Determinants

Reading: Strang 5.2

Learning objective: Be able to show $\det(A) = \det(A^T)$.

Review

Yesterday we started to look at the determinant of the transpose of a matrix.

If P is a permutation matrix then $\det(\mathrm{P}) = \det(\mathrm{P}^{\mathrm{T}})$.

Reason: $\det(P) \in \{-1, 1\}$ and $\det(P)\det(P^T) = 1$ because $P^{-1} = P^T$.

If U is an upper triangular matrix then $\det(U) = \det(U^T)$.

Reason: The determinant of an upper or lower triangular matrix is the product of its diagonal entries.

Determinant and Transpose

For any square matrix A

$$\boxed{\det(A) = \det(A^T)}$$

Proof: Any square matrix A has a decomposition as

$$PA = LU \implies A = P^T LU$$

where P is a permutation matrix, L is lower triangular, and U is upper triangular.

For any square matrix A

$$\boxed{\det(A) = \det(A^T)}$$

Proof:

$$A = P^T L U \implies \det(A) = \det(P^T)\det(L)\det(U)$$

$$A^T = U^T L^T P \implies \det(A^T) = \det(U^T)\det(L^T)\det(P)$$

We know that

$$\det(P^T) = \det(P), \det(L) = \det(L^T), \det(U) = \det(U^T)$$

thus

$$\det(A) = \det(A^T)$$

What goes for rows...

The last result means that all the row properties of the determinant also hold for columns.

Example:

Suppose that A has an all-zero column.

Then A^T has an all zero row, and so $\det(A^T) = 0$.

But $\det(A) = \det(A^T)$, so $\det(A) = 0$ as well.

What goes for rows…

§ If a matrix has two equal columns, then the determinant is zero.

§ The determinant is a linear function of each column separately.

§ Adding a multiple of one column to another does not change the determinant.

Review

Let's compute the determinant of this matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 7 \\ 1 & 1 & 1 & 5 \\ 2 & 5 & 7 & 9 \end{bmatrix}$$

We proceed by Gaussian elimination. Adding a multiple of one row to another does not change the determinant.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 7 \\ 1 & 1 & 1 & 5 \\ 2 & 5 & 7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$

Example

$$\begin{vmatrix}1&2&3&4\\2&4&6&7\\1&1&1&5\\2&5&7&9\end{vmatrix}=\begin{vmatrix}1&2&3&4\\0&0&0&-1\\0&-1&-2&1\\0&1&1&1\end{vmatrix}=-\begin{vmatrix}1&2&3&4\\0&-1&-2&1\\0&0&0&-1\\0&1&1&1\end{vmatrix}$$

swap rows 2 and 3

$$=-\begin{vmatrix}1&2&3&4\\0&-1&-2&1\\0&0&0&-1\\0&0&-1&2\end{vmatrix}=\begin{vmatrix}1&2&3&4\\0&-1&-2&1\\0&0&-1&2\\0&0&0&-1\end{vmatrix}$$

swap rows 3 and 4

$$=-1$$

Formulas for the determinant

Reading: Strang 5.2

Learning objective: Be able to compute the determinant by the cofactor formula.

Cofactor Expansion

Now we will see the cofactor formula for determinants.

Let's start with the 2-by-2 case.

By linearity in the first row:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{vmatrix} + a_{12} \begin{vmatrix} 0 & 1 \\ a_{21} & a_{22} \end{vmatrix}$$

$C_{11} =$ cofactor of entry a_{11}

cofactor of entry a_{12} $= C_{12}$

Simplifying Cofactors

$$C_{11} = \begin{vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & a_{22} \end{vmatrix} = a_{22}$$

cofactor of entry a_{11}

$$C_{12} = \begin{vmatrix} 0 & 1 \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ a_{21} & 0 \end{vmatrix} = - \begin{vmatrix} a_{21} & 0 \\ 0 & 1 \end{vmatrix} = -a_{21}$$

cofactor of entry a_{12}

Simplifying Cofactors

We get back to our usual formula for the 2-by-2 determinant.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{vmatrix} + a_{12} \begin{vmatrix} 0 & 1 \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

Cofactor Expansion

Now let's look at the 3-by-3 case.

