Neuroinformatics Journal Club
Edge-centric brain FC

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Today's paper

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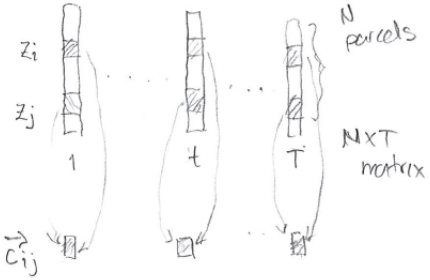
A mathematical perspective on edge-centric brain functional connectivity

Leonardo Novelli [1✉] & Adeel Razi [1,2,3]

Inconsistencies from the beginning

There are some major flaws with the paper. For once, the notation is confusing.

- So $c_{ij} = z_i(t)z_j(t)$. This is a vector.
- $C_{ij}(t)$ is a matrix?
- C_{ij} is a matrix in formula 12.



A vector then...

$$\vec{C_{ij}} = (z_i(1)\cdot z_j(1),\ z_i(2)\cdot z_j(2), \dots, z_i(T)\cdot z_j(T))$$

$$C_{ij}(t) = z_i(t)\cdot z_j(t) \quad (t^{\underline{th}} \text{ element of } \vec{C_{ij}})$$

A collection of vectors

N^2 elements

$$\vec{c}_{11} = (z_1(1)\cdot z_1(1), z_1(2)\cdot z_1(2), \dots, z_1(T)\cdot z_1(T))$$

$$\vec{c}_{12} = (z_1(1)\cdot z_2(1), z_1(2)\cdot z_2(2), \dots, z_1(T)\cdot z_2(T))$$

$$\vdots$$

$$\vec{c}_{1N} = (z_1(1)\cdot z_N(1), z_1(2)\cdot z_N(2), \dots, z_1(T)\cdot z_N(T))$$

$$\vec{c}_{21} = (z_2(1)\cdot z_1(1), z_2(2)\cdot z_1(2), \dots, z_2(T)\cdot z_1(T))$$

$$\vdots \qquad \vdots$$

$$\vec{c}_{2N} = (z_2(1)\cdot z_N(1), z_2(2)\cdot z_N(2), \dots, z_2(T)\cdot z_N(T))$$

$$\vdots$$

$$\vec{c}_{N1} = (z_N(1)\cdot z_1(1), z_N(2)\cdot z_1(2), \dots, z_N(T)\cdot z_1(T))$$

$$\vdots$$

$$\vec{c}_{NN} = (z_N(1)\cdot z_N(1), z_N(2)\, z_N(2), \dots, z_N(T)\cdot z_N(T)).$$

Edge Time Series?

EDGE TIME SERIES ??

$$\begin{bmatrix} z_1(t)\cdot z_1(t) & z_1(t)\cdot z_2(t) & \cdots & z_1(t)\, z_N(t) \\ z_2(t)\cdot z_1(t) & z_2(t)\cdot z_2(t) & \cdots & z_2(t)\, z_N(t) \\ \vdots & & & \\ z_N(t)\cdot z_1(t) & z_N(t)\cdot z_2(t) & \cdots & z_N(t)\, z_N(t) \end{bmatrix}$$

NxN symmetric.

only

$\binom{N}{2} = \frac{N\cdot(N-1)}{2}$.

are different.

$$c_{ij}(t) = z_i(t)\cdot z_j(t)$$

Definition of eFC?

$$eFC_{jk,lm} =: \sum_t c_{jk}(t)\, c_{lm}(t) \Big/ \Big(\qquad \Big)$$

$$= \sum_{t=1}^{T} z_j(t) \cdot z_k(t) \cdot z_l(t) \cdot z_m(t) \Big/ \Big(\qquad \Big)$$

$$r_{ij} = \frac{1}{T-1} \sum_{t=1}^{T} z_i(t) \cdot z_j(t)$$

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1N} \\ r_{21} & r_{22} & & r_{2N} \\ \vdots & & & \\ r_{N1} & r_{N2} & \cdots & r_{NN} \end{bmatrix}$$

Changing notation

$$M_1 = \begin{bmatrix} z_1^1 \cdot z_1^1 & z_1^1 \cdot z_2^1 & \cdots & z_1^1 \cdot z_N^1 \\ z_2^1 \cdot z_1^1 & z_2^1 \cdot z_2^1 & \cdots & z_2^1 z_N^1 \\ \vdots & & & \\ z_N^1 \cdot z_1^1 & z_N^1 \cdot z_2^1 & \cdots & z_N^1 \cdot z_N^1 \end{bmatrix}$$

