LECTURE NOTES

EE160: Introduction to Control (Fall 2023)

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ARTICLE HISTORY
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ABSTRACT
The 14th edition of a textbook is not making it a better material for "newbies". So I decided to draft my own lecture notes.

Contents

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1. Introduction to Control

In my first lecture on Sept. 26th, 2023, I was suggesting there should be a second episode for the Youtube video entitled "Animation vs. Math" featuring TSC, the sticker man.[1] But this time, we are going to further allow him to bring a new set of tools, including:

$$u, \text{ and } \frac{\mathrm{d}}{\mathrm{dt}} \tag{1}$$

where u grants TSC the ability to **control**, and $\frac{\mathrm{d}}{\mathrm{d}t}$ is the magic operator that brings to life some state variable $x \in \mathbb{R}$ describing a **system**, such that x begins to evolve with **time** t.

In this chapter, let's focus on explaining those three elements we have just mentioned: time, system, and control.

1.1. What is Time?

Is time really a thing?

First of all, let's define time. The time is measured in terms of periodic events. The sun rise and sun set, making a day. The SI unit second is defined in terms of the unperturbed transition frequency of the caesium 133 atom.[2] The positive direction of time elapse is defined in the second law of thermodynamics: "*The entropy of the universe tends to a maximum, or in loose terms, energy spreads out over time.*" The increase of the entropy of an isolated system indicates the direction of time.

But what is entropy?[3] From a macroscopic point of view, entropy (dented by S) changes whenever there is a transfer of heat:

$$\Delta S = \int_{t_0}^{t_1} \frac{-\mathrm{d}Q}{T(t)} \tag{2}$$

where $T(t)$ is the temperature when the dissipated *(note the negative sign)* heat $\mathrm{d}Q$ is made, and the differential change of heat energy $\mathrm{d}Q$ is the work done by the friction force:

$$\begin{aligned} \mathrm{d}Q &= \boldsymbol{F}_{\text{friction}} \mathrm{d}\boldsymbol{x} \text{ [J]} \\ Q &= \int_{\boldsymbol{x}_0}^{\boldsymbol{x}_1} \boldsymbol{F}_{\text{friction}} \mathrm{d}\boldsymbol{x} \text{ [J]} \end{aligned} \tag{3}$$

where $\boldsymbol{F}_{\text{friction}}$ is the friction force and $\boldsymbol{x}(t) \in \mathbb{R}^3$ is a trajectory in space. Think, if there is no longer transfer of heat in a universe, does this mean its time stops evolving?

1.2. System and its Block Diagram

"A system is a collection of interconnected parts that form a larger and more complex whole" [2]. It is widely accepted that a diagram of blocks and connecting arrows

[1]TSC stands for The Second Coming, the fourth stick figure that was created by Alan Becker.
[2]`https://en.wikipedia.org/wiki/Second`
[3]Fun fact: our yearbook is named "ENTROPY".

is useful for revealing the interconnection of different parts in a complex system. An example block diagram is shown in Fig. 1. We often call the physical system consisting of actuator and process, the **plant**.

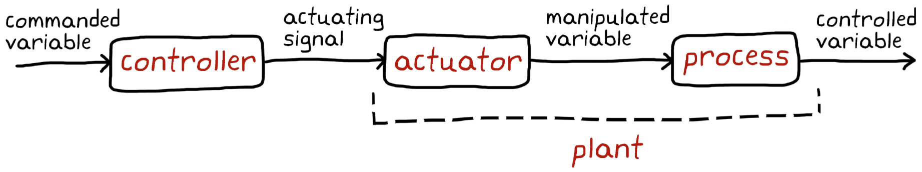


Figure 1. Block diagram showing controller, actuator, and process [2].

Generally speaking, in Fig. 1, a block is often representing an ordinary differential equation (ODE) and an arrow stands for a math operation that is rather bizarre: the convolution. We shall discuss convolution later, and for now you can take it for an integral transform (see Appendix).

1.2.1. Three Different Problems that can be Defined by Using a Block Diagram

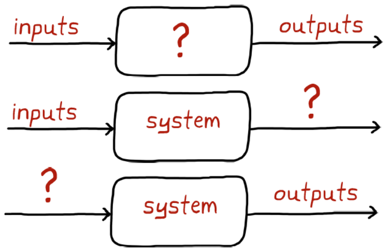


Figure 2. Three different problems arise in control systems [2].

