UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing

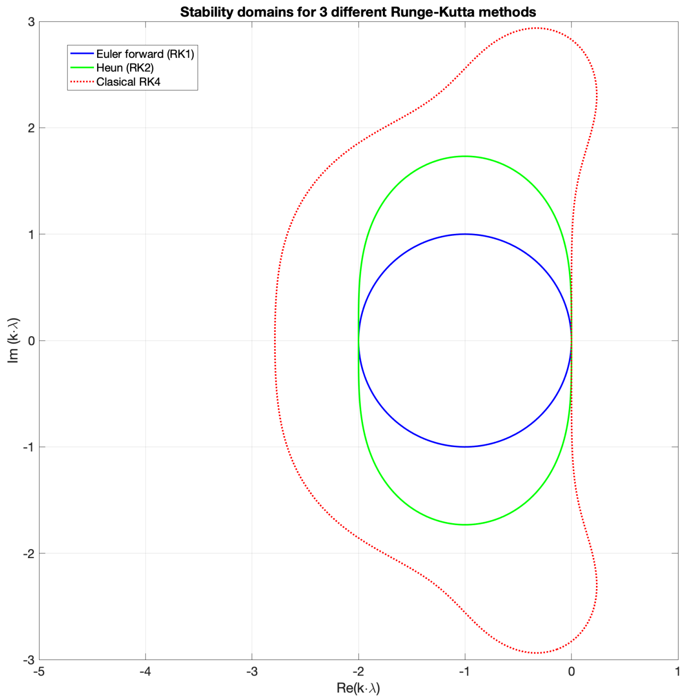
Continuous examination part 1, Scientific Computing for PDE, H22

Time: $13^{15} - 15^{45}$
Tools: Matlab, Python, Lecture notes in Studium.

You will login with "Tentainloggning" (you will see a grey background):
User: *ang-tenta* , **Password**: *tentamina*
Once logged in, create a folder under `c:\temp`, for example `c:\temp\StinaStudent`. Save all your flies in the created folder.

Make sure you motivate and present relevant analysis, and make sure to address all that is asked for. Make sure you have your name at each page that you hand in. The analysis and answers (except figures) are written down on paper that are handed in to the teacher. Script files (either Matlab or Python) and figures should be uploaded in Studium under ***CONTINUOUS EXAMINATION PART 1****. Notice also that you should upload your files no later than 16.00. There are two problems, each worth a total of 20p. Hence, the total number of points is 40. To pass this first examination you need at least 16 points (for grade 3). Grade 4 and 5 require 24 and 32 points, respectively. You may collect points on both problems.*

In the Figure the stability domains for a few Runge-Kutta methods are shown.



A difference operator $D_1 = H^{-1}\left(Q - \frac{1}{2}e_1 e_1^T + \frac{1}{2}e_m e_m^T\right)$ approximating $\partial/\partial x$ is a first-derivative SBP operator if $H = H^T > 0$, and $Q + Q^T = 0$. $e_1 = [1, 0, ..., 0]^T$ and $e_m = [0, ..., 0, 1]^T$.

A difference operator $D_2 = H^{-1}(-M - e_1 d_1 + e_m d_m)$ approximating $\partial^2/\partial x^2$ is said to be a second-derivative SBP operator if $H = H^T > 0$, $M = M^T \geq 0$, and $d_1 v \simeq u_x(x = x_l)$, $d_m v \simeq u_x(x = x_r)$ are finite difference approximations of the first derivatives at the left and right boundary points.

