Review: Dimension

Basis: Definition

Let V be a vector space. We say that the sequence of vectors $v_1, \ldots, v_n$ is a **basis** for V if and only if

- § $\mathrm{span}(\{v_1, \ldots, v_n\}) = V$
- § the sequence $v_1, \ldots, v_n$ is linearly independent.

Dimension Theorem

Lemma: Let V be a finite-dimensional vector space with $\text{span}(\{v_1, \ldots, v_m\}) = V$. If $n > m$ then any sequence of vectors $w_1, \ldots, w_n \in V$ is linearly dependent.

Theorem: Let V be a finite-dimensional vector space and $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be two bases for V. Then $m = n$.

In other words:

Any two bases for V have the same number of elements.

Dimension

Definition: Let V be a finite-dimensional vector space. The dimension of V is the number of elements in any basis for V.

This definition makes sense because of the dimension theorem.

When does a basis exist?

Having a basis is a very important property of a vector space—this is how we define the dimension.

Therefore we have to ask, when does a vector space have a basis?

Recall the definition of a finite-dimensional vector space:

V is finite-dimensional if there is a finite sequence of vectors $v_1, \ldots, v_n$ such that

$$\mathrm{span}(\{v_1, \ldots, v_n\}) = V$$

When does a basis exist?

Theorem: Any finite-dimensional vector space V has a basis.

Before proving the theorem, let's look at one special case: the smallest vector space $\{\mathbf{0}\}$.

A basis for $\{\mathbf{0}\}$ is given by the empty set.

This is why we defined the span of the empty set to be $\{\mathbf{0}\}$.

The empty set is linearly independent.

The dimension of $\{\mathbf{0}\}$ is 0 .

When does a basis exist?

Theorem: Any finite-dimensional vector space V has a basis.

We are going to prove this by giving an algorithm to construct a basis.

We are actually going to give two different algorithms.

Shrink algorithm: start with too many vectors and then successively throw away redundant ones.

Grow algorithm: start with the empty set and then successively add independent vectors until the span is V.

Shrink Algorithm

As V is finite-dimensional, by definition there is a finite sequence of vectors $v_1, \ldots, v_n$ such that

$$\operatorname{span}(\{v_1, \ldots, v_n\}) = V$$

Initialize $S = \{v_1, \ldots, v_n\}$.

While S is linearly dependent:

Take the first i satisfying

$$v_i = a_1 \cdot v_1 + \cdots + a_{i-1} \cdot v_{i-1} + a_{i+1} \cdot v_{i+1} + \cdots + a_n \cdot v_n$$

Set $S = S \setminus \{v_i\}$

End

Initialize $S = \{v_1, \ldots, v_n\}$ where $\mathrm{span}(S) = V$.

While S is linearly dependent:

Take the first i satisfying

$$v_i = a_1 \cdot v_1 + \cdots + a_{i-1} \cdot v_{i-1} + a_{i+1} \cdot v_{i+1} + \cdots + a_n \cdot v_n$$

Set $S = S \setminus \{v_i\}$

End

Key Fact: Throughout the algorithm $\mathrm{span}(S) = V$.

Reason: If $v_4 = a_1 \cdot v_1 + a_2 \cdot v_2 + a_3 \cdot v_3$ then

$$\mathrm{span}(\{v_1, v_2, v_3\}) = \mathrm{span}(\{v_1, v_2, v_3, v_4\})$$

Initialize $S = \{v_1, \ldots, v_n\}$ where $\mathrm{span}(S) = V$.

While S is linearly dependent:

Take the first i satisfying

$$v_i = a_1 \cdot v_1 + \cdots + a_{i-1} \cdot v_{i-1} + a_{i+1} \cdot v_{i+1} + \cdots + a_n \cdot v_n$$

Set $S = S \setminus \{v_i\}$

End

Key Fact: Throughout the algorithm $\mathrm{span}(S) = V$.

When the algorithm terminates S will be linearly independent, and the algorithm will always terminate as $\emptyset$ is linearly independent.

Grow Algorithm

Initialize $S = \emptyset.$

While $\mathrm{span}(S) \subsetneq V:$

Choose $w \in V \setminus \mathrm{span}(S)$

Set $S = S \cup \{w\}$

End

Key Fact: Throughout the algorithm S is linearly independent.

