Nullspace

Reading: Strang 3.2

Learning objective: Be able to find the nullspace of a matrix.

Nullspace

Let A be a m-by-n matrix.

Today we will talk about the nullspace of A, denoted $N(A)$.

$$N(A) = \{\vec{u} : A\vec{u} = \vec{0}_m\} \subseteq \mathbb{R}^n$$

The nullspace of A is the set of all solutions to the homogeneous equation $A\vec{x} = \vec{0}_m$.

Note that

$$C(A) \subseteq \mathbb{R}^m$$

$$N(A) \subseteq \mathbb{R}^n$$

Nullspace is a subspace

Let A be a m-by-n matrix.

Theorem: $N(A)$ is a subspace of $\mathbb{R}^n$.

Proof: $N(A)$ contains the all zero vector as

$$A\vec{0}_n = \vec{0}_m$$

No matter what A is, the nullspace of A always contains the zero vector.

Nullspace is a subspace

Let A be a m-by-n matrix.

Theorem: $N(A)$ is a subspace of $\mathbb{R}^n$.

Proof: $N(A)$ is closed under vector addition:

This follows by linearity of matrix-vector multiplication.

If $\vec{u}, \vec{v} \in N(A)$ then $A\vec{u} = A\vec{v} = \vec{0}_m$.

$$\begin{aligned} A(\vec{u} + \vec{v}) &= A\vec{u} + A\vec{v} \\ &= \vec{0}_m + \vec{0}_m = \vec{0}_m \end{aligned}$$

Thus $\vec{u} + \vec{v} \in N(A)$.

Nullspace is a subspace

Let A be a m-by-n matrix.

Theorem: $N(A)$ is a subspace of $\mathbb{R}^n$.

Proof: $N(A)$ is closed under scalar multiplication:

Again we use linearity of matrix-vector multiplication.

If $\vec{u} \in N(A)$ then $A\vec{u} = \vec{0}_m$. For any scalar $c \in \mathbb{R}$

$$\begin{aligned} A(c \cdot \vec{u}) &= c \cdot A\vec{u} \\ &= c \cdot \vec{0}_m = \vec{0}_m \end{aligned}$$

Thus $c \cdot \vec{u} \in N(A)$.

Example 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What is the nullspace?

Invertible Matrix

Say that A is an n-by-n invertible matrix.

What is its nullspace?

The nullspace of A is $N(A) = \{\vec{0}_n\}$.

What is its column space?

The column space of A is $C(A) = \mathbb{R}^n$.

The Big List

Let A be an n-by-n matrix. The following are equivalent:

- A is invertible.
- Gaussian elimination produces n pivots.
- $A\vec{x} = \vec{0}$ has a unique solution.
- A has a left inverse.
- A has a right inverse.
- The reduced row echelon form of A is the identity matrix.
- $\det(A) \neq 0$.
- $C(A) = \mathbb{R}^n$.
- $N(A) = \{\vec{0}_n\}$.

Example 2

Determine the nullspace of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix}$$

This is a problem we already know how to solve: find the general solution to $A\vec{x} = \vec{0}_3$.

We can proceed by Gaussian elimination.

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \xrightarrow[R_2' = R_2 - 2R_1]{} \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 7 & 13 \end{bmatrix} = B$$

Question: Do A and B have the same nullspace?

Note that $B = EA$ where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an invertible matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \xrightarrow[R_2' = R_2 - 2R_1]{} \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 7 & 13 \end{bmatrix} = B$$

Question: Do A and B have the same nullspace?

If $A\vec{u} = \vec{0}_3$ then $EA\vec{u} = E\vec{0}_3 = \vec{0}_3$.

This means $N(A) \subseteq N(B)$.

If $EA\vec{u} = \vec{0}_3$ then $A\vec{u} = E^{-1}\vec{0}_3 = \vec{0}_3$.

This means $N(B) \subseteq N(A)$.

Answer: Yes! The nullspace does not change under elementary row operations

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \xrightarrow[R_2' = R_2 - 2R_1]{} \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 7 & 13 \end{bmatrix} = B$$

Question: Do A and B have the same nullspace?

Answer: Yes! The nullspace does not change under elementary row operations.

We used two key facts:

§ Anything times the all-zero vector is the all-zero vector.

§ Elementary matrices are invertible.

$$
\begin{tikzcd}
A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \arrow[r, "R_2' = R_2 - 2R_1"'] & \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 7 & 13 \end{bmatrix} = B \arrow[d, "R_3' = R_3 - 3R_1"] \\
R = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & -2 \end{bmatrix} \arrow[l, "R_3' = R_3 + R_2"']
\end{tikzcd}
$$

The matrix R is in row echelon form.

