

## Hairer's 17 stage order 10 Runge-Kutta scheme

See: A Runge-Kutta Method of Order 10, E. Hairer, J. Inst. Maths Applics (1978) **21**, 47-59.

Following Ernst Hairer we use the following values for specific nodes and weights.

$$c_2 = \frac{1}{2}, \quad c_{16} = \frac{1}{2}, \quad c_{17} = 1,$$

$$b_2 = -\frac{1}{30}, \quad b_3 = -\frac{3}{25}, \quad b_4 = 0, \quad b_5 = 0, \quad b_6 = -\frac{13}{100}, \quad b_7 = -\frac{9}{50}, \quad b_8 = 0, \quad b_{13} = \frac{13}{100}, \quad b_{14} = \frac{9}{50}, \quad b_{15} = \frac{3}{25}, \quad b_{16} = \frac{1}{30}.$$

The node  $c_6$  is chosen to make three of the principal error terms (approximately) zero. Hairer's value is  $c_6 \simeq 0.7666539862535488$ , although a more accurate value is:

$$c_6 \simeq 0.7666539862535505911932668693560686601417881683220066628197494388337786524595845606945.$$

The nodes:

$$c_9 = \frac{1}{2} - \frac{\sqrt{147 - 42\sqrt{7}}}{42}, \quad c_{10} = \frac{1}{2} + \frac{\sqrt{147 + 42\sqrt{7}}}{42}, \quad c_{11} = \frac{1}{2} + \frac{\sqrt{147 - 42\sqrt{7}}}{42}, \quad c_{12} = \frac{1}{2} - \frac{\sqrt{147 + 42\sqrt{7}}}{42}$$

are the zeros of the derivative  $P'_5(x) = \frac{d}{dx} P_5(x)$  of the **Legendre polynomial**  $P_5(x)$  mapped linearly from the interval  $[-1, 1]$  to the interval  $[0, 1]$ .

They provide nodes for **Gauss-Lobatto integration** on the interval  $[0, 1]$ .

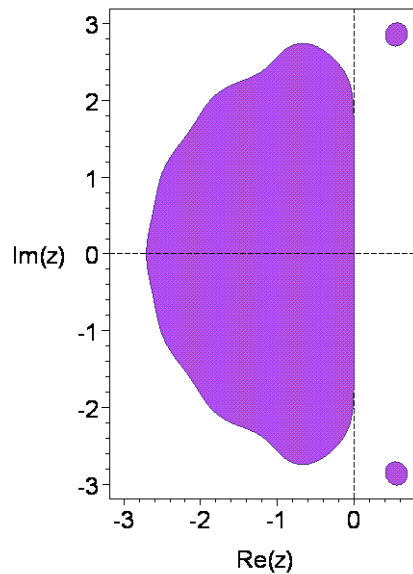
These values give rise to the following weights:

$$b_1 = \frac{1}{30}, \quad b_9 = \frac{7}{30} + \frac{\sqrt{7}}{60}, \quad b_{10} = \frac{7}{30} - \frac{\sqrt{7}}{60}, \quad b_{11} = \frac{7}{30} + \frac{\sqrt{7}}{60}, \quad b_{12} = \frac{7}{30} - \frac{\sqrt{7}}{60}, \quad b_{17} = \frac{1}{30}.$$

The principal error norm, that is, the 2-norm of the principal error terms is:  $0.5270474035 \times 10^{(-5)}$ .

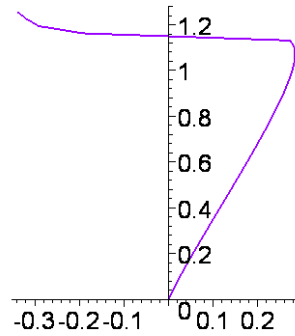
The maximum magnitude of the linking coefficients is: 1.054902178.

The stability region for the scheme is shown in the following picture.



The real stability interval of the scheme is  $[-2.7047, 0]$ .

The following picture shows the result of distorting the boundary curve of the stability region of the scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval  $[0, 1.1619]$ .

The coefficients to 85 digits are as follows.

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c[2]=.5233584004620047139632937023215170497515953383496781610502942213792083195573343614542,
c[3]=.5265091001416125727329516734775408259523008085442588621240322448552569517961160776999,
c[4]=.7897636502124188590994275102163112389284512128163882931860483672828854276941741165498,
c[5]=.3939235701256720143227738119996521367488350094732736567444747389961242969755195414769,
c[6]=.7666539862535505911932668693560686601417881683220066628197494388337786524595845606945,
c[7]=.2897636502124188590994275102163112389284512128163882931860483672828854276941741165498,
c[8]=.1084776892195672933536461100396272126536495082872530661076180009074872219675135232227,
c[9]=.3573842417596774518429245029795604640404982636367873040901247917361510345429002009092,
c[10]=.8825276619647323464255014869796690751828678442680521196637911779185276585194132570617,
c[11]=.6426157582403225481570754970204395359595017363632126959098752082638489654570997990908,
c[12]=.1174723380352676535744985130203309248171321557319478803362088220814723414805867429383,
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c[14]=.2897636502124188590994275102163112389284512128163882931860483672828854276941741165498,
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