Reason: If v_1, v_2, v_3 are linearly independent and

$$v_4 \notin \mathrm{span}(\{v_1, v_2, v_3\})$$

then $\{v_1, v_2, v_3, v_4\}$ are linearly independent.

Initialize $S = \emptyset$.

While $\mathrm{span}(S) \subsetneq V$:

Choose $w \in V \setminus \mathrm{span}(S)$

Set $S = S \cup \{w\}$

End

Key Fact: Throughout the algorithm S is linearly independent.

Reason: If v_1, v_2, v_3 are linearly independent and

$$v_4 \notin \mathrm{span}(\{v_1, v_2, v_3\})$$

then $\{v_1, v_2, v_3, v_4\}$ are linearly independent.

Question 6 on Problem Set 10.

Initialize $S = \emptyset$.

While $\mathrm{span}(S) \subsetneq V$:

Choose $w \in V \setminus \mathrm{span}(S)$

Set $S = S \cup \{w\}$

End

Key Fact: Throughout the algorithm S is linearly independent.

When the algorithm terminates $\mathrm{span}(S) = V$.

The algorithm will terminate as V is finite dimensional.

Recap

We have now seen that every finite-dimensional vector space has a basis.

This means that we can define the dimension of any finite-dimensional vector space.

We let $\dim(V)$ denote the dimension of the vector space V.

Basic Property

Let's look at a basic property of dimension.

Let V, W be two vector spaces with $W \subseteq V$. Then

$$\dim(W) \leq \dim(V)$$

Proof: Suppose $\dim(V) = n$ and $v_1, \ldots, v_n$ form a basis for V. Then $W \subseteq \operatorname{span}(\{v_1, \ldots, v_n\})$.

Thus there can be at most n linearly independent vectors in W. The dimension of W is at most n.

Dimensions of the Four Subspaces

Reading: Strang 3.6

Learning objective: Be able to find a basis for the row and column spaces of a matrix.

Four Subspaces

Let A be an m-by-n matrix.

There are four fundamental subspaces associated with A.

nullspace $N(A) \subseteq \mathbb{R}^n$

column space $C(A) \subseteq \mathbb{R}^m$

row space $C(A^T) \subseteq \mathbb{R}^n$

span of the rows of A

left nullspace $N(A^T) \subseteq \mathbb{R}^m$

$$\{\vec{u} : A^T\vec{u} = \vec{0}_n\}$$
$$=$$
$$\{\vec{u} : \vec{u}^T A = \vec{0}_n^T\}$$

We will see how to find a basis for each of these subspaces and determine their dimension.

Dimensions of the Four Subspaces

Let A be an m-by-n matrix.

Say after Gaussian elimination A has r pivots.

nullspace $N(A) \subseteq \mathbb{R}^n$

dimension: $n - r$

left nullspace $N(A^T) \subseteq \mathbb{R}^m$

dimension: $m - r$

row space $C(A^T) \subseteq \mathbb{R}^n$

dimension: r

column space $C(A) \subseteq \mathbb{R}^m$

dimension: r

Rank

Let A be an m-by-n matrix.

Say after Gaussian elimination A has r pivots.

row space $C(A^T) \subseteq \mathbb{R}^n$

dimension: r

column space $C(A) \subseteq \mathbb{R}^m$

dimension: r

The dimension of the column space of A is also called the rank of A.

The remarkable fact that the dimension of the row and column space is equal is referred to as

row rank = column rank

Reduced Row Echelon Form

The main tool we will use to figure out these dimensions is the reduced row echelon form.

Recall the $PA = LU$ factorization we saw earlier.

This factorization was useful for solving systems of linear equations.

The reduced row echelon form leads to another factorization of A that reveals the dimension of the column space and row space.

Reduced Row Echelon Form

- All nonzero rows are above any all-zero row.
- Every pivot is strictly to the right of any pivot of a row above it.
- Every pivot is one and is the only nonzero entry in its column.

The first two items are accomplished by Gaussian elimination, the third by Gauss-Jordan.

The reduced row echelon form is unique.