A and R have the same nullspace.

Let's find the general solution to $R\vec{x} = \vec{0}_3$.

$$R = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let's find the general solution to $R\vec{x} = \vec{0}_3$.

There are pivots in columns 1 and 3. Columns 2 and 4 are free.

Variables x_1 and x_3 are pivot variables.

We can introduce free variables for x_2 and x_4.

$$x_4 = t$$

$$2x_3 + 2t = 0 \implies x_3 = -t$$

$$x_2 = s$$

$$x_1 + 2s - 3t + 5t = 0 \implies x_1 = -2s - 2t$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A and R have the same nullspace.

$$N(A) = \{(-2s - 2t, s, -t, t) : s, t \in \mathbb{R}\}$$

We can express the nullspace of A in terms of a span of vectors.

$$\begin{bmatrix} -2s - 2t \\ s \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

This technique is called separation of variables.

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A and R have the same nullspace.

$$\begin{aligned} N(A) &= \{(-2s-2t, s, -t, t) : s, t \in \mathbb{R}\} \\ &= \operatorname{span}(\{(-2,1,0,0), (-2,0,-1,1)\}) \end{aligned}$$

Note that we can write the nullspace of A as the span of two vectors, and after Gaussian elimination there were two free columns.

This is a general fact as we see next.

Special Solutions

We can find vectors that span the nullspace in another way.

Let's look at the equation $R\vec{x} = \vec{0}_3$ again (in augmented matrix form).

$$\begin{bmatrix} 1 & 2 & 3 & 5 & \mathbf{0} \\ 0 & 0 & 2 & 2 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

Variables x_1 and x_3 are pivot variables.

Variables x_2 and x_4 are free variables.

Special Solutions

For each free variable, there is a corresponding special solution.

Set that free variable equal to one, and all other free variables equal to zero. Then solve for the pivot variables.

$$\begin{bmatrix} 1 & 2 & 3 & 5 & \mathbf{0} \\ 0 & 0 & 2 & 2 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

Special solution for variable 4:

$$x_4 = 1, x_2 = 0$$

$$x_3 = -1, x_1 - 3 + 5 = 0 \implies x_1 = -2$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 & \mathbf{0} \\ 0 & 0 & 2 & 2 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

Special solution for variable 4:

$$x_4 = 1, x_2 = 0$$

$$x_3 = -1, x_1 - 3 + 5 = 0 \implies x_1 = -2$$

$$\implies (-2, 0, -1, 1)$$

Special solution for variable 2:

$$x_2 = 1, x_4 = 0$$

$$x_3 = 0, x_1 = -2$$

$$\implies (-2, 1, 0, 0)$$

Special Solutions

Special solution for variable 4:

$$\Longrightarrow \ (-2, 0, -1, 1)$$

Special solution for variable 2:

$$\Longrightarrow \ (-2, 1, 0, 0)$$

Note that these are the same vectors that we obtained from the method of separation of variables.

The special solutions span the nullspace.

Special Solutions

Theorem: The nullspace of A is equal to the span of the special solutions to $A\vec{x} = \vec{0}$.

Proof: Suppose that after Gaussian elimination, A has k many free columns.

Let $\vec{u}_1, \ldots, \vec{u}_k$ be the special solutions corresponding to these free columns.

Then $\mathrm{span}(\{\vec{u}_1, \ldots, \vec{u}_k\}) \subseteq N(A)$ as each $\vec{u}_i \in N(A)$ and $N(A)$ is a subspace.

Special Solutions

Theorem: The nullspace of A is equal to the span of the special solutions to $A\vec{x} = \vec{0}$.

Proof: Now we want to show that

$$N(A) \subseteq \mathrm{span}(\{\vec{u}_1, \ldots, \vec{u}_k\})$$

The key point is the following: the setting of the free variables determines the values of the pivot variables.

Theorem: The nullspace of A is equal to the span of the special solutions to $A\vec{x} = \vec{0}$.