There are three problems that can be studied and they are respectively: system identification problem, simulation problem and control problem [2], depending on which part of the control system is unknown, as shown in Fig. 2.

1.3. What is Control?

The fundamental idea of control is simply we trying to modify the dynamics of any natural process, such that its entropy might decrease or increase at a different pace different from natural evolution.

In the class, we have been playing with those math tools in (1) to study how $x(t)$ evolves when its dynamics are one of the followings

$$\frac{\mathrm{d}}{\mathrm{d}t}x = 1;\quad \frac{\mathrm{d}}{\mathrm{d}t}x = -1;\quad \frac{\mathrm{d}}{\mathrm{d}t}x = x;\quad \frac{\mathrm{d}}{\mathrm{d}t}x = -x;\quad \frac{\mathrm{d}}{\mathrm{d}t}x = x^2;\quad \frac{\mathrm{d}}{\mathrm{d}t}x = -x^2 \tag{4}$$

In math, we tend to want to avoid diverging to infinity. The only system among the above that **always** gives a non-diverging response $x(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}x = -x \Rightarrow x\left(t\right) = x\left(0\right)\mathrm{e}^{-t}$$

This system evolves with time, and its response $x(t)$ converges towards 0 rather than infinity.

One important goal of control is to make system response to not diverge to infinity. We call a non-diverging system a **stable** system. Stability is a key property of a system and it implies there is an equilibrium point in the system where the dynamics become zero $\frac{\mathrm{d}}{\mathrm{d}t}x = 0$.

On the contrary, if a system is potential to grow into infinity, the system is **unstable**. For example, $\frac{\mathrm{d}}{\mathrm{d}t}x = -x^2$ is stable if the initial state is satisfies $0 \leq x(0)$, but for other initial state values, the $x(t)$ grows toward infinity as time elapses. We call the range $0 \leq x(0)$ that attracts $x(0)$ towards to a equilibrium point the region of attraction (ROA).

1.3.1. Feedforward Control

To change the dynamics, we need to further append the tool u to any of the former discussed systems, and it yields, e.g.,

$$\frac{\mathrm{d}}{\mathrm{d}t}x = x^2 + u \tag{5}$$

We want to get rid of the term x^2 by modifying the dynamics of the original system. Assuming x is known, such control goal is simply realized by setting $u = -x^2 + v$, leading to

$$\frac{\mathrm{d}}{\mathrm{d}t}x = v \tag{6}$$

where term v is yet designed. This means we are essentially treating the term x^2 as a disturbance to the system, and u is able to cancel the effect of such disturbance. Therefore, control is subjective: y^2 are essential the dynamics of the original system but it is treated as (internal) disturbance.

1.3.2. Proportional Control

We can further modify the dynamics (6) by designing v to be $v = -x$ to get that nicely behaving system $\frac{\mathrm{d}}{\mathrm{d}t}x = -x$ again. One might complain that $x(t) = \mathrm{e}^{-t}$ is converging too slow. To make the response faster, we can simply let $v = -K_P x$, with $K_P \in \mathbb{R}_+$. This is known as proportional control. Proportional control is the basic form of negative feedback control. **Feedback** refers to the practice to feed the system state (often measured) back to the control input u. **Negative** puts an emphasis that the modified dynamics v must make sure the sign of the exponent of the response $x(t)$ should be negative, e.g., $x(t) = \mathrm{e}^{-K_P t}$.

1.3.3. Model Predictive Control

The control input u is applied to an actual system by an actuator. The drawback of the proportional control is that it does not take full ability of the actuator, and the control input is proportional to the system state, requiring the actuator is able to produce an analog signal. In practice, however, the actuator is very likely to operate in a ON-OFF fashion. In this case, it makes more sense if we figure out how long the actuator should be turned ON.