Example

Gaussian elimination:

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 3 & 5 & 7\\ 2 & 4 & 6 & 8\end{bmatrix} \xrightarrow[\substack{R_2' = R_2 - R_1\\ R_3' = R_3 - 2R_1}]{} \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 2 & 4 & 6\\ 0 & 2 & 4 & 6\end{bmatrix} \xrightarrow[R_3' = R_3 - R_2]{} \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 2 & 4 & 6\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Gauss-Jordan phase makes pivots 1 and creates zeros above the pivots.

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 2 & 4 & 6\\ 0 & 0 & 0 & 0\end{bmatrix} \xrightarrow[R_2' = \frac{1}{2}R_2]{} \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{bmatrix} \xrightarrow[R_1' = R_1 - R_2]{} \begin{bmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

reduced row echelon form

Example

If we multiply together the elementary matrices implementing the row operations of Gaussian-Jordan elimination, we see

$$\underset{E}{\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{R}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Notice that as we did Gauss-Jordan elimination, E is not necessarily lower triangular.

Example

If we multiply together the elementary matrices implementing the row operations of Gaussian-Jordan elimination, we see

$$\underbrace{\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{R}$$

As E is the product of elementary matrices, it is invertible.

$$A = E^{-1} R$$

$$\underset{E}{\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{R}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

As E is the product of elementary matrices, it is invertible.

$$A = E^{-1} R$$

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{E^{-1}}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Question

For convenience, let's call $B = E^{-1}$.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Question: Are the columns of B linearly independent?

Answer

For convenience, let's call $B = E^{-1}$.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Question: Are the columns of B linearly independent?

Answer: Yes! As B is invertible its columns are linearly independent.

Question 2

For convenience, let's call $B = E^{-1}$.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Question: Are the first two columns of B linearly independent?

Answer

For convenience, let's call $B = E^{-1}$.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Question: Are the first two columns of B linearly independent?

Answer: Yes! Any subset of a linearly independent set is also linearly independent.

Subset of a linearly independent set

Any subset of a linearly independent set is also linearly independent.

If v_1, v_2, v_3 are linearly independent, then so are v_1, v_2.

We can see this via the contrapositive: if v_1, v_2 are linearly dependent then

$$a_1 \cdot v_1 + a_2 \cdot v_2 = \mathbf{0}$$

where a_1 or a_2 is not equal to zero. Then also

$$a_1 \cdot v_1 + a_2 \cdot v_2 + 0 \cdot v_3 = \mathbf{0}.$$

As E is the product of elementary matrices, it is invertible.

$$A = E^{-1}R$$

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{E^{-1}}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

As the last row of R is all zero, the product is not changed if we remove the last row of R and the last column of E^{-1}.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Rank Revealing Factorization

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}} \underset{C}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}}$$

This is the rank revealing factorization of A.

If the m-by-n matrix A has r pivots, we can write it as $A = BC$, where

B is a m-by-r matrix.

C is a r-by-n matrix.

Rank revealing factorization

If A has r many pivots, the rank revealing factorization expresses

$$A = \hat{B}\hat{R}$$

where

$\hat{B}$ is a m-by-r matrix.

$\hat{R}$ is a r-by-n matrix.

From the rank revealing factorization, we can find the dimension of the column space and the row space.

We can also find bases for these spaces.

Let's start with the column space.

Dimension of Column Space

Claim 1: If $A = \hat{B}\hat{R}$ then $C(A) \subseteq C(\hat{B})$.

Every column of A is a linear combination of the columns of $\hat{B}$.

Every column of A is in $C(\hat{B})$.

Dimension of Column Space

Because of the special structure of $\hat{R}$, we claim that also

$$C(\hat{B}) \subseteq C(A)$$

Every column of $\hat{B}$ is also a column of A.

Reason: $\hat{R}$ has an r-by-r identity matrix inside of it.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{\hat{B}}{\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}} \underset{\hat{R}}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}}$$

Pivot Columns

Claim: Every column of $\hat{B}$ is also a column of A .

Proof: For each $i = 1, \ldots, r$ there is a column of $\hat{R}$ (say the j^{th} column) that equals one in component i and zero everywhere else.

This means $A(:, j) = \hat{B}\hat{R}(:, j) = \hat{B}(:, i)$.

pivot columns

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}}_{\hat{B}} \underbrace{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}}_{\hat{R}}$$

Pivot Columns

Claim: Every column of $\hat{B}$ is also a column of A .

