Calculating Coefficients for Fourier Expansions of Siegel Modular Forms

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Plan of Attack

1. Modular Forms
   - Modular Forms and Periodicity
   - Generalizing Modular Forms
2. Siegel Modular Forms and their Coefficients
   - Properties of Siegel Modular Forms
   - Binary Quadratic Forms and Fourier Expansions
3. Our Work
   - Divisibility and Binary Quadratic Forms
   - Equivalence and Higher Levels
4. Conclusion

Modular Forms

- Generalize periodicity $f(x+kp) = f(x)$, where p is the period, $k \in \mathbb{R}$
- **Modular forms** are functions on complex numbers with a positive imaginary part.
- $\forall z \in H$ and for all matrices in $SL(2,\mathbb{Z})$, modular forms satisfy,

$$f\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = (cz+d)^k f(z)$$

- A convenient consequence: the periodicity allows for Fourier Expansions!

Fourier Expansions for Modular Forms

- What does a Fourier expansion look like?

$$f(\tau) = \sum_{n=0} c(n) e^{2\pi i n \tau}$$

- This is just sine and cosine waves added together
- Our object of interest is $c(n)$.
  - They reveal a lot about how $f(\tau)$ "looks"
  - We have lots of these documented on websites like LMFDB

Generalizing Modular Forms

- **Siegel modular forms** are functions on complex symmetric matrices with positive imaginary parts
- It also satisfies some behavior, but where modular forms are periodic on a subset of $SL(2, \mathbb{Z})$, Siegel modular forms are periodic on a subset of the **symplectic group**:

$$\Gamma_g(N) = \left\{ \gamma \in GL_{2g}(\mathbb{Z}) | \gamma^{\mathrm{T}} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}, \gamma \equiv I_{2g} \mod N \right\}$$

- We call g the **genus**
- This sets us up to describe periodicity on Siegel modular forms

Convenient transition

- Our work focuses on Siegel paramodular forms
  - This just means a Siegel modular form of genus 2.
- Since we have periodicity on some subset of $\Gamma_g(N)$ (trust us, we do), we have a Fourier expansion for our Siegel paramodular forms
- And now, we are interested in learning about these expansions and their Fourier coefficients

Coefficients and Binary Quadratic Forms

- Our new Fourier expansion looks like:

$$F(Z) = \sum_{S \in A(N)^+} a(S) e^{2\pi i tr(SZ)}$$

- Our coefficients, $a(S)$, are a different this time: they're indexed on $\begin{pmatrix} a & b \\ b & d \end{pmatrix} \in A(N)^+$, matrices where $N|a$, $b \in \frac{1}{2}\mathbb{Z}$, $a > 0$, and $ac - b^2 > 0$
- It is common to think of these as **binary quadratic forms**, equations of the form $ax^2 + bxy + cy^2$

Calculating New Coefficients

- We have: sets of coefficients $a(n)$ and their respective indexes $\begin{pmatrix} a & b \\ b & d \end{pmatrix} \in A(N)^{+}$
- We have: five formulas which produce new coefficients for subsets of our current coefficients. Example 1:

$$a_{\chi}(S) = p^{1-k}\chi(2\beta) \sum_{b \in (\mathbb{Z}/p\mathbb{Z})^{\times}} \chi(b) a\left(S[\begin{bmatrix} 1 & -bp^{-1} \\ & p \end{bmatrix}]\right)$$

Determining Which Coefficients Have Solutions

- There are many matrix operations occurring here
- We know our matrices have integer values
- It turned out a minority of observed samples have solutions for our example!

An Example Claim

Claim:

Let S be a binary quadratic form such that $S = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ with discriminant D such that $p^4 | x$ and $p \nmid 2(bx + py)/p^2$,

$$S' = (S[\begin{bmatrix} 1 & -bp^{-1} \\ 0 & p \end{bmatrix}]^{-1}) = \begin{bmatrix} x & (bx + py)/p^2 \\ (bx + py)/p^2 & (b^2x + 2bpy + p^2z)/p^4 \end{bmatrix}$$

. Then S' is integer-valued $\implies$ $p^2 || D$ and $p \nmid z$

Equivalence and Generally Making Things Harder

- Not all Siegel modular forms are so easy!
- Once we know which coefficients map to new coefficients, we must also be sure they are unique
- This depends on the **level** of the Siegel modular form, but the problem is analogous to the number theory problem of equivalence of binary quadratic forms
- For more information, see David Cox's "Primes of the form $x^2 + ny^2$"

Thanks for your patience!

Thank you for your time! We hope you enjoyed!