Proof: Now we want to show that

$$N(A) \subseteq \operatorname{span}(\{\vec{u}_1, \ldots, \vec{u}_k\})$$

The key point is the following: the setting of the free variables determines the values of the pivot variables.

$$N(A) \ni \vec{v} = \begin{array}{|c|l}
\hline
 & \\
\hline
t_1 & \longleftarrow \text{ first free variable} \\
\hline
 & \\
\hline
t_2 & \longleftarrow \text{ second free variable} \\
\hline
\vdots & \\
\hline
t_k & \longleftarrow \; k^{th} \text{ free variable} \\
\hline
\end{array}$$

$$N(A) \ni \vec{v} = \begin{bmatrix} \ \\ t_1 \\ \ \\ t_2 \\ \vdots \\ t_k \end{bmatrix} \begin{array}{l} \ \\ \longleftarrow \text{ first free variable} \\ \ \\ \longleftarrow \text{ second free variable} \\ \ \\ \longleftarrow k^{th} \text{ free variable} \end{array}$$

$$\vec{u}_1 = \begin{bmatrix} \ \\ 1 \\ \ \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} \ \\ 0 \\ \ \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad \cdots \qquad \vec{u}_k = \begin{bmatrix} \ \\ 0 \\ \ \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$N(A) \ni \vec{v} = \begin{array}{|c|}\hline \\ \hline t_1 \\ \hline \\ \hline t_2 \\ \hline \vdots \\ \hline t_k \\ \hline \end{array} \qquad \vec{u}_1 = \begin{array}{|c|}\hline \\ \hline 1 \\ \hline \\ \hline 0 \\ \hline \vdots \\ \hline 0 \\ \hline \end{array} \qquad \vec{u}_2 = \begin{array}{|c|}\hline \\ \hline 0 \\ \hline \\ \hline 1 \\ \hline \vdots \\ \hline 0 \\ \hline \end{array} \qquad \cdots \; \vec{u}_k = \begin{array}{|c|}\hline \\ \hline 0 \\ \hline \\ \hline 0 \\ \hline \vdots \\ \hline 1 \\ \hline \end{array}$$

We see that $t_1 \cdot \vec{u}_1 + t_2 \cdot \vec{u}_2 + \cdots + t_k \cdot \vec{u}_k$ agrees with $\vec{v}$ in all the free variables.

Therefore

$$\vec{v} = t_1 \cdot \vec{u}_1 + t_2 \cdot \vec{u}_2 + \cdots + t_k \cdot \vec{u}_k$$

as the value of the free variables determines the value of the pivot variables.

Special Solutions

Theorem: The nullspace of A is equal to the span of the special solutions to $A\vec{x} = \vec{0}$.

Proof: As we have shown that for an arbitrary $\vec{v} \in N(A)$

$$\vec{v} = t_1 \cdot \vec{u}_1 + t_2 \cdot \vec{u}_2 + \cdots + t_k \cdot \vec{u}_k$$

this implies

$$N(A) \subseteq \operatorname{span}(\{\vec{u}_1, \ldots, \vec{u}_k\})$$

and finishes the proof.

Corollary

Theorem: The nullspace of A is equal to the span of the special solutions to $A\vec{x} = \vec{0}$.

Corollary: If after Gaussian elimination A has k many free columns, then there are k vectors that span the nullspace.

This corollary will be important when we start talking about dimension.

4 Fundamental Spaces

Let A be a m-by-n matrix.

There are 4 fundamental subspaces associated with A.

nullspace $N(A) \subseteq \mathbb{R}^n$

column space $C(A) \subseteq \mathbb{R}^m$

row space $C(A^T) \subseteq \mathbb{R}^n$

span of the rows of A

left nullspace $N(A^T) \subseteq \mathbb{R}^m$

$\{\vec{u} : \vec{u}^T A = \vec{0}_n^T\}$

Dimension: Motivation

Dimension

We use the word "dimension" in everyday speech.

