Linear Algebra Practice Questions
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Exercises: Vectors

Question 1: Let $\boldsymbol{x} = (1, 2)'$ and $\boldsymbol{y} = (2, 1)'$:
(1.1) Find $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y}$
(1.2) Find $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{y}$
(1.3) Draw a diagram of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{u}$ and $\boldsymbol{v}$
(1.4) Find norms of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{u}$ and $\boldsymbol{v}$
(1.5) When can the (Euclidean) norm of a vector be zero or negative?
(1.6) Find the inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$
(1.7) Are $\boldsymbol{x}$ and $\boldsymbol{y}$ linearly independent or dependent?

Question 2: Let $\boldsymbol{x}_1 = (1, 1, 1)'$ and $\boldsymbol{x}_2 = (1, 2, 0)'$ (note that subscripts 1 and 2 are indexes which distinguish two different vectors, and do not refer to size of the matrices):
(2.1) Can a weak inequality between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be established?
(2.2) Are $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ linearly independent or dependent?
(2.3) Is the function of vectors $\boldsymbol{u}(\boldsymbol{x}_1) = \mathbf{1}_{3\times 1} + 2\boldsymbol{x}_1$ a linear or nonlinear transformation of $\boldsymbol{x}_1$? What is the dimension of $\boldsymbol{u}$?
(2.4) What is the dimension of $\boldsymbol{v}$ if $\boldsymbol{v}(\boldsymbol{x}_1, \boldsymbol{x}_2) = ||\boldsymbol{x}_1||.||\boldsymbol{x}_2|| + ||\boldsymbol{x}'_1.\boldsymbol{x}_2||$? Is it an N-to-1 mapping or an N-to-N mapping? How do you depict $\boldsymbol{v}$ on a Cartesian space?
(2.5) Is the function of vector $\boldsymbol{y}(\boldsymbol{x}_1, \boldsymbol{x}_2) = b_0\mathbf{1}_{3\times 1} + b_1\boldsymbol{x}_1 + b_2\boldsymbol{x}_2$ a linear transformation of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$? what is the dimension of $\boldsymbol{y}$? what is the numerical value of vector $\boldsymbol{y}$?
(2.6) What is the univariate analogue of $\boldsymbol{x}'\mathbf{1}_{3\times 1}$?

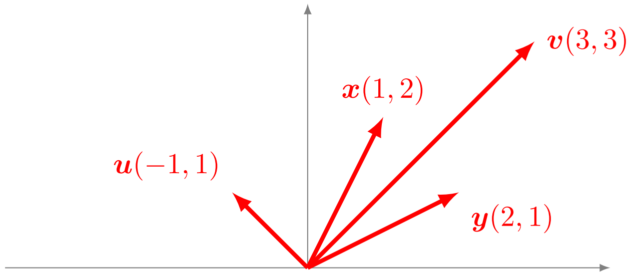
Solution to Exercises (Vectors)

Question 1:

(1.1) Find $\boldsymbol{v} = (1, 2) + (2, 1) = (3, 3)'$

(1.2) Find $\boldsymbol{u} = (1, 2) - (2, 1) = (-1, 1)'$

(1.3) Draw a diagram of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{u}$ and $\boldsymbol{v}$



(1.4) $||\boldsymbol{x}|| = \sqrt{5}$, $||\boldsymbol{y}|| = \sqrt{5}$ and $||\boldsymbol{u}|| = \sqrt{2}$ and $||\boldsymbol{v}|| = \sqrt{3^2 + 3^2} = \sqrt{18}$

(1.5) Euclidean norm is always a non-negative number (for real-valued vectors). It is zero iff the vector is a null vector.

(1.6) $\boldsymbol{x}'.\boldsymbol{y} = 1 \times 2 + 2 \times 1 = 4$

(1.7) Vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent because there is <u>no</u> such combination $\boldsymbol{x} = \lambda \boldsymbol{y}$, for $\lambda \neq 0$, can be established.

Question 2:

(2.1) No, because neither of the vectors is element-wise weakly greater or less than the other one.

(2.2) Vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are linearly independent since $\boldsymbol{x} = \lambda \boldsymbol{y}$, for $\lambda \neq 0$ cannot be established.

(2.3) Function $\boldsymbol{u}(\boldsymbol{x}_1)$ is a linear transformation of $\boldsymbol{x}_1$, and dimensions coincide with those of $\boldsymbol{x}_1$.

