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A STABILITY CRITERION FOR SEPARATRIX POLYGONS IN THE PHASE PLANE

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INTRODUCTION

In the book of ANDRONOV et al. on bifurcation theory [2] a stability criterion is given relative to the integral curves near a saddle to saddle loop. Two types of loops may be distinguished. In both types a separatrix connects the saddle point with itself; in the "small" loop (Fig. 1), the region within the loop does not contain the remaining separatrices, whereas for the "large" loop (Fig. 2), the opposite statement can be made.

Suppose the autonomous system, in which such loops occur, is represented by

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where P(x,y) and Q(x,y) are continuously differentiable functions. Then the following statement can be made ([2],p.304). If by σ_s is denoted the value of div(P,Q) in the saddle point (where $\sigma_s = \lambda+\mu$, and $\mu < 0$, $\lambda > 0$ are the eigenvalues of the locally linearized system (1) in the saddle point), then for $\sigma_s > 0$ the saddle to saddle loop is unstable, whereas for $\sigma_s < 0$ the loop is stable. This means that $\sigma_s > 0$ the loop is an α limit continuum for integral curves near the loop and and for $\sigma_s < 0$ the loop is an ω limit continuum for such curves. For a "small" loop the integral curves are inside the loop and for a "large" loop outside of it; these regions are indicated by regions I in Figs. 1 and 2.

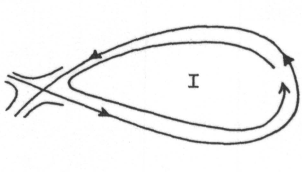

Fig. 1 "Small" saddle to saddle loop.

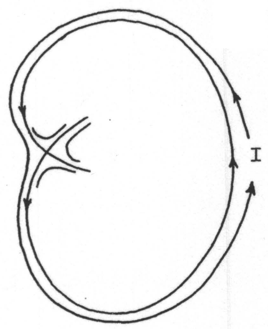

Fig. 2 "Large" saddle to saddle loop.

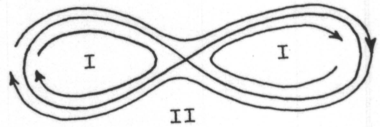

Fig. 3 Two "small" loops.

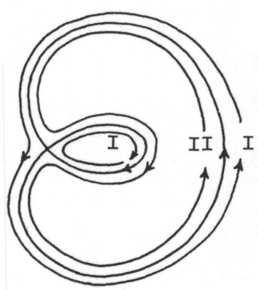

Fig. 4 A "small" and a "large" loop.

If $\sigma_s = 0$, in [2] examples are given where the loop is stable, unstable or has a neighbourhood with closed integral curves. If with all the separatrices in the saddle point loops are formed there arise two possibilities: two "small" loops (Fig. 3) and a "small" and a "large" loop (Fig. 4). It was noted by ANDRONOV et. al. [2] as an immediate corrolory of the previous statement, that for $\sigma_s \neq 0$, the two loops are both stable or unstable limit continua for integral curves in the regions I. It may easily be seen by a trivial extension of the arguments given in [2], that the same conclusion may be reached for the integral curves in the regions II in Figs. 3 and 4. In the case of the two "small" loops this region is outside of these loops, in the other case it is in the region in between the "small" loop and the "large" loop.

In this paper we wish to extent the statement with regard to the stability of a saddle to saddle loop to separatrix polygons in the phase plane of system (1). These are polygons, the corner points of which are saddle points, and the sides are formed by the separatrices connecting these saddle points. Polygons with two saddle points are shown in Fig. 5; those with three saddle points in Fig. 6. "Small" and "large" polygons may again be distinguished in an obvious way. The stability criterion which will be derived only depends on the eigenvalues of the locally linearized system in the saddle points.

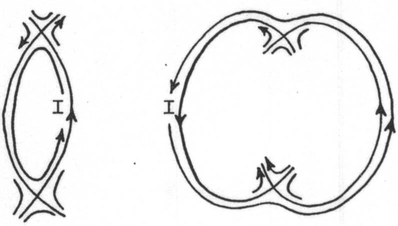

Fig. 5 Separatrix polygons with two saddle points.

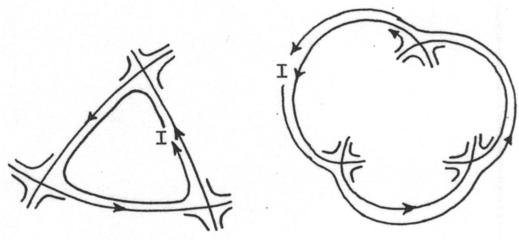

Fig. 6 Separatrix triangles.

Apart from the charming fact that local properties of the functions $P(x,y)$ and $Q(x,y)$ make a global statement possible, the criterion is of interest if bifurcation of the (structurally unstable) polygon is studied. Its derivation uses the succession function, in relation to which first some properties of integral curves near a saddle point are derived.

BEHAVIOUR OF INTEGRAL CURVES AND THE DIVERGENCE INTEGRAL NEAR A SADDLE POINT

Without loss of generality we may study saddle points of system (1) as being located in the origin of the x,y coordinate system. Then (1) may be written as

$$(2)\qquad \frac{dx}{dt} = ax + by + \phi(x,y) \equiv P(x,y), \quad \frac{dy}{dt} = cx + dy + \psi(x,y) \equiv Q(x,y)$$

where $ad-bc < 0$, and $\phi(x,y)$ and $\psi(x,y)$ are Lipschitz continuously differentiable functions, such that $\phi(0,0) = \psi(0,0) = \phi_x(0,0) = \phi_y(0,0) = \psi_x(0,0) = \psi_y(0,0)$.