We again use linearity in the first row.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Cofactor Expansion

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$C_{11} =$ cofactor of entry a_{11}

$C_{12} =$ cofactor of entry a_{12}

$C_{13} =$ cofactor of entry a_{13}

Cofactor Formula

We arrive at the formula for cofactor expansion in the first row:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$C_{11} = \begin{vmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$C_{12} = \begin{vmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$C_{13} = \begin{vmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Same goes, fellow rows!

There is nothing special about the first row...

Using linearity of the determinant in the second row:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{21}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$C_{21} =$ cofactor of entry a_{21}

$C_{22} =$ cofactor of entry a_{22}

$C_{23} =$ cofactor of entry a_{23}

Cofactor Formula

Here is the formula for cofactor expansion in the second row:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$C_{21} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$C_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$C_{22} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

General Formula

In general, for an n-by-n matrix A, cofactor expansion in the i^{th} row looks like this:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

where C_{ik} is the cofactor of entry a_{ik} .

Here $C_{ik} = \det(A'_{ik})$ where A'_{ik} is the matrix that agrees with A outside of row i .

In row i , A'_{ik} is zero everywhere except for the (i,k) entry, which is 1.

Simplifying Cofactors

To make this formula more useful, let's simplify the cofactors.

We want to express cofactors in terms of the determinant of a smaller matrix.

Start with C_{11}

$$C_{11} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Using AMORTA, which does not change the determinant, we can make the first column below entry $(1, 1)$ all zero.

Simplifying Cofactors

Claim:

$$C_{11} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Reason: Let

$$M_{11} = \begin{bmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Say that Gaussian elimination brings M_{11} to an upper triangular matrix U using p many row swaps.

Simplifying Cofactors

Reason: Let

$$M_{11} = \begin{bmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Say that Gaussian elimination brings M_{11} to an upper triangular matrix U with p many row swaps.

Then $\det(M_{11}) = (-1)^p \det(U)$.

Now do the same operations on the matrix.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Simplifying Cofactors

Now do the same operations on the matrix.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

It will again transform the lower right submatrix to U with p many row swaps.

$$\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (-1)^p \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & u_{11} & \cdots & u_{1,n-1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & u_{n-1,n-1} \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (-1)^p \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & u_{11} & \cdots & u_{1,n-1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & u_{n-1,n-1} \end{vmatrix}$$

The matrix on the right is upper triangular. Thus its determinant is

$$(-1)^p \cdot 1 \cdot u_{11} \cdot u_{22} \cdots u_{n-1,n-1} = (-1)^p \det(U)$$

This shows

$$C_{11} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Second Cofactor

in the first row

Now let's look at the cofactor of entry a_{12}.

$$C_{12} = \begin{vmatrix} 0 & 1 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} 0 & 1 & \cdots & 0 \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & 0 & \cdots & a_{nn} \end{vmatrix}$$

Using AMORTA, which does not change the determinant, we can make the column below entry $(1, 2)$ all zero.

Joke

Reduce to previous case

$$C_{12} = \begin{vmatrix} 0 & 1 & \cdots & 0 \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & 0 & \cdots & a_{nn} \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{21} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

what goes for rows, goes for columns: exchanging two columns multiplies the determinant by -1.

$$C_{12} = -1 \cdot \begin{vmatrix} a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

Second Cofactor

in the first row

$$C_{12} = -1 \cdot \begin{vmatrix} a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

Start with A and erase the first row and second column.

