Uppsala University

Lecture Notes

An Introduction to Matching and Stable Matching in Bipartite Graphs

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Note

This class was delivered in Hungarian for high school student in 24th of July 2025 as part of the Eötvös József College's Natural Science Camp Math section.

Disclaimer

As this document is mainly **AI translated** some **inaccuracies or errors may occur**. The notes are more mathematical, then the delivered lecture was. It aimed to follow the story of matching boys and girls throughout the whole session including the proofs as well making it more accessible to high school students.

Bipartite Graphs, Matchings and Hall's Theorem

Short problem statements

Problem 1. Let $X = \{\text{András, Béla, Csaba}\}$ be the set of boys and $Y = \{\text{Dóri, Eszter, Flóra}\}$ the set of girls. Mutual sympathies are represented by a bipartite graph $G = (X \cup Y, E)$.

(a) *Csaba liked nobody.* Question: is there a perfect matching that pairs every boy?

(b) *András and Dóri mutually liked everyone.* Question: is there a perfect matching?

(c) *As in (b), and additionally Béla and Flóra liked each other.* Question: is there a perfect matching?

Precise definitions

Definition 1 (Bipartite graph)**.** A graph $G = (V, E)$ is *bipartite* if there exist two disjoint sets X, Y such that $V = X \cup Y$ and $E \subseteq X \times Y$. Then X and Y are called the *parts* of the graph.

Definition 2 (Matching, completeness, saturated vertex)**.** Let $G = (X \cup Y, E)$ be bipartite. A set of edges $M \subseteq E$ is a *matching* if for every $v \in X \cup Y$, $\deg_M(v) \leq 1$ (where $\deg_M$ denotes degree with respect to M). A matching is called X*-saturating* if every $x \in X$ is incident with an edge of M. If every vertex of both X and Y is matched, we speak of a *perfect matching*.

Definition 3 (Neighborhood)**.** For $S \subseteq X$, the *neighborhood* of S in G is $\mathrm{N}_G(S) := \{y \in Y : \exists x \in S \text{ with } xy \in E\}$. We write $\mathrm{N}(S)$ when the graph is clear.

Necessary condition

Problem 2. Given 4 boy and 4 girl draw graphs of sympathies, where there is no complete matching. What is common in these cases? What should be necessary to guarantee a complete matching?

Lemma 1 (Necessity of Hall's condition)**.** *If there exists an X-saturating matching M, then for every $S \subseteq X$ we have $|\mathrm{N}(S)| \geq |S|$.*

Proof. The matching M maps each element of S to a distinct element of Y, so $|\mathrm{N}(S)| \geq |S|$. $\square$

Hall's marriage theorem

Theorem 1 (Kőnig–Hall)**.** *Let $G = (X \cup Y, E)$ be bipartite. Then there exists an X-saturating matching if and only if for all $S \subseteq X$*

$$|\,\mathrm{N}(S)| \geq |S|. \tag{$*$}$$

Proof (classical induction + two cases). Necessity was shown in the lemma. For sufficiency we proceed by induction on $|X|$.
Base case: $|X| = 0$ is trivial; $|X| = 1$ implies the sole vertex x has at least one neighbor, hence one edge forms a saturating matching.
Inductive step: Assume the statement holds for all $|X| \leq n$, and let $|X| = n + 1$.
Two cases arise.
Case 1. There exists nonempty $S \subset X$ with $|\,\mathrm{N}(S)| = |S|$. Consider $G_1 := G[S \cup \mathrm{N}(S)]$ and $G_2 := G[(X \setminus S) \cup (Y \setminus \mathrm{N}(S))]$.

- G_1 satisfies Hall: for all $T \subseteq S$, $\mathrm{N}_{G_1}(T) = \mathrm{N}_G(T)$, so $|\,\mathrm{N}_{G_1}(T)| \geq |T|$. By induction there is an S-saturating matching M_1 in G_1.

- G_2 also satisfies Hall: take $U \subseteq X \setminus S$. Then $\mathrm{N}_{G_2}(U) = \mathrm{N}_G(U) \setminus \mathrm{N}(S)$. From $(*)$, $|\,\mathrm{N}_G(U \cup S)| \geq |U| + |S|$. But $\mathrm{N}_G(U \cup S) \subseteq \mathrm{N}_G(U) \cup \mathrm{N}(S)$, hence $|\,\mathrm{N}_G(U) \setminus \mathrm{N}(S)| = |\,\mathrm{N}_G(U) \cup \mathrm{N}(S)| - |\,\mathrm{N}(S)| \geq |U|$. Thus Hall holds for G_2. By induction there is an $X \setminus S$-saturating matching M_2.