Let the eigenvalues of the coefficient matrix

$$(3)\qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be denoted by $\mu(\mu < 0)$ and $\lambda(\lambda > 0)$. Then

$$\mu = \tfrac{1}{2}(a+d) - \tfrac{1}{2}\sqrt{(a+d)^2 - 4(ad-bc)},$$

(4)

$$\lambda = \tfrac{1}{2}(a+d) + \tfrac{1}{2}\sqrt{(a+d)^2 - 4(ad-bc)}.$$

The corresponding eigenvectors are directed along the separatrices for which the polar angles θ_μ and θ_λ with the x axis, are given by

$$\tan\theta_\mu = \frac{1}{2b}[d - a - \sqrt{(a+d)^2 - 4(ad-bc)}],$$

(5)

$$\tan\theta_\lambda = \frac{1}{2b}[d - a - \sqrt{(a+d)^2 - 4(ad-bc)}].$$

For a particular choice of the coefficients in (2) the integral curves near the saddle point are as sketched in Fig. 7. Note that for increasing t the motion on the separatrix corresponding to μ is towards the saddle point and along the other separatrix is away from the saddle point. Let L be an integral curve of (2) entering a (small) neighbourhood of the saddle point $\rho < \rho_*$ where ρ is the polar radius, and let A be the intersection point of L with $\rho = \rho_*$ when entering this neighbourhood and B the intersection point when leaving it. Let furthermore α be the arc length, measured along $\rho = \rho_*$ from A to the μ separatrix, and β the arc length from B to the λ separatrix. Obviously when $\alpha \to 0$, $\beta \to 0$. For the derivation of the stability criterion for separatrix polygons we need to know the limiting behaviour of the function $\beta = \beta(\alpha)$ for $\alpha \to 0$. This behaviour is given in the following lemma.

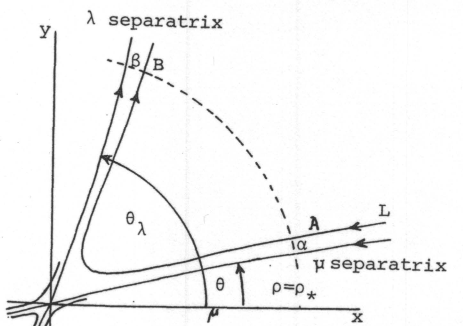

Fig. 7 Integral curves near a saddle point.

LEMMA 1. *For* $\alpha \to 0$, *the function* $\beta(\alpha)$ *may be expressed as*

(6) $$\beta(\alpha) = A\alpha^{-\mu/\lambda}[1 + a(\alpha)]$$

Where $A > 0$ *and* $a(\alpha) \to 0$ *for* $\alpha \to 0$.

PROOF. First we bring (2) into canonical form in the well known way (see for instance [1,p.119]) by means of a non-singular transformation

(7) $\bar{x} = p_{11}x + p_{12}y, \quad \bar{y} = p_{21}x + p_{22}y, \quad p_{11}p_{22} - p_{12}p_{21} \neq 0.$

Then the system (1) may be written as

(8) $$\frac{d\bar{x}}{dt} = \mu\bar{x} + \bar{\phi}(\bar{x},\bar{y}), \quad \frac{d\bar{y}}{dt} = \lambda\bar{y} + \bar{\psi}(\bar{x},\bar{y})$$

where $\bar{\phi}(\bar{x},\bar{y})$ and $\bar{\psi}(\bar{x},\bar{y})$ are Lipschitz continuously differentiable functions in a neighbourhood of the origin, such that $\bar{\phi}(0,0) = \bar{\psi}(0,0) = \bar{\phi}_{\bar{x}}(0,0) = \bar{\phi}_{\bar{y}}(0,0) = \bar{\psi}_{\bar{x}}(0,0) = \bar{\psi}_{\bar{y}}(0,0) = 0$. The separatrices may now be represented as

(9) μ separatrix: $\bar{y} = f(\bar{x}), \quad f(0) = f'(0) = 0,$

(10) λ separatrix: $\bar{x} = g(\bar{y}), \quad g(0) = g'(0) = 0,$

where $f(\bar{x})$ and $g(\bar{y})$ are Lipschitz continuously differentiable functions on an interval containing the origin. This result may be obtained starting from the observation that $\bar{\phi},\bar{\psi} \in C'$ implies $f,g \in C'$ (see [6,p.333,thm.4.2]) and checking the Lipschitz continuity of f' and g' directly in (8) on the basis of the Lipschitz continuity of the first derivatives of $\bar{\phi}$ and $\bar{\psi}$.

Introduce new variables through

(11) $\bar{\bar{x}} = \bar{x} - g(\bar{y}), \quad \bar{\bar{y}} = \bar{y} - f(\bar{x});$

then the system (8) becomes

(12) $$\frac{d\bar{\bar{x}}}{dt} = \mu\bar{\bar{x}}[1 + \bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})], \quad \frac{d\bar{\bar{y}}}{dt} = \lambda\bar{\bar{y}}[1 + \bar{\bar{\psi}}(\bar{\bar{x}},\bar{\bar{y}})]$$

where $\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})$ and $\bar{\bar{\psi}}(\bar{\bar{x}},\bar{\bar{y}})$ are Lipschitz continuous functions, such that $\bar{\bar{\phi}}(0,0) = \bar{\bar{\psi}}(0,0) = 0$ and the separatrices now fall along the coordinate axes. This may be shown as follows. Substitution of (11) in (8) leads in first instance to:

$$\frac{d\bar{\bar{x}}}{dt} = \mu\bar{\bar{x}} + \mu g\{\bar{y}(\bar{\bar{x}},\bar{\bar{y}})\} + \bar{\phi}[\bar{\bar{x}},\bar{\bar{y}}] - g'\{\bar{y}(\bar{\bar{x}},\bar{\bar{y}})\}[\lambda\bar{y}(\bar{\bar{x}},\bar{\bar{y}}) + \bar{\psi}[\bar{\bar{x}},\bar{\bar{y}}]],$$