Take the determinant and multiply by -1

Third Cofactor

in the first row

$$C_{13} = \begin{vmatrix} 0 & 0 & 1 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & 0 & \cdots & a_{nn} \end{vmatrix}$$

$$= -1 \cdot \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ a_{21} & 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{n1} & 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \ddots & \vdots \\ 0 & a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Third Cofactor

in the first row

$$C_{13} = \begin{vmatrix} 0 & 0 & 1 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & 0 & \cdots & a_{nn} \end{vmatrix}$$

To compute C_{13}

Start with A and erase the first row and third column.

Then take the determinant.

General Cofactor

in the first row

Let M_{1j} be the matrix A with the first row and j^{th} column removed.

$$C_{1j} = (-1)^{j-1}\det(M_{1j})$$

We have to do $j-1$ swaps of adjacent columns to bring column j to column 1

Plugging this into expansion along the first row:

$$\det(A) = \sum_{j=1}^{n} (-1)^{j-1} a_{1j} \det(M_{1j})$$

General Cofactor

Now we consider a cofactor from any row.

Let M_{ij} be the matrix A with the i^{th} row and j^{th} column removed.

$$C_{ij} = (-1)^{i-1}(-1)^{j-1}\det(M_{ij})$$

Now we also have to do $i-1$ adjacent row swaps to reduce to the first row case.

$$C_{ij} = (-1)^{i+j}\det(M_{ij})$$

Cofactor Formula

Let M_{ij} be the matrix A with the i^{th} row and j^{th} column removed.

Cofactor expansion along row i

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$$

Remember that $\det(A) = \det(A^T)$

$\longrightarrow$ we can also do cofactor expansion along any column.

Example

Do cofactor expansion along the first row:

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + (-1) \cdot (-1) \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 1 \cdot 1 + 1 \cdot (-1)$$

$$= 0$$

This matrix is singular.

Midterm Review

The midterm will take place Tuesday October 10 at 2:30pm in Exam Hall C.

The midterm is closed book. You will have 50 minutes.

The midterm will cover the material through lecture 6b. There will be no determinants on the midterm.

Midterm

Please know
this number
for the midterm

Tutorial Number	Time	Location	Tutor
T2	08.30-09.30	SPMS-TR+2	Chen Ziwen
T3	08.30-09.30	SPMS-TR+8	Zafer Selcuk Aygin
T4	08.30-09.30	SPMS-TR+9	Farm Hui Jia
T5	09.30-10.30	SPMS-TR+1	Xia Tingfei
T6	09.30-10.30	SPMS-TR+2	Chen Ziwen
T7	09.30-10.30	SPMS-TR+8	Zafer Selcuk Aygin
T11	13.30-14.30	SPMS-TR+8	Zhou Zhou
T12	13.30-14.30	SPMS-TR+9	Farm Hui Jia
T13	14.30-15.30	SPMS-TR+1	Troy Lee
T14	14.30-15.30	SPMS-TR+2	Chen Ziwen

We distribute the graded midterms back to you through the tutorials. If the tutorial number is missing on the answer sheet you will have to pick the midterm up from my office.

Major Topics

§ Vectors

Addition, scalar multiplication, taking linear comb.

Dot Product

Geometric view of sets of vectors

§ Solving systems of linear equations

Gaussian elimination

Determining the number of solutions

LU Decomposition

Major Topics

§ Matrix Multiplication

Column picture, Row picture

Row-Column picture (dot products)

Column-Row picture (outer products)

§ Invertibility

Finding the inverse (Gauss-Jordan Elimination)

Conditions Equivalent to Invertibility

Dot Product

Let $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3)$. The dot product of $\vec{a}$ and $\vec{b}$ is

$$\langle \vec{a}, \vec{b} \rangle = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$$

$$\begin{aligned} \langle \vec{a}, \vec{b} \rangle &= \vec{a}^T \vec{b} \\ &= \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \end{aligned}$$

Length and Angle

The dot product connects to the length and angle between vectors.

$$\|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle}$$

$$\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$$

where $\vec{a}, \vec{b}$ make an angle θ .

