Chapter 5

Combinatorics

> *"I think you're begging the question," said Haydock, "and I can see looming ahead one of those terrible exercises in probability where six men have white hats and six men have black hats and you have to work it out by mathematics how likely it is that the hats will get mixed up and in what proportion. If you start thinking about things like that, you would go round the bend. Let me assure you of that!" (Agatha Christie, The Mirror Crackd)*

This chapter is dedicated to combinatorics, which refers broadly to different ways of counting objects.

Suppose for example that you have two slots to be filled, and for the first slot, there are n_1 choices, while there are n_2 choices for the second slot. How many ways are there to fill up both slots? Well, for the first slot, we have n_1 choices, now for each of these, we still have n_2 choices, for a total of $n_1 n_2$ choices.

More generally, if there are k slots, and n_1 choices for the 1st slot, n_2 choices for the second slot, until n_k choices for the kth slot, we get a total of $n_1 \cdot n_2 \cdots n_k$ choices.

Example 47. Suppose you have 11 choices of fruits, 16 choices of crunches, and 15 choices of sauces. How many choices of yoghurts do you get, if you can choose one fruit, one crunch, and one sauce? Well, you can pick any of the 11 fruits, so 11 choices here. Next, for fruit 1, you can choose 16 crunches, then for fruit 2, you can choose 16 crunches, etc finally for fruit 11, you can still choose 16 crunches. Then for each choice of a fruit and a crunch, you can choose 15 sauces, which makes it a total of $11 \cdot 16 \cdot 15$.

Principle Of Counting (I)

- There are two slots to be filled, there are n_1 *choices* for slot 1 and n_2 *choices* for slot 2.
  - E.g., you have 3 choices for the main course and 2 choices for dessert.
- The total number of unique choices to fill the slots is $n_1 n_2$

Principle Of Counting (II)

- In general: n_1, n_2, … n_k choices for *k-slots*
- $n_1 * n_2 * \ldots * n_k$ ways
  - (cardinality of the cartesian product of sets)

Example: Llao Llao, one yoghurt with 1 fruit, 1 crunch, and 1 sauce

- 11 fruits
- 16 crunches
- 15 sauces

Now $n_1 \cdot n_2 \cdots n_k$ is also the cardinality of the cartesian product of k sets, where the set i has n_i elements. We will see the formal definition of cartesian product in the next chapter. For now, we will just give an example.

Example 48. Suppose you have 3 choices for the main course, and 2 choices for the dessert. How many choices of menus do you have? For the main course, you have 3 choices, now for main course 1, you have 2 choices, for main course 2, you have 2 choices and for main course 3, you still have 2 choices, thus a total of 6. An alternative way to view this question is to explicitly list all the choices:

main course 1, dessert 1	main course 1, dessert 2
main course 2, dessert 1	main course 2, dessert 2
main course 3, dessert 1	main course 3, dessert 2

This makes a total of 6 menus. You notice that when we list all the options, for every element in the set { main course }, we list all the elements in the set { dessert }. We get what is called a cartesian product of two sets, the set { main course 1, main course 2, main course 3 }, and the set { dessert 1, dessert 2 }, which by definition is { (main course 1, dessert 1),(main course 1, dessert 2),(main course 2, dessert 1),(main course 2, dessert 2),(main course 3, dessert 1),(main course 3, dessert 2)}.

Given a set A, the power set $P(A)$ is the set of all subsets of A. We will count the number of elements in $P(A)$. Write $A = \{a_1, \ldots, a_n\}$. Now list all subsets of A, and to each subset, associate a binary vector of length n, where every coefficient is either 0 or 1: the first coefficient is 1 if a_1 is in the subset, and 0 otherwise, similarly, the second coefficient is 1 if a_2 is in the subset, and 0 otherwise, and so on and so forth. Since every element is in a given subset or not, we do obtain all possible binary vectors of length n, and there are 2^n of them. We will see later another derivation using the so-called binomial theorem.

Example 49. Consider the set $A = \{1, 2\}$.

$$\begin{array}{c|l} \varnothing & 00 \\ \{1\} & 10 \\ \{2\} & 01 \\ A & 11 \end{array}$$

Cardinality Of Power Set

- Consider a set A={a_1,…,a_n} with n elements.
- List all subsets of A. Create a table

All subsets of A	Binary vectors
{a_1}	10…0

- Each of these n elements are *either in a subset of A or not:* 2 choices
  - Such a choice needs to be made for *each of the n elements*
- Thus 2*2*…*2=2^n choices.
  - Another derivation: the Binomial theorem.

