Statistical Machine Learning
p-values and Multiple Testing

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Hypothesis test

- A *statistical hypothesis* H ia a conjecture about the probability distribution of a population.
- A hypothesis H is said to be *simple* if the distribution of the population is completely specified by H. If not, then H is called a *composite* hypothesis.
- H_0 is called the *null hypothesis* and H_1 the *alternative hypothesis*.
- We say that we commit a *type I error* if we decide to accept H_1, whereas in reality H_0 is true.
- The probability of committing a type I error will be denoted by α.
- Acceptance of H_0 whereas H_1 is true is called a *type II error*.
- The probability of committing a type II error will be denoted by β.

Error types

	True State of Nature	
Decision	H_0 is TRUE	H_1 is TRUE
Reject H_0 Accept H_1	TYPE I Error	Correct Decision
Accept H_0 Reject H_1	Correct Decision	TYPE II Error

Example

A factory has packets of coffee with and adjusted weight of 500 grams. We assume that the weight of the packages is $N(500, 50)$-distributed. Coffee packages are stored in containers (of unknown number). An error in a scale, some of the containers are filled up with packages of weight $N(490, 50)$-distributed. To determine containers that have incorrectly weight the packages, we draw a sample of two packages from the given container. Based on the outcome of a 2-vector (X_1, X_2) we will make a conjecture about the packages in that container. Namely

$$H_0\text{: the population is } N(500, 50) - \text{distributed}$$
$$H_1\text{: the population is } N(490, 50) - \text{distributed}$$

Let's suppose our decision as
If both X_1 *and* $X_2 \leq 496$ *then we accept* H_1*, otherwise we accept* H_0*.*

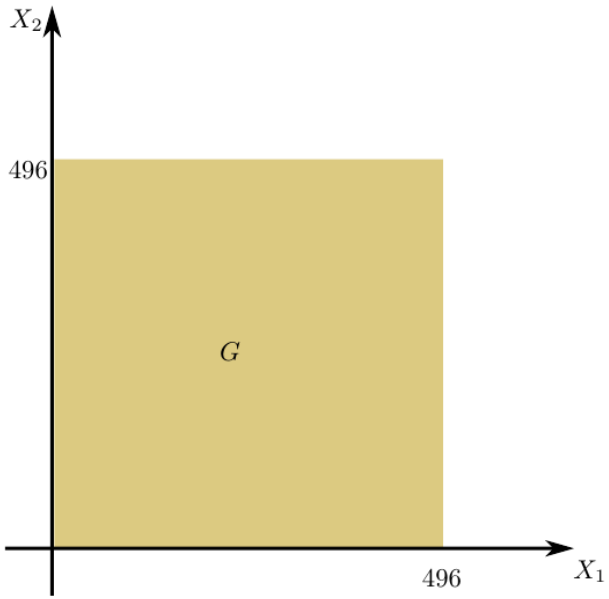
Example (cont)

Let's define our *critical region* G as

$$G = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1, x_2 \leq 496\}$$

Then, we can formulate our decision as

$$= \begin{cases} \text{if } (X_1, X_2) \in G & \text{then we choose } H_1 \text{ as our conjecture.} \\ \text{if } (X_1, X_2) \notin G & \text{then we choose } H_0 \text{ as our conjecture.} \end{cases}$$



Calculating α and β

$$
\begin{aligned}
\alpha &= P(\text{acceptance of } H_1 | H_0 \text{ is true}) \\
&= P((X_1, X_2) \in G | \mu = 500, \sigma^2 = 50) \\
&= P(X_1 \leq 496 | \mu = 500, \sigma^2 = 50) \cdot P(X_2 \leq 496 | \mu = 500, \sigma^2 = 50) \\
&= 0.081
\end{aligned}
$$

$$
\begin{aligned}
\beta &= P(\text{acceptance of } H_0 | H_1 \text{ is true}) \\
&= P((X_1, X_2) \notin G | \mu = 490, \sigma^2 = 50) \\
&= 1 - P((X_1, X_2) \in G | \mu = 490, \sigma^2 = 50) \\
&= 1 - P(X_1 \leq 496 | \mu = 490, \sigma^2 = 50) \cdot P(X_2 \leq 496 | \mu = 490, \sigma^2 = 50) \\
&= 0.356
\end{aligned}
$$

Hypothesis test

A *Hypothesis test* is a collection

$$(X_1, \ldots, X_n; H_0; H_1 : G)$$

where $X_1, \ldots, X_n$ is a sample, H_0 and H_1 hypotheses concerning the probability distribution of the population and $G \subset \mathbb{R}^n$ a Borel set (meaning a collection of open sets)
If H_0 is a simple statistical hypothesis. The *level of significance* of the hypothesis test $(X_1, \ldots, X_n; H_0; H_1; G)$ is understood to be the number

$$\alpha = P^{H_0}_{X_1,\ldots,X_n}(G)$$

Thus, we say that α represents the probability of committing a type I error.