Example

Find a unit vector perpendicular to the vector $(1, -2, 1)$.

A vector $\vec{u} = (u_1, u_2, u_3)$ will be perpendicular to $(1, -2, 1)$ if and only if

$$\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u_1 - 2u_2 + u_3 = 0$$

Find a unit vector perpendicular to the vector $(1, -2, 1)$.

$$u_1 - 2u_2 + u_3 = 0$$

The general solution is

$$u_3 = s$$

$$u_2 = t \qquad s(-1, 0, 1) + t(2, 1, 0) \qquad s, t \in \mathbb{R}$$

$$u_1 = 2t - s$$

As we just need one vector we can set $t = 0$, and choose s to obtain a unit vector.

$$\frac{1}{\sqrt{2}}(-1, 0, 1)$$

Systems of Linear Equations

To solve a system of linear equations $A\vec{x} = \vec{b}$ we can do Gaussian elimination on the augmented matrix

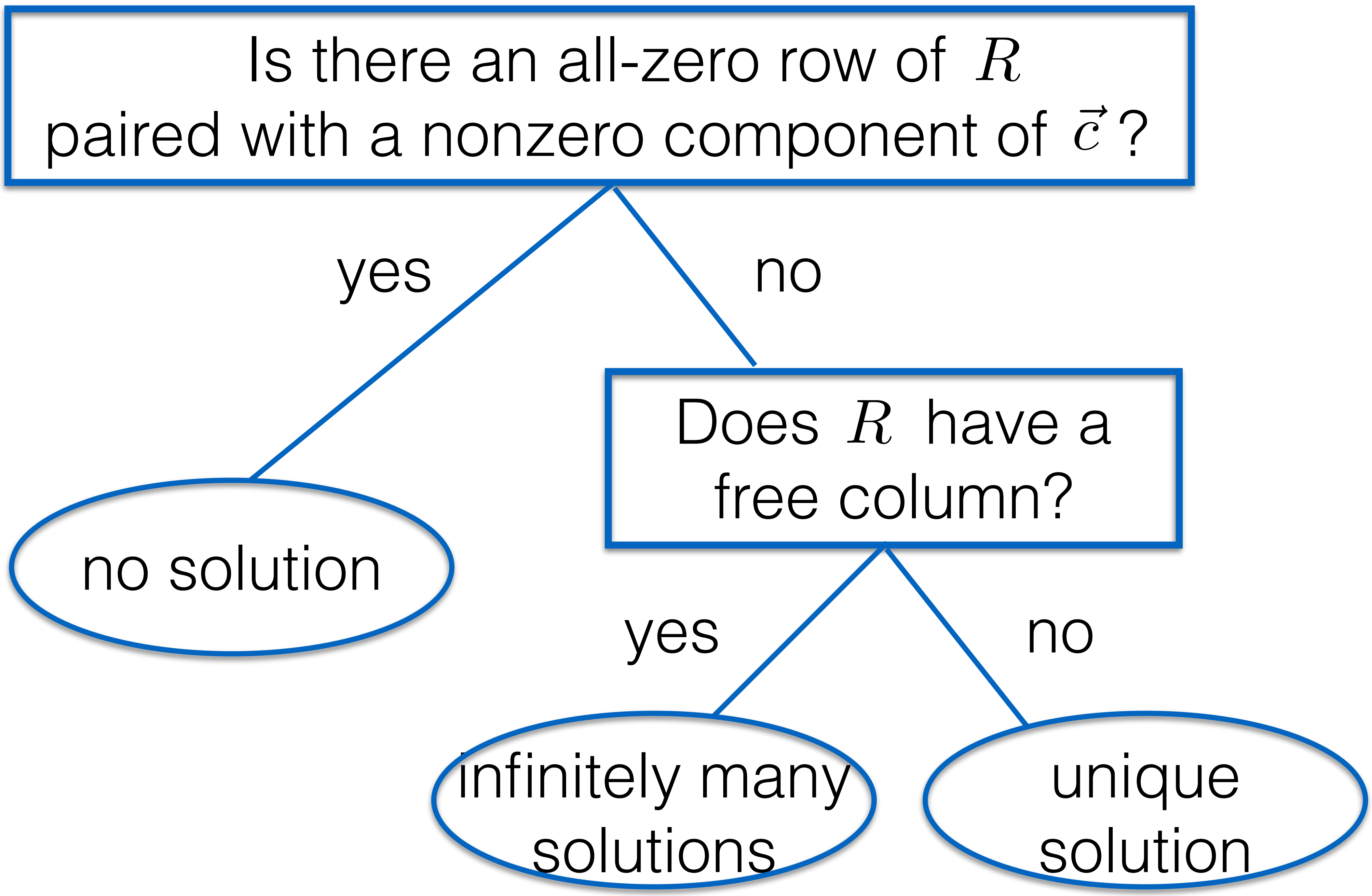
$$[A \mid \vec{b}] \xrightarrow{\text{G.E.}} [R \mid \vec{c}]$$

