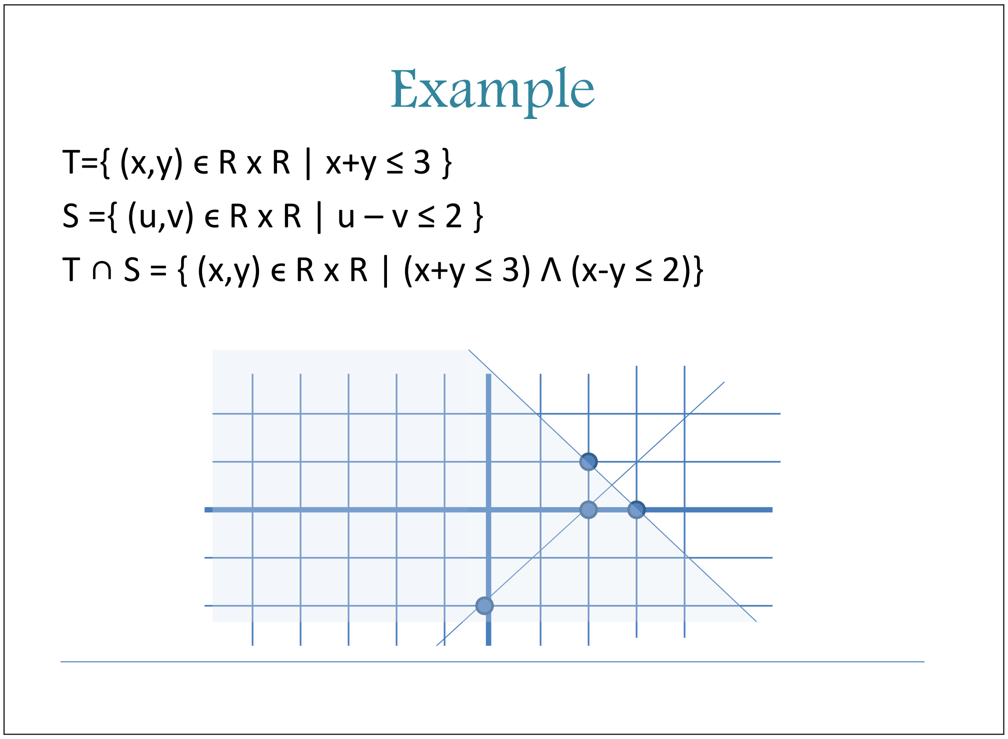
Let R, S ⊆ $A_1 \times \cdots \times A_n$ be two relations.

R ∩ S is the relation such that

$$(a_1, a_2, a_3, \ldots, a_n) \in R \cap S \Leftrightarrow$$
$$(a_1, a_2, a_3, \ldots, a_n) \in R \wedge (a_1, a_2, a_3, \ldots, a_n) \in S$$

Example. A = {1, 2,3}, B = {1,2,3, 4} and R = {(1, 1), (2, 2),(3,3)}, S={(1,1),(1,2),(1,3),(1,4)} then

R ∩ S = {(1, 1)}



Example 89. Consider the set $\mathbb{R} \times \mathbb{R}$, with the following two relations:

$$T = \{(x, y) \in \mathbb{R} \times \mathbb{R},\ x + y \leq 3\}, S = \{(u, v) \in \mathbb{R} \times \mathbb{R},\ u - v \leq 2\}.$$

Then the intersection of T and S is

$$T \cap S = \{(x, y) \in \mathbb{R} \times \mathbb{R},\ (x + y \leq 3) \wedge (x - y \leq 2)\}.$$

To visualize this intersection, draw the 2-dimensional real plane. Then draw the two lines $x + y = 3$ and $u - v = 2$. Then identify the points that belong to T (you know they are on one side of the line, pick an easy example, for example the point $(0, 0)$ to know on which side), and identify the points that belong to S similarly. The intersection of T and S is then easily identified as the region that belongs to both T and S.

Exercises for Chapter 8

Exercise 70. Consider the sets $A = \{1, 2\}$, $B = \{1, 2, 3\}$ and the relation $(x, y) \in R \iff (x-y)$ is even. Compute the inverse relation R^{-1}. Compute its matrix representation.

