

Neuroinformatics Journal Club

Dynamic Mode Decomposition

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Computational Neuroscience

Extracting spatial-temporal coherent patterns in large-scale neural recordings using dynamic mode decomposition

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Background

What are eigenvalues and eigenvectors?

Let A be a $n \times n$ matrix (real), a number $\lambda \in \mathbb{C}$ is an **eigenvalue** if there is a nonzero vector $x \in \mathbb{C}^n$ for which $Ax = \lambda x$. We call such a vector an **eigenvector** of A associated with λ .

The collection of all eigenvalues of A is called the **spectrum** of A .

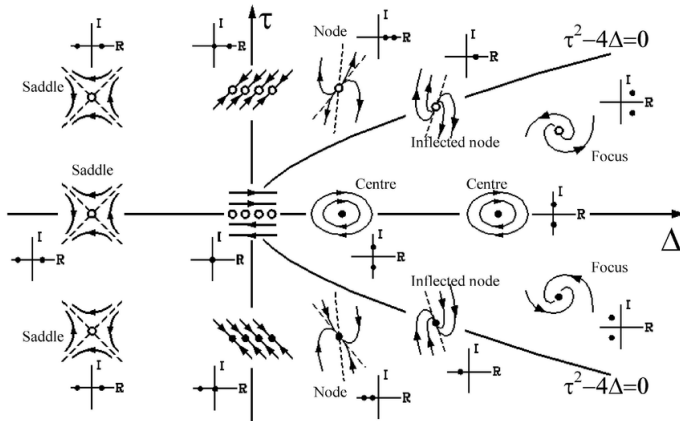
The fundamental theorem for linear systems: If A is a $n \times n$ matrix. For any $x_0 \in \mathbb{R}^n$, the initial value problem

$$\dot{x} = Ax \quad x(0) = x_0$$

has a unique solution

$$x(t) = \exp(A^t)x_0$$

Eigenvalues in Dynamical Systems



<https://www.researchgate.net/profile/Marco-Altosole/publication/245387409/figure/fig1/AS:392836487892996@1470670925920/Types-of-phase-portrait.png>

Singular Value Decomposition

If $A \in M_{m,n}$ has rank k , then it may be written in the form

$$A = V\Sigma W^*$$

where $V \in M_m$ and $W \in M_n$ are unitary ($W^*W = I_n$). The matrix $\Sigma = \text{diag}\{\sigma_1, \dots, \sigma_q\}$ are the non-negative square roots of the eigenvalues of AA^* . The columns of V are eigenvectors of AA^* , and the columns of W are eigenvectors of A^*A . If $m \leq n$ and if AA^* has distinct eigenvalues, then V is determined up to a right diagonal factor $D = \text{diag}(\exp(i\theta_1), \dots, \exp(i\theta_n))$ where $\theta_j \in \mathbb{R}$.

Main Premise

The DMD algorithm seeks a best-fit linear matrix A that approximately advances the state of a system $x \in \mathbb{R}^n$ forward in time according to the linear system

$$x_{k+1} = Ax_k$$

where $x_k = x(k\Delta t)$ and Δt denotes a fixed time step that is small enough to resolve the highest frequencies in the dynamics.

DMD setup

$$(3.2a) \quad \mathbf{X} = \begin{bmatrix} \mathbf{x}(t_1) & \mathbf{x}(t_2) & \cdots & \mathbf{x}(t_m) \end{bmatrix},$$

$$(3.2b) \quad \mathbf{X}' = \begin{bmatrix} \mathbf{x}(t'_1) & \mathbf{x}(t'_2) & \cdots & \mathbf{x}(t'_m) \end{bmatrix}.$$