row echelon form

Don't forget to also modify $\vec{b}$ by the row operations.

The solution set to $A\vec{x} = \vec{b}$ is the same as the solution set to $R\vec{x} = \vec{c}$.

Say that R is in row echelon form. Here is a flowchart to determine the number of solutions to $R\vec{x} = \vec{c}$.



Solving Linear Equations

For what values of b does the following system have a solution?

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 1 & 2 & 3 & \mathbf{b} \\ 1 & 3 & 5 & \mathbf{1} \end{bmatrix}$$

For what values of b does the following system have a solution?

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 1 & 2 & 3 & \mathbf{b} \\ 1 & 3 & 5 & \mathbf{1} \end{bmatrix}$$

Do Gaussian elimination:

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 1 & 2 & 3 & \mathbf{b} \\ 1 & 3 & 5 & \mathbf{1} \end{bmatrix} \xrightarrow[\substack{R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1}]{} \begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 0 & 1 & 2 & \mathbf{b-1} \\ 0 & 2 & 4 & \mathbf{0} \end{bmatrix}$$

$$\Big\downarrow \; R_3' = R_3 - 2R_2$$

This has a solution iff

$$b = 1$$

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 0 & 1 & 2 & \mathbf{b-1} \\ 0 & 0 & 0 & \mathbf{2(1-b)} \end{bmatrix}$$

Elementary Matrices

We have now seen all the different types of elementary row operations and corresponding elementary matrices.

- Add a multiple of a row to another
- Exchange two rows
- Multiply a row by a nonzero constant

We can form the corresponding elementary matrix by performing the row operation on the identity matrix.

Midterm 14/15

Write down two elementary matrices E_1, E_2 such that

$$E_1 E_2 \neq E_2 E_1$$

Think about the operational interpretation.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2' = R_2 + aR_1]{} \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3' = R_3 + bR_1]{} \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3' = R_3 + bR_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \xrightarrow[R_2' = R_2 + aR_1]{} \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$$

The order here doesn't matter.

Midterm 14/15

Write down two elementary matrices E_1, E_2 such that

$$E_1 E_2 \neq E_2 E_1$$

Think about the operational interpretation.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2' = R_2 + aR_1]{} \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3' = R_3 + bR_2]{} \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ab & b & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3' = R_3 + bR_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{bmatrix} \xrightarrow[R_2' = R_2 + aR_1]{} \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & b & 1 \end{bmatrix}$$

Here the order does matter matter.

