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Dynamic Mode Decomposition

Horacio Gómez-Acevedo
Department of Biomedical Informatics
University of Arkansas for Medical Sciences

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Computational Neuroscience

Extracting spatial–temporal coherent patterns in large-scale neural recordings using dynamic mode decomposition

Bingni W. Brunton[a,b,c,*], Lise A. Johnson[d,e], Jeffrey G. Ojemann[e], J. Nathan Kutz[b]

[a] Department of Biology, University of Washington, Seattle, WA 98195, USA
[b] Department of Applied Mathematics, University of Washington, Seattle, WA 98195, USA
[c] Institute for Neuroengineering, University of Washington, Seattle, WA 98195, USA
[d] Center for Sensorimotor Neural Engineering, University of Washington, Seattle, WA 98195, USA
[e] Department of Neurological Surgery, University of Washington, Seattle, WA 98195, USA

Background

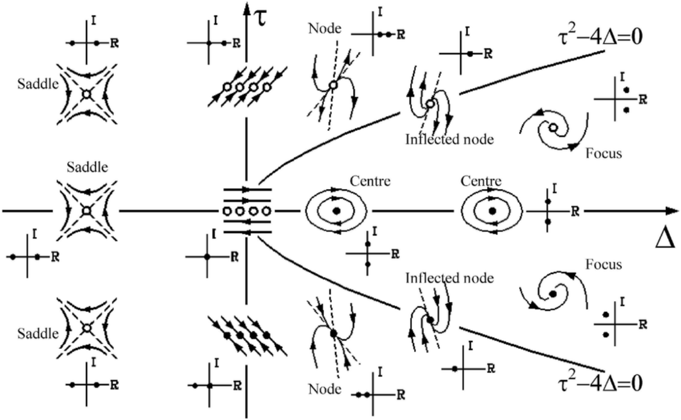
What are eigenvalues and eigenvectors?
Let A be a $n \times n$ matrix (real), a number $\lambda \in \mathbb{C}$ is an **eigenvalue** if there is a nonzero vector $x \in \mathbb{C}^n$ for which $Ax = \lambda x$. We call such a vector an **eigenvector** of A associated with λ.
The collection of all eigenvalues of A is called the **spectrum** of A.
The fundamental theorem for linear systems: If A is a $n \times n$ matrix. For any $x_0 \in \mathbb{R}^n$, the initial value problem

$$\dot{x} = Ax \qquad x(0) = x_0$$

has a unique solution

$$x(t) = \exp(A^t)x_0$$

Eigenvalues in Dynamical Systems



https://www.researchgate.net/profile/Marco-Altosole/publication/245387409/figure/fig1/AS:392836487892996@1470670925920/Types-of-phase-portrait.png

Singular Value Decomposition

If $A \in M_{m,n}$ has rank k, then it may be written in the form

$$A = V \Sigma W^*$$

where $V \in M_m$ and $W \in M_n$ are unitary $(W^* W = I_n)$. The matrix $\Sigma = diag\{\sigma_1, \ldots, \sigma_q\}$ are the non-negative square roots of the eigenvalues of AA^*. The columns of V are eigenvectors of AA^*, and the columns of W are eigenvectors of A^*A. If $m \leq n$ and if AA^* has distinct eigenvalues, then V is determined up to a right diagonal factor $D = diag(\exp(i\theta_1), \ldots, \exp(i\theta_n)$ where $\theta_j \in \mathbb{R}$.

Main Premise

The DMD algorithm seeks a best-fit linear matrix A that approximately advances the state of a system $x \in \mathbb{R}^n$ forward in time according to the linear system

$$x_{k+1} = Ax_k$$

where $x_k = x(k\Delta t)$ and Δt denotes a fixed time step that is small enough to resolve the highest frequencies in the dynamics.

DMD setup

$$\mathbf{X} = \begin{bmatrix} | & | & & | \\ \mathbf{x}(t_1) & \mathbf{x}(t_2) & \cdots & \mathbf{x}(t_m) \\ | & | & & | \end{bmatrix}, \tag{3.2a}$$

$$\mathbf{X}' = \begin{bmatrix} | & | & & | \\ \mathbf{x}(t_1') & \mathbf{x}(t_2') & \cdots & \mathbf{x}(t_m') \\ | & | & & | \end{bmatrix}. \tag{3.2b}$$