Exercise 71. Consider the sets $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation $(x, y) \in R \iff x \mid y$. Compute the matrix of the inverse relation R^{-1}.

Exercise 72. Let R be a relation from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $xRy \leftrightarrow 2|(x - y)$. Show that if n is odd, then n is related to 1.

Exercise 73. This exercise is about composing relations.

1. Consider the sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$ with the following relations R from A to B, and S from B to C:

$$R = \{(a_1, b_1), (a_1, b_2)\},\ S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}.$$

   What is the matrix of $R \circ S$?

2. In general, what is the matrix of $R \circ S$?

Exercise 74. Consider the relation R on $\mathbb{Z}$, given by $aRb \iff a - b$ divisible by n. Is it symmetric?

Exercise 75. Consider a relation R on any set A. Show that R symmetric if and only if $R = R^{-1}$.

Exercise 76. Consider the set $A = \{a, b, c, d\}$ and the relation

$$R = \{(a, a), (a, b), (a, d), (b, a), (b, b), (c, c), (d, a), (d, d)\}.$$

Is this relation reflexive? symmetric? transitive?

Exercise 77. Consider the set $A = \{0, 1, 2\}$ and the relation $R = \{(0, 2), (1, 2), (2, 0)\}$. Is R antisymmetric?

Exercise 78. Are symmetry and antisymmetry mutually exclusive?

Exercise 79. Consider the relation R given by divisibility on positive integers, that is $xRy \leftrightarrow x|y$. Is this relation reflexive? symmetric? antisymmetric? transitive? What if the relation R is now defined over non-zero integers instead?

Exercise 80. Consider the set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Show that the relation $xRy \leftrightarrow 2|(x - y)$ is an equivalence relation.

Exercise 81. Show that given a set A and an equivalence relation R on A, then the equivalence classes of R partition A.

Exercise 82. Consider the set $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the relation

$$xRy \leftrightarrow \exists c \in \mathbb{Z},\ y = cx.$$

Is R an equivalence relation? is R a partial order?

Examples for Chapter 8

We saw the definition of partial order R on a set S. When a set S has a partial order R, we can give a special name to this set.

Definition 50. A set S together with a partial order R is called a partially ordered set (or poset) (S, R).

To write that a is related to b, that is aRb, in the case where the relation R is actually a partial order, we introduce the notation $a \preceq b$, and $a \prec b$ when a is different from b.

Definition 51. If $(S, \preceq)$ is a poset, and every two elements of S satisfy that either $a \preceq b$ or $b \preceq a$, then S is called a totally ordered set and $\preceq$ is called a total order.

Example 90. For example $(\mathbb{Z}, \leq)$ is a totally ordered set. To check this, we need to first check that we have a partial order, namely that $\leq$ satisfies reflexivity, antisymmetry and transitivity:

1. reflexivity: $\forall x \in \mathbb{Z}$, $x \leq x$ holds.
2. antisymmetry: $\forall x, y \in \mathbb{Z}$, $x \leq y \wedge y \leq x \to x = y$ holds.
3. transitivity: $\forall x, y, z \in \mathbb{Z}$ if $x \leq y \wedge y \leq z$, then we have that $x \leq y \leq z$ therefore it is true that $x \leq z$. So we have a partial order, or $(\mathbb{Z}, \leq)$ is a poset. Then we have that for every two integers x, y, we can decide whether $x \leq y$ or $y \leq x$, thus we have a totally ordered set.

Our first example of relations will be to define the notion of lexicographic order.

Definition 52. Given two posets $(A, \preceq_1)$ and $(B, \preceq_2)$, the lexicographic order on $A \times B$ is defined as

$$(a, b) \preceq (a', b') \iff (a \prec_1 a') \vee (a = a' \wedge b \preceq_2 b').$$

Notice that when we usually use the term "lexicographic order", it suggests alphabetical order. Indeed, if $\preceq_1$ and $\preceq_2$ are both the alphabetical order, and $A = B$, the definition gives us the ordering that one would find in a dictionary (with the convention that "no character" is less than "a").

Posets

A set S together with a partial order R is called a partially ordered set, or poset (S,R).