Filling *r* Slots With *n* Choices

- There are n elements, with which to fill *r* slots.
- When elements can be repeated:
  - Using the principle of counting: n*n*…*n = **n^r choices**
- When elements cannot be repeated:
  - n choices for first slot,
  - n-1 choices for second slot,…
  - n-(r-1) choices for last slot
  - In total: **n(n-1)(n-2)…(n-r+1)** choices

  - **E.g., sequence of choice of cards from a deck of cards**

Now suppose that there are n elements, to be put in r slots. If elements can be repeated, we are in the scenario we have just seen, and there are n^r choices. Now if elements cannot be repeated, then, we have n choices for the first slot, $n-1$ choices for the second slot, and so on and so forth, until $n-(r-1)$ choices for the last slot. We thus get

$$n(n-1)(n-2)\cdots(n-r+1). \tag{5.1}$$

This is for example what happens when picking cards from a deck of cards, once the cards are not put back in the deck.

We call *permutation* of n elements a rearrangement of these elements. We also call permutation a rearrangement of only part of the n elements.

To count the number of permutations of n elements, see that you have n slots, and you have n ways to assign one element in the first slot. Then you have a second slot, and this time $n-1$ elements to choose from. Then you have a third slot, and now $n-2$ elements to choose from, etc, until you reach the last slot, where the last element is put. The total number of permutations of n elements is

$$n(n-1)(n-2)\cdots 2\cdot 1 = n!.$$

Alternatively, you can take the formula (5.1) for $r=n$, by noticing that all the n elements are attributed to the n slots, which gives a permutation of the n elements. This also shows that the number of permutations of n elements is

$$n(n-1)(n-2)\cdots 2\cdot 1 = n!.$$

The above general scenario, when there are n elements but only r slots, and elements cannot be repeated, is called *permutations of n objects taken r at a time*, that is an arrangement where ordering matters, and the number $P(n,r)$ of such permutations is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Example 50. In how many ways can we award a 1st, a 2nd and a 3rd prize to 8 contestants? For the first prize, any of the 8 can be awarded. But then once one contestant gets the first prize, only 7 contestants are left (because one contestant does not get two prizes!!) So for the second prize, we have 7 choices. And then for the 3rd prize, only 6 choices, so the total is $8\cdot 7\cdot 6 = 336$.

Permutation: n!

A permutation is an arrangement of all or part of a set of objects, *with* regard to the order of the arrangement.

(1st position: 3 choices, 2nd position 2 choices, 3rd position 1 choice)

Number of permutations of n objects

n*(n-1)*(n-2)…*2*1 = n!

where n! is called n factorial.

Permutation: P(n,r)

Filling r slots with n choices and no repetition:

n(n-1)(n-2)…(n-r+1)

Permutations of n objects: **n!**

Number of permutations of n objects taken r at a time

(n objects, the number of ways in which r items can be ordered)

P(n,r) = n(n-1)(n-2)…(n-r+1) = n!/(n-r)!

where n!=n*(n-1)*(n-2)*…*2*1 (called n factorial).

Permutations can be defined also if some of the objects are repeated, that is, we still have n elements, but $n = k_1 + k_2 + \cdots + k_r$, that is, there are k_1 elements of type 1, k_2 elements of type 2, until k_r elements of type r. How many permutations do we have in this case? To count this, we can proceed as follows: place k_1 elements out of n places, then place k_2 elements in $n-k_1$ places, etc until you place k_r elements in the remaining places.

This means that we have

$$\binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{k_r}{k_r} \tag{5.2}$$

where we recall that $\binom{n}{k}$ counts the number of ways of choosing k elements out of n.

We also call $\binom{n}{r}$ a *combination*, that is a say of selecting objects without considering the order of the selection. We have that

$$\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!}.$$

Indeed, recall that when we had r slots and n objects, we have $P(n,r)$ ways of placing the objects, where

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Now we still have r slots, n objects to choose from, but this time, we do not care about the ordering, and there are $r!$ possible ordering for each combination:

$$r!C(n,r) = P(n,r).$$

Example 51. From a committee of 8 people, in how many ways can you choose a chair and a vice-chair (one person cannot hold more than one position)? It is $P(8,2) = \frac{8!}{6!} = 8 \cdot 7$. Indeed, once the chair is chosen (8 choices), we have 7 choices for the vice chair.