(2.4) Function $\boldsymbol{v}$ is an N-to-1 mapping because the right-hand-side is sum of scalars (norm of any size vectors is a scalar operator). The graphical illustration would be only one single point on the real line $\boldsymbol{y}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sqrt{1^2 + 1^2 + 1^2} \times \sqrt{1^2 + 2^2 + 0^2} + \sqrt{(1 \times 1)^2 + (1 \times 2)^2 + (1 \times 0)^2} = 2\sqrt{5}$

(2.5) Function $\boldsymbol{y}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is a linear transformation of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Dimensions are 3×1 and the numerical value of elements of this vector are defined in terms of parameters $(b_0 + b_1 + b_2,\ b_0 + b_1 + 2b_2,\ b_0 + b_1)'$.

(2.6) Univariate analogue refers to operation that yields the same results but without vector/matrix operations. Vector operation $\boldsymbol{x}'\mathbf{1}_{3\times 1}$ is an inner product (giving sum of element-wise product of all elements from both vectors). In this question, we have a special case of inner product because one of the vectors is a one vector (e.g. $\mathbf{1}_{3\times 1}$). Particularly, this implies that the inner product of any vector $\boldsymbol{x}$ by a $\mathbf{1}$ vector only sums up the individual elements of vector $\boldsymbol{x}$. This operation can also be written using a sum:

$$\sum_{i=1}^{n} x_i \quad = \quad 1 + 1 + 1$$

Exercises: Matrices

Question 1: Let

$$\boldsymbol{X} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Y} = \begin{bmatrix} 5 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(1.1) What are the transpose transformations $\boldsymbol{X}'$ and $\boldsymbol{Y}'$?

(1.2) Is matrix $\boldsymbol{X}'$ symmetric?

(1.3) In how many ways can $\boldsymbol{X}$ and $\boldsymbol{Y}$ (or their transpose transformations) can be multiplied (general multiplication)?

(1.4) How many linearly independent vectors can you find within $\boldsymbol{X}$?

(1.5) What is the determinant of $\boldsymbol{X}$, $det(\boldsymbol{X})$?

(1.6) Find the rank of $\boldsymbol{X}$, $rank(\boldsymbol{X})$?

(1.7) Find the rank of $\boldsymbol{YX}$?

(1.8) What is the univariate analogue of $\mathbf{1}'_{3\times 1}\boldsymbol{Y}'\mathbf{1}_{4\times 1}$?

(1.9) Find the norm of $\boldsymbol{X}$?

(1.10) Is $\boldsymbol{X}$ invertible, why?

(1.11) Is $\boldsymbol{X}'\boldsymbol{X}$ invertible, why?

(1.12) Is $\boldsymbol{X}$ a singular matrix?

Question 2: Matrix inversion and rank:

(2.1) Find the inverse form of $\boldsymbol{I}_2$

(2.2) Find the inverse form of $\mathbf{1}_{2\times 2}$

(2.3) What is the rank of $\mathbf{0}_{n\times n}$?

Question 3: Let

$$\boldsymbol{X} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

(3.1) Find the rank and determinant of $\boldsymbol{X}$.

(3.2) Is $\boldsymbol{X}$ an idempotent matrix?

Question 4: Let

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(4.1) What is $(\boldsymbol{X}'\boldsymbol{X})^{-1}$.

(4.2) Find $(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$.

(4.3) Denote $\boldsymbol{M} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ and find $\boldsymbol{MY}$.

(4.4) Find $\boldsymbol{Y} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$

Solution to Exercises (Matrices)

Question 1:

(1.1) Transpose transformations $\boldsymbol{X}'$ and $\boldsymbol{Y}'$ are:

$$\boldsymbol{X}' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Y}' = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 4 & 1 & 2 & 0 \\ 3 & 1 & 2 & 1 \end{bmatrix}$$

(1.2) Matrix $\boldsymbol{X}'$ is symmetric because $\boldsymbol{X} = \boldsymbol{X}'$.

(1.3) $\boldsymbol{X}$ is 3×3 and $\boldsymbol{Y}$ is 4×3, noting the conformity property, we can have $\boldsymbol{YX}$, $\boldsymbol{XY}'$ and $\boldsymbol{X}'\boldsymbol{Y}'$.

(1.4) None, since the determinant is equal to $2 = 2 \times 1 \times 1$ which is non-zero.

(1.5) $det(\boldsymbol{X}) = 2$.

(1.6) $rank(\boldsymbol{X}) = 3$ because this matrix has non-zero determinant (it is full-rank or non-singular), hence there exists three linearly independent column (row) vectors within the matrix.

(1.7) Second and third rows of $\boldsymbol{Y}$ are linearly dependent. After eliminating one of vectors of $\boldsymbol{Y}$, there exists only three linearly independent vectors within $\boldsymbol{Y}$ which indicates $rank(\boldsymbol{Y}) = 3$. The product $\boldsymbol{YX}$ is 3×3 and $rank(\boldsymbol{YX}) = 3$ since the product of two matrices, each with $rank = 3$ amounts to a $rank = 3$ matrix.

