Macro-Finance:
Class 3 (Solow Model)

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Today's Outline

- Solow model recap
- Predictions
- Exercises
- Problems for next week

You are encouraged to try the problems at the end beforehand. Let me know if you have any queries:

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- Office hours: Thursdays 3-4pm

Why a macroeconomic model?

- add a systematic structure to data, economic behaviour, etc.
- in general is a method of accounting but augmented to account for risk, etc.
- provide implications beyond personal opinion, based on common grounds
- can be debated, agreed on in terms of assumptions, facts, etc.
- assumptions simplify the complexity of an economic system
- present an understandable version of real-world
- macro models are useful for long-term prediction

Parsimony, empirical consistency, and testability are important features.

Solow Model

- capital accumulates over time
- more capital generates more output and economic welfare
- indicators such as aggregate saving, depreciation, etc. affect this accumulation
- over-saving or over-consuming can both be equally detrimental to welfare
- what is the ideal trade-off between consumption and saving?

The Model

The fundamental Solow equation

$$C_t + K_{t+1} = f(K_t, L_t) + (1 - \delta - n)K_t$$

The output (real units) or income each period is

$$\begin{aligned} Y_t &= f(K_t, L_t) \\ C_t &= (1 - s)Y_t \end{aligned}$$

Combine to obtain

$$\Delta K_{t+1} = sf(K_t, L_t) - (\delta + n)K_t$$

- production inputs are capital and labour $\{K_t, L_t\}$
- aggregate consumption is C_t
- saving rate s (exhibits observed optimal behaviour)
- depreciation rate δ
- population growth n

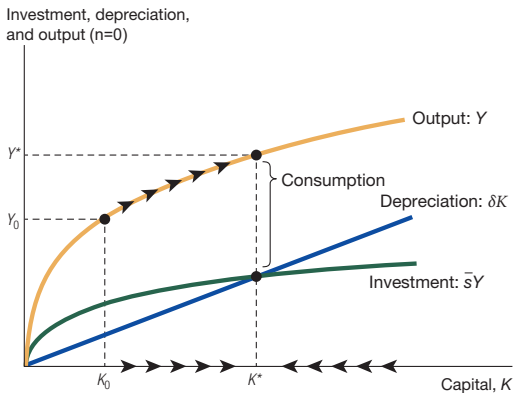
Steady State

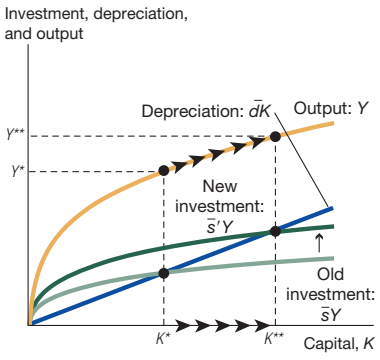
Consider a Cobb-Douglas technique $f(.,.) = AK_t^{\alpha} L_t^{1-\alpha}$ for $\alpha \in [0,1]$ then $\Delta K_{t+1} = 0$:

$$k_t^* = \frac{K_t^*}{L_t} = \left(\frac{sA}{\delta + n}\right)^{\frac{1}{1-\alpha}}$$

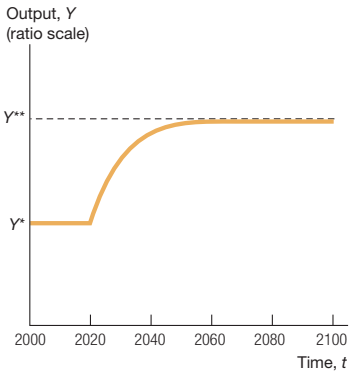
where $k_t = K_t / L_t$ is the per capita stock of capital.

- What is the pairwise relationship between k_t^* and the following s, δ, n, A and α?
- Discuss the economic reason why is population growth negatively proportional to stock of capital.

