Chapter 2

Propositional Logic

> *"Contrariwise,' continued Tweedledee, 'if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic". (Lewis Carroll, Alice's Adventures in Wonderland and Through the Looking-Glass)*

This chapter is dedicated to one type of logic, called *propositional logic*. The world *logic* refers to the use and study of valid reasoning. Logic contains rules and techniques to formalize statements, to make them precise. Logic is studied by philosophers, mathematicians and computer scientists. Logic appears in different areas of computer science, such as programming, circuits, artificial intelligence and databases.

It is useful to represent knowledge precisely and to help extract information. This last sentence may not be clear at this point, but hopefully it will become once we progress.

The term *propositional logic* thus refers to a logic which relies on *propositions*, which is defined as follows:

Definition 10. A proposition is a statement either true or false, but not both.

Example 10.
1. The statement "$1 + 1 > 3$" is false, while the statement "$5 > 3$" is true. Both statements are propositions.
2. The statement "What a great book!" is not a proposition. Someone is expressing an opinion.

Logic

- Accepted rules for making precise statements.
- Logic for computer science: programming, artificial intelligence, logic circuits, database.
- Logic
  - Represents knowledge precisely
  - Helps to extract information (inference)

Proposition

A statement that is **either true or false** but not **both** is called a proposition.

- Examples of propositions
  - “1 + 1 = 2” · · · True
  - “1 + 1 > 3” · · · False
  - “Singapore is in Europe.” · · · False

```
gap> (5>3);
true
gap> (1>3);
false
gap>
```

- Examples (which are not propositions)
  - “1 + 1 > x” · · · ×
  - “What a great book!” . . . ×
  - “Is Singapore in Asia?” . . .×

We just said that a proposition is a statement which is either true or false. We next give a definition for a statement which cannot be assinged a truth value.

Definition 11. A paradox is a statement that cannot be assigned a truth value.

Thus a paradox cannot be a proposition!

Example 11. Here is a paradox called the liar paradox: "This statement is false". Suppose that "This statement is false" takes a true value, then it must be that the statement is false, but then if "this statement is false" is false, then the statement is true, and we can iterate this process which will never lead to any conclusion.

To formalize statements, we will use *symbols*, and these symbols will have the same truth values as the statements. For example, if we consider the proposition "$1 + 1 > 5$", we will denote it by p: $p =$"$1 + 1 > 5$". Since "$1 + 1 > 5$" is false, p will take the truth value false (F).

Sometimes, we will use *logical operators* to combine statements. Here are three basic operators that we will discuss into details next:

$$\begin{array}{cc} \wedge & \text{conjunction (and)} \\ \vee & \text{disjunction (or)} \\ \neg & \text{negation (not)} \end{array}$$

The $\neg$ operator is sometimes denoted $\sim$ instead.

We are particularly interested in combining propositions (statements that either true or false).

Definition 12. A compound proposition is a statement obtained by combining propositions with logical operators.

Let us start with the negation operator:

$$\boxed{\neg \text{ negation (not)}}.$$

We will use a truth table to describe how $\neg$ operates on a proposition p:

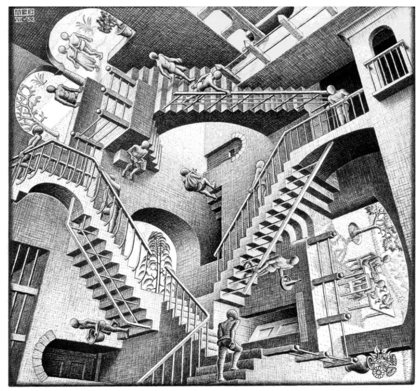
p	$\neg p$
T	F
F	T

The way the truth table is read is: on the first row, if $p = T$, then $\neg p = F$, on the second row, if $p = F$, then $\neg p = T$.

Paradox

A paradox is a statement that *cannot be assigned a truth value*.

- A paradox is not a proposition.
- Example: the liar paradox
  "This statement is false"



Art Work by Escher ("Relativity")

Symbolic Logic

- Use symbols to represent statements (both have *the same truth values*)
- Use logical operators to combine statements
  - Compound propositions = propositions combined with logical operator(s)
- Three basic operators
  - $\wedge$: conjunction (and)
  - $\vee$: disjunction (or)
  - $\neg$: negation (not, alternatively ~)

Here is an example: if $p =$"you shall pass", $\neg p =$"you shall not pass".

Next we have the disjunction operator:

$$\boxed{\vee \text{ disjunction (or)}}$$

It is an operator that combines two propositions p and q as described in the following truth table:

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

The disjunction operator returns T when at least one of the two propositions p,q is true. This is why whenever there is at least one T on a row, $p \vee q$ is true. We notice that $p \vee q$ and $q \vee p$ have the same truth table. In this case, we will say that the compound propositions $p \vee q$ and $q \vee p$ are equivalent propositions. We also say that the operator $\vee$ is commutative.

Finally we have the conjunction operator:

$$\boxed{\wedge \text{ conjunction (and)}}$$

It is an operator that combines two propositions p and q as described in the following truth table:

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

This time, for $p \wedge q$ to be true, we need both p and q to be true, which is why there is only one row for which $p \wedge q$ is true. Note that $\wedge$ is also commutative.

We now have three operators $\neg, \vee, \wedge$, that we can apply on propositions p, q. Our first big result of this chapter is called De Morgan's Law, and states connections between these three operators.

Theorem 1 (De Morgan's Law.)**.** *We have*

$$\neg(p \wedge q) \equiv \neg p \vee \neg q, \;\; \neg(p \vee q) \equiv \neg p \wedge \neg q.$$

Negation ¬

- Negation (not) of p: ¬p (~ p is also used)

p	¬p
T	F
F	T

Truth Table

- p: You shall pass

  ¬p: You shall not pass

Picture from the movie Lord of the Rings

Disjunction V

- Disjunction (or) of p with q: pVq

p	q	pVq
T	T	T
T	F	T
F	T	T
F	F	F

qVp
T
T
T
F

True when
\`at least one'
of them is
true

- p V q ≡ q V p, i.e. operator V commutes

means "equivalent"

```
gap>
gap> (5>3) or (1>5);
true
gap>
```

Conjunction ∧

- Conjunction (and) of p with q: p ∧ q

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

True only when `both' of them are true

- ∧ is also commutative: p ∧ q ≡ q ∧ p

```
gap> (5>3) and (7>5);
true
gap>
gap>
gap> (5>3) and (1>5);
false
```

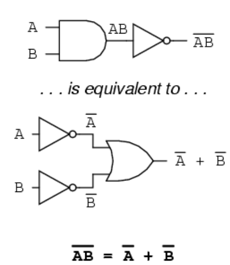
De Morgan's Law

$\neg (p \wedge q) \equiv \neg p \vee \neg q$

$\neg (p \vee q) \equiv \neg p \wedge \neg q$

Augustus De Morgan (1806-1871)

p q	¬p ¬q p ∧ q	¬(p ∧ q)	¬p V¬q
T T	F F T	F	F
T F	F T F	T	T
F T	T F F	T	T
F F	T T F	T	T



Picture from Wikipedia

Contradiction

A statement that is always false is called a contradiction.