(1.8) Re-write $\mathbf{1}_{3\times 1}\boldsymbol{Y}'\mathbf{1}_{4\times 1}$ operation without using vectors/matrices operations. The special point about this is that for matrices we have to pre- and post-multiply matrix $\boldsymbol{Y}'$ with vectors of ones (e.g. $\mathbf{1}'_{3\times 1}$ and $\mathbf{1}_{4\times 1}$):

$$\begin{aligned} \mathbf{1}'_{3\times 1}\boldsymbol{Y}'\mathbf{1}_{4\times 1} &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 2 & 0 \\ 4 & 1 & 2 & 0 \\ 3 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5+1+2+0 \\ 4+1+2+0 \\ 3+1+2+1 \end{bmatrix} \end{aligned}$$

where from the first line to the second, we post-multiplied only $\boldsymbol{Y}'\mathbf{1}_{4\times 1}$. As we can see:

$$\begin{bmatrix} 5+1+2+0 \\ 4+1+2+0 \\ 3+1+2+1 \end{bmatrix}$$

is, first, a 3×1 vector with each element equal to sum of row elements of $\boldsymbol{Y}'$. If we apply the pre-multiplication $\mathbf{1}'_{3\times 1}\,(\boldsymbol{Y}'\mathbf{1}_{4\times 1})$:

$$\begin{aligned} \mathbf{1}'_{3\times 1}\boldsymbol{Y}'\mathbf{1}_{4\times 1} &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5+1+2+0 \\ 4+1+2+0 \\ 3+1+2+1 \end{bmatrix} \\ &= \begin{bmatrix} 5+1+2+0+4+1+2+0+3+1+2+1 \end{bmatrix} \end{aligned}$$

Last line is only the sum of all elements of the original matrix $\boldsymbol{Y}$, which we can re-do using the sum operation:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} x_{x_{ij}} = 5+1+2+0+4+1+2+0+3+1+2+1$$

Note that we need to sum over rows, and then sum over columns (or vice versa), that is why we have the double sum operator.

(1.9) There are two ways of working out the norm of a matrix, if the matrix is a diagonal matrix (which is also square because it is diagonal). The quadratic form under the square root $\boldsymbol{X}'\boldsymbol{X}$ is:

$$\begin{aligned} \boldsymbol{X}'\boldsymbol{X} &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 1^2 & 0 \\ 0 & 0 & 1^2 \end{bmatrix} \end{aligned}$$

and then:

$$\begin{aligned} \sqrt{tr(\boldsymbol{X}'\boldsymbol{X})} &= \left[tr \left(\begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 1^2 & 0 \\ 0 & 0 & 1^2 \end{bmatrix} \right) \right]^{\frac{1}{2}} \\ &= \sqrt{2^2 + 1^2 + 1^2} \end{aligned}$$

which is equivalent to $||\boldsymbol{X}|| = \sqrt{2^2 + 1^2 + 1^2}$.

(1.10) Yes, because $det(\boldsymbol{X}) \neq 0$.

(1.11) Yes, because $\boldsymbol{X}'\boldsymbol{X}$ is full rank and therefore invertible.

(1.12) $\boldsymbol{X}$ is non-singular because $det(\boldsymbol{X}) \neq 0$.

Question 2:

(2.1) The inverse form of $\boldsymbol{I}_2$ is $\boldsymbol{I}_2$ because it is an identity matrix.

(2.2) The inverse form of $\mathbf{1}_{2\times 2}$ is not defined because matrix one, has all elements equal to one which makes all of its columns (rows) linearly dependent and thus it is not full rank, and non-invertible.

(2.3) The null matrix is by definition a rank-zero matrix: $rank(\mathbf{0}_{n\times n}) = 0$

Question 3:

(3.1) $det(\boldsymbol{X}) = 0$, and $rank(\boldsymbol{X}) = 2$ (a singular matrix or not full rank). The maximum rank possible for this matrix is 3, however, since the determinant is zero, there exists at least one linear combination among rows or columns (in this case $C_3 = C_2 - 2C_1$). After establishing this linear combination, no further linear combination is possible and rank becomes 2, as there are only two linearly independent vectors within $\boldsymbol{X}$, $rank(\boldsymbol{X}) = 2$

(3.2) Yes, because $\boldsymbol{X}\boldsymbol{X} = \boldsymbol{X}$.

Question 4: Let

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(4.1) $(\boldsymbol{X}'\boldsymbol{X})^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

(4.2) $(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} = (-1, -1, 3)'$.

(4.3) $\boldsymbol{MY} = (1, 2, 3)'$.
(4.4) $\boldsymbol{Y} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} = (0, 0, 0)'$.