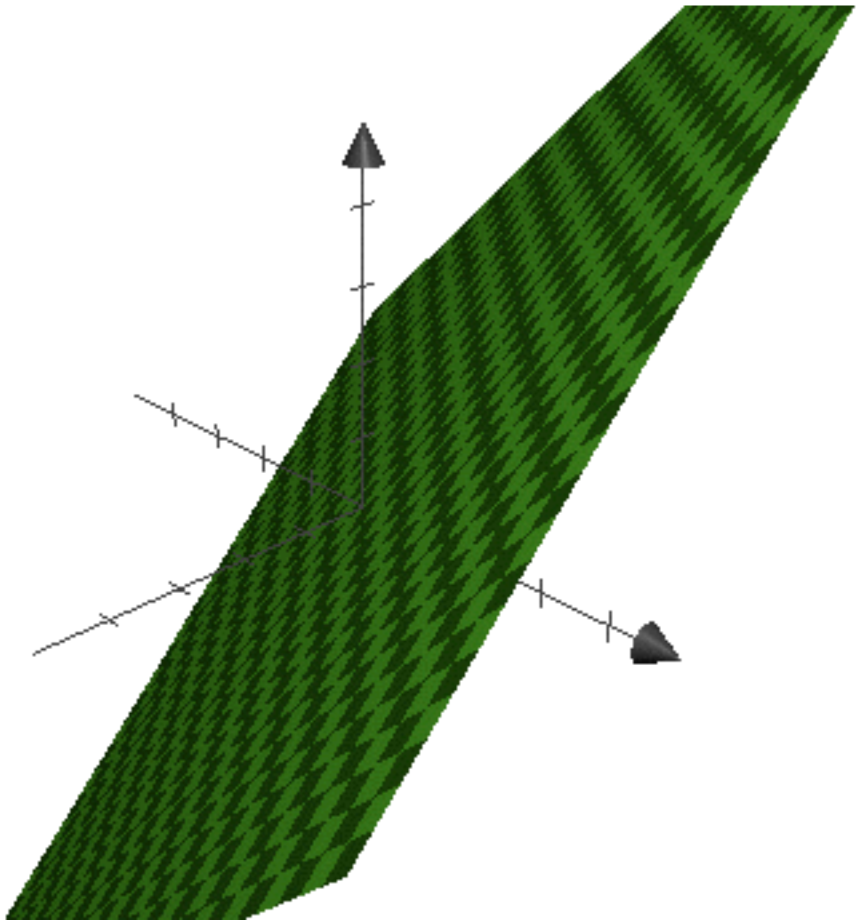
Intuitively, we understand what 2D and 3D mean.

Now, we are going to define "dimension" mathematically!

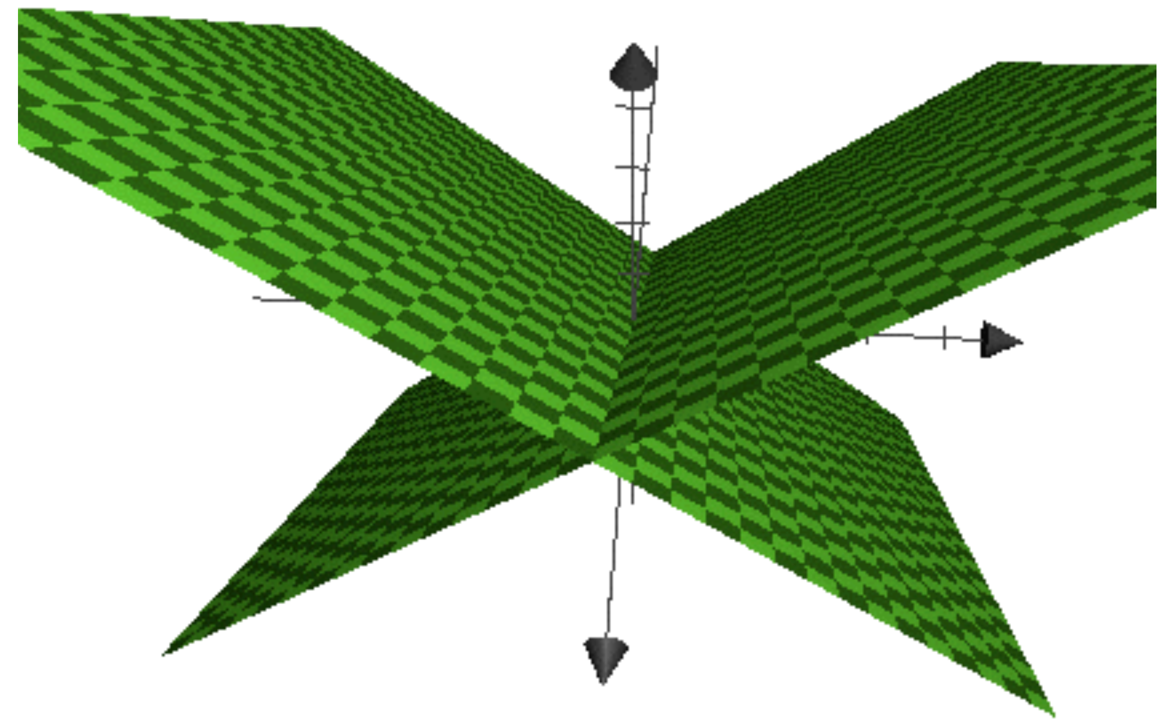
And we won't stop at 3D... you will say with confidence

"That space is eight dimensional!"