The power function

With our previous setup the β could not be used for composite hypothesis. Thus, we have to define something more general.

- Let $f(\cdot, \theta))_{\theta \in \Theta}$ be a family of probability densities
- Let's assume that the population $X_1, \ldots, X_n$ has a probability density $f(\cdot, \theta)$ where $\theta \in \Theta$.
- Let's assume that H_0 and H_1 are statements of the type

$$H_0 : \theta \in \Theta_0 \quad \text{and} \quad H_1 : \theta \in \Theta_1$$

where $\Theta_0 \cup \Theta_1 = \Theta$ and $\Theta_1 \cap \Theta_1 = \emptyset$.

For a fixed $\theta \in \Theta$, the probability distribution of the population is completely specified. Then, we define for every $\theta \in \Theta_1$

$$\beta(\theta) = P^{\theta}_{X_1,\ldots,X_n}(G^c)$$

The expression $1 - \beta(\theta)$ is called the *power function* for $\theta \in \Theta_1$.

Example (cont)

From our previous example, given is a $N(\mu, 50)$-distributed population, where $\mu \leq 500$. The family of probability densities $f(\cdot, \mu)$ where $\mu \leq 500$ and

$$f(x, \mu) = \frac{1}{\sqrt{100\pi}} \exp\left(\frac{-(x-\mu)^2}{100}\right)$$

The parameter space is defined by $\Theta = (-\infty, 500]$.
If we draw a sample X_1, X_2 of size 2 from this population. We can define $\Theta_0 = \{500\}$ and $\Theta_1 = (-\infty, 500)$. This corresponds to the following hypotheses

$$H_0 : \mu = 500 \quad \text{against} \quad H_1 : \mu < 500$$

If we choose the following critical region

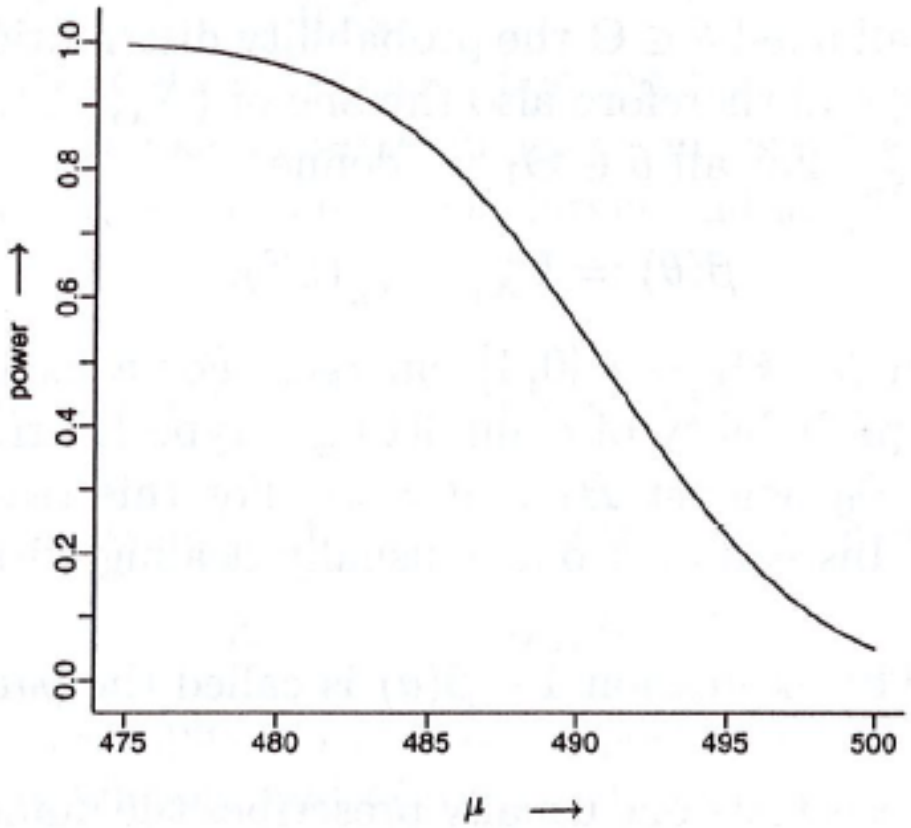
$$G = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \leq 494.63\}$$