Equation (3.1) may be written in terms of these data matrices as

$$\mathbf{X}' \approx \mathbf{A}\mathbf{X}. \tag{3.3}$$

DMD trick

computing A in (3.4), we may project A onto the first r POD modes in $\mathbf{U}_r$ and approximate the pseudoinverse using the rank-r SVD approximation $\mathbf{X} \approx \mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^*$:

$$\tilde{\mathbf{A}} = \mathbf{U}_r^* \mathbf{A} \mathbf{U}_r \tag{3.6a}$$

$$= \mathbf{U}_r^* \mathbf{X}' \mathbf{X}^\dagger \mathbf{U}_r \tag{3.6b}$$

$$= \mathbf{U}_r^* \mathbf{X}' \mathbf{V}_r \mathbf{\Sigma}_r^{-1} \mathbf{U}_r^* \mathbf{U}_r \tag{3.6c}$$

$$= \mathbf{U}_r^* \mathbf{X}' \mathbf{V}_r \mathbf{\Sigma}_r^{-1}. \tag{3.6d}$$

The leading spectral decomposition of $\mathbf{A}$ may be approximated from the spectral decomposition of the much smaller $\tilde{\mathbf{A}}$:

$$\tilde{\mathbf{A}} \mathbf{W} = \mathbf{W} \mathbf{\Lambda}. \tag{3.7}$$

The diagonal matrix $\mathbf{\Lambda}$ contains the *DMD eigenvalues*, which correspond to eigenvalues of the high-dimensional matrix $\mathbf{A}$. The columns of $\mathbf{W}$ are eigenvectors of $\tilde{\mathbf{A}}$ and provide a coordinate transformation that diagonalizes the matrix. These columns may be thought of as linear combinations of POD mode amplitudes that behave linearly with a single temporal pattern given by the corresponding eigenvalue λ.

The eigenvectors of $\mathbf{A}$ are the *DMD modes* $\mathbf{\Phi}$, and they are reconstructed using the eigenvectors $\mathbf{W}$ of the reduced system and the time-shifted data matrix $\mathbf{X}'$:

$$\mathbf{\Phi} = \mathbf{X}' \tilde{\mathbf{V}} \tilde{\mathbf{\Sigma}}^{-1} \mathbf{W}. \tag{3.8}$$

DMD expansion

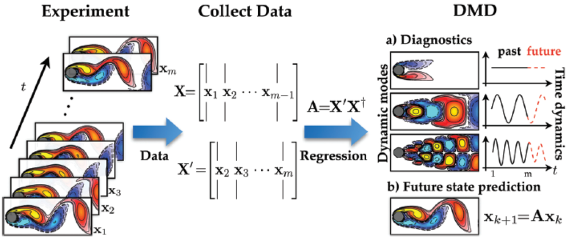


Fig. 3.1 *Overview of DMD illustrated on the fluid flow past a circular cylinder at Reynolds number 100. Reproduced with permission from Kutz et al.* [225].

3.1.1. Spectral Decomposition and the DMD Expansion. Once the DMD modes and eigenvalues are computed, it is possible to represent the system state in terms of the DMD expansion

$$x_k = \sum_{j=1}^{r} \phi_j \lambda_j^{k-1} b_j = \Phi \Lambda^{k-1} b, \tag{3.9}$$

where ϕ_j are eigenvectors of $\mathbf{A}$ (DMD modes), λ_j are eigenvalues of $\mathbf{A}$ (DMD eigenvalues), and b_j are the mode amplitudes. The DMD expansion (3.9) is directly analo-

DMD expansion (cont)

The spectral expansion in (3.9) may be converted to continuous time by introducing the continuous eigenvalues $\omega = \log(\lambda)/\Delta t$:

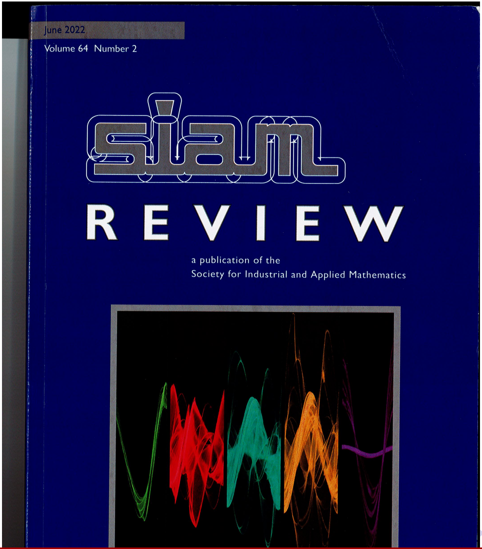
$$\mathbf{x}(t) = \sum_{j=1}^{r} \phi_j e^{\omega_j t} b_j = \mathbf{\Phi} \exp(\mathbf{\Omega} t)\mathbf{b}, \tag{3.14}$$

where $\mathbf{\Omega}$ is a diagonal matrix containing the continuous-time eigenvalues ω_j. Thus, the data matrix $\mathbf{X}$ may be represented as

(3.15)

$$\mathbf{X} \approx \begin{bmatrix} | & & | \\ \phi_1 & \cdots & \phi_r \\ | & & | \end{bmatrix} \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_r \end{bmatrix} \begin{bmatrix} e^{\omega_1 t_1} & \cdots & e^{\omega_1 t_m} \\ \vdots & \ddots & \vdots \\ e^{\omega_r t_1} & \cdots & e^{\omega_r t_m} \end{bmatrix} = \mathbf{\Phi}\mathrm{diag}(\mathbf{b})\mathbf{T}(\boldsymbol{\omega}).$$

Colophon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