Example

- In how many ways can we award a 1st, 2nd and 3rd prize to 8 contestants?
- For the 1st prize, any of the 8, then for the 2nd prize, any of the 7 left, and for the 3rd prize, any of the 6 left: 8*7*6=336

Permutation

In general: The number of distinguishable permutations from a collection of objects, where first object appears (repeats) k_1 times, second object k_2 times, … for r distinct objects:

$$\boxed{n!/(k_1!\, k_2! \ldots k_r!)}$$

Example: How many permutations in "MISSISSIPPI"?

M appears k_1 =1, I appears k_2 =4, S appears k_3 =4, P appears k_4 =2:

11!/(*1*! *4*!.4!2!)

Example 52. From a committee of 8 people, in how many ways can you choose a subcommittee of 2 people: it is $C(8,2) = \frac{8!}{2!6!} = 28$. Indeed, in this case, any 2 persons among the 8 people will do, irrespectively of the ordering. This means that we choose person 1 with person 2, person 1 with person 3, etc until person 1 with person 8 (7 possibilities),or person 2 with person 3, etc until person 2 with person 8 (6 possibilities), or person 3 with person 4, ..., person 3 with person 8 (5 possibilities), and by continuing the list, we get $7+6+5+4+3+2+1 = 28$.

This example also illustrates that $r!C(n,r) = P(n,r)$. Indeed, we know that $P(8,2)$ takes into account the ordering, therefore, if say person 1 is chair and person 2 is vice chair, it counts for 1 choice, while it counts for 2 choices for the subcommittee, since person 1 and person 2, and person 2 and person 1, represent the same subcommittee.

We now finish the computations of (5.2):

$$\binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{k_r}{k_r} = \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{k_2!(n-k_1-k_2)!}\cdots\frac{(n-k_1-\cdots k_{r-1})!}{k_r!}$$

where we notice that we can cancel out numerator and denominator to finally obtain

$$\frac{n!}{k_1!k_2!\cdots k_r!}.$$

Example 53. Suppose we want the number of permutations in "MISSISSIPPI". We have that the letter M appears once ($k_1 = 1$), the letter I appears $k_2 = 4$ times, the letter S appears $k_3 = 4$ times, and the letter P appears $k_4 = 2$ times. Thus the number of permutations is

$$\frac{11!}{1!4!4!2!}.$$

Combination: C(n,r) or $\binom{n}{r}$

A combination is a selection of all or part of a set of objects, *without* regard to the order in which objects are selected.

E.g. Team of 4 people from a group of 10

Number of combinations of n objects taken r at a time

$$\boxed{\binom{n}{r} = \mathbf{C(n,r) = n!/r!(n\text{-}r)!}}$$

- There are r! possible orderings within each combination
- So r! C(n,r) = P(n,r) by definition of permutation

Example

From a committee of 8 people, in how many ways can you choose:

- a chair and vice-chair (one person cannot hold more than one position?)
- P(8,2)
- a subcommittee of 2 people?
- C(8,2)

Exercises for Chapter 5

Exercise 42. A set menu proposes 2 choices of starters, 3 choices of main dishes, and 2 choices of desserts. How many possible set menus are available?

Exercise 43.
- In a race with 30 runners where 8 trophies will be given to the top 8 runners (the trophies are distinct, there is a specific trophy for each place),in how many ways can this be done?
- In how many ways can you solve the above problem if a certain person, say Jackson, must be one of the top 3 winners?

Exercise 44. In how many ways can you pair up 8 boys and 8 girls?

Exercise 45. How many ternary strings of length 4 have zero ones?

Exercise 46. How many permutations are there of the word "repetition"?

Examples for Chapter 5

We are often advised, when creating passwords, to use long passwords, with symbols and numerical digits. We will compute a few examples of how quickly the number of possible passwords grow using combinatorics.

Example 54. Suppose that a password is 9 characters long, and each character in the password can be an upper case letter, a lower case letter, a digit, or one of the six special characters $*$, $<$, $>$, !, $+$, $=$. How many different passwords are available for this computer system?