Example: This course is easy
`and' this course is not easy

$p \land (\neg p) \equiv F$

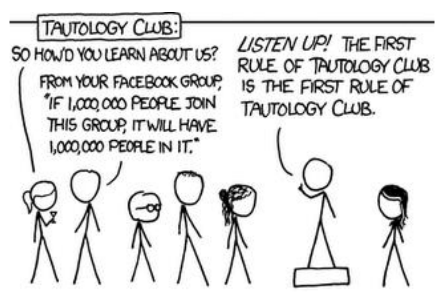
Tautology

An expression that always gives a true value is called a tautology .

Example: $p \lor (\neg p) \equiv T$

p	¬p	p∨¬p
T	F	T
F	T	T

Always true!



Somewhat similar to "Head I win, Tail you lose"

Proof. The proof consists of computing the truth table. We need to show that the truth table for $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are the same, which will say that both compound propositions are equivalent. Let us compute in details the first row of the truth table for $\neg(p \wedge q)$ and $\neg p \vee \neg q$: if both p and q are true, then $\neg p$ and $\neg q$ are false, and $p \wedge q$ is true, since it is true exactly when both p and q are true. Then $\neg(p \wedge q)$ is the negation of $p \wedge q$, that is $\neg(p \wedge q)$ is false. We only need to show that $\neg p \vee \neg q$ is false too, which is the case, since both $\neg p$ and $\neg q$ are false. The other rows are filled in similarly. The proof of $\neg(p \vee q) \equiv \neg p \wedge \neg q$ is done in Exercise 12. □

So far, we have seen two types of statements: (1) a proposition, which is a statement either always true, or always false, and (2) a paradox, which is a statement whose truth value cannot be assigned. Here are two new types of statements:

Definition 13. A contradiction is a statement that is always false.

Example 12. A typical example of a contradiction is $p \wedge (\neg p)$. Why? recall that $p \wedge q$ takes the truth value true only when both p and q are true at the same time...but it is not possible for both p and $\neg p$ to be true both at the same time.

If a statement which is always false is called a contradiction, you may wonder whether there is a name for a statement which is always true?

Definition 14. A tautology is a statement that always gives a true value.

Example 13. For an example of contradiction, we choose $\wedge$, because $p \wedge q$ is true only when both p and q is true. Now we may similarly get an example of tautology. Pick $\vee$ instead. For $p \vee q$ to be true, it is enough that either p or q is true. Thus $p \vee (\neg p)$ is a tautology.

We already said that two statements are *equivalent* if their truth tables are the same. Here is an example of three statements which are equivalent:

$$\neg h \wedge \neg b \equiv \neg b \wedge \neg h \equiv \neg(b \vee h).$$

We notice that the first equivalence follows from the commutativity of $\wedge$. For the second equivalence, this is one of De Morgan law! (which can be seen through a truth table).

Equivalent Expressions

Consider the following three statements

- Alice is not married but Bob is not single

  $\neg h \wedge \neg b$

- Bob is not single and Alice is not married

  $\neg b \wedge \neg h$

- Neither Bob is single nor Alice is married

  $\neg(b \vee h)$

- These three statements are equivalent

  $\neg h \wedge \neg b \equiv \neg b \wedge \neg h \equiv \neg(b \vee h)$

Equivalent Expressions

The three statements

$\neg h \wedge \neg b \equiv \neg b \wedge \neg h \equiv \neg(b \vee h)$

are equivalent.

b h	$\neg b$	$\neg h$	$b \vee h$	$(\neg h \wedge \neg b)$	$(\neg b \wedge \neg h)$	$\neg(b \vee h)$
T T	F	F	T	F	F	F
T F	F	T	T	F	F	F
F T	T	F	T	F	F	F
F F	T	T	F	T	T	T

The term logical equivalence (law) is new to us, but in fact, we already saw several examples of such equivalences. We speak of logical equivalences to describe laws that express transformations from one logical expression to another equivalent one. Let us summarize those we know already:

- De Morgan laws: $\neg(p \wedge q) \equiv \neg p \vee \neg q$, and $\neg(p \vee q) \equiv \neg p \wedge \neg q$,
- Commutativity of $\vee$ and $\wedge$: $p \wedge q \equiv q \wedge p$, $p \vee q \equiv q \vee p$,
- The negation of true is false, the negation of false is true: $\neg T = F$, $\neg F = T$.

Here are two other ones that we saw implicitly. Suppose that $\mathcal{T}$ is a tautology (statement always true) and C is a contradiction (statement always false). Then

- The negation of a tautology is a contradiction: $\neg \mathcal{T} \equiv C$.
- The negation of a contradiction is a tautology: $\neg C \equiv \mathcal{T}$.

Here are three new logical equivalences:

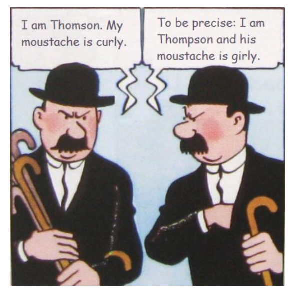
- Double negation: $\neg(\neg p) \equiv p$.
- Idempotent: $p \wedge p \equiv p$, $p \vee p \equiv p$.
- Absorption: The first one is $p \vee (p \wedge q) \equiv p$. Let us see why, using a truth table:

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

The section absorption law is $p \wedge (p \vee q) \equiv p$ (see Exercise 13).