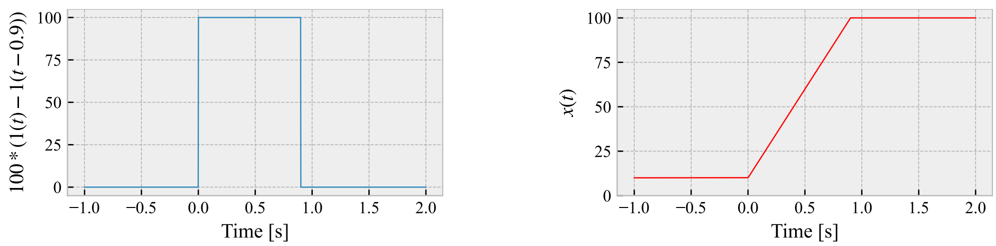
For example, let's assume a motor's speed is $x(0) = 10$ rad/s at $t = 0$ s. We are asked to make the motor speed to go up to $x(1) = 100$ rad/s at $t = 1$ s. Let's assume when the actuator is ON, the control input is 100 s^{-2}, which cannot be changed. In this case, the just-exact control input is defined by a Heaviside step function as follows

$$v = 100\left[\mathbf{1}\left(t\right) - \mathbf{1}\left(t - 0.9\right)\right] \tag{7}$$

The solution to this control system is

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}x = v \\ &\Rightarrow x = 100\int_0^t \left[\mathbf{1}\left(t\right) - \mathbf{1}\left(t - 0.9\right)\right]\mathrm{d}t \end{aligned} \tag{8}$$

which states that $x(t)$ will ramp up between $[0, 0:9]$ s, as shown in Fig. 1b. Applying



(a) Excitation in (7) (b) Response in (8)

Figure 3. Simple example showing the spirit of model predictive control.

control effort in a period of time just enough to exactly reach the goal describes the spirit of model predictive control. The exact amount of excitation applied to the system is predicted and calculated based on the model of the system.

1.3.4. Sliding Mode Control

Another idea to apply maximum control effort is the sliding mode control (SMC). In simple terms, the control law is designed to be a signum function as follows

$$v = 100 \ \mathrm{sgn}\left(100 - x\right)$$

or

$$v = \begin{cases} 100, & \text{if } 100 - x > 0 \\ -100, & \text{others} \end{cases}$$

A similar idea is the bang-bang control or hysteresis control, which sets a region of no control instead of the signum function:

$$v = \begin{cases} 100, & \text{if } 100 - x > 3 \\ -100, & \text{if } 100 - x < -3 \end{cases}$$

which gives a non-responsive region having width of 6 rad/s around the goal, 100 rad/s.

1.3.5. Dynamic Control / Integral Control / Adaptive Control

If we view the control input v as the output of some dynamical system: $\frac{\mathrm{d}}{\mathrm{d}t}v = f(x)$, the resulting control law is called dynamic control.

A well known dynamic control is the integral control, which simply designs v as an integral of the control error:

$$v = K_I \int_0^t (100 - x)\,\mathrm{d}t$$

or in a form of dynamical system:

$$\frac{\mathrm{d}}{\mathrm{d}t}v = K_I\,(100 - x)$$

where $K_I \in \mathbb{R}_+$.

Adaptive control also belongs to dynamic control, but with some further considerations for guaranteed stability.

1.3.6. Linear Quadratic Regulator (LQR)

For a simple first-order linear system:

$$\frac{\mathrm{d}}{\mathrm{d}t}x = ax$$

the LQR is

$$u = -(a + \sqrt{a^2 + \psi})x$$

with the tuning button $\psi \in \mathbb{R}$. The optimization objective is

$$J_{\mathrm{LQR}} = \int_0^{t_f} \left[\psi x(t)^2 + u^2\right] \mathrm{d}t$$

where t_f is the final time to end the control. The continuous-time differential Riccati equation will be needed to derive the control law [3, Section 22.4].

1.3.7. Using States as Control Input

For systems of higher order than first-order, using states as control input is a principle that is at the center of control. The idea has been used in (integral) back-stepping control and input-output linearizing control. In addition, the nested loop control is designed based on the same principle, but it often assumes the control transients of the inner loop are short enough.

1.4. Book Recommendations and Other Resources

In addition to the books I recommended in the class, please see others' opinions on what control theory is about. Among them, I recommend to watch Brian Douglas's video on Mar. 2nd, 2015 (`https://www.youtube.com/watch?v=oBc_BHxw78s`): "Why

Learn Control Theory". To summarize his video, control theory is widely used in electrical engineering, mechanical engineering, communication engineering, civil engineering, industrial engineering, and aerospace engineering. Control theory is essentially a subject of applied mathematics, it is building models of your systems, it is simulating model to make predictions, it is to understand dynamics and how it interact with environment, it is filtering noises and rejecting disturbances, and it is selecting, building and testing hardware to make sure it has expected performance in an unexpected environment. It is a tool that every engineer should learn to understand his/her system.