In this case, we have 9 slots, and for each of these slots, we can choose one character. There are 68 different possibilities for each character (26 upper case letters, 26 lower case letters, 10 digits, and 6 special characters), thus the answer is 68^9.

Example 55. Suppose that a password is 9 characters long, and each character in the password can be an upper case letter, a lower case letter, a digit, or one of the six special characters $*$, $<$, $>$, !, $+$, $=$. How many different passwords contain at least one occurrence of at least one of the six special characters?

To solve this, what we will do is count the total number of passwords, and then remove some passwords, namely those who do not contain any special character. We already know from the above example that the total is 68^9. Now we need to count those which do not contain any special character. So now we have still 9 slots to be filled, but we can use only 62 characters, thus we have 62^9 choices, and the answer is $68^9 - 62^9$.

The above example teaches us that sometimes, it is easier to count by "removing" the elements we are not interested in, rather than to count directly those we are interested in!

Example 56. Suppose that a password is again 9 characters long, and each character in the password can be an upper case letter, a lower case letter, or a digit. How many different passwords are possible if a password must include at least one uppercase letter, one lower-case letter, and one digit?

To solve this, we will use the technique as in the above example. First, we count the total number of passwords, and then remove those we do not want. The total number of passwords, well, we have 9 slots, each of them can take 62 values, as already done above, so this makes 62^9. Next we start counting the strings that are not valid.

Counting Passwords (I)

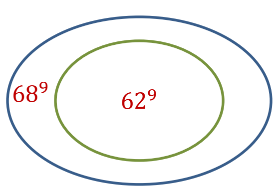
- A password is 9 characters long
- each character can be an upper case letter, a lower case letter, a digit, or one of the six special characters , <, >, !, +, =
- How many different passwords are available ? 68^9

26+26+10+6=68

Counting Passwords (II)

- A password is 9 characters long
- each character can be an upper case letter, a lower case letter, a digit, or one of the six special characters , <, >, !, +, =
- How many different passwords contain at least one occurrence of at least one of the six special characters?

$68^9 - 62^9$



http://www.pcworld.com/article/2010058/password-management-future-technology.html

A password must include at least one upper case letter, so we have to remove all the passwords made using no upper case letter, and we have 36^9 of them. Then we have to remove the passwords made using only lower-case letter, also 36^9 of them. We further remove those made without any digit, 52^9 of them. This would give

$$62^9 - 36^9 - 36^9 - 52^9.$$

But there is a catch! In the above example, we only removed one set of invalid passwords. But here, we are actually removing 3 sets, and the problem in doing that, is that we could be counting elements several times! To check whether this is happening, we need to check whether the sets of invalid passwords intersect.

- All digit strings both have no upper case letter, and no lower case letter, so they were counted twice, and there are 10^9 of them.
- All lower case letter strings both have no digit and no upper case letter, so they were counted twice, and there are 26^9 of them.
- All upper case letter strings both have no digit and no lower case letter, so they were counted twice, and there are 26^9 of them.

Note that there is no string that satisfy the three conditions. So the final number of valid passwords is

$$62^9 - 36^9 - 36^9 - 52^9 + 10^9 + 26^9 + 26^9.$$

Counting Passwords (III)

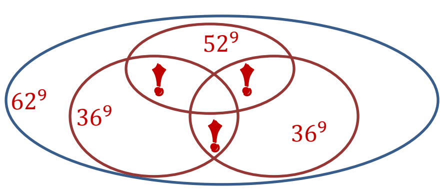
- A password is 9 characters long
- each character can be an upper case letter, a lower case letter, a digit.
- How many different passwords are possible if a password must include at least one uppercase letter, one lower-case letter, and one digit?

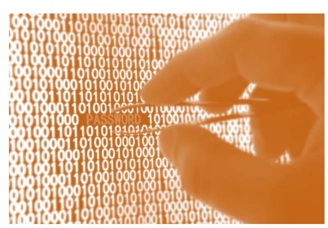
http://www.pcworld.com/article/2010058/password-management-future-technology.html

Counting Passwords (III)

- A password is 9 characters long
- each character can be an upper case letter, a lower case letter, a digit.
- How many different passwords are possible if a password must include at least one uppercase letter, one lower-case letter, and one digit?

$62^9 - 52^9 - 36^9 - 36^9 + 10^9 + 26^9 + 26^9$





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