The columns of $\hat{B}$ appear in A in the columns corresponding to pivot columns of $\hat{R}$.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}}_{\hat{B}} \underbrace{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}}_{\hat{R}}$$

pivot columns

Basis of the Column Space

We have now shown that $C(\hat{B}) = C(A)$.

Theorem: The columns of $\hat{B}$ form a basis for $C(A)$.

Proof: The columns of $\hat{B}$ span $C(\hat{B}) = C(A)$.

The columns of $\hat{B}$ are linearly independent, as they are a subset of the columns of B , an invertible matrix.

Note: the columns of $\hat{B}$ are the first r columns of E^{-1}, where $EA = R$.

Basis of the Column Space

Theorem: The columns of $\hat{B}$ form a basis for $C(A)$.

Corollary: The columns of A corresponding to the pivot columns of $\mathrm{rref}(A)$ form a basis for $C(A)$.

Reason: These are the columns of $\hat{B}$.

Corollary: The dimension of $C(A)$ is the number of pivots of A.

Reason: This is the number of columns of $\hat{B}$.

Row Space

Now we turn our attention to the row space.

We will similarly show that the rows of $\hat{R}$ form a basis for the row space of A .

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}}_{\hat{B}} \underbrace{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}}_{\hat{R}}$$

This will show that the dimension of the row space equals the number of pivots (and the dimension of the column space).

Row Space

Let us recall how we arrived at the rank revealing factorization.

For $R = \mathrm{rref}(A)$ we had $EA = R$.

Similarly to Claim 1, this means the row space of R is contained in the row space of A .

Every row of R is a linear combination of the rows of A .

$$C(R^T) = C(A^T E^T) \subseteq C(A^T)$$

Row Space

Let us recall how we arrived at the rank revealing factorization.

For $R = \mathrm{rref}(A)$ we had $EA = R$.

The matrix E is invertible.

This means $A = E^{-1}R$.

The row space of A is contained in the row space of R.

Each row of A is a linear combination of the rows of R.

The row space of A and R is the same.

Row Space and Elementary Operations

This argument shows more generally that A and GA have the same row space for an invertible matrix G.

The row space does not change under elementary row operations.

A matrix A and its reduced row echelon form R have the same row space.

Row Space

Let us recall how we arrived at the rank revealing factorization.

From $R = \mathrm{rref}(A)$ we created $\hat{R}$ by removing any all-zero rows.

$$A = E^{-1}R \longrightarrow A = \hat{B}\hat{R}$$

The row space of R and $\hat{R}$ is the same.

To find a basis for the row space of A it suffices to find a basis for the row space of $\hat{R}$.

Row Space

To find a basis for the row space of A it suffices to find a basis for the row space of $\hat{R}$.

Claim: The rows of $\hat{R}$ are a basis for the row space of $\hat{R}$, and therefore also for the row space of A.

Proof: The rows of $\hat{R}$ clearly span the row space of $\hat{R}$.

We need to show that they are linearly independent.

Linear Independence

Claim: The rows of $\hat{R}$ are a basis for the row space of $\hat{R}$, and therefore also for the row space of A.

Proof: We need to show the rows of $\hat{R}$ are linearly independent.

Here is the $\hat{R}$ matrix from our previous example.

$$\hat{R} = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Each row is one in a position where all the other rows are zero.

This will always be the case by the definition of reduced row echelon form.

Easy Test

The rows of $\hat{R}$ satisfy the easy test for linear independence.

Easy Test: If $\vec{v_1}, \ldots, \vec{v_k} \in \mathbb{R}^n$ are such that, for each $\vec{v_i}$ there is a coordinate j where

$$\vec{v_i}(j) \neq 0$$
$$\vec{v_t}(j) = 0 \text{ for all } t \neq i$$

then $\vec{v_1}, \ldots, \vec{v_k}$ are linearly independent.

Example of Easy Test

$$a_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 0 \\ 0 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By considering each unique nonzero coordinate in turn, we see that all coefficients must be zero.

Dimension of the Row Space

Claim: The rows of $\hat{R}$ are a basis for the row space of $\hat{R}$, and therefore also for the row space of A.

The rows of $\hat{R}$ satisfy the easy test and are therefore linearly independent, and a basis for the row space of A.