Examples



a plane

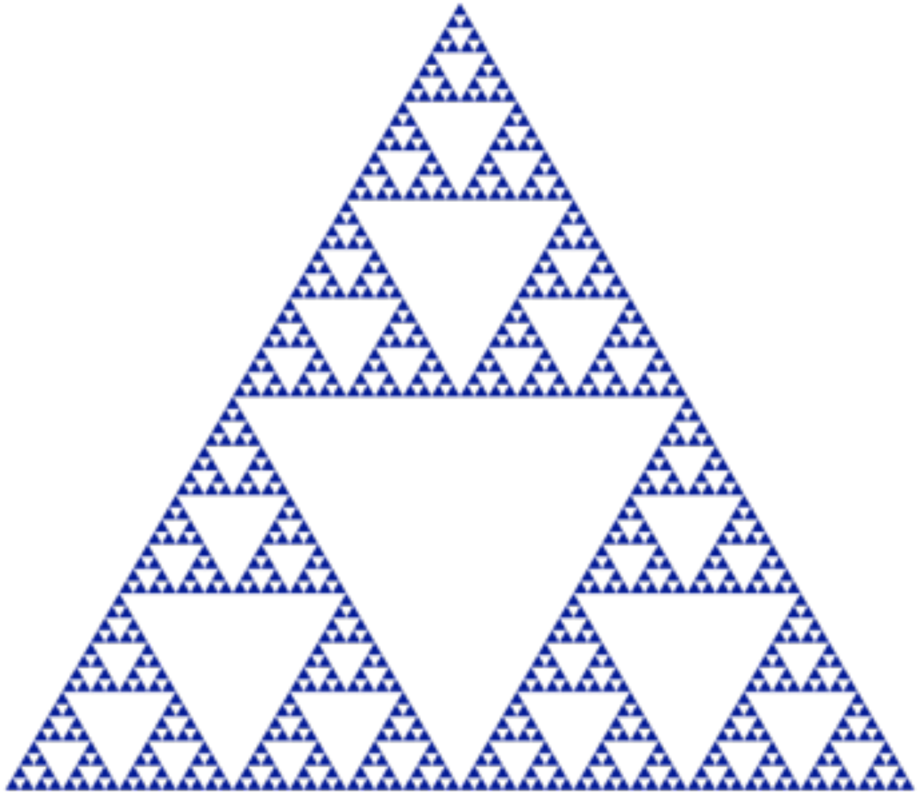


intersection
of two planes

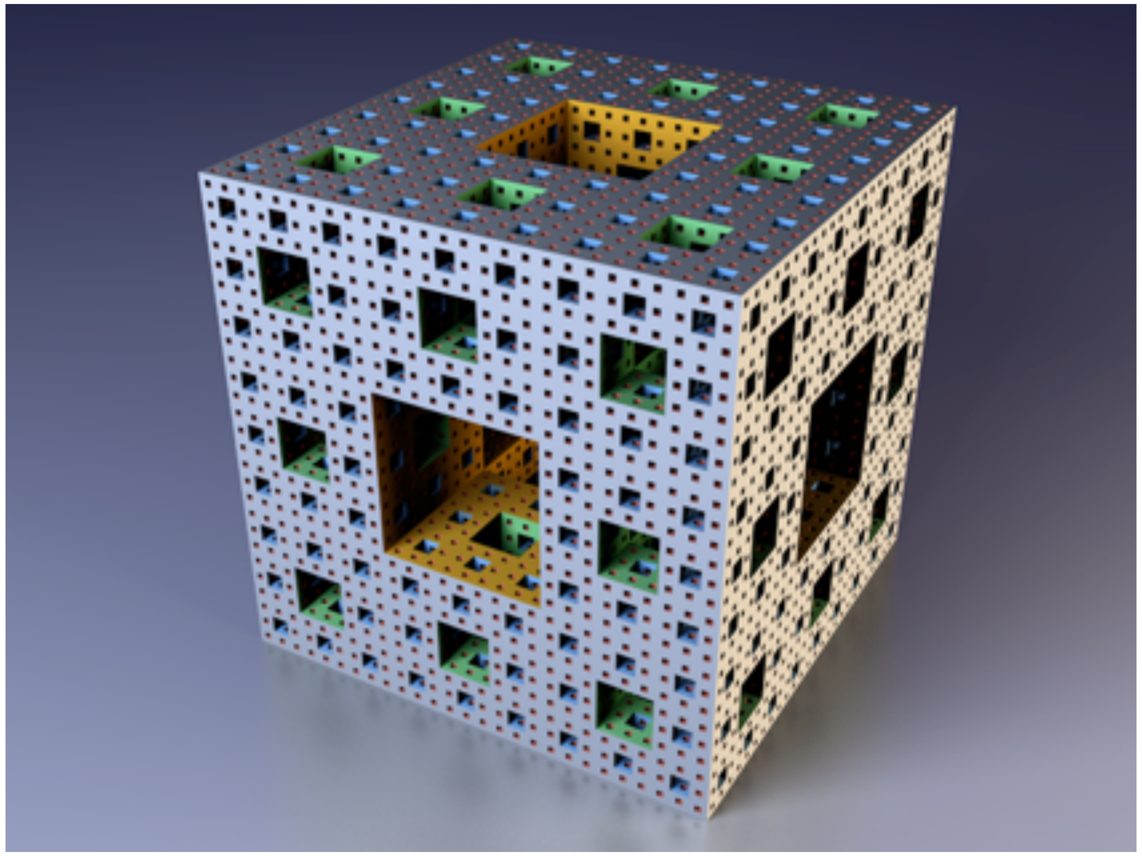
We will define the dimension of things like this...