2. Mathematic Model of Linear Systems

This chapter, however, does not treat control like what we have done (in time domain) in Chapter 1, as most of those controlled dynamics can only be solved using numerical integrations.

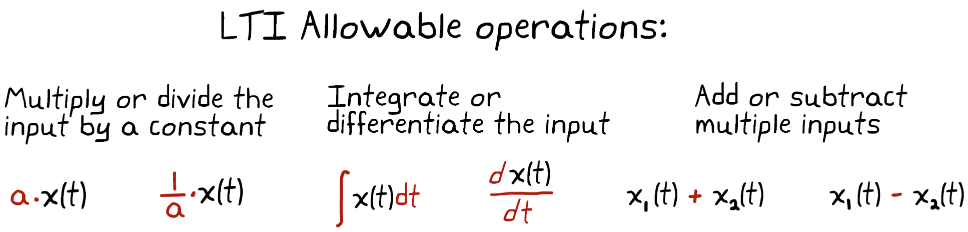


Figure 4. Allowable operations for building a linear time-invariant (LTI) system.

In order to potentially have a closed form solution, we need to study a class of simple systems originated from physics laws. From their governing ODEs, we realize they happen to only consist of the linear operations shown in Fig. 4, hence they are called linear systems. Linear system allows us to apply a series of impulse excitations one by one, and then sum up those impulse responses to produce the output of any arbitrary input, supposing an arbitrary input function can be represented as a series of impulse functions.

2.1. Across Variable and Through Variable

Across variable and through variables are concepts proposed in [1] for helping take abstract of various different physics systems. They are not very useful in this course, and it is sufficient to pay attention to the force-current analogy and force-voltage analogy.

2.2. Analogue Systems and Analogue Variables [1, Section 2.2]

We are going to show that systems that stem from different physics laws end up being very analogue in terms their dynamical equations.

2.2.1. Force-current analogy

The analogy between a damper-spring-mass system and RLC circuit is called force-current analogy.

Kirchhoff's current law states that all currents owing into a node must be equal to the current flowing out of it (as a consequence of charge conservation):

$$\frac{v}{R} + C\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{1}{L}\int_0^t v\mathrm{d}t = i\,(t)$$

where symbols are defined in [1, Fig. 2.3].

From [1, Fig. 2.2]. Newtonian mechanics state that the change of momentum equals to the sum of forces applied to the particle with mass M:

$$M\frac{\mathrm{d}^2}{\mathrm{d}t^2}y + b\frac{\mathrm{d}}{\mathrm{d}t}y + ky = F\left(t\right)$$

where M is mass, b is friction/viscous coefficient, k is a spring constant, and F is the force applied to the system. Think which location has been used to define $y = 0$?[4]

2.2.2. Force-voltage analogy

Table 1. Analogy between a particle and a charge.

Particle	Charge
r	Q
$v = \frac{\mathrm{d}r}{\mathrm{d}t}$	$i = \frac{\mathrm{d}Q}{\mathrm{d}t}$
m	L
$p = mv$	$\psi = Li$
$F = m\frac{\mathrm{d}v}{\mathrm{d}t}$	$e = L\frac{\mathrm{d}i}{\mathrm{d}t}$

The analogy between Newton's second law of motion and Faraday's law of induction is called force-current analogy, which implies there is an analogy between a particle and a charge, as summarized in Table 1:

- particle's position r and charge (that passes through a cross-sectional area) Q;
- velocity v and current i;
- inertial mass m and inductance L;
- momentum p and flux linkage ψ;
- force F and voltage e.

Furthermore, the active power in electrical circuit corresponds to the increase in velocity amplitude, and the reactive power corresponds to the change in the direction of the velocity (note velocity should be a vector in space).

2.3. System as Excitation and Response

Alternative to the math equations like ODEs, a system can also be solely defined by its inputs and outputs. This fact actually serves as the foundation of system identification. Input and output are also known as excitation and response.