(13) $$\frac{d\bar{\bar{y}}}{dt} = \lambda\bar{\bar{y}} + \lambda f\{\bar{x}(\bar{\bar{x}},\bar{\bar{y}})\} + \bar{\psi}[\bar{\bar{x}},\bar{\bar{y}}] - f'\{\bar{x}(\bar{\bar{x}},\bar{\bar{y}})\}[\mu\bar{x}(\bar{\bar{x}},\bar{\bar{y}}) + \bar{\phi}[\bar{\bar{x}},\bar{\bar{y}}]],$$

$$\bar{\phi}[\bar{\bar{x}},\bar{\bar{y}}] = \bar{\phi}\{\bar{x}(\bar{\bar{x}},\bar{\bar{y}}),\bar{y}(\bar{\bar{x}},\bar{\bar{y}})\}, \quad \bar{\psi}[\bar{\bar{x}},\bar{\bar{y}}] = \bar{\psi}\{\bar{x}(\bar{\bar{x}},\bar{\bar{y}}),\bar{y}(\bar{\bar{x}},\bar{\bar{y}})\}.$$

Here $\bar{x}(\bar{\bar{x}},\bar{\bar{y}})$, $\bar{y}(\bar{\bar{x}},\bar{\bar{y}})$ are continuously differentiable functions in a neighbourhood of the origin, obtained by solving (11). The condition, that $\bar{\bar{x}} \equiv 0$ and $\bar{\bar{y}} \equiv 0$ are solutions of (13), leads to:

$$0 = \mu g\{\bar{y}(0,\bar{\bar{y}})\} + \bar{\phi}[0,\bar{\bar{y}}] - g'\{\bar{y}(0,\bar{\bar{y}})\}[\lambda\bar{y}(0,\bar{\bar{y}}) + \bar{\psi}[0,\bar{\bar{y}}]],$$

(14)

$$0 = \lambda y\{\bar{x}(\bar{\bar{x}},0)\} + \bar{\psi}[\bar{\bar{x}},0] - f'\{\bar{x}(\bar{\bar{x}},0)\}[\mu\bar{x}(\bar{\bar{x}},0) + \bar{\phi}[\bar{\bar{x}},0]].$$

Subtraction of (14) from (13) and use of the smallness and differentiability properties of $\bar{\phi},\bar{\psi},f,g,\bar{x}$ and $\bar{y}$ then leads to (12) with the statements with regard to the functions $\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})$ and $\bar{\bar{\psi}}(\bar{\bar{x}},\bar{\bar{y}})$. In particular there is

(15) $$|\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})| \le K[|\bar{\bar{x}}|+|\bar{\bar{y}}|], \quad |\bar{\bar{\psi}}(\bar{\bar{x}},\bar{\bar{y}})| \le K[|\bar{\bar{x}}|+|\bar{\bar{y}}|]$$

for some constant $K > 0$. It may be seen, that because of the regularity of the transformations (7) and (11), the points A and B map onto $(\bar{\bar{a}},\bar{\bar{\alpha}})$ and $(\bar{\bar{\beta}},\bar{\bar{b}})$, respectively, where

(16) $$\bar{\bar{\alpha}} = K_1\alpha, \quad \bar{\bar{\beta}} = K_2\beta$$

and $K_1 > 0$ for $\alpha \ge 0$, $K_2 > 0$ for $\beta \ge 0$, $\bar{\bar{a}} > 0$, $\bar{\bar{b}} > 0$. Furthermore let $\alpha \to 0$, $\bar{\bar{a}} \to a > 0$, $\bar{\bar{b}} \to b > 0$.

The behaviour of the integral curves of system (12) for $\bar{\bar{\phi}} \equiv \bar{\bar{\psi}} \equiv 0$ now suggests a further transformation, which was also used by TER MORSCHE [3], and in fact we will follow the line of analysis in [3] for a completion of the proof of the lemma. Thus let this transformation be given by

(17) $$u = \bar{\bar{x}}^{-\lambda/\mu} - \bar{\bar{y}}, \quad v = \bar{\bar{x}}^{-\lambda/\mu}\,\bar{\bar{y}};$$

then the inverse transformation is given by

$$(18)\qquad \bar{\bar{x}} = (\tfrac{1}{2}u + \tfrac{1}{2}\sqrt{u^2 + 4v})^{-\mu/\lambda}, \quad \bar{\bar{y}} = -\tfrac{1}{2}u + \tfrac{1}{2}\sqrt{u^2 + 4v}.$$

The separatrix along the positive x axis is mapped onto the positive u axis and the separatrix along the positive y axis onto the negative u axis.

The system (12) transforms by means of (18) to

$$(19)\qquad \frac{du}{dt} = -\lambda\sqrt{u^2+4v}\,[1+\xi_1(u,v)] - \lambda u\xi_2(u,v), \quad \frac{dv}{dt} = -\lambda v\eta(u,v),$$

where $\xi_1(0,0) = \xi_2(0,0) = \eta(0,0) = 0$.