Equation (3.1) may be written in terms of these data matrices as

$$(3.3) \quad \mathbf{X}' \approx \mathbf{A}\mathbf{X}.$$

DMD trick

computing \mathbf{A} in (3.4), we may project \mathbf{A} onto the first r SVD modes in \mathbf{U}_r and approximate the pseudoinverse using the rank- r SVD approximation $\mathbf{X} \approx \mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^*$:

$$(3.6a) \quad \tilde{\mathbf{A}} = \mathbf{U}_r^* \mathbf{A} \mathbf{U}_r$$

$$(3.6b) \quad = \mathbf{U}_r^* \mathbf{X}' \mathbf{X}^\dagger \mathbf{U}_r$$

$$(3.6c) \quad = \mathbf{U}_r^* \mathbf{X}' \mathbf{V}_r \mathbf{\Sigma}_r^{-1} \mathbf{U}_r^* \mathbf{U}_r$$

$$(3.6d) \quad = \mathbf{U}_r^* \mathbf{X}' \mathbf{V}_r \mathbf{\Sigma}_r^{-1}.$$

The leading spectral decomposition of \mathbf{A} may be approximated from the spectral decomposition of the much smaller $\tilde{\mathbf{A}}$:

$$(3.7) \quad \tilde{\mathbf{A}} \mathbf{W} = \mathbf{W} \mathbf{\Lambda}.$$

The diagonal matrix $\mathbf{\Lambda}$ contains the *DMD eigenvalues*, which correspond to eigenvalues of the high-dimensional matrix \mathbf{A} . The columns of \mathbf{W} are eigenvectors of $\tilde{\mathbf{A}}$ and provide a coordinate transformation that diagonalizes the matrix. These columns may be thought of as linear combinations of POD mode amplitudes that behave linearly with a single temporal pattern given by the corresponding eigenvalue λ .

The eigenvectors of \mathbf{A} are the *DMD modes* $\mathbf{\Phi}$, and they are reconstructed using the eigenvectors \mathbf{W} of the reduced system and the time-shifted data matrix \mathbf{X}' :

$$(3.8) \quad \mathbf{\Phi} = \mathbf{X}' \tilde{\mathbf{V}} \tilde{\mathbf{\Sigma}}^{-1} \mathbf{W}.$$

DMD expansion

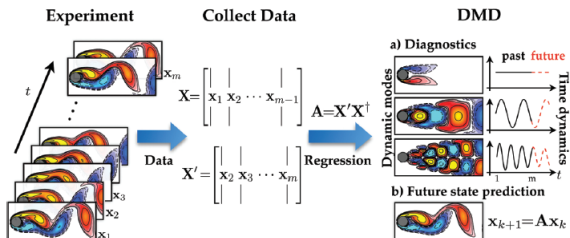


Fig. 3.1 Overview of DMD illustrated on the fluid flow past a circular cylinder at Reynolds number 100. Reproduced with permission from Kutz et al. [225].

3.1.1. Spectral Decomposition and the DMD Expansion. Once the DMD modes and eigenvalues are computed, it is possible to represent the system state in terms of the DMD expansion

$$(3.9) \quad x_k = \sum_{j=1}^r \phi_j \lambda_j^{k-1} b_j = \Phi \Lambda^{k-1} b,$$

where ϕ_j are eigenvectors of A (DMD modes), λ_j are eigenvalues of A (DMD eigenvalues), and b_j are the mode amplitudes. The DMD expansion (3.9) is directly analo-

DMD expansion (cont)

The spectral expansion in (3.9) may be converted to continuous time by introducing the continuous eigenvalues $\omega = \log(\lambda)/\Delta t$:

$$(3.14) \quad \mathbf{x}(t) = \sum_{j=1}^r \phi_j e^{\omega_j t} b_j = \Phi \exp(\Omega t) \mathbf{b},$$

where Ω is a diagonal matrix containing the continuous-time eigenvalues ω_j . Thus, the data matrix \mathbf{X} may be represented as

$$(3.15) \quad \mathbf{X} \approx \begin{bmatrix} | & & | \\ \phi_1 & \cdots & \phi_r \\ | & & | \end{bmatrix} \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_r \end{bmatrix} \begin{bmatrix} e^{\omega_1 t_1} & \cdots & e^{\omega_1 t_m} \\ \vdots & \ddots & \vdots \\ e^{\omega_r t_1} & \cdots & e^{\omega_r t_m} \end{bmatrix} = \Phi \text{diag}(\mathbf{b}) \mathbf{T}(\omega).$$

Colophon