For the RLC system we just introduced, when the excitation current is described by a Heaviside step function:

$$i(t) = \mathbf{1}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{others} \end{cases} \tag{9}$$

its response is

$$v\left(t\right) = K_2 \mathrm{e}^{-\alpha_2 t} \cos\left(\beta_2 t + \phi_2\right)$$

whose time domain plot is shown in Fig. 5.[5]

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[4] It is the equilibrium position of the spring where the force $ky = 0$.

[5] Think why step current excitation produces an impulse response?

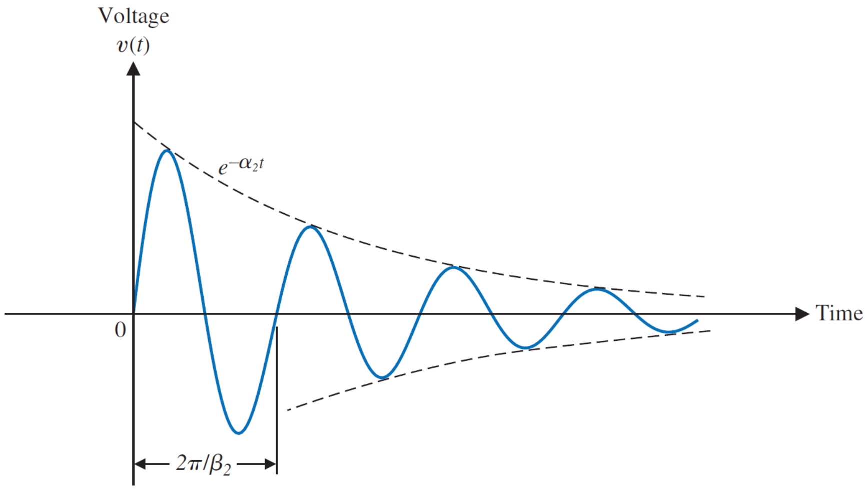


Figure 5. Impulse response of parallel RLC circuit when step current is applied [1, FIGURE 2.3].

For the damper-spring-mass system to have the same response (in waveform), we need to apply an excitation force as follows:

$$F(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{1}(t) = \delta(t) \tag{10}$$

which is known as the Dirac delta function. The Dirac delta function is also known as impulse function. It is equivalent to we using a hammer to apply an impact force of 1 N to the system.

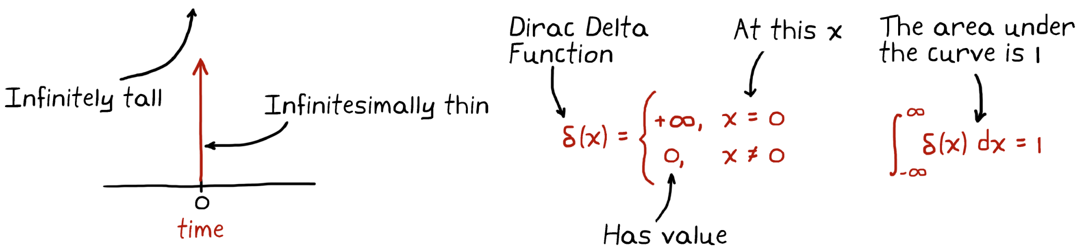


Figure 6. Dirac delta function properties [2].

The Dirac delta function is not a regular function, and some key properties are elaborated in Fig. 6. Recall Heaviside step function is the integral of Dirac delta function, and its derivative of $\mathbf{1}(t)$ at $t = 0$ is infinite. This infinite derivative does not make Heaviside step function to grow into an arbitrary large number, implying such an "impulse" has finite amount of energy.

2.3.1. Impulse Response

Impulse response is the response of an ODE when an impulse function $\delta(t)$ is applied as input. We will see very soon why impulse response is fully representative of an ODE in time-domain. In other words, the ODE and impulse response are equivalent representations of a linear system.

Note even though the response in Fig. 5 is a result of applying a step current in

(9), we still call it the impulse response of this RLC circuit, because the current is the input to a differential-and-integral equation rather than an ODE.