$N \times N$. symmetric

$\binom{N}{2} \times \binom{N}{2}$

$$\cdots \quad M_\tau = \begin{bmatrix} z_1^\tau \cdot z_1^\tau & z_1^\tau \cdot z_2^\tau & \cdots & z_1^\tau \cdot z_N^\tau \\ z_2^\tau \cdot z_1^\tau & z_2^\tau \cdot z_2^\tau & \cdots & z_2^\tau \cdot z_N^\tau \\ \vdots & & & \\ z_N^\tau \cdot z_1^\tau & z_N^\tau \cdot z_2^\tau & \cdots & z_N^\tau z_N^\tau \end{bmatrix}$$

$$\boxed{Z_{\text{parcel}}^{\text{time}}}$$

$$\sum_{t=1}^{T} z_1^t \cdot z_1^t \qquad \sum_{t=1}^{T} z_1^t \cdot z_2^t$$

Fourth order statistics eFC

$$
\begin{array}{cccc|l}
\overbrace{\hspace{6em}}^{N\text{ terms}} & \overbrace{\hspace{6em}}^{N-1\text{ terms}} & \overbrace{\hspace{6em}}^{2\text{-terms}} & \overbrace{\hspace{3em}}^{1\text{-term}} & \\
\sum_{\tau=1}^{T} z_1^\tau z_1^\tau z_1^\tau z_1^\tau \cdots \sum_{\tau=1}^{T} z_1^\tau z_1^\tau z_1^\tau z_N^\tau & \sum_{\tau=1}^{T} z_1^\tau z_1^\tau z_2^\tau z_2^\tau \cdots \sum_{\tau=1}^{T} z_1^\tau z_1^\tau z_2^\tau z_N^\tau & \cdots \sum_{\tau=1}^{T} z_1^\tau z_1^\tau z_{N-1}^\tau z_{N-1}^\tau , \sum_{\tau=1}^{T} z_1^\tau z_1^\tau z_{N-1}^\tau z_N^\tau & \sum_{\tau=1}^{T} z_1^\tau z_1^\tau z_N^\tau z_N^\tau & \\
\sum_{\tau=1}^{T} z_1^\tau z_2^\tau z_1^\tau z_1^\tau \cdots \sum_{\tau=1}^{T} z_1^\tau z_2^\tau z_1^\tau z_N^\tau & \sum_{\tau=1}^{T} z_1^\tau z_2^\tau z_2^\tau z_2^\tau \cdots \sum_{\tau=1}^{T} z_1^\tau z_2^\tau z_2^\tau z_N^\tau & \cdots \sum_{\tau=1}^{T} z_1^\tau z_2^\tau z_{N-1}^\tau z_{N-1}^\tau , \sum_{\tau=1}^{T} z_1^\tau z_2^\tau z_{N-1}^\tau z_N^\tau & \sum_{\tau=1}^{T} z_1^\tau z_2^\tau z_N^\tau z_N^\tau & N\text{ rows.} \\
\vdots \qquad \vdots & & & & \\
\sum_{\tau=1}^{T} z_1^\tau z_N^\tau z_1^\tau z_1^\tau \cdots \sum_{\tau=1}^{T} z_1^\tau z_N^\tau z_1^\tau z_N^\tau & \sum_{\tau=1}^{T} z_1^\tau z_N^\tau z_2^\tau z_2^\tau \cdots \sum_{\tau=1}^{T} z_1^\tau z_N^\tau z_2^\tau z_N^\tau & \cdots \sum_{\tau=1}^{T} z_1^\tau z_N^\tau z_{N-1}^\tau z_{N-1}^\tau , \sum_{\tau=1}^{T} z_1^\tau z_N^\tau z_{N-1}^\tau z_N^\tau & \sum_{\tau=1}^{T} z_1^\tau z_N^\tau z_N^\tau z_N^\tau & \\
\sum_{\tau=1}^{T} z_2^\tau z_2^\tau z_1^\tau z_1^\tau \cdots \sum_{\tau=1}^{T} z_2^\tau z_2^\tau z_1^\tau z_N^\tau & \sum_{\tau=1}^{T} z_2^\tau z_2^\tau z_2^\tau z_2^\tau \cdots \sum_{\tau=1}^{T} z_2^\tau z_2^\tau z_2^\tau z_N^\tau & \cdots \sum_{\tau=1}^{T} z_2^\tau z_2^\tau z_{N-1}^\tau z_{N-1}^\tau , \sum_{\tau=1}^{T} z_2^\tau z_2^\tau z_{N-1}^\tau z_N^\tau & \sum_{\tau=1}^{T} z_2^\tau z_2^\tau z_N^\tau z_N^\tau & \\
\vdots \qquad \vdots & \vdots \qquad \vdots & \vdots \qquad \vdots & \vdots & N-1\text{ rows.} \\
\sum_{\tau=1}^{T} z_2^\tau z_N^\tau z_1^\tau z_1^\tau \cdots \sum_{\tau=1}^{T} z_2^\tau z_N^\tau z_1^\tau z_N^\tau & \sum_{\tau=1}^{T} z_2^\tau z_N^\tau z_2^\tau z_2^\tau \cdots \sum_{\tau=1}^{T} z_2^\tau z_N^\tau z_2^\tau z_N^\tau & \cdots \sum_{\tau=1}^{T} z_2^\tau z_N^\tau z_{N-1}^\tau z_{N-1}^\tau , \sum_{\tau=1}^{T} z_2^\tau z_N^\tau z_{N-1}^\tau z_N^\tau , & \sum_{\tau=1}^{T} z_2^\tau z_N^\tau z_N^\tau z_N^\tau & \\
\vdots \qquad \vdots & \vdots \qquad \vdots & \vdots \qquad \vdots & \vdots & \\
\sum_{\tau=1}^{T} z_{N-1}^\tau z_{N-1}^\tau z_1^\tau z_1^\tau \cdots \sum_{\tau=1}^{T} z_{N-1}^\tau z_{N-1}^\tau z_1^\tau z_N^\tau & \sum_{\tau=1}^{T} z_{N-1}^\tau z_{N-1}^\tau z_2^\tau z_2^\tau \cdots \sum_{\tau=1}^{T} z_{N-1}^\tau z_{N-1}^\tau z_2^\tau z_N^\tau & \cdots \sum_{\tau=1}^{T} z_{N-1}^\tau z_{N-1}^\tau z_{N-1}^\tau z_{N-1}^\tau ; \sum_{\tau=1}^{N} z_{N-1}^\tau z_{N-1}^\tau z_{N-1}^\tau z_N^\tau , & \sum_{\tau=1}^{T} z_{N-1}^\tau z_{N-1}^\tau z_N^\tau z_N^\tau & \\
\sum_{\tau=1}^{T} z_{N-1}^\tau z_N^\tau z_1^\tau z_1^\tau \cdots \sum_{\tau=1}^{T} z_{N-1}^\tau z_N^\tau z_1^\tau z_N^\tau & \sum_{\tau=1}^{T} z_{N-1}^\tau z_N^\tau z_2^\tau z_2^\tau \cdots \sum_{\tau=1}^{T} z_{N-1}^\tau z_N^\tau z_2^\tau z_N^\tau & \cdots \sum_{\tau=1}^{T} z_{N-1}^\tau z_N^\tau z_{N-1}^\tau z_{N-1}^\tau ; \sum_{\tau=1}^{T} z_{N-1}^\tau z_N^\tau z_{N-1}^\tau z_N^\tau , & \sum_{\tau=1}^{T} z_{N-1}^\tau z_N^\tau z_N^\tau z_N^\tau & 2\text{ rows} \\
\sum_{\tau=1}^{T} z_N^\tau z_N^\tau z_1^\tau z_1^\tau \cdots \sum_{\tau=1}^{T} z_N^\tau z_N^\tau z_1^\tau z_N^\tau & \sum_{\tau=1}^{T} z_N^\tau z_N^\tau z_2^\tau z_2^\tau \cdots \sum_{\tau=1}^{T} z_N^\tau z_N^\tau z_2^\tau z_N^\tau & \cdots \sum_{\tau=1}^{T} z_N^\tau z_N^\tau z_{N-1}^\tau z_{N-1}^\tau ; \sum_{\tau=1}^{T} z_N^\tau z_N^\tau z_{N-1}^\tau z_N^\tau , & \sum_{\tau=1}^{T} z_N^\tau z_N^\tau z_N^\tau z_N^\tau & 1\text{ row}
\end{array}
$$