Logical Equivalences

- Useful laws to transform one logical expression to an equivalent one.
- Axioms (*T* ≡ tautology, *C* ≡ contradiction):
  $\neg T \equiv F$ $\neg F \equiv T$ $\neg T \equiv C \equiv F$ $\neg C \equiv T \equiv T$
- De Morgan: $\neg(p \wedge q) \equiv \neg p \vee \neg q$
  $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- Commutativity:
  $p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$



Logical Equivalence Laws

- double negation: $\neg(\neg p) \equiv p$
- idempotent: $p \wedge p \equiv p$ and $p \vee p \equiv p$
- absorption: $p \vee (p \wedge q) \equiv p$ and $p \wedge (p \vee q) \equiv p$

We will next add three more logical operators. We already know 3 of them:

$\wedge$	conjunction (and)
$\vee$	disjunction (or)
$\neg$	negation (not)

Our fourth logic operator is a conditional statement:

$$\boxed{\rightarrow \text{ if then}}$$

that is $p \rightarrow q$ means "if p then q". You have to be a bit careful with this operator! It sounds a lot like a statement you may have encountered while programming, but actually it is a bit different: if p is true, then $p \rightarrow q$ takes the truth value of q, this part is not too surprising. However, if p is false, then it is assumed that $p \rightarrow q$ is true by default, also called vacuously true!!

Now that we have a new logic operator, let us see how it relates to those we know.

Theorem 2 (The Conversion Theorem.)**.** *We have*

$$p \rightarrow q \equiv \neg p \vee q.$$

Proof. This can be seen using the truth tables:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Alternatively

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

and then use De Morgan law. $\square$

Conditional Statement $\rightarrow$

Known operators: $\wedge$ conjunction (and), $\vee$ disjunction (or), $\neg$ negation.

- if p then q: $p \rightarrow q$

By definition, when p is false, $p \rightarrow q$ is true. This is called *vacuously true* or *true by default*.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

```
gap>
gap> a:=10;; if (a>5) then Print("yes"); fi;
yes
gap> a:=1;; if (a>5) then Print("yes"); fi;
gap>
gap>
```

Not really the same!

Conversion Theorem

Theorem: $p \rightarrow q \equiv \neg p \vee q$

Proof

- $p \rightarrow q$ means $\neg(p \wedge \neg q)$ (it cannot be that p is true but q is false).
- Apply DeMorgan's law

p q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T T	T	F	T
T F	F	F	F
F T	T	T	T
F F	T	T	T

Given the conditional statement if $p \to q$, it can be modified in three different ways to give raise to some other statements:

- The converse of $p \to q$ is $q \to p$.
- The inverse of $p \to q$ is $\neg p \to \neg q$.
- The contrapositive of $p \to q$ is $\neg q \to \neg p$.

The notion of converse and contrapositive often appear in the context of proof techniques. The reason why the contrapositive appears in the context of proof is because of this theorem:

Theorem 3. *We have*

$$\neg q \to \neg p \equiv p \to q,$$

that is the contrapositive of $p \to q$ is equivalent to $p \to q$!

Proof. Recall the Conversion Theorem: $p \to q \equiv \neg p \vee q$. Let us start with $\neg q \to \neg p$:

$$\begin{array}{ll} \neg q \to \neg p & \\ \equiv \neg(\neg q) \vee \neg p & \text{Conversion Theorem} \\ \equiv q \vee \neg p & \text{double negation} \\ \equiv \neg p \vee q & \text{commutativity} \\ \equiv p \to q & \text{Conversion Theorem.} \end{array}$$

Another proof using truth tables is found in Exercise 16. □

The contrapositive of $p \to q$ is also called only if, which is our 5th logic operator ($\triangleq$ means "equal by definition")

$$\text{only if} \triangleq \neg q \to \neg p,$$

even though it is equivalent to $p \to q$.

Example 14. The formulas are saying

$$(p \text{ only if } q) = (\neg q \to \neg p) \equiv (p \to q).$$

Let us see how this translates into sentences:

$$\underbrace{\text{“Bob pays taxes}}_{p} \text{ only if } \underbrace{\text{his income is more than 1000 SD”.}}_{q}$$

We notice where is the "only if". Then $\neg q \to \neg p$ is

$$\underbrace{\text{“if Bob's income is less than 1000 SD,}}_{\neg q} \text{ then } \underbrace{\text{he does not pay taxes.}}_{\neg p}$$

Conditional Statements

The converse of $p \rightarrow q$ is $q \rightarrow p$, the inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$, the contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

Theorem: $\neg q \rightarrow \neg p \equiv p \rightarrow q$

Proof

$$\begin{aligned} & \neg q \rightarrow \neg p \\ \equiv\ & \neg(\neg q) \vee \neg p \\ \equiv\ & q \vee \neg p \\ \equiv\ & \neg p \vee q \\ \equiv\ & p \rightarrow q \end{aligned}$$

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Only If

- p only if $q \triangleq \neg q \rightarrow \neg p$
- $\neg q \rightarrow \neg p$ is the contrapositive of $p \rightarrow q$.
- (if not q then not p) $\equiv$ ($p \rightarrow q$). Why?

- Example: 'Bob pays taxes only if his income $\geq$ \$1000'

$\triangleq$ 'if Bob's income < \$1000 then he does not pay taxes'

$\equiv$ 'if Bob pays tax then his income $\geq$ \$1000'

Finally $p \to q$ is

$$\underbrace{\text{“if Bob pays taxes,}}_{p} \text{ then } \underbrace{\text{his income is more than 1000 SD.}}_{q}$$

The terms “sufficient” and “necessary” might sound more or less like having the same meaning, but they have different specific meanings in the world of logic:

Definition 15. When $p \to q$, we call p a sufficient condition for q, while q is called a necessary condition for p.

Our list of logical operators will now be complete, with the addition of the biconditionial operator:

$$\boxed{p \leftrightarrow q \text{ biconditional of } p \text{ and } q.}$$

By definition $p \leftrightarrow q$ means $(p \to q) \wedge (q \to p)$. This statement, also referred to as if and only if, appears very often during proofs.

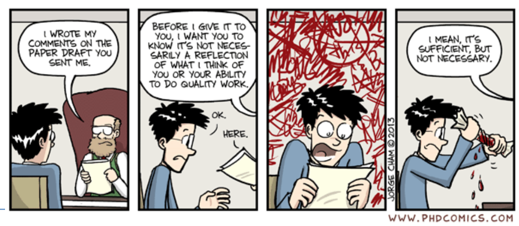
Finally, when combining logic operators, there may be doubts on which operator to apply first. The use of parentheses helps in deciding which operators should be computed first. If several $\neg$ are used, start with the rightmost one (the one that applies directly on the proposition). If several $\to$ are used instead (or other logical operators connecting two propositions), start from the leftmost one.