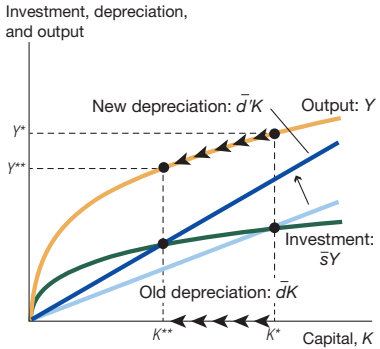


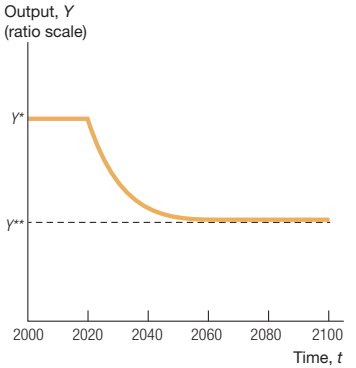


(a) The Solow diagram with output.



(b) Output over time.





Return to Scale

Return to scale implies that for $\lambda > 0$:

$$\lambda f(K_t, L_t) \quad = \quad f(\lambda K_t, \lambda L_t)$$

- increasing (IRS) if $\lambda > 1$
- constant (CRS) if $\lambda = 1$
- decreasing (DRS) if $\lambda < 1$
- e.g. double the input, obtain more than double the output ($\lambda > 1$)

Determine the return to scale for the following:

- $K_t^{\frac{1}{2}} L_t^{\frac{1}{2}}$
- $AK_t + L_t$
- $K_t + K^{\frac{1}{3}} L^{\frac{2}{3}}$
- $K_t + K^{\frac{1}{3}} L^{\frac{1}{3}}$
- $K^{\frac{1}{3}} L^{\frac{2}{3}} - A$
- $[\alpha K^{\frac{1}{\theta}} + (1-\alpha) L^{\frac{1}{\theta}}]^{\theta}$

Golden Rule

The golden rule is:

$$\begin{aligned} \text{MPK} &= (n+\delta) \\ \alpha A k_t^{\alpha-1} &= . \end{aligned}$$

where MPK $= f_K$, (MPL $= f_L$) is the marginal product of capital (labour), measures contribution of an additional unit of capital (labour) to production, where f_K and f_L are derivative of the production function with respect to inputs. Simplify to obtain

$$k_t^g = \left(\frac{\alpha A}{\delta + n}\right)^{\frac{1}{1-\alpha}}$$

then $\alpha \lessgtr s$ determines $k_t^* \lessgtr k_t^g$. Steady state depends on savings rate:

- $s = 100\%$: all income goes to investment capital for future production ($C_t = 0$).
- $s = 0\%$: no new investment capital is created (capital depreciates without replacement)
- Golden Rule level of savings: the savings propensity such that per-capita consumption is at its maximum possible constant value.

Technology

Technology, but also institutions, contract law and its enforcements, legal efficiency, infrastructure, financial market efficiency, etc.

- in 1990s one coaxical cable handled one household's network traffic.
- in 2010 the *very* same cable handles 50,000 households' network traffic

Technology grows over time, which enables production at lower cost (real units):

- Discuss if firms should do R&D in this model?

Equilibrium

Capital market, labour market and goods market:

$$\text{saving} = \text{investment} \tag{1}$$
$$\text{labour force} = \text{employment opportunities} \tag{2}$$
$$\text{consumption demand} = \text{real units produced} \tag{3}$$

Equilibrium Implications

Discuss the following:

- more saving
- more labour force
- more output

Nominal Indicators and Rigidities

Discuss the following implications:

- inflation versus deflation
- labour contracts

Excel/R

1) Use Solow.xls to calculate what happens to macroeconomic outcomes when the saving rate drops to 10% from 25%. Why does this happen?