2.3.2. System as Operator

With the concept of impulse response, we are now ready to view system as an operator $f(\cdot)$. In simple terms, the system transfers the impulse excitation $\delta(t)$ into another signal $x(t) = f(\delta(t))$. In general case, when the input is a signal $u(t)$, the system's output becomes $x(t) = f(u(t))$. We are going to show $f(u(t))$ is a convolution of $u(t)$ and $f(\delta(t))$.

An operator should have no memory, otherwise the output will become dependent on the operator's internal states. For example, the impulse response in Fig. 5 would be different, if the capacitor is already charged to some extent at $t = 0$ s. Therefore, to view system as an operator, we need to assume the initial conditions of the system states to be null: $v(t) = \frac{\mathrm{d}}{\mathrm{d}t}v(t) = 0$, i.e., capacitor voltage and inductance current should be equal to zero.

2.3.3. Signal Analysis

Any signal can be decomposed into a series of delta functions:

$$u\left(t\right) \approx \sum_{k=-\infty}^{+\infty} \left[u\left(k\Delta t\right)\Delta t\right]\delta\left(t - k\Delta t\right) \tag{11}$$

with k an integer, where Δt is the sampling period. Motivation is from Calculus: the integral of a signal is the sum of rectangle area under the curve, so we can view any signal $x(t)$ as a bunch of thin rectangles with different heights $x(k\Delta t)$ but the same width Δt. A visualization of (11) is shown in Fig. 7.

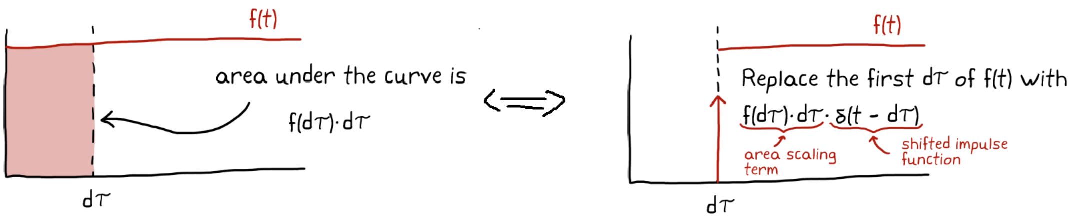


Figure 7. Signal can be equivalent represented using delta function times area under curve.

Since system can be viewed as an operator $f(\cdot)$, applying the operator to both sides of (11) yields

$$x\left(t\right) = f\left(u\left(t\right)\right) \approx \sum_{k=-\infty}^{+\infty} f\left(u\left(k\Delta t\right)\delta\left(t - k\Delta t\right)\right) \tag{12}$$

which shows that the system response of some arbitrary input function $u(t)$ can be approximately calculated as the sum of a series of impulse responses of delta function excitation of different amplitude of $u(k\Delta t)$.

We need some convenient property for the system operator $f(\cdot)$ so that we can extract the coefficient outside of the operator $f(\cdot)$ to get following results:

$$x(t) = f(u(t)) \approx \sum_{k=-\infty}^{+\infty} f(u(k\Delta t)\,\delta(t-k\Delta t)) = \sum_{k=-\infty}^{+\infty} u(k\Delta t)\, f(\delta(t-k\Delta t))$$

If this holds, we can safely say that a system can be described by its impulse response $f(\delta(t-0))$. The operator that makes the above result valid is known as the LTI operator, which has been summarized in Fig. 4 before. System's response to any input can be derived as a sum of a series scaled impulse responses, as shown in Fig. 8. This is why impulse response is important because an arbitrary response can be represented as the sum of a series of impulse responses.

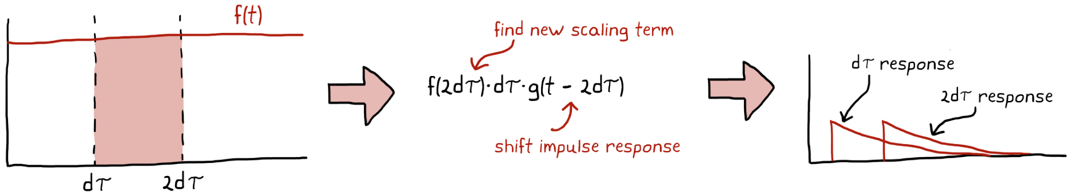


Figure 8. Apply the input slice by slice and sum up the resulting impulse response to get final response, where f in figure is input u in text, $g(\cdot)$ in figure is impulse response $f(\delta(\cdot))$ in text, and $\mathrm{d}\tau$ in figure is Δt in text.