Corollary: The dimension of the row space of A is equal to the number of pivots of A.

Corollary: The dimension of the row space of A is equal to the dimension of the column space of A.

Summary

The rank revealing factorization gives us a basis for the column space of A and the row space of A, and shows that these dimensions are equal.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{\hat{B}}{\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}} \underset{\hat{R}}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}}$$

The columns of $\hat{B}$ form a basis for the column space of A.

The rows of $\hat{R}$ form a basis for the row space of A.

Application: Image Compression

Compression

The rank revealing factorization can be used for compression.

This matrix is of size 369-by-590.

To store the value of each pixel takes one byte.

In total, we have to remember $369 \cdot 590 \approx 218$ Kb.

Compression

Say that we could find a small factorization of this image:

590

369 

A

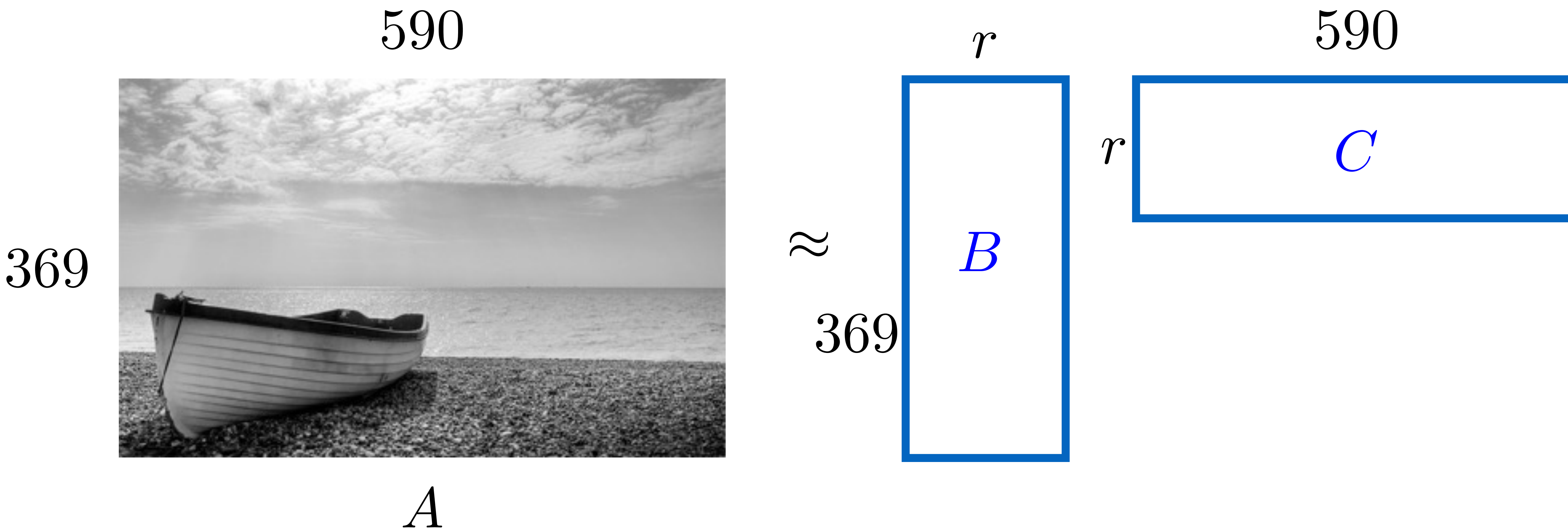
$$= \underset{369 \times r}{B} \; \underset{r \times 590}{C}$$

To store B and C requires $r \cdot (369 + 590)$ bytes

From B and C we can reconstruct A.

Compression

In practice, we look for an approximate factorization.