Midterm 14/15

Write down two elementary matrices E_1, E_2 such that

$$E_1 E_2 \neq E_2 E_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \qquad\qquad\qquad\qquad =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & b & 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ab & b & 1 \end{bmatrix}$$

07/08 Midterm

$$A = \begin{bmatrix} 1 & 7 & -1 \\ 2 & 2 & 10 \\ 3 & 9 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 43 & -37 \\ 0 & -12 & 12 \\ 0 & -12 & 12 \end{bmatrix}$$

Find elementary matrices $E_1, \ldots, E_k$ such that

$$B = E_k \cdots E_1 A$$

Start doing some elementary row operations on A

$$B = \begin{bmatrix} 1 & 43 & -37 \\ 0 & -12 & 12 \\ 0 & -12 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 & -1 \\ 2 & 2 & 10 \\ 3 & 9 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 7 & -1 \\ 0 & -12 & 12 \\ 3 & 9 & 9 \end{bmatrix}$$

$$R_2' = R_2 - 2R_1$$

What elementary matrix implements this (by left multiplication)?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2' = R_2 - 2R_1$$

$$B = \begin{bmatrix} 1 & 43 & -37 \\ 0 & -12 & 12 \\ 0 & -12 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 & -1 \\ 0 & -12 & 12 \\ 3 & 9 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 7 & -1 \\ 0 & -12 & 12 \\ 0 & -12 & 12 \end{bmatrix}$$

$$R_3' = R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$R_3' = R_3 - 3R_1$$

$$B = \begin{bmatrix} 1 & 43 & -37 \\ 0 & -12 & 12 \\ 0 & -12 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 & -1 \\ 0 & -12 & 12 \\ 0 & -12 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 43 & -37 \\ 0 & -12 & 12 \\ 0 & -12 & 12 \end{bmatrix}$$

$$R_1' = R_1 - 3R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1' = R_1 - 3R_2$$

Putting it all together

$$\begin{bmatrix}1 & 43 & -37\\0 & -12 & 12\\0 & -12 & 12\end{bmatrix} = \begin{bmatrix}1 & -3 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-3 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 7 & -1\\2 & 2 & 10\\3 & 9 & 9\end{bmatrix}$$

Invertibility

We have many conditions equivalent to invertibility.

- A is invertible.
- Gaussian elimination produces a full set of pivots.
- $A\vec{x} = \vec{0}$ has a unique solution.
- A has a left inverse.
- A has a right inverse.
- The reduced row echelon form of A is the identity matrix.

Which one is best to use will depend on the problem.

Example

A square matrix A satisfies the equation

$$A^4 - 3A^3 + 2A^2 - I = 0$$

Is A invertible?

In this case we can find a right (or left) inverse.

$$A(A^3 - 3A^2 + 2A) = I$$

Example 2

A square matrix A satisfies

$$A^2 = \mathbf{0}_{n\times n}$$

Is A invertible?

There are several approaches. I like to use the condition

$$\vec{u} \neq \vec{0} \text{ and } A\vec{u} = \vec{0} \implies A \text{ not invertible}$$

Example 2

A square matrix A satisfies

$$A^2 = \mathbf{0}_{n \times n}$$

Is A invertible?

Answer: No.

Case 1: A is the all-zero matrix. Then A is not invertible.

Case 2: A is not the all-zero matrix. Then some column of A is not equal to $\vec{0}$.

Say $A(:, i) \neq \vec{0}$.

Example 2

A square matrix A satisfies

$$A^2 = \mathbf{0}_{n \times n}$$

Is A invertible?

Case 2: A is not the all-zero matrix. Then some column of A is not equal to $\vec{0}$.

Say $A(:, i) \neq \vec{0}$.

Then

$$A(A(:, i)) = \mathbf{0}_{n \times n}(:, i) = \vec{0}$$

thus A is not invertible.

Example 3

$$A = \begin{bmatrix} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & ? \end{bmatrix}$$

where ? stands for anything (not necessarily the same).

Is A invertible?

Example 3

$$A = \begin{bmatrix} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & ? \end{bmatrix}$$

Is A invertible?

Let's use the condition that A is invertible iff after Gaussian elimination it has a full set of pivots.

The bottom three rows cannot all have pivots.

Pivots must appear in separate columns.

There will be at most 4 pivots and A is singular.