- GDP falls because investment falls. By 2050, GDP reduces from 15.49 to 10.28 (drops $33.6\% = \frac{10.28-15.49}{15.49}$)

$$K_2 \quad = \quad s \times \underbrace{f(K_1, L_1)}_{4.08} - (1 - \underbrace{\delta}_{6\%} + \underbrace{n}_{0\%}) \underbrace{K_1}_{8.23}$$

- given $n = 0$ labour force remains constant but K_2 changes. Iterating forward yields:

$$\begin{aligned} f(K_{50}, L; s = 10\%) &= \text{GDP}_{50} = 10.28 \\ f(K_{50}, L; s = 15\%) &= \text{GDP}_{50} = 15.49 \end{aligned}$$

2) What happens after 20 years when the TFP growth rate drops from 3% to 2%? What does this tell us about the determinants of long term living standards and asset prices?

- use compound interest given an initial value 4.08 in 2010
- GDP rises to 8.3 (2%) by 2030, but to 10.53 (3%)
- GDP growth is greater than the TFP growth because it multiplies a growing capital stock.

3) If output, capital and labour are equal to 10, 20 and 5, respectively, what is the TFP when $\alpha = \frac{1}{3}$ using a Cobb-Douglas function $f = AK^{\alpha}L^{1-\alpha}$? ($A \approx 1.26$)

Secular Stagnation

Inadequate demand and low growth in long-term (as opposed to business cycles or short-term recessions):

- over-saving behaviour generates price appreciation in financial assets or real estate
- economy (production firms) operates at below capacity
- particularly, inadequate demand for (e.g. domestic) innovation-driven products dampens technological progressions

`larrysummers.com/category/secular-stagnation/`

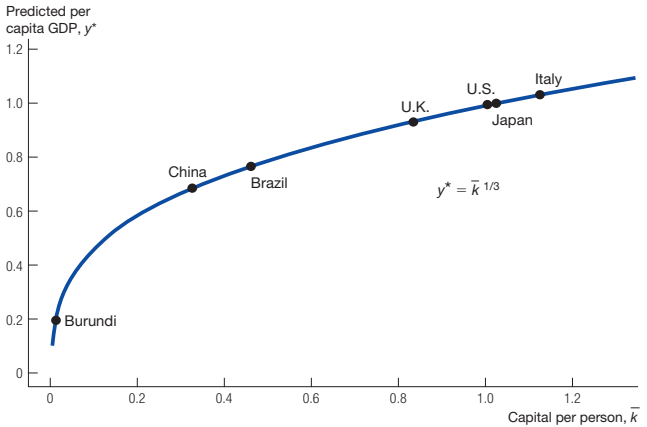
- recession vs stagnation vs stagflation
- inflation, deflation, disinflation and hyperinflation

Predictions

Convergence — Growth

Convergence

- developed economies grow more slowly than the developing
- gap between economies with different development phases narrows



Divergence — Wealth Heterogeneity

Welfare (inter alia) requires both GDP growth & wealth distribution considerations:

- persons own as much as poor 50% (oxfam 2019)
- taxation is a transfer
- Piketty's Capital provides evidence in support of worsening distribution

Wealth concentration

- one of the reasons behind limited stock market participation
- political economy bias
- income inequality, shifts more money to the wealthy, who tend to save it rather than spend it, thus increasing savings and perhaps driving up financial asset prices

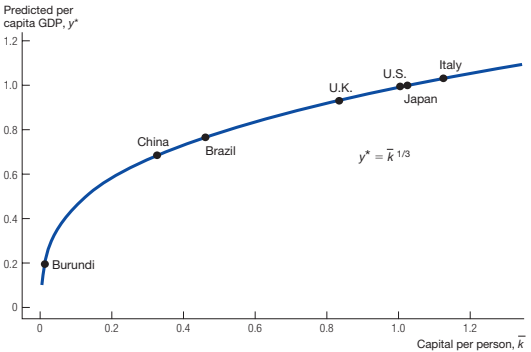
Cross-economy (intergenerational) mobility