Example (cont)

Then, our 5-tuple $(X_1, X_2; H_0; H_1; G)$ constitutes a hypothesis test. The size α of the critical region G is $\alpha = 0.05$, and the power function

$$\begin{aligned} 1 - \beta(\mu) &= 1 - P^{\mu}_{X_1,X_2}(G^c) = P^{\mu}_{X_1,X_2}(G) = P((X_1, X_2) \in G) \\ &= P(X_1 \leq 494.63 \text{ and } X_2 \leq 494.63 | \mu = \mu, \sigma^2 = 50) \\ &= P(X_1 \leq 494.63 | \mu = \mu, \sigma^2 = 50) \\ &\quad \cdot P(X_2 \leq 494.63 | \mu = \mu, \sigma^2 = 50) \end{aligned}$$

Power function example



Normally Distributed Case

Suppose we are dealing with a $N(\mu, \sigma^2)$-distributed population where both μ and σ are unknown. If we wish to test the hypothesis

$$H_0 : \mu = \mu_0 \text{ against } H_1 : \mu \neq \mu_0$$

Critical regions based on the likelihood ratio are of type

$$G = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \left| \frac{\overline{x} - \mu_0}{s/\sqrt{n}} \right| \geq c\}$$

where $s^2 = \frac{1}{n-1} \sum (x_i - \overline{x})^2$
The outcome of the variable

$$T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

is decisive in the choice to reject H_0 or not. This is called the *Test statistic* for this hypothesis test. Under the H_0 the test statistics is t-distributed with $n - 1$ degrees of freedom.

Normally distributed case

The decision becomes

$$\text{if } |\frac{\overline{X} - \mu_0}{S/\sqrt{n}}| \geq c, \text{ then assume } H_1$$

$$\text{if } |\frac{\overline{X} - \mu_0}{S/\sqrt{n}}| < c, \text{ then assume } H_0$$

And we have the following equivalence

$$P(|\frac{\overline{X} - \mu_0}{S/\sqrt{n}}| \geq u) \leq \alpha \iff u \geq c$$

If the outcome u of $|\frac{\overline{X}-\mu_0}{S/\sqrt{n}}|$ satisfies

$$P(|\frac{\overline{X} - \mu_0}{S/\sqrt{n}}| \geq |u|) \leq \alpha$$

then, we accept H_1.

p-values

The expression

$$P(|\frac{\overline{X} - \mu_0}{S/\sqrt{n}}| \geq |u|)$$

is called the *p*-value associated with the outcome u of the test statistic. Therefore, we normally represent the decision rule

$$\text{if } P\text{-value} \leq \alpha, \text{ then assume } H_1$$

$$\text{if } P\text{-value} > \alpha, \text{ then assume } H_0$$

Cherry Picking Problem

Let's suppose we use the same data set and apply different test statistics at the α level of significance:

- W_α a Wilcoxon test,
- P_α a permutation test, and
- T_α a t-test.

In this scenario, it may be possible that W_α may be true when P_α and T_α are not.

$$P(W_\alpha \text{ or } T_\alpha \text{ or } P_\alpha | H) \geq P(W_\alpha | H) = \alpha$$

we will have an **inflated** type I error by picking and choosing after the fact which test to report.

Cherry Picking Problem

Similarly, if our intent was to conceal a side effect by reporting the results were not significant, we will inflate the Type II error and **deflate** the power β of our test by an after-the-fact choice as