A good exercise to see whether the different logical equivalence laws seen so far are handled is to prove that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$. The details are given in Exercise 17.

Sufficient and Necessary Conditions

When $p \rightarrow q$, p is called a ***sufficient* condition** for q , q is a ***necessary* condition** for p.

- Being Japanese is a sufficient condition for being Asian.
  ≡ if someone is Japanese then s/he will be an Asian
- Being Asian is a necessary condition for being Japanese.
  ≡ 'if someone is not Asian, he can not be Japanese'



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Example

- Let f: 'you fix my ceiling', p: 'I pay my rent'.
  - 'you fix my ceiling or I won't pay my rent'
    - $f \vee \neg p \quad \equiv p \rightarrow f$
  - `If you do not fix my ceiling then I won't pay my rent'
    - $\neg f \rightarrow \neg p \equiv p \rightarrow f$
  - 'I will pay my rent only if you fix my ceiling'
    - $\neg f \rightarrow \neg p \equiv p \rightarrow f$

Biconditional $\leftrightarrow$

- The biconditional of p and q: $p \leftrightarrow q \triangleq (p \rightarrow q) \wedge (q \rightarrow p)$
  – True only when p and q have identical truth values
- *If and only if (iff)*

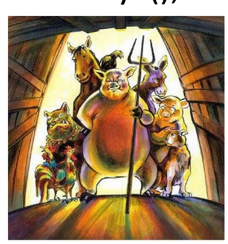
Known operators:
$\wedge$ conjunction (and),
$\vee$ disjunction (or),
$\neg$ negation,
$\rightarrow$ conditional (if then)

p q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T T	T	T	**T**
T F	F	T	**F**
F T	T	F	**F**
F F	T	T	**T**

Operator Precedence

- From high to low: ( ) , $\neg$, $\wedge\vee$, $\rightarrow$, $\leftrightarrow$
- When equal priority instances of *binary connectives* are not separated by (), the *leftmost one has precedence*. E.g. $p \rightarrow q \rightarrow r \equiv (p \rightarrow q) \rightarrow r$
- When instances of $\neg$ are not separated by (), the *rightmost one has precedence*:

  E.g. $\neg\neg\neg p \equiv \neg(\neg(\neg p))$



All animals are equal, but some are more equal than others – George Orwell, Animal Farm

© to the artist, http://pda-animalfarm.wikispaces.com/Animalism

Example

- Show that $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

$p \vee q \rightarrow r$

$\equiv (p \vee q) \rightarrow r$ (operator precedence)

$\equiv \neg(p \vee q) \vee r$ (why?)

$\equiv (\neg p \wedge \neg q) \vee r$ (DeMorgan's)

$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$ (why?)

$\equiv (p \rightarrow r) \wedge (q \rightarrow r)$ (why?)

Summary

- Useful logical equivalence laws
  - Proving equivalence using these laws
- Conditional & Biconditional statements
  - Sufficient and necessary conditions
- Operator precedence

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Before continuing, it is probably a good time to summarize shortly what we did. We defined 6 logical operators:

$\wedge$	conjunction (and)	$p \wedge q$
$\vee$	disjunction (or)	$p \vee q$
$\neg$	negation (not)	$\neg p$
$\rightarrow$	conditional (if then)	$p \rightarrow q$
	only if	p only if q
$\leftrightarrow$	biconditional (if and only if)	$p \leftrightarrow q$

We also saw several important logical equivalences, De Morgan Laws and the Conversion Theorem:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q, \;\; \neg(p \wedge q) \equiv \neg p \vee \neg q, \;\; p \rightarrow q \equiv \neg p \vee q.$$

Now that we have these basic tools, we will use them to construct more sophisticated logical constructs, such as *arguments*.

Definition 16. An argument is a sequence of statements. The last statement is called the conclusion, all the previous statements are premises, or assumptions/hypotheses.

Definition 17. A valid argument is an argument where the conclusion is true if the premises are true.

We may rephrase this definition of valid argument in the language of logic: a series of statements forms a valid argument if and only if the conjunction of premises (that is, premises connected by a $\wedge$) implying the conclusion is a tautology. Recall that a tautology is a proposition which is always true. Here is an example of conjunction of premises implying a conclusion:

$$((\text{premise}) \wedge (\text{premise})) \rightarrow \text{conclusion}$$

Thus if the premises are true, then the truth value of

$$((\text{premise}) \wedge (\text{premise}))$$

is true, and thus the truth value of

$$((\text{premise}) \wedge (\text{premise})) \rightarrow \text{conclusion}$$

is true as well.

Valid Argument

An argument is a sequence of statements. The last statement is called the conclusion, all the previous statements are premises (or assumptions/ hypotheses).

A valid argument is an argument where the *conclusion is true if the premises are true.*

Example:
'if you pay up in full then I will deliver it'; 'you pay up in full'; — premises
'I will deliver it'; — conclusion

Valid Argument

A valid argument is an argument where the *conclusion is true if the premises are true.*

- A series of statements form a valid argument if and only if `the conjunction of premises implying the conclusion' is a tautology
- ((premise)∧(premise)) → conclusion

If instead any of the premises is false, then

$$((\text{premise}) \wedge (\text{premise}))$$

takes the value false, and by definition of the conditional operator $\to$, the truth value of

$$((\text{premise}) \wedge (\text{premise})) \to \text{conclusion}$$

is true as well. Thus

$$((\text{premise}) \wedge (\text{premise})) \to \text{conclusion}$$

always takes true as truth value, and is indeed a tautology.

Here is a typical template of what an argument looks like:

$$\boxed{\begin{array}{l} p \to q; \\ p; \\ \therefore q; \end{array}}$$

The premises are $p \to q; p;$, and the conclusion is q.

How do we know that this argument is valid? We check that it is indeed a tautology...which is something that we do as usual, using a truth table, where the first columns contain the premises, and the last column contains $((p \to q) \wedge p) \to q$, and we need that the last column contains the truth value true.

In fact, it is even easier than a normal truth table, because here, the only case we really care about is the case where all premises are true (the corresponding rows are called critical rows)! Again, this is because if any premise is wrong, then the truth value of $((p \to q) \wedge p) \to q$ is always true by default.