Examples

We won't define the dimension of things like this...



Sierpinski Triangle



Menger Sponge

Subspaces

We will define dimension for subspaces.

There are two key ingredients that go into the definition of dimension.

The first ingredient you already know, the notion of span.

The second ingredient, linear independence, we introduce now.

Linear Independence

Reading: Strang 3.5

Learning objective: Be able to tell if a sequence of vectors is linearly independent in $\mathbb{R}^n$.

Linear independence

We start with an m-by-n matrix A.

Definition: The columns of A are linearly independent if and only if the only solution to $A\vec{x} = \vec{0}_m$ is $\vec{x} = \vec{0}_n$.

In other words, the columns of A are linearly independent iff $N(A) = \{\vec{0}_n\}$.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Let's find the solutions to $A\vec{x} = \vec{0}_3$.

$$\begin{bmatrix} 1 & 1 & \mathbf{0} \\ 1 & -1 & \mathbf{0} \\ 1 & 1 & \mathbf{0} \end{bmatrix} \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - R_1} \begin{bmatrix} 1 & 1 & \mathbf{0} \\ 0 & -2 & \mathbf{0} \\ 0 & 0 & \mathbf{0} \end{bmatrix} \longrightarrow \begin{aligned} x_2 &= 0 \\ x_1 &= 0 \end{aligned}$$

The only solution to $A\vec{x} = \vec{0}_3$ is $\vec{x} = \vec{0}_2$.

The columns of A are linearly independent.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Now let's find the solutions to $A^T\vec{x} = \vec{0}_2$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & \mathbf{0} \\ 1 & -1 & 1 & \mathbf{0} \end{array}\right] \xrightarrow[R_2' = R_2 - R_1]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & \mathbf{0} \\ 0 & -2 & 0 & \mathbf{0} \end{array}\right] \longrightarrow \begin{aligned} x_3 &= t \\ x_2 &= 0 \\ x_1 &= -t \end{aligned}$$

There are infinitely many solutions to $A^T\vec{x} = \vec{0}_2$.

The columns of A^T are not linearly independent.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

The columns of A^T are not linearly independent.

How could we have known this without doing any computation?

After Gaussian elimination A^T can have at most 2 pivots, yet there are 3 columns. There will be a free column.

If a matrix has more columns than rows then the columns will not be linearly independent.

General Definition

Linear independence does not just apply to the columns of a matrix.

We can define it in a general vector space.

Definition: Let V be a vector space. The sequence of vectors $v_1, \ldots, v_n \in V$ is linearly independent if and only if

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \cdots + a_n \cdot v_n = \mathbf{0}$$

implies $a_1 = a_2 = \cdots = a_n = 0$.

The only linear combination of $v_1, \ldots, v_n$ giving the zero vector is when all coefficients are zero.

General Definition

If there is a linear combination

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \cdots + a_n \cdot v_n = \mathbf{0}$$

with some $a_i \neq 0$, then we say the sequence $v_1, \ldots, v_n$ is linearly dependent.

If the sequence $v_1, \ldots, v_n$ is linearly dependent, then one of the vectors can be written as a linear combination of the others.

Note that any sequence containing the zero vector $\mathbf{0}$ will always be linear dependent.

Language

We will say "the sequence of vectors $v_1, \ldots, v_n$ is linearly independent," or, more informally, "the vectors $v_1, \ldots, v_n$ are linearly independent."

For example, we can say "the columns of the matrix A are linearly independent."

It does not make sense to say "the matrix A is independent."

Examples

The vectors $(1,1,1),(1,-1,1)$ are linearly independent.

The vectors $(1,1,1),(1,-1,1),(0,0,0)$ are linearly dependent.

The vectors $(1,0,0),(0,1,0),(0,0,1)$ are linearly independent.

Examples

The vectors $(1,1),(1,-1),(-1,-1)$ are linearly dependent.