Instead of (19) we may also consider the equation:

$$(20)\qquad \frac{1}{v}\frac{dv}{du} = \frac{\bar{\bar{\phi}}\{\bar{\bar{x}}(u,v),\bar{\bar{y}}(u,v)\} - \bar{\bar{\psi}}\{\bar{\bar{x}}(u,v),\bar{\bar{y}}(u,v)\}}{\{\bar{\bar{x}}(u,v)\}^{-\lambda/\mu}[1+\bar{\bar{\phi}}\{\bar{\bar{x}}(u,v),\bar{\bar{y}}(u,v)\}] + \bar{\bar{y}}(u,v)[1+\bar{\bar{\psi}}\{\bar{\bar{x}}(u,v),\bar{\bar{y}}(u,v)\}]}$$

$$\equiv F(u,v),$$

which integrated between the points $A(\bar{\bar{a}}^{-\lambda/\mu}-\bar{\bar{\alpha}},\ \bar{\bar{a}}^{-\lambda/\mu}\bar{\bar{\alpha}})$ and $B(\bar{\bar{\beta}}^{-\lambda/\mu}-\bar{\bar{b}},\ \bar{\bar{b}}\bar{\bar{\beta}}^{-\lambda/\mu})$ yields the relation

$$(21)\qquad \frac{\bar{\bar{\beta}}^{-\lambda/\mu}}{\bar{\bar{\alpha}}} = \frac{\bar{\bar{a}}^{-\lambda/\mu}}{\bar{\bar{b}}} \exp \int_{\bar{\bar{a}}^{-\lambda/\mu}-\bar{\bar{\alpha}}}^{\bar{\bar{\beta}}^{-\lambda/\mu}-\bar{\bar{b}}} F\{\xi,v(\xi)\}d\xi.$$

We wish to determine the limiting behaviour of this expression for $\bar{\bar{\alpha}} \to 0$ $(\alpha \to 0)$. Let in the interval $\bar{\bar{\alpha}} \geq 0$ be considered, be

$$p = \max(\bar{\bar{a}}^{-\lambda/\mu} - \bar{\bar{\alpha}}, \alpha^{-\lambda/\mu}), \qquad q = \min(\bar{\bar{\beta}}^{-\lambda/\mu} - \bar{\bar{b}}, b)$$

and define

$$(22)\qquad \begin{aligned} &\Phi(u) = F\{u,v(u)\} \text{ on } \bar{\bar{\beta}}^{-\lambda/\mu} - \bar{\bar{b}} \leq u \leq \bar{\bar{a}}^{-\lambda/\mu} - \bar{\bar{\alpha}} \\ &\Phi(u) = 0 \quad \text{on} \quad q \leq u \leq \bar{\bar{\beta}}^{-\lambda/\mu} - \bar{\bar{b}} \quad \text{and} \quad \bar{\bar{a}}^{-\lambda/\mu} - \bar{\bar{\alpha}} \leq u \leq p; \end{aligned}$$

then we will show the existence of

$$(23)\qquad \lim_{\bar{\bar{\alpha}}\to 0} \int_p^q \Phi(\xi)d\xi = \lim_{\bar{\bar{\alpha}}\to 0} \int_{\bar{\bar{a}}^{-\lambda/\mu}-\bar{\bar{\alpha}}}^{\bar{\bar{\beta}}^{-\lambda/\mu}-\bar{\bar{b}}} F\{\xi,v(\xi)\}d\xi.$$

First we note that $F(u,v)$ is continuous outside of the origin and since $v(u) > 0$ for $\bar{\bar{\alpha}} > 0$, $\Phi(u)$ has possibly only two jump discontinuities between p and q; thus the integral exists for $\bar{\bar{\alpha}} > 0$. Secondly, if $\bar{\bar{\alpha}} = 0$, the unique solution of (20) is $v(u) \equiv 0$, whereas

$$\text{for } u < 0:\ F(u,0) = \frac{\bar{\bar{\phi}}\{0,\bar{\bar{y}}(u,0)\} - \bar{\bar{\psi}}\{0,\bar{\bar{y}}(u,0)\}}{\bar{\bar{y}}(u,0)[1+\bar{\bar{\psi}}\{0,\bar{\bar{y}}(u,0)\}]}$$

(24)

$$\text{for } u > 0:\ F(u,0) = \frac{\bar{\bar{\phi}}\{\bar{\bar{x}}(u,0),0\} - \bar{\bar{\psi}}\{\bar{\bar{x}}(u,0),0\}}{\{\bar{\bar{x}}(u,0)\}^{-\lambda/\mu}[1+\bar{\bar{\phi}}\{\bar{\bar{x}}(u,0),0\}]}.$$

From (15) follows on $q \le u \le p$, $v \ge 0$ (for ρ_* small enough) the estimate:

$$|\Phi(u)| < M_1 + M_2|u|^{-1-\mu/\lambda}, \quad M_1 > 0,\ M_2 > 0. \tag{25}$$

Since $\int_p^q \{M_1+M_2|u|^{-1-\lambda/\mu}\}du$ exists, we may write

$$\lim_{\bar{\bar{\alpha}}\to 0} \int_p^q \Phi(\xi)d\xi = \int_p^q \lim_{\bar{\bar{\alpha}}\to 0} \Phi(\xi)d\xi = \int_{a^{-\mu/\lambda}}^b F(\xi,0)d\xi. \tag{26}$$

With the aid of (16), (21) and (26) the result of the lemma is now readily obtained. □

For the derivation of the stability criterion for separatrix polygons yet another result is wanted which follows in the following lemma.

LEMMA 2. *Let the integral curve* L *through* A *and* B *be indicated by* $x = x(t)$, $y = y(t)$; *then for* $\alpha \to 0$

$$\int_A^B \operatorname{div}[P\{x(t),y(t)\},Q\{x(t),y(t)\}]dt = -\left(1+\frac{\mu}{\lambda}\right)\ln\alpha + B(\alpha), \tag{27}$$

where $B(\alpha)$ *has a finite limit for* $\alpha \to 0$.