Let us compute the first row of the truth table:

p	q	$(p \to q)$	$(p \to q) \wedge q$	$((p \to q) \wedge p) \to q$
T	T	T	T	T

This first row is a critical row, because all premises namely $p \to q$ and p, both are true. There is only one critical row, because the only other entry with p true is when q is false, in which case the other premise $p \to q$ is false. On this first row, the truth value of the conclusion is true, this is thus a tautology, and we do have a valid argument.

Valid Argument Template

p → q;
p; } premises

∴ q conclusion

- By definition, a valid argument satisfies: "If the premises are true, then the conclusion is true"
- Also: ((p→q)∧p) → q is a tautology.

truth values of premises and of the conclusion

critical rows are rows with *all premises true*

if in all critical rows the conclusion is true, then the *argument is valid* (otherwise it is *invalid).*

p q	p→q	(p→q))∧p	((p→q))∧p) → q
T T	T	T	T
T F	F	F	T
F T	T	F	T
F F	T	F	T

No need to calculate

Counter Example

If in all critical rows the conclusion is true, then the *argument is valid* (otherwise it is *invalid).*

A *critical row with a false conclusion* is a *counter example* that invalidates the argument (=makes the argument not valid).

– A counter example indicates a situation where the conclusion does not follow from the premises.

Note that we could use another simplification, which is to replace the last column by the conclusion, namely to compute

p	q	$(p \to q)$	$(p \to q) \wedge q$	q
T	T	T	T	T

Why? because once all the premises are true, then the truth value of $((p \to q) \wedge p) \to q$ is that of q...this is true in general, once

$$((\text{premise}) \wedge (\text{premise}))$$

is true, then the truth value of

$$((\text{premise}) \wedge (\text{premise})) \to \text{conclusion}$$

is exactly the truth value of the conclusion!

It is enough to have one critical row with a truth value false, to invalide the arugment: this row provides a counter-example, that is a situation where all the premises are true, yet the conclusion does not follow.

Example 15. Consider the argument:

$$\boxed{\begin{array}{l} S = (f \wedge a \to r); \\ f; \\ \neg a; \\ \therefore \neg r; \end{array}}$$

To check whether this argument is valid, we need to check the truth tables when the premises are all true. The premises are S, f and $\neg a$. When is S true? it is true when $f \wedge a$ is false, and when $f \wedge a$ is true and r is true.

a	r	f	$\neg a$	$f \wedge a$	S	$\neg r$
				F	T	
	T			T	T	

Note that we put the value of $\neg r$, namely the conclusion, in the last column.

Then we need f and $\neg a$ to be true, that is f is true and a is false. This means that $f \wedge a$ is false, thus the second row in the above table cannot be a critical row. Now the first row is one, irrespectively of the truth value of r.

Invalid Argument Example

'if it is falling *and* directly above me then I'll run ' — premises
'It is falling'
'it is *not* directly above me
'I will *not* run' — conclusion

$S = (f \wedge a \rightarrow r)$; f; ¬a; — premises
∴ ¬r — conclusion

a	r	f	¬a	f ∧ a	S	¬r
T	T	T	F	T	T	F
T	T	F	F	F	T	F
T	F	T	F	T	F	T
T	F	F	F	F	T	T
F	T	T	T	F	T	**F**
F	T	F	T	F	T	F
F	F	T	T	F	T	**T**
F	F	F	T	F	T	T

Critical rows

Invalid argument : conclusion on 5th row is false!

Fallacy

A fallacy is an error in reasoning that results in an invalid argument.

Fallacy 1: converse error.

Example

- If it is Christmas, then it is a holiday.
- It is a holiday. Therefore, it is Christmas!

$p \rightarrow q$;
q;
$\therefore\ p$

Fallacy 2: inverse error.

Example

- If it is raining, then I will stay at home.
- It is not raining. Therefore I would not stay at home!

$p \rightarrow q$;
$\neg p$;
$\therefore\ \neg q$

Thus the two critical rows that we need to consider are:

a	r	f	$\neg a$	$f \wedge a$	S	$\neg r$
F	T	T	T	F	T	F
F	F	T	T	F	T	T

and the first row gives a counter example!

Definition 18. A fallacy is an error in reasoning that results in an invalid argument.

Here are examples of fallacies:
A converse error consists of

$$p \to q; q; \therefore p.$$

This is not a valid argument because the row of

p	q	$p \to q$
F	T	T

gives a counter-example.
An inverse error consists of

$$p \to q; \neg p; \therefore \neg q.$$

This is not a valid argument because the row of

p	q	$p \to q$	$\neg p$	$\neg q$
F	T	T	T	F

gives a counter-example.
It is also possible that an argument is invalid, but still it may lead to a correct conclusion, e.g. by coincidence...

Definition 19. A rule of inference is a logical construct which takes premises, analyzses their synthax, and returns a conclusion.

We already saw

$$\boxed{p \to q; p; \therefore q; \text{ Modus Ponens}}$$

Similarly, we have

$$\boxed{p \to q; \neg q; \therefore \neg p; \text{ Modus Tollens.}}$$

Invalid Argument, Correct Conclusion

- An argument may be invalid, but it may still draw a correct conclusion (e.g. by coincidence)
- Example
  - If New York is a big city, then New York has tall buildings
  - New York has tall buildings
    - So New York is a big city
- So what happened?
  - We have just made an invalid argument
    - Converse error!
  - But conclusion is true (a fact true by itself)

Inference Rules

A rule of inference is a logical construct which takes premises, analyzes their syntax and returns a conclusion.

We already saw

$$\begin{array}{r} p \rightarrow q; \\ p; \\ \therefore q \end{array}$$

Modus ponens
(method of affirming)

$$\begin{array}{r} p \rightarrow q; \\ \neg q; \\ \therefore \neg p \end{array}$$

Modus tollens
(method of denying)

Indeed, the premises are $p \to q$ and $\neg q$, the conclusion is $\neg p$ and

p	q	$(p \to q)$	$\neg q$	$\neg p$
T	T	T	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	T

and the only critical row is the last one, for which the conclusion is true. We did not fill up the first three rows on purpose, since they are not critical.