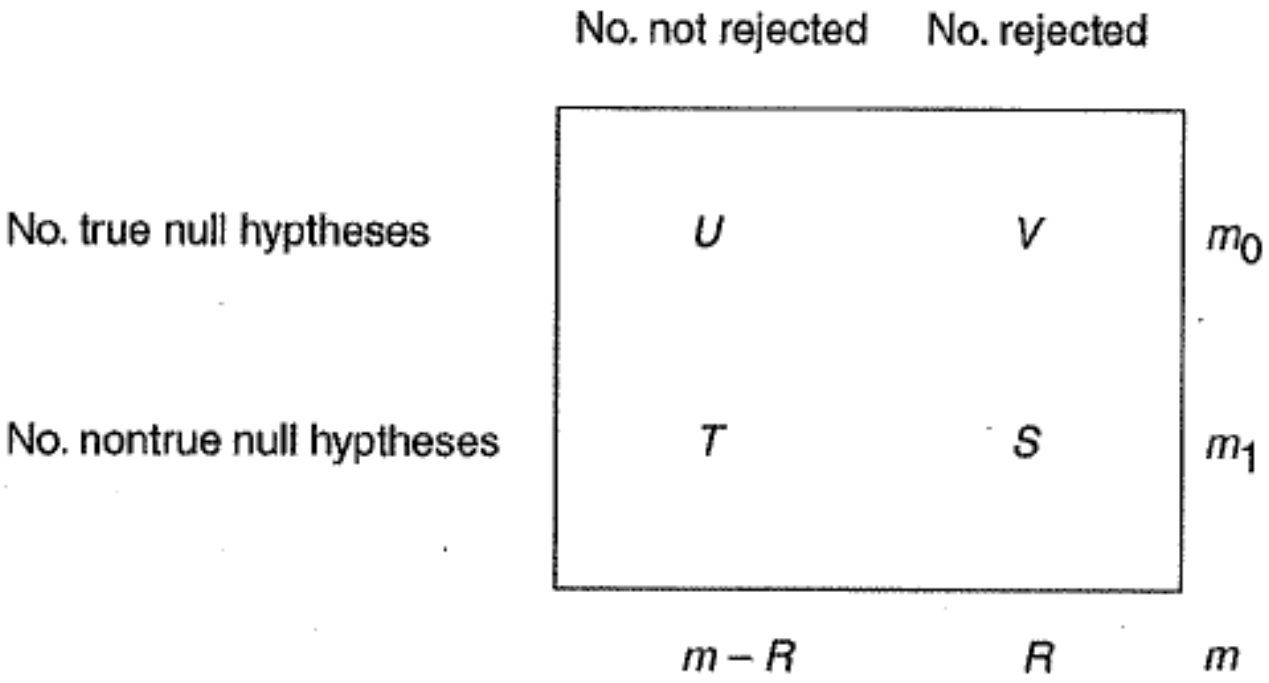
$$\beta = P(not(W_\alpha \text{and} P_\alpha \text{ and } T_\alpha)|K) \leq (P(\text{ not } W_\alpha|K))$$

This quote from (**commonerrstat**) is important to remember.

> *We are not free to pick and choose among tests; any such conduct is unethical.* **Both the comparison and the test statistic must be specified in advance of examining the data.**

Multiple Testing

Consider the problem of simultaneously testing m null hypotheses H_j, $j = 1, \ldots, m$ and denote by R the number of rejected hypothesis. In the frequentist setting, the situation can be summarized as



Multiple Testing (cont)

The m hypotheses are assumed to be known in advance, while the number m_0 and m_1 of true and false null hypothesis are unknown parameter, R is an observable random variable, and S, T, U, and V are unobservable random variables.
In general, we want to minimize the number V of false positives, or type I errors, and the number T of false negative or type II errors. The standard approach is to pre-specify an acceptable type I error rate α that seek tests that minimize the type II error rate, within the class of tests with type I error rate α.

Test for multiple comparisons

In terms of these random variables, we can define the main rates used in the present context. When we are facing testing hypotheses of possibly thousands of significance tests, there are a number of alternatives dealing

- The per-comparison error rate (PCER). The expected value of the number of type I error over the number of hypotheses

$$PCER = \mathbb{E}(V)/m$$

- The family-wise error rate (FWER). The probability of at least one type I error

$$P(V \geq 1)$$

- The false discovery rate (FDR) is the expected proportion of type I error among rejected hypotheses

$$FDR = \mathbb{E}(V/R; R > 0) = \mathbb{E}(V/R|R > 0)P(R > 0)$$

Ajusted *p*-values

To account for multiple hypothesis testing, one may calculate the adjusted *p*-values. Given a test procedure, the adjusted *p*-value correspoinding to the test of a single hypothesis H_j can be defined as the level of the entire test procedure at which H_j would just be rejected, given the values of all test statistics involved.
Control of the FWER at level α, the Bonferroni procedure rejects any hypothesis H_j with *p*-value less than or equal to α/m.

FDR control

The adjusted p-value goes according to this formula

$$\tilde{p}_{r_i} = \min_{k=i,\ldots,m} \{\min(mp_{r_k}, 1)\}$$

This adjustment leads to strong control of the FDR under the additional assumption of independence of the test statistics.

References

Materials and some of the pictures are from (**pestman**), and (**speed**).
I have used some of the graphs by hacking TiKz code from StakExchange, Inkscape for more aesthetic plots and other old tricks of TEX