One of the vectors can be written as a linear combination of the others:

$$(1,1) = 0 \cdot (1,-1) + (-1) \cdot (-1,-1)$$

Note, however, that $(1,-1)$ is not a linear combination of $(1,1)$ and $(-1,-1)$.

Question

Are the vectors $(1,1,1,1),(1,-1,1,-1),(1,1,-1,-1)$ linearly independent?

Let's set up the equation to test for independence

$$a_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can equivalently express this in matrix form.

Question

Are the vectors $(1, 1, 1, 1), (1, -1, 1, -1), (1, 1, -1, -1)$ linearly independent?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The vectors are linearly independent iff the only solution to this equation is $a_1 = a_2 = a_3 = 0$.

Our general definition agrees with the definition of when the columns of a matrix are independent.

Question

Are the vectors $(1,1,1,1),(1,-1,1,-1),(1,1,-1,-1)$ linearly independent?

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{0} \\ 1 & -1 & 1 & \mathbf{0} \\ 1 & 1 & -1 & \mathbf{0} \\ 1 & -1 & -1 & \mathbf{0} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & \mathbf{0} \\ 0 & -2 & 0 & \mathbf{0} \\ 0 & 0 & -2 & \mathbf{0} \\ 0 & -2 & -2 & \mathbf{0} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & \mathbf{0} \\ 0 & -2 & 0 & \mathbf{0} \\ 0 & 0 & -2 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

The only solution is $a_1 = a_2 = a_3 = 0$.

These vectors are linearly independent.

Method

When we are asked if a sequence of vectors $v_1, \ldots, v_n \in \mathbb{R}^m$ is linearly independent, we can make an m-by-n matrix A with these vectors as its columns and solve the equation $A\vec{x} = \vec{0}_m$.

The sequence $v_1, \ldots, v_n$ is linearly independent iff the only solution is $\vec{x} = \vec{0}_n$.

Question

How many vectors can a sequence of linearly independent vectors in $\mathbb{R}^m$ have?

Say we have a sequence of vectors $v_1, \ldots, v_n \in \mathbb{R}^m$.

To test if they are linearly independent, we can make an m-by-n matrix A whose columns are $v_1, \ldots, v_n$ and see how many solutions the equation $A\vec{x} = \vec{0}_m$ has.

If $n > m$ then A will always have a free column after Gaussian elimination, thus $A\vec{x} = \vec{0}_m$ will have infinitely many solutions.

Question

How many vectors can a sequence of linearly independent vectors in $\mathbb{R}^m$ have?

If $n > m$ then any sequence of n vectors $v_1, \ldots, v_n \in \mathbb{R}^m$ will be linearly dependent.

Note that we can have a sequence of m many linearly independent vectors in $\mathbb{R}^m$.

If we take any invertible m-by-m matrix, the columns will be linearly independent.

The Big List

Let A be an n-by-n matrix. The following are equivalent:

- A is invertible.
- Gaussian elimination produces n pivots.
- $A\vec{x} = \vec{0}$ has a unique solution.
- A has a left inverse.
- A has a right inverse.
- The reduced row echelon form of A is the identity matrix.
- $\det(A) \neq 0$.
- $C(A) = \mathbb{R}^n$.
- $N(A) = \{\vec{0}_n\}$.
- The columns of A are linearly independent.

Linear independence in general vector spaces

Reading: Strang 3.5

Learning objective: Be able to tell if a sequence of vectors is linearly independent in the space of matrices and real-valued functions.

Linear Independence

Although linear independence is a new definition, to check that a sequence of vectors $v_1, \ldots, v_n \in \mathbb{R}^m$ is linearly independent reduces to something we have done all semester: solving a system of linear equations.

In general vector spaces, checking linear independence can be less obvious.

Now we look at some techniques to prove vectors are linearly independent in other vector spaces.

Matrix Spaces

Show that the following matrices are linearly independent.

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

Let's set up the equation for checking independence:

$$a_1 \cdot \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} + a_2 \cdot \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} + a_3 \cdot \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} + a_4 \cdot \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix} + a_5 \cdot \begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix} + a_6 \cdot \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Matrix Spaces

Let's set up the equation for checking independence:

$$a_1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Simplifying, this means

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which implies $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0.$

Thus these 6 matrices are linearly independent.

Matrix Spaces

Show that the following matrices are linearly independent.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What made this question easy?

Each matrix has a one in an entry where all the other matrices are zero.

Matrix Spaces

Let's look at a more interesting example.

Is this sequence of matrices linearly independent?

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Let's set up the equation for checking independence:

$$a_1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix Spaces

Is this sequence of matrices linearly independent?