We write $a \preccurlyeq b$ to say that (a,b) is a partial order and $a \prec b$ when a is different from b.

If (S, $\preccurlyeq$) is a poset, and every two elements of S satisfy that either $a \preccurlyeq b$ or $b \preccurlyeq a$, then S is called a totally ordered set, and $\preccurlyeq$ is called a total order.

Example of totally ordered set? (Z, ≤)

Lexicographic Order (I)

Given two posets (A, $\preccurlyeq_1$) and (B, $\preccurlyeq_2$), the lexicographical order $\preccurlyeq$ on the Cartesian product $A \times B$ is defined as

$(a,b) \preccurlyeq (a',b')$ if and only if $a \prec_1 a'$ or ($a = a'$ and $b \preccurlyeq_2 b'$).

- What is the connection between this definition of lexicographical order and the usual one?

The usual one is alphabetical order, starting with the 1st character in the string, then the 2nd (if the first was equal) etc., considering "no character" for shorter words to be less than "a". It is a particular case of this definition.

- Generalize this definition to the Cartesian product of n sets.

Define $\preccurlyeq$ on $A_1 \times \ldots \times A_n$ by $(a_1,\ldots a_n) \prec (b_1,\ldots,b_n)$ if $a_1 \prec b_1$ or $a_1 = b_1, \ldots, a_i = b_i$ and $a_{i+1} \prec b_{i+1}$

The way it works is: you first compare the first character of two words, say *an* and *be*, and if one character is before the other, e.g. here $a \prec b$, then the word starting with a, that is *an* will come before the word *be*: $an \preceq be$. If the first character is the same for both words, say *an* and *aw*, then we look at the second one. In our definition, we consider only words of length 2, so if the first character is the same, we just need $n \preceq w$, so $an \preceq aw$. When dealing with words, we could compare *an* to *a*, so we have $a \preceq an$ because of the convention that "no character" is smaller than "a".

Note that the definition of lexicographic order is more general, in that both sets A and B could be different, and the partial orders on each set could be distinct too.

Now we can state this definition for more than two sets.

Definition 53. Given n posets $(A_1, \preceq_1), \ldots, (A_n, \preceq_n)$, the lexicographic order on $A_1 \times \ldots \times A_n$ is defined as

$$(a_1, \ldots, a_n) \preceq (b_1, \ldots, b_n) \iff (a_1 \prec_1 b_1) \vee (a_1 = b_1, \ldots, a_i = b_i \wedge a_{i+1} \prec_{i+1} b_{i+1}).$$

The lexicographic order is a partial order. This is because this property is inherited from the partial order on each component. For example, to check reflexivity, we need to see that

$$\forall (a,b) \in (A, \preceq_1) \times (B, \preceq_2), (a,b) \preceq (a,b).$$

By definition, $(a,b) \preceq (a,b) \iff a \prec_1 a \vee (a = a \wedge b \preceq_2 b)$ which is true, because $\preceq_2$ is reflexive. For antisymmetry, we need to see that

$$\forall (a,b), (a',b') \in (A, \preceq_1) \times (B, \preceq_2), (a,b) \preceq (a',b') \wedge (a',b') \preceq (a,b) \rightarrow (a,b) = (a',b').$$

If $(a \prec_1 a')$, it cannot be that $a = a'$, and we need to consider $(a' \prec a)$. The case $(a \prec_1 a') \wedge (a' \prec_1 a)$ takes the value false, therefore giving a truth value of true to the property of antisymmetry of the lexicographic order. Now for the case $(a = a')$, if $b \prec_2 b'$, then $(b \prec_2 b') \wedge (b' \prec_2 b)$ takes the value false, and again the truth value of antisymmetry is true, or $b = b'$ in which case $(a,b) = (a',b')$ as desired. For transitivity

$$\forall (a,b), (a',b'), (a'',b'') \in (A, \preceq_1) \times (B, \preceq_2), (a,b) \preceq (a',b') \wedge (a',b') \preceq (a'',b'') \rightarrow (a,b) \preceq (a'',b'').$$

We have 4 cases: $(a = a') \wedge (a' = a'')$, $(a = a') \wedge (a' \prec a'')$, $(a \prec a') \wedge (a' = a'')$, $(a \prec a') \wedge (a' \prec a'')$. For the first case, we get that $a = a' = a''$, and for the three other cases, we get by transitivity on A that $a \prec_1 a''$ as needed. Next we look at $a = a''$, to check the condition on $b, b''b''$. But again, for the 4 possible cases involving them, transitivity on B will give $b \preceq_2 b''$ as needed.