Here are some more inference rules:

$$\boxed{p \wedge q; \therefore q.} \tag{2.1}$$

In its truth table, we only care about the entry when $p \wedge q$ is true, for which it must be that both p and q are true, thus q is true as well and the argument is valid.

$$\boxed{p; q; \therefore p \wedge q.}$$

In its truth table, we only care about the row where both p and q are true. In this case, $p \wedge q$ is true.

$$\boxed{p; \therefore p \vee q.}$$

If p is true, then $p \vee q$ is true (irrespectively of the truth value of q).

$$\boxed{p \vee q; \neg p; \therefore q.}$$

Again, in its truth table, we care about the rows where $p \vee q$ is true. There are three such rows (exclude the one where both p and q are false). Now the row for which $\neg p$ is true is the one where p is false, this means we have only one critical row, and if p is false, q has to be true (for $p \vee q$ to be true). Two more such rules are found in Exercises 20 and 21.

The dilemma inference rule is:

$$\boxed{p \vee q; p \to r; q \to r; \therefore r.}$$

For $p \vee q$ to be true, we exclude both p and q false. When p is true, then r has to be true. When p is false, r may take any value since $p \to r$ is automatically true.

More Inference Rules

Conjunctive Simplification (particularizing)	$p \wedge q$; ∴ p
Conjunctive Addition (specializing)	p; q; ∴ $p \wedge q$
Disjunctive addition (generalization)	p; ∴ $p \vee q$
Disjunctive Syllogism (case elimination)	$p \vee q$; $\neg p$; ∴ q
Rule of contradiction	$\neg p \to C$; ∴ p
Alternative rule of contradiction	$\neg p \to F$; ∴ p

Inference Rule: Dilemma

Dilemma (case by case discussions)

$p \vee q$;
$p \to r$;
$q \to r$;
∴ r

p	q	r	$(p \to r)$
T	T	T	T
T	F	T	T
F	T	T	T
F	T	F	T

Among these rows, the 4rth one is not critical, since when q is true and r is false, then $q \to r$ is false. The 3 first rows are all critical, and the conclusion r takes the value true, thus the argument is valid.

The hypothetical syllogism rule is:

$$\boxed{p \to q; q \to r; \therefore p \to r.}$$

For a change, we will prove this rule using equivalences instead of a truth table.

$$\begin{aligned}
& (p \to q) \wedge (q \to r) \\
\equiv\ & (p \to q) \wedge (\neg q \vee r) \text{ conversion Theorem} \\
\equiv\ & [(p \to q) \wedge \neg q)] \vee [(p \to q) \wedge r] \text{ distributivity} \\
\equiv\ & ([(p \to q) \wedge \neg q)] \vee (p \to q)) \wedge ([(p \to q) \wedge \neg q)] \vee r) \ \text{ distributivity}
\end{aligned}$$

Let us simplify the first term $([(p \to q) \wedge \neg q) \vee (p \to q))$: we see that $(p \to q)$ appears twice, let us call give it a name, say $a = (p \to q)$ to see better what this statement looks like:

$$(a \wedge \neg q) \vee a.$$

We see that whenever a is true, this expression is true (because of $\vee a$). Now whenever a is false, then $a \wedge \neg q$ is false, no matter what is $\neg q$. Thus

$$(a \wedge \neg q) \vee a \equiv a$$

and

$$([(p \to q) \wedge \neg q)] \vee (p \to q)) \equiv p \to q.$$

Thus we can go back to our computations, and find that

$$(p \to q) \wedge (q \to r) \equiv (p \to q) \wedge ([(p \to q) \wedge \neg q)] \vee r)\,.$$

We now look at the other term, namely

$$[(p \to q) \wedge \neg q)] \vee r.$$

Inference Rule: Hypothetical Syllogism

Hypothetical syllogism:

$p \to q$;
$q \to r$;
$\therefore p \to r$

- Example

  If I do not wake up, then I cannot go to work.

  If I cannot go to work, then I will not get paid.

  Therefore, if I do not wake up, then I will not get paid.

Proof of Hypothetical Syllogism

$p \to q;\ q \to r;\ \therefore\ p \to r$

$(p \to q) \wedge (q \to r)$ (hypotheses; assumed true)

$\equiv (p \to q) \wedge (\neg q \vee r)$ (Conversion Theorem)

$\equiv [(p \to q) \wedge \neg q\,] \vee [(p \to q) \wedge r]$ (Distributive)

$\equiv [((p \to q)\ \wedge \neg q\,) \vee (p \to q)] \wedge [((p \to q) \wedge \neg q\,) \vee r]$ (Distributive)

$\equiv (p \to q) \wedge [((p \to q) \wedge \neg q\,) \vee r]$ (Recall absorption law: $a \vee (a \wedge b) \equiv a$,

hence $[((p \to q) \wedge \neg q\,) \vee (p \to q)] \equiv p \to q$)

$\equiv (p \to q) \wedge [((\neg p \vee q) \wedge \neg q\,) \vee r]$ (Conversion)

$\equiv (p \to q) \wedge [((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee r]$ (Distributive)

$\equiv (p \to q) \wedge [((\neg p \wedge \neg q) \vee F) \vee r]$ (Negation)

$\equiv (p \to q) \wedge [(\neg p \wedge \neg q) \vee r]$ (Unity)

$\equiv (p \to q) \wedge\ [(\neg p \vee r)\ \wedge\ (\neg q \vee r)]$ (Distributive)

$\equiv [(p \to q) \wedge (\neg q \vee r)] \wedge\ (\neg p \vee r)$ (Commutative; Associative)

$\therefore\ (\neg p \vee r) \equiv p \to r$ (Conjunctive simplification; conversion)

Using the conversion theorem, we have $(p \to q) \equiv \neg p \vee q$, thus

$$\begin{aligned} & [(p \to q) \wedge \neg q)] \vee r \\ \equiv\ & [(\neg p \vee q) \wedge \neg q)] \vee r \\ \equiv\ & [(\neg p \wedge \neg q) \vee (q \wedge \neg q)] \vee r \text{ distributivity} \end{aligned}$$

We can now simplify $(q \wedge \neg q)$, since this expression is always false (a statement and its contrary cannot be true at the same time). Then we have

$$(\neg p \wedge \neg q) \vee F,$$

which is true exactly when $(\neg p \wedge \neg q)$ is true, and false exactly when $(\neg p \wedge \neg q)$ is false, thus we get

$$\begin{aligned} & [(p \to q) \wedge \neg q)] \vee r \\ \equiv\ & (\neg p \wedge \neg q) \vee r \\ \equiv\ & (\neg p \vee r) \wedge (\neg q \vee r) \text{ distributivity.} \end{aligned}$$