PROOF.

$$\int_A^B \operatorname{div}[P\{x(t),y(t)\},Q\{x(t),y(t)\}]dt = \tag{28}$$

$$\int_A^B (a+d+\phi_x\{x(t),y(t)\} + \psi_y\{x(t),y(t)\})dt =$$

$$= (\lambda+\mu) \int_A^B dt + \int_A^B (\phi_x\{x(t),y(t)\} + \psi_y\{x(t),y(t)\})dt.$$

Consider first the second integral and perform the transformations, which were also used in the proof of Lemma 1. Using the properties of the functions $\phi(x,y)$ and $\psi(x,y)$ it may then be seen that

$$\int_A^B (\phi_x\{x(t),y(t)\} + \psi_y\{x(t),y(t)\})dt =$$

(29)

$$= -\frac{1}{\lambda}\int_{\bar{\bar{a}}^{-\lambda/\mu}-\bar{\bar{\alpha}}}^{\bar{\bar{\beta}}^{-\lambda/\mu}-\bar{\bar{b}}} \frac{f\{\bar{\bar{x}}(u,v),\bar{\bar{y}}(u,v)\}}{\{\bar{\bar{x}}(u,v)\}^{-\lambda/\mu}[1+\bar{\bar{\phi}}\{\bar{\bar{x}}(u,v),\bar{\bar{y}}(u,v)\}]+\bar{\bar{y}}(u,v)[1+\bar{\bar{\psi}}\{\bar{\bar{x}}(u,v),\bar{\bar{y}}(u,v)\}]}du$$

where the function $f\{\bar{\bar{x}}(u,v),\bar{\bar{y}}(u,v)\}$ has the same properties as the nominator in (20), and in fact the analysis following (20) may be used to show that the integral (29) only contributes to the function $B(\alpha)$, having a finite limit for $\alpha \to 0$. In order to determine the limiting behaviour of the first integral in (28), we integrate system (12) along the integral curve L. We integrate from A to S and S to B, where S is the intersection point of L with the curve $\bar{\bar{y}} = \bar{\bar{x}}^{-\lambda/\mu}$.

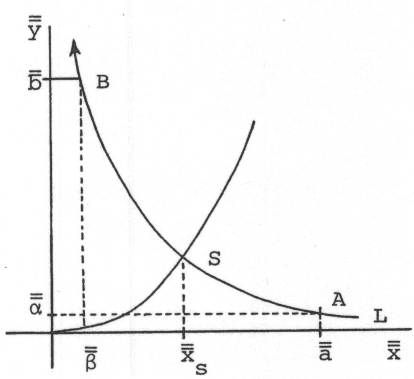

Fig. 8 Integral curves near a saddle point.

Then

$$(30) \qquad \int_A^B dt = \int_A^S \frac{d\bar{\bar{x}}}{\mu\bar{\bar{x}}[1+\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})]} + \int_S^B \frac{d\bar{\bar{y}}}{\lambda\bar{\bar{y}}[1+\bar{\bar{\psi}}(\bar{\bar{x}},\bar{\bar{y}})]}.$$

Now

$$\int_A^S \frac{d\bar{\bar{x}}}{\mu\bar{\bar{x}}[1+\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})]} = \frac{1}{\mu}\int_A^S \frac{d\bar{\bar{x}}}{\bar{\bar{x}}} - \frac{1}{\mu}\int_A^S \frac{\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})\,d\bar{\bar{x}}}{\bar{\bar{x}}[1+\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})]} . \tag{31}$$

With (15) the second integral may be estimated by

$$\left|\int_A^S \frac{\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})}{\bar{\bar{x}}[1+\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})]}\,d\bar{\bar{x}}\right| \le K\int_A^S \frac{d\bar{\bar{x}}}{|1+\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})|} + K\int_A^S \frac{\bar{\bar{y}}}{\bar{\bar{x}}}\,\frac{d\bar{\bar{x}}}{|1+\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})|}$$

which since $\bar{\bar{y}}(\bar{\bar{x}}) \le \bar{\bar{x}}^{-\lambda/\mu}$ on L leads to

$$\int_A^S \frac{d\bar{\bar{x}}}{\mu\bar{\bar{x}}[1+\bar{\bar{\phi}}(\bar{\bar{x}},\bar{\bar{y}})]} = \frac{1}{\mu}\ln \bar{\bar{x}}_s + A_1(\bar{\bar{\alpha}}), \tag{32}$$

where $A_1(\bar{\bar{\alpha}})$ is a function with a finite limit for $\bar{\bar{\alpha}} \to 0$ $(\alpha \to 0)$.

Similarly

$$\int_S^B \frac{d\bar{\bar{y}}}{\lambda\bar{\bar{y}}[1+\bar{\bar{\psi}}(\bar{\bar{x}},\bar{\bar{y}})]} = \frac{-1}{\lambda}\ln \bar{\bar{y}}_s + A_2(\bar{\bar{\alpha}}), \tag{33}$$

where $A_2(\bar{\bar{\alpha}})$ has the same limiting behaviour as $A_1(\bar{\bar{\alpha}})$. As a result (30) may be written as

$$\int_A^B dt = \frac{1}{\mu}\ln \bar{\bar{x}}_s - \frac{1}{\lambda}\ln \bar{\bar{y}}_s + A(\bar{\bar{\alpha}}) = \frac{2}{\mu}\ln \bar{\bar{x}}_s + A(\bar{\bar{\alpha}}), \tag{34}$$

with $A(\bar{\bar{\alpha}})$ again having a finite behaviour for $\bar{\bar{\alpha}} \to 0$ $(\alpha \to 0)$. In order to derive the dependency of $\bar{\bar{x}}_s$ on α we proceed as in Lemma 1. In fact by replacing B by S in relation (21) we obtain

$$\frac{\bar{\bar{x}}_s^{\,-\lambda/\mu}}{\bar{\bar{\alpha}}} = \frac{\bar{\bar{a}}^{-\lambda/\mu}}{\bar{\bar{x}}_s^{\,-\lambda/\mu}} \exp\int_{\bar{\bar{a}}^{-\lambda/\mu}-\bar{\bar{\alpha}}}^{0} F\{\xi,v(\xi)\}d\xi \tag{35}$$

and the argument following (21) may be used to show that:

$$\bar{\bar{x}}_s(\alpha) = B\alpha^{-\mu/2\lambda}[1+b(\alpha)], \tag{36}$$

where $B > 0$ and $b(\alpha) \to 0$ for $\alpha \to 0$. Substituting this expression in (34) and (29) and (34) in (28) yields the result of Lemma 2. □