Lexicographic Order (II)

Given two posets $(A, \preccurlyeq_1)$ and $(B, \preccurlyeq_2)$, the lexicographical order $\preccurlyeq$ on the Cartesian product $A \times B$ is defined as

$(a,b) \preccurlyeq (a',b')$ if and only if $a \prec_1 a'$ or ($a = a'$ and $b \preccurlyeq_2 b'$).

- Is the lexicographic order a partial order? Yes it is!
- Is the lexicographic order a total order? If A and B are totally ordered, then yes, otherwise not necessarily.

Lexicographic Order (III)

Given two posets $(A, \preccurlyeq_1)$ and $(B, \preccurlyeq_2)$, the lexicographical order $\preccurlyeq$ on the Cartesian product $A \times B$ is defined as

$(a,b) \preccurlyeq (a',b')$ if and only if $a \prec_1 a'$ or ($a = a'$ and $b \preccurlyeq_2 b'$).

- Let A=(a,b,c) with $a \preccurlyeq_1 b \preccurlyeq_1 c$ and let B = {0,1} with $0 \preccurlyeq_2 1$. Compute the lexicographic ordering on AxB.

The lexicographic ordering on AxB is:
$(a,0) \preccurlyeq (a,1) \preccurlyeq (b,0) \preccurlyeq (b,1) \preccurlyeq (c,0) \preccurlyeq (c,1)$

The property of total order says that we must be able to compare any (a, b) and (a', b'). By definition, this comparison relies on componentwise comparison on A and B, so we need to have a total order on A and B.

Example 91. Suppose $A = \{a, b, c\}$ with $a \preceq_1 b \preceq_1 c$ and $B = \{0, 1\}$ with $0 \preceq_2 1$. A lexicographic order on $A \times B$ is obtained as follows. The pairs starting with a come first, namely $(a, 0)$ and $(a, 1)$. Then $(a, 0)$ comes first because 0 comes before 1 in the order we have on B:

$$(a, 0) \preceq (a, 1) \preceq (b, 0) \preceq (b, 1) \preceq (c, 0) \preceq (c, 1).$$

We now look at another example of sets and relations, which is that of scheduling. The set A we consider is a set of tasks, and there are constraints on this set, to specify the dependencies among the different tasks. In order to obtain a scheduling for the set A, we need to find an order in which to perform all the tasks, one at a time, but it must be such that the different constraints are respected. Finding a total ordering which is consistent with a partial order is called topological sorting. Formally:

Definition 54. A topological sort on a set A is a total order $\sqsubset$ on A such that $a \preceq b$ implies $a \sqsubset b$.

A standard toy example, which can be found in many references and which we will use here as well, is that of getting dressed, where the different tasks are putting on a specific clothing, and the partial order is given on the diagram shown on the slide. We want to compute a scheduling for getting dressed, which respects the given constraints.

One method to obtain such a scheduling is to use the notion of minimal/minimum element.

Definition 55. Let $\preceq$ be a partial order on a set A. An element a in A is minimum if and only if it is $\preceq$ every other element of A The element a is minimal if and only if no other element is $\preceq a$.