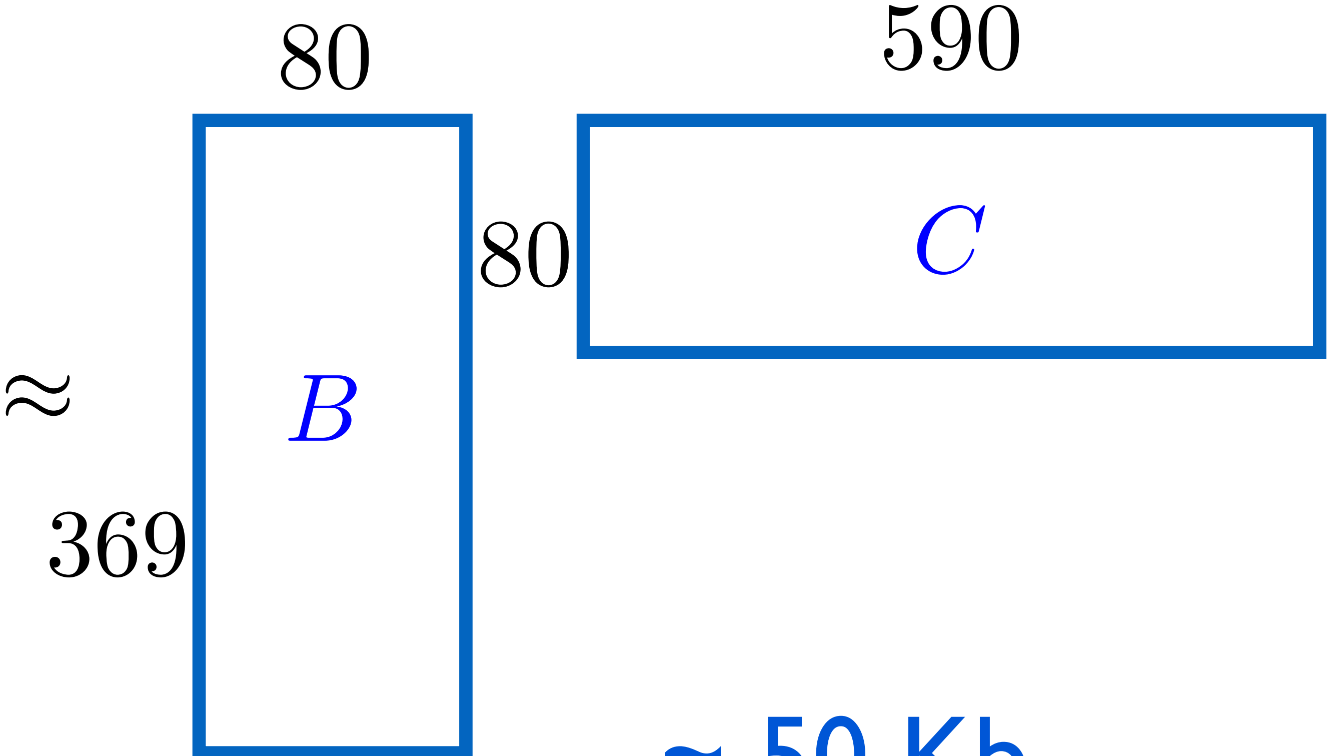
This is called lossy compression.

From B and C we cannot perfectly reconstruct A, but we try to get close.

Compression

In practice, we look for an approximate factorization.

$\approx$



~ 50 Kb

We created this image with an r value around 80.

Dimension of the Nullspace

Reading: Strang 3.6

Learning objective: Be able to find a basis for the nullspace of a matrix.

Dimension of the Nullspace

Let A be an m-by-n matrix with r many pivots.

Recall that $N(A) \subseteq \mathbb{R}^n$.

Now we will see that $\dim(N(A)) = n - r$.

The special solutions to $A\vec{x} = \vec{0}_m$ form a basis for the nullspace of A.

Special Solutions

Let's go back to our running example.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Elementary operations do not change the nullspace.

Thus A and $R = \mathrm{rref}(A)$ have the same nullspace.

Let's look at the special solutions to $R\vec{x} = \vec{0}_3$.

$$\begin{array}{cc} & \begin{array}{cc} \text{pivot columns} & \text{free columns} \end{array} \\ R = & \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

There is a special solution associated with each free column.

To construct it, we set the associated free variable equal to one, and all other free variables equal to zero.

We then solve for the pivot variables.