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Let's set up the equation for checking independence:

$$a_1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which implies

$$\begin{bmatrix} a_1 + a_2 & a_1 + a_3 \\ a_1 - a_3 & -a_1 - a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Is this sequence of matrices linearly independent?

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Let's set up the equation for checking independence:

$$\begin{bmatrix} a_1 + a_2 & a_1 + a_3 \\ a_1 - a_3 & -a_1 - a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This is equivalent to a system of linear equations!

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Is this sequence of matrices linearly independent?

$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

This is equivalent to a system of linear equations!

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & 0 & -1\\ -1 & -1 & 0\end{bmatrix} \begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

Now we know how to find solutions to this

$$\begin{bmatrix}1 & 1 & 0 & \mathbf{0}\\ 1 & 0 & 1 & \mathbf{0}\\ 1 & 0 & -1 & \mathbf{0}\\ -1 & -1 & 0 & \mathbf{0}\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1 & 0 & \mathbf{0}\\ 0 & -1 & 1 & \mathbf{0}\\ 0 & -1 & -1 & \mathbf{0}\\ 0 & 0 & 0 & \mathbf{0}\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1 & 0 & \mathbf{0}\\ 0 & -1 & 1 & \mathbf{0}\\ 0 & 0 & -2 & \mathbf{0}\\ 0 & 0 & 0 & \mathbf{0}\end{bmatrix}$$

Is this sequence of matrices linearly independent?

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

After Gaussian elimination we arrive at

$$\begin{bmatrix} 1 & 1 & 0 & \mathbf{0} \\ 0 & -1 & 1 & \mathbf{0} \\ 0 & 0 & -2 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

This has a unique solution: $a_3 = 0, a_2 = 0, a_1 = 0$.

This sequence of matrices is linearly independent.

Matrix Spaces

Is this sequence of matrices linearly independent?

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

A similar method will work in general for checking if a sequence of matrices is independent.

We can reduce the problem to solving a system of linear equations.

Function Spaces

Now let's look at a linear independence problem in the vector space of real valued functions.

Question: Are the functions $1, x, x^2$ linearly independent?

Let's set up the equation for checking independence.

The zero element in this vector space is the constant zero function $\mathbf{0}(x) = 0 \text{ for all } x \in \mathbb{R}$.

Function Spaces

Now let's look at a linear independence problem in the vector space of real valued functions.

Question: Are the functions $1, x, x^2$ linearly independent?

Let's set up the equation for checking independence.

$$a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 = \mathbf{0}$$

We can also reduce this problem to solving a system of linear equations, by a technique called sampling.

Function Spaces

Question: Are the functions $1, x, x^2$ linearly independent?

Let's set up the equation for checking independence.

$$a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 = \mathbf{0}$$

This equation must hold for every $x \in \mathbb{R}$.

It must hold for $x = 0 \implies a_0 = 0$

It must hold for $x = 1 \implies a_0 + a_1 + a_2 = 0$

It must hold for $x = 2 \implies a_0 + 2a_1 + 4a_2 = 0$

Question: Are the functions $1, x, x^2$ linearly independent?

$$a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 = \mathbf{0}$$

This equation must hold for every $x \in \mathbb{R}$.

It must hold for $x = 0 \implies a_0 = 0$

It must hold for $x = 1 \implies a_0 + a_1 + a_2 = 0$

It must hold for $x = 2 \implies a_0 + 2a_1 + 4a_2 = 0$

This gives us a system of linear equations:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Question: Are the functions $1, x, x^2$ linearly independent?

$$a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 = \mathbf{0}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix on the left is a Vandermonde matrix:

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

with $a = 0, b = 1, c = 2$. We know that this matrix is invertible whenever a, b, c are all distinct.

Thus $1, x, x^2$ are linearly independent.

Polynomials

We can use the same argument to show more generally that the functions $1, x, x^2, \dots, x^{n-1}$ are linearly independent.

$$a_0 \cdot 1 + a_1 \cdot x + \cdots + a_{n-1} \cdot x^{n-1} = \mathbf{0}$$

By evaluating this expression at $x = 0, x = 1, \dots, x = n-1$, we get a system of n linear equations in n unknowns.

The coefficient matrix is an invertible Vandermonde matrix, by last week's problem set.

Thus these functions are linearly independent.

Polynomials

We can use the same argument to show more generally that the functions $1, x, x^2, \ldots, x^{n-1}$ are linearly independent.

As a consequence of this result, we see that the only polynomial

$$a_0 + a_1 \cdot x + \cdots + a_{n-1}x^{n-1}$$

which is equal to the constant zero function, is when all coefficients $a_0, a_1, \ldots, a_{n-1}$ are equal to zero.