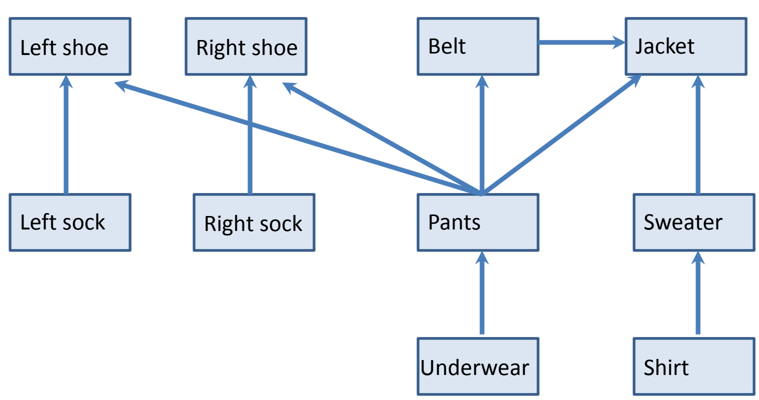
The difference between the two definitions is subtle. Note that for a minimum element, it should be comparable to every other element of the set, and this comparison may just not exist if the order is not total! If the order is total, then both definitions are the same.

Scheduling

- Suppose you have a set A of tasks, and a set of constraints specifying that a given task depends on the completion of another task.
- You are interested in a scheduling, that is an order in which to perform all the tasks one at a time while respecting the dependency constraints.
- Finding a total ordering that is consistent with a partial order is called topological sorting.

A topological sort of a partial order $\preccurlyeq$ on a set A is a total order [on A such that a $\preccurlyeq$ b implies a [b.

Topological Sort (I)



Compute a topological sort.

http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-spring-2010/readings/MIT6_042JS10_chap07.pdf

Let us look at our getting dressed scheduling problem:

1. We have four elements that are minimal, "left sock", "right sock", "underwear" and "shirt". So let us pick one of them, say "shirt".

2. Then we look at the left over elements, now "sweater" becomes a minimal element as well. So we can pick "sweater". This gives us "shirt" $\sqsubset$ "sweater".

3. Now the the minimal elements are "left sock", "right sock", "underwear", so let us choose "underwear".

4. Next the minimal elements are "left sock", "right sock", "pants". So we pick "left sock". This gives us "shirt" $\sqsubset$ "sweater" $\sqsubset$ "underwear" $\sqsubset$ "left sock".

5. The minimal elements are "right sock", "pants". Let us pick "right sock".

6. The minimal element is "pants". This gives: "shirt" $\sqsubset$ "sweater" $\sqsubset$ "underwear" $\sqsubset$ "left sock" $\sqsubset$ "right sock" $\sqsubset$ "pants".

7. We then pick "left shoe", after which we can pick "right shoe", and finally "belt" and "jacket", to get as final scheduling: "shirt" $\sqsubset$ "sweater" $\sqsubset$ "underwear" $\sqsubset$ "left sock" $\sqsubset$ "right sock" $\sqsubset$ "pants" $\sqsubset$ "left shoe" $\sqsubset$ "right shoe" $\sqsubset$ "belt" $\sqsubset$ "jacket".

Topological Sort (II)

An example (not unique) of topological sort is:

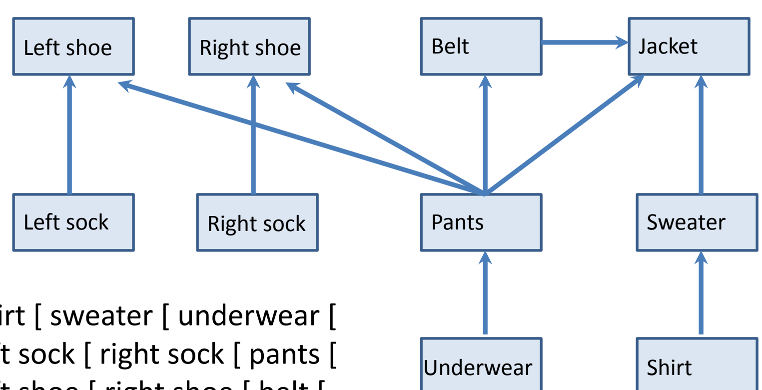
shirt [sweater [underwear [left sock [right sock [pants [left shoe [right shoe [belt [jacket

Let ≼ be a partial order on the set A. An element a in A is **minimum** iff it is ≼ every other element of A.

The element a is **minimal** iff no other element is ≼ a.

Minimum = minimal in total order but not otherwise!

Topological Sort (V)



shirt [sweater [underwear [left sock [right sock [pants [left shoe [right shoe [belt [jacket

http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-spring-2010/readings/MIT6_042JS10_chap07.pdf