SUCCESSION FUNCTION AND STABILITY CRITERION FOR SEPARATRIX POLYGONS

Let us consider autonomous systems in the phase plane, represented by

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{37}$$

where P(x,y) and Q(x,y) are continuously differentiable functions. For such a system consider a separatrix polygon with n saddle points, and number these points in the clockwise direction, taking any saddle point as the first. Let the direction of increasing t along the separatrices also result in a clockwise motion along the polygon. If, moreover, a "small" polygon is considered, the situation may be sketched as in Fig. 9. This situation may be considered to be typical for all separatrix polygons in the sense that the stability criterion to be derived also applies to polygons with any numbering of the saddle points, counter-clockwise motion on the separatrices, and to "large" polygons. Let M be some point on the separatrix connecting the saddle points 1 and n, and let l be a transversal through M, represented by the parametric equations

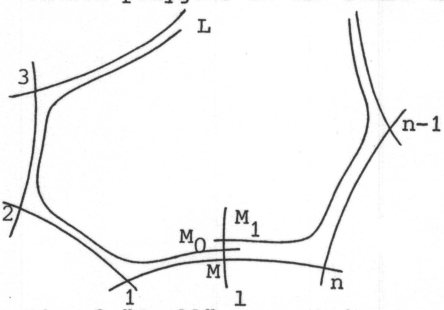

Fig. 9 "Small" separatrix polygon with n sides.

(38) $x = g(\varepsilon)$, $y = h(\varepsilon)$

where $g,h \in C^1$; $\varepsilon = 0$ corresponds to the point M and $\varepsilon > 0$ to points on l inside the polygon. Consider points on l inside the polygon and close to M and let M_0 be one such point, corresponding to $\varepsilon = \varepsilon_0$. Let L be the integral curve of system (37) through M_0 and

(39) $x = x(t)$, $y = y(t)$

the motion along this curve, for which the point M_0 corresponds to $t = t_0$. Let $t = t_1 > t_0$ be the lowest value of t for which L again intersects the transversal l and indicate this point with M_1 and $\varepsilon = \varepsilon_1$. Then the mapping f: $\varepsilon_0 \to \varepsilon_1$ for all points on l ($\varepsilon_0 \geq 0$) is called the succession function. Obviously $f(0) = 0$; if $f(\varepsilon) > \varepsilon$ ($< \varepsilon$) for $\varepsilon > 0$ the integral curves near the separatrix polygon approach it for $t \to -\infty$ ($t \to +\infty$) and the polygon is unstable (stable), whereas if $f(\varepsilon) \equiv \varepsilon$ the integral curves form an anular region of closed paths. In order to study $f(\varepsilon)$ near $\varepsilon = 0$, the limiting behaviour of its derivative for $\varepsilon \to 0$ is of interest. As is shown in [2], for the derivative of the succession function in M_0 may be derived the expression:

$$f'(\varepsilon_0) = \frac{\Delta(M_0)}{\Delta(M_1)} \exp \int_{t_0}^{t_1} \operatorname{div}[P\{x(t),y(t)\},\ Q\{x(t),y(t)\}]dt \tag{40}$$

where $\Delta(M_0) = P(x_0,y_0)h'(\varepsilon_0) - Q(x_0,y_0)g'(\varepsilon_0)$, $\Delta(M_1) = P(x_1,y_1)h'(\varepsilon_1) - Q(x_1,y_1)g'(\varepsilon_1)$. Obviously

$$\lim_{\substack{P_0,P_1\to P\\ \varepsilon_0\to 0}} \frac{\Delta(M_0)}{\Delta(M_1)} = 1. \tag{41}$$

<u>THEOREM</u>. *Suppose the system* $\frac{dx}{dt} = P(x,y)$, $\frac{dy}{dt} = Q(x,y)$, *where* $P(x,y)$, $Q(x,y)$ *are Lipschitz continuously differentiable functions, has a separatrix polygon in the region* G. *Choose any numbering of the corner points (saddle points) of the polygon and denote by* $\mu_i < 0$, $\lambda_i > 0$ *the eigenvalues of the locally linearized system in the i-th saddle point. Let* $\Sigma = 1 - \left|\frac{\mu_1\mu_2\cdot\cdots\cdot\mu_n}{\lambda_1\lambda_2\cdot\cdots\cdot\lambda_n}\right|$; *then if* $\Sigma > 0$, *the separatrix polygon is unstable, if* $\Sigma < 0$ *stable.*