Let us now combine everything together:

$$\begin{aligned} & (p \to q) \wedge (q \to r) \\ \equiv\ & (p \to q) \wedge ([(p \to q) \wedge \neg q)] \vee r) \\ \equiv\ & (p \to q) \wedge (\neg p \vee r) \wedge (\neg q \vee r) \\ \equiv\ & [(p \to q) \wedge (\neg q \vee r)] \wedge (\neg p \vee r) \\ \therefore\ & (\neg p \vee r) \\ \equiv\ & p \to r \text{ conversion theorem.} \end{aligned}$$

The second last "therefore" statement follows from the inference rule (2.1):

$$\boxed{p \wedge q; \therefore q.}$$

Exercises for Chapter 2

Exercise 11. Decide whether the following statements are propositions. Justify your answer.

1. $2 + 2 = 5$.
2. $2 + 2 = 4$.
3. $x = 3$.
4. Every week has a Sunday.
5. Have you read "Catch 22"?

Exercise 12. Show that

$$\neg(p \vee q) \equiv \neg p \wedge \neg q.$$

This is the second law of De Morgan.

Exercise 13. Show that the second absorption law $p \wedge (p \vee q) \equiv p$ holds.

Exercise 14. These two laws are called distributivity laws. Show that they hold:

1. Show that $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$.
2. Show that $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$.

Exercise 15. Verify $\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) \equiv \neg p$ by

- constructing a truth table,
- developing a series of logical equivalences.

Exercise 16. Using a truth table, show that:

$$\neg q \to \neg p \equiv p \to q.$$

Exercise 17. Show that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$.

Exercise 18. Are $(p \to q) \vee (q \to r)$ and $p \to r$ equivalent statements?

Exercise 19. Prove or disprove the following statement:

$$p \wedge (\neg(q \to r)) \equiv (p \to r).$$

Exercise 20. Show that this argument is valid:

$$\boxed{\neg p \to F; \therefore p.}$$

Exercise 21. Show that this argument is valid, where C denotes a contradiction.

$$\boxed{\neg p \to C; \therefore p.}$$

Exercise 22. 1. Prove or disprove the following statement:

$$(p \wedge q) \to p \equiv T.$$

2. Decide whether the following argument is valid.

$\neg d \to h;$
$\neg h \to d;$
$\therefore \neg d \vee \neg h$

Exercise 23. Determine whether the following argument is valid:

$$\begin{aligned}
&\neg p \to r \wedge \neg s\\
&t \to s\\
&u \to \neg p\\
&\neg w\\
&u \vee w\\
&\therefore t \to w.
\end{aligned}$$

Exercise 24. Determine whether the following argument is valid:

$$\begin{aligned}
&p\\
&p \vee q\\
&q \to (r \to s)\\
&t \to r\\
&\therefore \neg s \to \neg t.
\end{aligned}$$

Exercise 25. Decide whether the following argument is valid:

$(p \vee q) \to \neg r;$
$\neg r \to s;$
$p;$
$\therefore s$

Examples for Chapter 2

To practice some of the logical concepts we have studied so far, we will consider the well known puzzles of the island of knights and knaves.

The assumptions are that on this island:

- Knights always tell the truth.
- Knaves always lie.

As a visitor, you will be told some statements, and you have to decide whether the islanders you are speaking with are knights or knaves.

The general method will be to rewrite the claims of the islanders using propositional logic, and then to use truth tables to figure out the truth!

Example 16. You meet two islanders A and B. A says: " I am a knave but he is not". You need to decide what are A and B.

To do so, we will use logic. Let us call p the statement "A is a knight", and q the statement "B is a knight". With that, we rephrase the statement of A.

A says: "$\neg p \wedge q$". The truth table of $\neg p \wedge q$ is

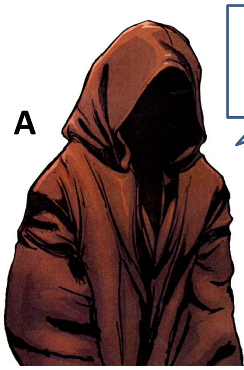
p	$\neg p$	q	$\neg p \wedge q$
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F

Now either A is a knight, or A is a knave. If A is a knight, then the statement p is true, and A must always tell the truth, thus $\neg p \wedge q$ must be true as well. This cannot be from the truth table.

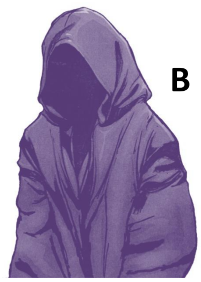
If A is a knave, then the statement p is not true, and A must always lie, thus $\neg p \wedge q$ must be false as well. This happens in the last row of the table, where p is false, q is false, and $\neg p \wedge q$ is false, thus we conclude that A **is a knave, and** B **is a knave**.

Knights & Knaves I

A



I am a knave
but he is not.



B

Art belongs to the artist

Knights & Knaves I

A

I am a knave
but he is not.
¬p ∧ q

- p=“A is a knight”
- q=“B is a knight”

p	q	¬p ∧ q
T	T	F
T	F	F
F	T	T
F	F	F

- If A is a knight, then p = true, and ¬p ∧ q must be true.
- If A is a knave, then p=false, and ¬p ∧ q must false.

Example 17. You meet two islanders A and B. A says: " If I am a knight then so is he". You need to decide what are A and B.

Again, let us call p the statement "A is a knight", and q the statement "B is a knight". Then A says: "$p \to q$", with truth table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

If A is a knight, then the statement p is true, and A must always tell the truth, thus $p \to q$ must be true as well. This is possible with the first row

If A is a knave, then the statement p is not true, and A must always lie, thus $p \to q$ must be false as well. This cannot happen. We conclude that A **is a knight, and** B **is a knight**.

Example 18. You meet two islanders A and B. A says: " I am a knave or B is a knight". You need to decide what are A and B.