Then $M := M_1 \cup M_2$ is an X-saturating matching in G.
Case 2. For every nonempty $S \subset X$, $|\,\mathrm{N}(S)| \geq |S| + 1$. Pick any $x \in X$ and a neighbor $y \in \mathrm{N}(\{x\})$. Remove x and y to form G'. For nonempty $T \subseteq X \setminus \{x\}$, $|\,\mathrm{N}_{G'}(T)| \geq |\,\mathrm{N}_G(T)| - 1 \geq |T|$, so G' satisfies Hall. By induction G' admits a matching saturating $X \setminus \{x\}$; adding xy yields an X-saturating matching in G. □

Problems

Problem 3. At a ball, 10 boys and 10 girls would like to dance with each other. It is known that each person is willing to dance with exactly 4 members of the opposite sex. Can they be paired up so that everyone is happy with their partner?

Problem 4. At a ball, 10 boys and 10 girls would like to dance with each other. It is known that each person is willing to dance with exactly 4 members of the opposite sex. They would like to dance in 4 rounds, such that in each round everyone dances with a new partner whom they like. Is this possible?

Problem 5. Among 100 boys and 100 girls, each person is willing to dance with at least 50 members of the opposite sex. Can they be paired up so that everyone dances with a partner they like?

Structure of a Bipartite Graph with Respect to a Matching and Alternating Reachability

Definition 4 (Alternating and augmenting paths)**.** Let $G = (X \cup Y, E)$ be a bipartite graph and let $M \subseteq E$ be a matching. A path is *alternating (w.r.t. M)* if its edges alternate between $E \setminus M$ and M. An alternating path whose two endpoints are unmatched by M is an *augmenting path*; flipping the edges on such a path increases the size of the matching by 1.

Definition 5 (Partition via alternating reachability)**.** Let

$C_1 := \{x \in X : x \text{ is free (unmatched) in } M\}, \qquad C_2 := \{y \in Y : y \text{ is free (unmatched) in } M\}.$

Now perform a BFS/DFS of alternating paths starting from the vertices in C_1. Define:

$$B_1 := \{x \in X : x \text{ is matched and reachable via an alternating path from } C_1\},$$

$$B_2 := \{y \in Y : y \text{ is matched and reachable via an alternating path from } C_1\}.$$

The remaining vertices are

$$A_1 := X \setminus (B_1 \cup C_1), \qquad A_2 := Y \setminus (B_2 \cup C_2).$$

Thus $X = A_1 \cup B_1 \cup C_1$ and $Y = A_2 \cup B_2 \cup C_2$ form a disjoint partition.

Theorem 2 (Forbidden edge types)**.** *With the partition above, the following hold:*

(i) There is no edge between A_1 and B_2.

(ii) There is no edge between B_1 and A_2.

Consequently, the only possible edge types are

$$A_1 - A_2, \quad B_1 - B_2, \quad C_1 - B_2, \quad A_1 - C_2.$$

Proof. (i) Suppose $x \in A_1$, $y \in B_2$, and $xy \in E$. Since $y \in B_2$, there exists an alternating path from C_1 to y. If $xy \notin M$, then this extends the alternating path to x, so x would be in B_1, contradiction. If $xy \in M$, then y's matched edge leads directly to x, so x would also be reached, again contradiction.
(ii) Symmetric: if $x \in B_1$ and $y \in A_2$ are joined, then if $xy \notin M$ we could extend the path to y, contradiction; if $xy \in M$, then y would be reached together with x, contradiction.
Thus only the listed edge types can exist. □

Corollary (Berge's Lemma / Maximality via absence of augmenting paths)*.* If there is no alternating path from C_1 to C_2, then M is a maximum matching.

Proof. If a larger matching M' existed, the symmetric difference $M \triangle M'$ would contain an alternating path starting and ending at free vertices of M, i.e. an augmenting path—contradiction. Alternatively, Theorem 2 shows that when no augmenting path exists, the set $K := A_1 \cup B_2$ is a vertex cover of size $|M|$. By Kőnig's theorem, the size of the maximum matching equals the size of a minimum vertex cover, so M is maximum. □

Remark (Motivation for the Hungarian algorithm). This partition explains why iteratively searching for augmenting paths works: either a new augmenting path is found and the matching can be increased, or no such path exists, in which case the partition $(A_1, B_1, C_1; A_2, B_2, C_2)$ immediately yields a minimum vertex cover certifying maximality. Thus, instead of checking all $2^{|X|}$ subsets required by Hall's condition, the Hungarian method finds a maximum matching in polynomial time.

Stable Marriage Problem and the Gale–Shapley Algorithm

Exercises on Stable Matching

Problem 6. Four boys A, B, C, D each wish to marry one of the four girls E, F, G, H. Each boy and each girl has a strict preference ordering over the other side. We want to arrange marriages so that no boy and girl would rather run away with each other than stay with their assigned partner.