<u>PROOF</u>. We investigate the limiting behaviour of (40) for $\varepsilon \to 0$ and divide therefore the t interval $t_0 < t < t_1$ into 2n intervals, namely n intervals, each of which lies in a small neighbourhood of a saddle point and another n intervals which are the remaining intervals. Thus, if L enters the small neighbourhood of the i-th saddle point at $t = \tau_i$ and leaves it again at $t = \bar{\tau}_i$, the integral in (40) may be written as

$$\int_{t_0}^{t_1} \operatorname{div}[P\{x(t),y(t)\},Q\{x(t),y(t)\}]dt =$$

$$\int_{t_0}^{\tau_1} \operatorname{div}[P\{x(t),y(t)\},Q\{x(t),y(t)\}]dt + \tag{42}$$

$$+ \sum_{i=1}^{n} \int_{\tau_i}^{\bar{\tau}_i} \operatorname{div}[P\{x(t),y(t)\},Q\{x(t),y(t)\}]dt +$$

$$+ \sum_{i=1}^{n-1} \int_{\bar{\tau}_i}^{\tau_{i+1}} \operatorname{div}[P\{x(t),y(t)\},Q\{x(t),y(t)\}]dt +$$

$$+\int_{\bar{\tau}_n}^{t_1} \operatorname{div}[P\{x(t),y(t)\},Q\{x(t),y(t)\}]dt.$$

For $\varepsilon_0 \to 0$ the integrals outside of the saddle point neighbourhoods have a finite limit. In each of the saddle point neighbourhoods it follows from Lemma 2 that the integral has a singular behaviour. According to Lemma 2 the divergent part of (42) may be written as:

$$(43) \qquad \sum_{i=1}^{n} -\left(1+\frac{\mu_i}{\lambda_i}\right) \ln \alpha_i,$$

where α_i is the value of α of the i-th saddle point and τ_i and $\bar{\tau}_i$ are chosen to correspond with the points A and B for the i-th saddle point. Let furthermore β_i be the value of β of the i-th saddle point. Then the continuous dependency of the solution from initial data assures that may be written

$$(44) \qquad \begin{aligned} &\alpha_1 = c_1\varepsilon_0, \quad \alpha_2 = c_2\beta_1, \ldots \ \alpha_n = c_n\beta_{n-1}; \\ &c_i > 0 \quad \text{for} \quad \varepsilon_0 \geq 0, \ \beta_i \geq 0 \qquad\qquad i = 1,\ldots,n. \end{aligned}$$

With (44) and Lemma 1, (43) may be written as

$$\left[-\left(1+\frac{\mu_1}{\lambda_1}\right)-\frac{\mu_1}{\lambda_1}\left[-\left(1+\frac{\mu_2}{\lambda_2}\right)-\frac{\mu_2}{\lambda_2}\left[\ldots -\frac{\mu_{n-1}}{\lambda_{n-1}}\left[-\left(1+\frac{\mu_n}{\lambda_n}\right)\right]\underbrace{\ldots}_{n}\right]\ln \varepsilon_0 =$$

$$(45) \qquad \left[1+\frac{\mu_1}{\lambda_1}-\frac{\mu_1}{\lambda_1}-\frac{\mu_1\mu_2}{\lambda_1\lambda_2}+\frac{\mu_1\mu_2}{\lambda_1\lambda_2}+\ldots+(-1)^{n-2}\frac{\mu_1\mu_2\ldots\mu_{n-1}}{\lambda_1\lambda_2\ldots\lambda_{n-1}}-(-1)^{n-2}\cdot\right.$$

$$\left.\cdot\frac{\mu_1\mu_2\ldots\mu_{n-1}}{\lambda_1\lambda_2\ldots\lambda_{n-1}}+(-1)^{n-1}\frac{\mu_1\mu_2\ldots\mu_n}{\lambda_1\lambda_2\ldots\lambda_n}\right](-\ln \varepsilon_0) = \Sigma(-\ln \varepsilon_0).$$

Thus it may be seen from (45), (40) and (41) that if $\Sigma > 0$ there follows $f'(0) = +\infty$, and since $f(0) = 0$, $f(\varepsilon) > 0$ for $\varepsilon > 0$, it also follows that $f(\varepsilon) > \varepsilon$ for $\varepsilon > 0$. Thus the separatrix polygon is unstable. If $\Sigma < 0$ it may be shown similarly that $f'(0) = 0$, and since $f(\varepsilon) > 0$ for $\varepsilon > 0$ as well as $f(0) = 0$, it is seen that $f(\varepsilon) < \varepsilon$ for $\varepsilon > 0$. Thus the separatrix polygon is stable. □

EXAMPLES WHEN $\Sigma = 0$

For the special case of the saddle to saddle node (n = 1) the criterion reduces to the one given in [2], since for n = 1 there is $\Sigma = \frac{\lambda+\mu}{\lambda} = \frac{\sigma_s}{\lambda}$ and $\Sigma \gtrless 0$ if and only if $\sigma_s \gtrless 0$. In [2] examples were given for $\sigma_s = 0$ such that the loop was either stable, unstable or had a neighbourhood with only closed integral curves. We will now similarly show for $n \geq 2$, that if $\Sigma = 0$, there may be cases where an arbitrary small neighbourhood of the separatrix polygon contains closed paths, as well as cases where the polygon is stable or unstable.

As an example for n = 2, consider the system

$$\frac{dx}{dt} = y + \gamma(y^2 - 2\cos x - 2)\sin x,$$
(46)
$$\frac{dy}{dt} = -\sin x + \gamma(y^2 - 2\cos x - 2)y.$$

For $\gamma = 0$ (46) reduces to the well known pendulum equation, the solutions of which may be written as

(47) $$y^2 - 2\cos x = C,$$

where C is a constant. For C = 2 eq. (47) yields the equation for the separatrix polygon with n = 2. For $-2 < C < 2$ eq. (47) gives the closed integral curves inside the polygon, and C = -2 corresponds to the centerpoint at the origin.

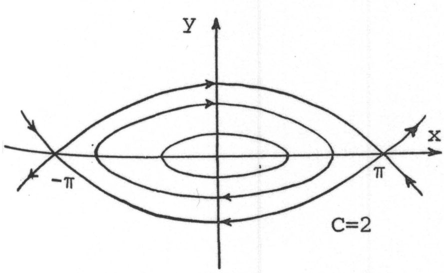

Fig. 10 Integral curves for the pendulum equation.