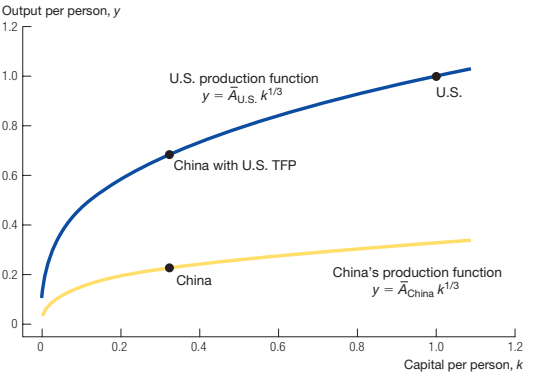
labour can move. Labour with same characteristics receives higher wages because

$$\text{MPL}_t \quad = \quad (1-\alpha)AK_t^{\alpha}L_t^{-\alpha}$$

- higher capital
- higher technology
- higher contribution share
- lower labour force

Capital Flow





Example: China vs US

on the demand side, China is saving-driven whereas the US is consumption-driven:

- difficult to get people to consume to stimulate the output sector
- social security, pension, education, healthcare

on the supply side, China is production whereas the US is innovation driven

- China has one of the best infrastructure, roads, railways, telecomm, etc.
- investment is more readily consumable relative to innovation

shifts toward technology, R&D, etc.

- after WWII China spent 5cents per dollar and now spends 95cents relative to the US
- however, technological progression takes time (e.g. semiconductors)

Exercises

$\mathbb{Q}$ — In the Cobb-Douglas production function $f = K^{\alpha}L^{1-\alpha}$, if $\alpha = \frac{1}{4}$, then:

1. capital's share of GDP is one-fourth.
2. labor's share of GDP is half.
3. capital's share of GDP is three-fourths.
4. labor's share of income is one-fourth.
5. capital's share of GDP is four.

$\mathbb{Q}$ — The marginal product of labor is defined as:

1. output divided by labor.
2. the additional output generated by hiring an additional unit of labor.
3. the additional output generated by hiring an additional unit of labor and capital.
4. the additional output generated by hiring an additional unit of capital.
5. the additional revenue generated by hiring an additional unit of labor.

$\mathbb{Q}$ — A production function exhibits constant returns to scale when you:

1. hold inputs constant—obtain double the output.
2. double each input—obtain more than double the output.
3. double each input—obtain less than double the output.
4. double one input—obtain double the output.
5. double each input—obtain double the output.

$\mathbb{Q}$ — In the Solow model, if investment is ... depreciation, the capital stock ... .

1. less than; grows
2. equal to; declines
3. greater than; grows
4. equal to; grows
5. greater than; declines

$\mathbb{Q}$ — Immediately following the increase in the saving rate, output grows rapidly. As the economy approaches its new steady state, the growth rate:

1. gradually increases.
2. is negative.
3. gradually declines.
4. is constant
5. None of these answers is correct.

$\mathbb{Q}$ — Suppose the population growth is zero then if we define the saving rate as s, output as $f(K, L)$, and the depreciation rate as δ, and if $sf - \delta k = 0$, the economy is:

1. contracting
2. in its short-run equilibrium.
3. growing
4. at the steady state
5. None of these answers is correct.

A two-country model of equilibrium interest rates

You are not required to hand in this problem but prior attempt is encouraged. We will review this in two weeks (next week: text data analysis).

Suppose now that there are two countries (domestic and foreign) and in each country there is a representative agent that lives for two periods and has preferences given by

$$U(c_1^j, c_2^j) \quad = \quad \ln c_1^j + \beta \ln c_2^j \tag{4}$$

where $j = d$ for the domestic country and $j = f$ for the foreign country Assume also that:

- the domestic representative agent receives as income y_1^f units of the consumption good in period 1, and zero in period 2
- the foreign representative agent receives zero income in period 1 and y_2^f units of the consumption good in period 2;
- agents can borrow or lend at the world gross interest rate $1 + r$.

1. Find the optimal consumption for each of the two agents as a function of r and β and their own incomes.
2. Find the level of utility of each agent as a function of r and β and their own incomes. How does the utility of the two agents change as r changes? Why?
3. Assume that these are the only two countries in the world and solve for the equilibrium interest rate as a function of β and the endowments. (Hint: you should impose the market clearing condition that total consumption in each period is equal to the total endowment in that period). How does r change a y_2^f changes? Why?