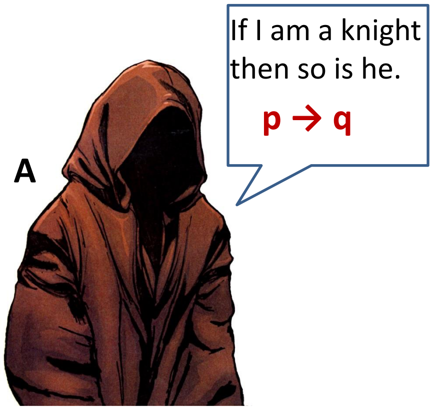
As twice above, let us call p the statement "A is a knight", and q the statement "B is a knight". Then A says: "$\neg p \vee q$". Its truth table is

p	$\neg p$	q	$\neg p \vee q$
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T

If A is a knight, then the statement p is true, and A must always tell the truth, thus $\neg p \vee q$ must be true as well. This happens in the first row of the table.

If A is a knave, then the statement p is not true, and A must always lie, thus $\neg p \vee q$ must be false as well. This cannot happen, thus we conclude that A **is a knight, and** B **is a knight**.

Knights & Knaves II

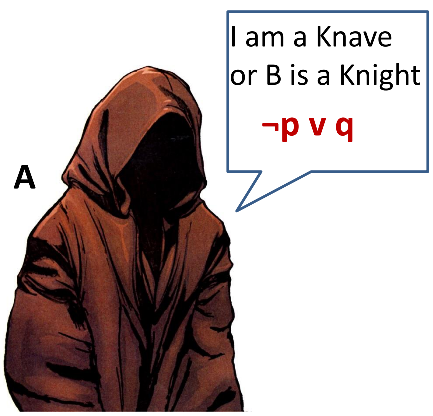


- p="A is a knight"
- q="B is a knight"

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

- If A is a knight, then p = true, and p → q must be true.
- If A is a knave, then p=false, and p → q must false.

Knights & Knaves III



- p="A is a knight"
- q="B is a knight"

p	q	¬p v q
T	T	T
T	F	F
F	T	T
F	F	T

- If A is a knight, then p = true, and ¬p v q must be true.
- If A is a knave, then p=false, and ¬p v q must false.

Example 19. You meet one islander A. A says: " I am a knave". You need to decide what is A. This case is actually similar to the "liar's paradox". You can write the truth table as above, and see that nothing can be decided from it, which is the definition of a paradox! If A is knight, he tells the truth and says that he is a knave, which is not possibe. So A must be a knave. But if A is a knave, he lies, and so he must be a knight, which is not possible either.

We next give another example of how to combine different statements with inference rules to extract information:

Example 20. During a murder investigation, you have gathered the following clues:

1. if the knife is in the store room, then we saw it when we cleared the store room;
2. the murder was committed at the basement or inside the apartment;
3. if the murder was committed at the basement, then the knife is in the yellow dust bin;
4. we did not see a knife when we cleared the store room;
5. if the murder was committed outside the building, then we are unable to find the knife;
6. if the murder was committed inside the apartment, then the knife is in the store room.

The question is: "where is the knife?"

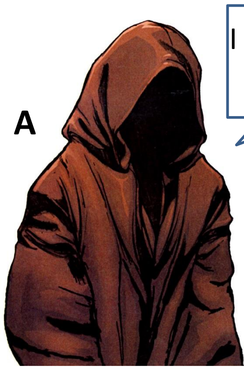
First, we assigned symbols to the above clues:

s	:	the knife is in the store room;
c	:	we saw the knife when we clear the store room;
b	:	the murder was committed at the basement;
a	:	murder was committed inside the apartment;
y	:	the knife is in the yellow dust bin;
o	:	the murder was committed outside the building;
u	:	we are unable to find the knife;

Knights & Knaves IV

- p="A is a knight"

A

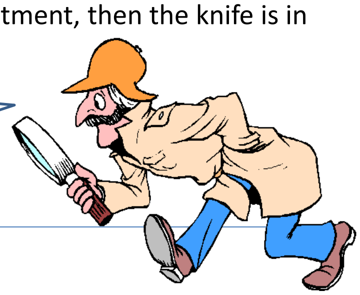


p	¬p
T	F
F	T

It's a paradox!

The Murder Clues

1. if the knife is in the **s**tore room, then we saw it when we **c**leared the store room;
2. the murder was committed at the **b**asement or inside the **a**partment;
3. if the murder was committed at the **b**asement, then the knife is in the **y**ellow dust bin;
4. we did not see a knife when we **c**leared the store room;
5. if the murder was committed **o**utside the building, then we are **u**nable to find the knife;
6. if the murder was committed inside the **a**partment, then the knife is in the **s**tore room.



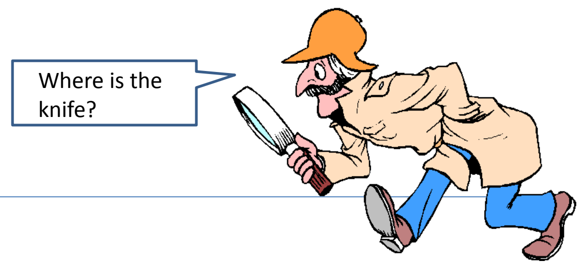
Next we rewrite the clues inside these symbols, and logical operators:

1. $s \to c$
2. $b \vee a$
3. $b \to y$
4. $\neg c$
5. $o \to u$
6. $a \to s$

From 1. and 4., we deduce that $\neg s$. From 6. and $\neg s$, we deduce $\neg a$. Once we have $\neg a$, with 2., we deduce b. Once we have b, with 3., we deduce y, that is, the knife is in the yellow bin!

Statements

1. if the knife is in the **s**tore room, then we saw it when we **c**leared the store room;
2. the murder was committed at the **b**asement or inside the **a**partment;
3. if the murder was committed at the **b**asement, then the knife is in the **y**ellow dust bin;
4. we did not see a knife when we **c**leared the store room;
5. if the murder was committed **o**utside the building, then we are **u**nable to find the knife;
6. if the murder was committed inside the **a**partment, then the knife is in the **s**tore room.



Applying Inference Rules

1. if **s**, then **c**;
2. **b** or **a**;
3. if **b**, then **y**;
4. not **c**;
5. if **o**, then **u**;
6. if **a**, then **s**

1.	$s \to c$	
2.	$b \lor a$	
3.	$b \to y$	
4.	$\neg c$	
5.	$o \to u$	
6.	$a \to s$	
7.	$\neg s$	1, 4; modus tollens
8.	$\neg a$	6, 7; modus tollens
9.	b	2, 8; case elimination
$\therefore$	y	3, 9; modus ponens