It may easily be seen that $y^2 - 2\cos x = 2$ also is a separatrix polygon for eq. (46) in the case that $\gamma \neq 0$. In fact the vector field in (46) may be obtained from the case $\gamma = 0$ by rotating the vector in each point over an angle $\arctan\gamma(y^2 - 2\cos x - 2)$. The curves (47) with $-2 < C < 2$ are therefore curves without contact for system (46) and $\gamma \neq 0$. It may then easily be concluded that for $\gamma > 0$ the separatrix polygon is unstable and for $\gamma < 0$ stable. Moreover $\Sigma = 0$ for all values of γ.

As an example for $n \geq 3$, we consider the system

$$\frac{dx}{dt} = -\sum_{m=1}^{n} \sin(2m-1)\frac{\pi}{n} \prod_{\substack{k=1\\k\neq m}} \{\cos\frac{\pi}{n} - x\cos(2k-1)\frac{\pi}{n} - y\sin(2k-1)\frac{\pi}{n}\} +$$

$$- \gamma \prod_{k=1}^{n} \{\cos\frac{\pi}{n} - x\cos(2k-1)\frac{\pi}{n} - y\sin(2k-1)\frac{\pi}{n}\} \cdot$$

$$(48) \quad \cdot \left[\sum_{m=1}^{n} \cos(2m-1)\frac{\pi}{n} \prod_{\substack{k=1\\k\neq m}}^{n} \{\cos\frac{\pi}{n} - x\cos(2k-1)\frac{\pi}{n} - y\sin(2k-1)\frac{\pi}{n}\}\right],$$

$$\frac{dy}{dt} = \sum_{m=1}^{n} \cos(2m-1)\frac{\pi}{n} \prod_{\substack{k=1\\k\neq m}}^{n} \{\cos\frac{\pi}{n} - x\cos(2k-1)\frac{\pi}{n} - y\sin(2k-1)\frac{\pi}{n}\} +$$

$$- \gamma \prod_{k=1}^{n} \{\cos\frac{\pi}{n} - x\cos(2k-1)\frac{\pi}{n} - y\sin(2k-1)\frac{\pi}{n}\} \cdot$$

$$\cdot \left[\sum_{m=1}^{n} \sin(2m-1)\frac{\pi}{n} \prod_{\substack{k=1\\k\neq m}}^{n} \{\cos\frac{\pi}{n} - x\cos(2k-1)\frac{\pi}{n} - y\sin(2k-1)\frac{\pi}{n}\right].$$

For $\gamma = 0$ eq. (48) may be solved to yield the solutions

$$(49) \qquad \prod_{k=1}^{n} \{\cos\frac{\pi}{n} - x\cos(2k-1)\frac{\pi}{n} - y\sin(2k-1)\frac{\pi}{n}\} = C,$$

where C is a constant. For C = 0 eq. (49) yields a separatrix polygon, which is a regular n sided polygon with cornerpoints on the unit circle. For C > 0 eq. (49) yields the closed integral curves inside the polygon whereas $C = \cos^n \pi/n$ corresponds to the centerpoint (0,0). It may be seen that the polygon is also a separatrix polygon for (48) when $\gamma \neq 0$. Similarly as in the previous example, the vector field in (48) for $\gamma \neq 0$ may be obtained from the case $\gamma = 0$ by rotating the vector in each point. The curves (49) inside the polygon are therefore curves without contact for system (48) with $\gamma \neq 0$. Depending on the sign of γ the separatrix polygon will then be stable or unstable. Moreover $\Sigma = 0$ for all values of γ.

SEPARATRIX POLYGONS EXTENDING TO INFINITY

System (1) may be said to have a separatrix polygon extending to infinity if the system obtained from (1) by the transformation

$$(50) \qquad \bar{x} = \frac{x}{\sqrt{1+x^2+y^2}}\,, \qquad \bar{y} = \frac{y}{\sqrt{1+x^2+y^2}}$$

has a separatrix polygon, having one or more points in common with the unit circle in the $\bar{x},\bar{y}$ plane. Obviously the stability properties of a separatrix polygon are invariant with respect to eq. (50). As a result, the stability of a separatrix polygon extending to infinity may be studied in the $\bar{x},\bar{y}$ plane, where the theorem stated in this paper applies.

CONCLUDING REMARKS

1. We have chosen to follow the line of arguments given in [2] and used the expression (40) for the derivative of the succession function. For that we need Lemma 1. Once Lemma 1 is obtained, an alternative route may be taken avoiding this expression and Lemma 2. If Lemmma 1 is applied at all the saddle points and (44) is used, for the succession function there may be obtained

$$(51) \qquad f(\varepsilon) = C\varepsilon^{1-\Sigma}[1 + c(\varepsilon)],$$

where $C > 0$ is a constant and $c(\varepsilon) \to 0$ for $\varepsilon \to 0$. The theorem then follows from the remarks following (39).

2. When the stability theorem, derived above, was presented at the International Conference on Non-linear Oscillations at Prague, September 1978, prof. K.R. Schneider called my attention to a paper by L.A. CHERKAS [4] in which mention is made of the same stability criterion in relation to previous work of H. DULAC [5]. In this very lengthy paper [5] H. Dulac studies the differential equation $P(x,y)dx + Q(x,y)dy = 0$, where $P(x,y)$ and $Q(x,y)$ are analytic functions, using transformations and series solutions. In fact it appears that Lemma 1 and expression (51) were known to him, so that a stability criterion could have been derived once the differential equation was written as a dynamic system. The present account relaxes the conditions on the functions $P(x,y)$ and $Q(x,y)$ and gives a direct and elementary proof of the stability theorem.

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