Let's go back to our running example.

$$R = \begin{array}{cccc} \text{pivot columns} & & \text{free columns} & \\ \end{array}$$

$$R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Special solution associated with column 3:

$$x_3 = 1, x_4 = 0$$

$$x_2 = -2$$

$$x_1 = 1$$

Special solution associated with column 4:

$$x_3 = 0, x_4 = 1$$

$$x_2 = -3, x_1 = 2$$

Let's go back to our running example.

pivot columns free columns

$$R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Special solution associated with column 3: $(1, -2, 1, 0)$

Special solution associated with column 4: $(2, -3, 0, 1)$

Notice that for a matrix in reduced row echelon form, the special solutions are easy to read off.

Special Solutions

Let A be an m-by-n matrix with r many pivots.

How many special solutions will A have?

If there are r pivots and n columns, then there will be $n - r$ free columns.

Theorem: The special solutions form a basis for the nullspace of A.

Proof: We argued in a previous lecture that the special solutions span the nullspace of A.

Theorem: The special solutions form a basis for the nullspace of A.

Proof: Now let's see that the special solutions are linearly independent.

The special solutions also pass the easy test for linear independence.

By construction, the special solution for column i is nonzero in the i^{th} component, and all other special solutions are zero in the i^{th} component.

Example:

Special solution associated with column 3: $(1, -2, 1, 0)$

Special solution associated with column 4: $(2, -3, 0, 1)$

Dimension of the Nullspace

Theorem: The special solutions form a basis for the nullspace of A.

Corollary: If the m-by-n matrix A has r pivots, the dimension of the nullspace of A is $n - r$.

Reason: The number of special solutions is $n - r$.

Dimension of the Left Nullspace

Reading: Strang 3.6

Learning objective: Be able to find a basis for the left nullspace of a matrix.

Left Nullspace

Let A be an m-by-n matrix with r many pivots.

The left nullspace of A is the set

$$\{\vec{u} : \vec{u}^T A = \vec{0}_n^T\} = \{\vec{u} : A^T \vec{u} = \vec{0}_n\}$$
$$= N(A^T)$$

Because $r = \dim(C(A)) = \dim(C(A^T))$ the n-by-m matrix A^T also has r pivots.

Thus $\dim(N(A^T)) = m - r$.

Basis for the Left Nullspace

We could find a basis for the left nullspace by taking the special solutions to $A^T\vec{x} = \vec{0}_n$.

Let's see another way.

For $R = \text{rref}(A)$ there is an invertible matrix E such that

$$EA = R$$

In the case of our running example:

$$\underbrace{\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{R}$$

Basis for the Left Nullspace

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In the case of our running example

$$\underset{E}{\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}} = \underset{R}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

In general, the last $m - r$ rows of R will be all-zero.

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In general, the last $m - r$ rows of R will be all-zero.

The i^{th} row of R is equal to $E(i,:)A$.

This means the last $m - r$ rows of E are in the left nullspace of A.

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Theorem: The last $m - r$ rows of E form a basis for the left nullspace of A.

Basis for the Left Nullspace

For $R = \mathrm{rref}(A)$ there is an invertible matrix E such that

$$EA = R$$

Theorem: The last $m - r$ rows of E form a basis for the left nullspace of A.

Proof: We have seen that the last $m - r$ rows of E are in the left nullspace of A.

Moreover, they are linearly independent.

Because E is invertible, its rows are linearly independent.

We also know the dimension of the left nullspace is $m - r$.

?

Basis for the Left Nullspace

For $R = \mathrm{rref}(A)$ there is an invertible matrix E such that

$$EA = R$$

Theorem: The last $m - r$ rows of E form a basis for the left nullspace of A.

Proof: We have $m - r$ linearly independent vectors in a subspace of dimension $m - r$.

These vectors must span the left nullspace of A.

If a vector $\vec{v} \in N(A^T)$ was not in their span, then we would have a sequence of $m - r + 1$ linearly independent vectors in $N(A^T)$, a contradiction.

Summary

Let A be an m-by-n matrix with r pivots.

$$EA = R = \operatorname{rref}(A)$$

nullspace $N(A) \subseteq \mathbb{R}^n$

dimension: $n - r$

basis: special solutions

left nullspace $N(A^T) \subseteq \mathbb{R}^m$

dimension: $m - r$

basis: last $m - r$ rows of E.

row space $C(A^T) \subseteq \mathbb{R}^n$

dimension: r

basis: nonzero rows of R.

column space $C(A) \subseteq \mathbb{R}^m$

dimension: r

basis: first r columns of E^{-1}.