	1	2	3	4
A	E	F	G	H
B	F	E	H	G
C	E	G	H	F
D	H	G	F	E

	1	2	3	4
E	B	A	D	C
F	A	D	C	B
G	D	C	A	B
H	C	D	A	B

1. Find one stable set of marriages.
2. Are there multiple stable matchings? If so, list them all. Otherwise show them it is unique.

Problem 7. Three boys A, B, C each wish to marry one of the three girls D, E, F. Again, we want to arrange the marriages so that no boy and girl would prefer to leave their assigned partners to be with each other.

	1	2	3
A	D	E	F
B	E	D	F
C	F	D	E

	1	2	3
D	C	B	A
E	A	B	C
F	C	A	B

1. Find one stable matching between the boys and the girls.
2. Are there multiple stable matchings? If so, list them all. Otherwise show them it is unique.

Problem 8. Two boys A, B and two girls C, D want to marry. Their preferences are given as follows:

	1	2
A	C	D
B	D	C

	1	2
C	B	A
D	A	B

1. Find a stable matching.
2. Are there multiple stable matchings? If so, list them all. Otherwise show them it is unique.

1 Problem Formulation

We consider a bipartite matching problem between two equally sized sets:

- A set $F = \{f_1, f_2, \ldots, f_n\}$ of "men" (or generally, proposers),
- A set $C = \{c_1, c_2, \ldots, c_n\}$ of "women" (or generally, receivers).

Each agent has a strict preference ordering over the members of the opposite set.

Definition 6 (Matching)**.** A *matching* M is a set of pairs (f, c) such that each $f \in F$ and each $c \in C$ appears in at most one pair.

Definition 7 (Stability)**.** A matching M is *stable* if there does not exist a pair (f, c) such that:

a) f and c are not matched together in M, and

b) f prefers c over his partner in M, and c prefers f over her partner in M.

Such a pair (f, c) is called a *blocking pair*.

2 The Gale–Shapley Algorithm

The algorithm proceeds iteratively with proposers making offers:

Algorithm: Gale–Shapley (Proposer-Oriented)

1. Each free man f proposes to the most preferred woman c to whom he has not yet proposed.
2. The woman c considers the proposal:
   - She *tentatively accepts* the proposal from the man she prefers most,
   - and *rejects* all others.
3. Rejected men become free and propose again in the next round.
4. The process repeats until every person is matched.

Theorem 3 (Gale–Shapley, 1962)**.** *The Gale–Shapley algorithm always terminates with a stable matching.*

Proof. Termination follows since each man proposes to each woman at most once, hence the process takes at most n^2 steps. To prove stability, assume for contradiction that a blocking pair (f, c) exists. Then f must have proposed to c before proposing to his final partner (since c is preferred). If c rejected f, it was because she held a better or equal proposal, and she never drops a more preferred partner. Thus c cannot prefer f over her final partner, contradiction. $\square$

3 Properties

Theorem 4 (Proposer Optimality)**.** *The Gale–Shapley algorithm yields the best possible stable partner for every proposer.*

Proof. Suppose man f does not receive his best stable partner c^*. Then there exists a stable matching M^* in which f is matched with c^*. In the algorithm, f must have proposed to c^* before his final partner. If c^* rejected him, she must have held a man f' that she prefers to f. But then c^* prefers f' in both the algorithm's outcome and in M^*. Therefore, (f, c^*) cannot be a pair in any stable matching, contradiction. Thus each proposer receives the best partner possible. $\square$

Theorem 5 (Receiver Pessimality)**.** *The Gale–Shapley algorithm yields the worst possible stable partner for every receiver.*

Proof. Assume some receiver c receives a partner better than her worst stable partner. Then there exists another stable matching M' in which she is matched with a less preferred man. But by the proposer-optimality result, all men are matched with their best stable partners in the Gale–Shapley outcome. Thus no receiver can improve without some proposer losing his optimal partner, which is impossible. Hence every receiver is matched to her least-preferred stable partner. $\square$

4 Historical Background

The Stable Marriage Problem was introduced by David Gale and Lloyd Shapley in 1962 in their seminal paper *College Admissions and the Stability of Marriage*. The Gale–Shapley algorithm became the foundation of matching theory and market design.
In 2012, **Lloyd Shapley** and **Alvin E. Roth** were awarded the **Nobel Memorial Prize in Economic Sciences**. Shapley was recognized for his theoretical contributions, and Roth for applying these ideas to real-world markets, such as the **U.S. National Resident Matching Program** (introduced in the 1950s) and school choice systems.

5 Notes

- More general problems
  - Cases with unequal set sizes: some agents may remain unmatched.
  - Cases with multiple slots (e.g., universities with several openings).
- Real-world use: U.S. National Resident Matching Program (since the 1950s).
- Other Applications: College admissions (students vs. universities), School choice systems