Total rows $N+(N-1)+\cdots+1 = \binom{N}{2}$

Derivation of edge FC

$$E[C_{jk}(t)C_{jl}(t)] = E[Z_i(t)Z_k(t)Z_j(t)Z_l(t)] + \text{sum of products of two} \tag{1}$$

.

The assumption of $\kappa(E[Z_i(t)Z_k(t)Z_j(t)Z_l(t)]) = 0$ is unrealistic. Confussion at its best... $E(Z_i) = 0$, Ok, but if they are assuming independence the whole thing is zero!

Ergodicity

This condition is very strong, and it refers to this scary looking condition

$$\lim_{t\to\infty} E[C_{jk}(t)C_{jl}(t)] \text{ exists!} \tag{2}$$

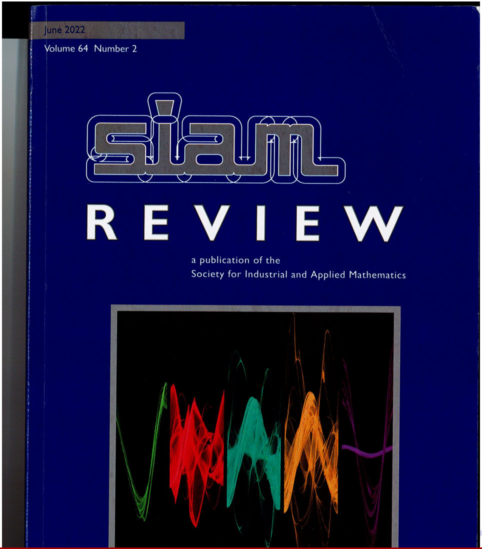
They are clever enough to put it under the carpet by saying "the ergodic assumption guarantees that sample estimates converge to the ensemble expectation as the number of time frame increases". Equation 10 holds in the limit case... otherwise it is an estimate!

Distance? right!

The equation 12 needs to be divided by $(T-1)^2$, furthermore there are tons of repetitions in the equation (remember we eliminated the lower half of the matrices)

This is what I need to do

Colophon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