Problem 1: Consider the following problem,

$$\begin{array}{lll} \mathbf{u}_t = \mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u} + \mathbf{F}, & -1 \leq x \leq 1,\ t \geq 0, & \\ L_l \mathbf{u} = g_l, & x = -1,\ t \geq 0, & \\ L_r \mathbf{u} = g_r, & x = 1,\ t \geq 0, & \\ \mathbf{u} = \mathbf{f}(x), & -1 \leq x \leq 1,\ t = 0, & \end{array} \tag{1}$$

where $\mathbf{F} = \mathbf{F}(x, t)$ is a forcing function, $\mathbf{B}$=$\mathbf{B}(x)$ is bounded, and $\mathbf{f} = \mathbf{f}(x)$ the initial data. The matrix $\mathbf{A}$ is given by,

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ \beta & 0 & 1 \end{bmatrix} \text{, where } \beta \text{ is a real constant.} \tag{2}$$

a) What are the requirements for a general problem on the form given by (1) to be well-posed, disregarding the boundary conditions, i.e. here assume $BT \leq 0$. (**2p**)

b) Consider now five different choices of β: (i) $\beta = -1$, (ii) $\beta = 1$, (ii) $\beta = -1/4$, (iv) $\beta = 2$, (v) $\beta = 3$. Motivate which of these cases leads to a well-posed problem, and for these cases also derive the correct number of boundary conditions for each boundary. (**5p**)

c) Consider the case where $\beta = 1$. Derive three sets of well-posed boundary conditions for (1). This means finding L_l and L_r. (**3p**)

d) Derive an SBP-Projection approximation of (1), with one of the well-posed sets of boundary conditions derived in c). (**2p**)

e) Show stability for the SBP-Projection approximation in d), for the case where $\mathbf{B} = 0$. (**3p**)

D) Implement the SBP-Projection approximation in Matlab or Python, using the 6th order accurate SBP operator, and use RK4 as time-integrator. Initialize with $u^{(1)} = \exp(-(6x)^2)$, $u^{(2)} = -\exp(-(6x)^2)$ and $u^{(3)} = 0$. Here we set $\mathbf{F} = 0$ and $\mathbf{B} = 0$. Set $m = 201$ and present stable solution plots for $t = 0.2$, $t = 0.4$, $t = 0.6$. Present also the chosen time-step (here denoted k). Upload your figures and script files in Studium. Use the format *png* or *jpg* for the figures. (**5p**)

The second problem is on the next page

Problem 2: Consider the following problem,

$$
\begin{array}{ll}
i\, u_t = -u_{xx} + V(x)u, & -3 \leq x \leq 1,\ t \geq 0, \\
L_l u = g_l, & x = -3,\ t \geq 0, \\
L_r u = g_r, & x = 1,\ t \geq 0, \\
u = f(x), & -3 \leq x \leq 1,\ t = 0,
\end{array}
\tag{3}
$$

where $V(x) = \alpha \cdot \exp\left(-(6(x-1))^2\right)$, for some real parameter $\alpha \geq 0$, and i is the imaginary unit.

a) Show that (3) yields energy-conservation, disregarding the boundary conditions, i.e. here assume $BT = 0$. (**2p**)

b) Derive two different sets of well-posed boundary conditions for (3) that leads to energy-conservation, i.e. $BT = 0$. (**2p**)

c) Derive an SBP-Projection approximation of (3), with one of the well-posed sets of boundary conditions derived in b). (**2p**)

d) Show stability for the SBP-Projection approximation in c). (**2p**)

e) Derive a set of well-posed boundary conditions for (3) that leads to damping of energy at both boundaries, i.e. $BT < 0$. (**3p**)

f) Derive an SBP-Projection approximation of (3), with the well-posed set of boundary conditions derived in e). (**2p**)

g) Show stability for the SBP-Projection approximation in f). (**3p**)

H) Explain why Euler forward (RK1) is not a suitable time-integrator for the SBP-Projection approximation in c). Motivate how to chose the time-step k (in terms of grid-spacing h) if you instead use the RK4 time-integrator, i.e. derive a bound $k \leq p(h)$ for some function p. Derive the functions $p(h)$ when applying the 4th and 6th order accurate SBP operators and $\alpha = 100$. *Hint: Study the eigenvalues to the discretization (or solution) matrix.* (**4p**)