2.4. Convolution

Finally, let's formalize the signal decomposition (11) by performing the limit $\Delta t \to 0$, so sum becomes an integral

$$(11) \overset{\Delta t \to 0}{\Longrightarrow} u(t) = \int_{-\infty}^{+\infty} u(\tau)\,\delta(t-\tau)\,\mathrm{d}\tau \triangleq \mathrm{conv}(u(t), \delta(t)) \tag{13}$$

where $k\Delta t$ has been replaced with the variable τ over which the integral (i.e., the sum) is performed, and Δt has been replaced with the differential of time, $\mathrm{d}t$. The integrand in (13) simply means to pick the value of signal $u(t)$ at $t=\tau$.

Now we are ready to define the math operation that an arrow in a block diagram (e.g., Fig. 1) represents. An arrow in a block diagram applies a system's response to another system as input in order to get its response. Therefore, the arrow convolutes the previous block's output with the impulse response of the next block.

As another useful property of convolution, a signal $u(t)$ convolutes with the delta function $\delta(t-t_1)$ would experience a time shift and becomes $u(t-t_1)$:

$$\mathrm{conv}(u(t), \delta(t-t_1)) = u(t-t_1) \tag{14}$$

In our experiment class, you will find that multiplying two polynomials together can be accomplished by performing the discrete convolution of the polynomial coefficients [2]:

```
f = [1 2 3];
g = [3 0 1];
w = conv(f,g)
```

```
It prints out: w = 3 6 10 2 3
```

2.5. Linear System

Linear system is a wider concept than linear time-invariant (LTI) system. If a system satisfies homogeneity and superposition, it is a linear system. See Fig. 9 for an intuitive definition.

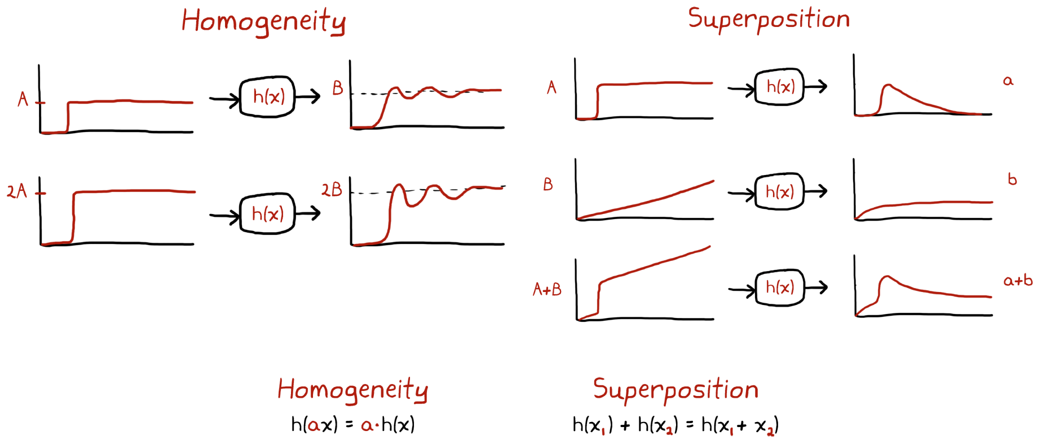


Figure 9. Homogeneity and superposition are two necessary properties of a linear system [2].

If a linear system further sanctifies the property of time-invariance (see Fig. 10), it is then called an LTI system.

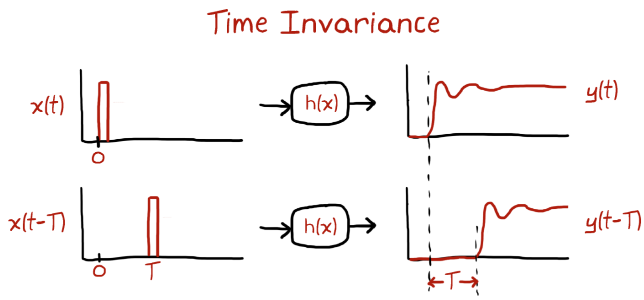


Figure 10. Time invariance states the system should give the same response regardless of the time when excitation is applied [2].

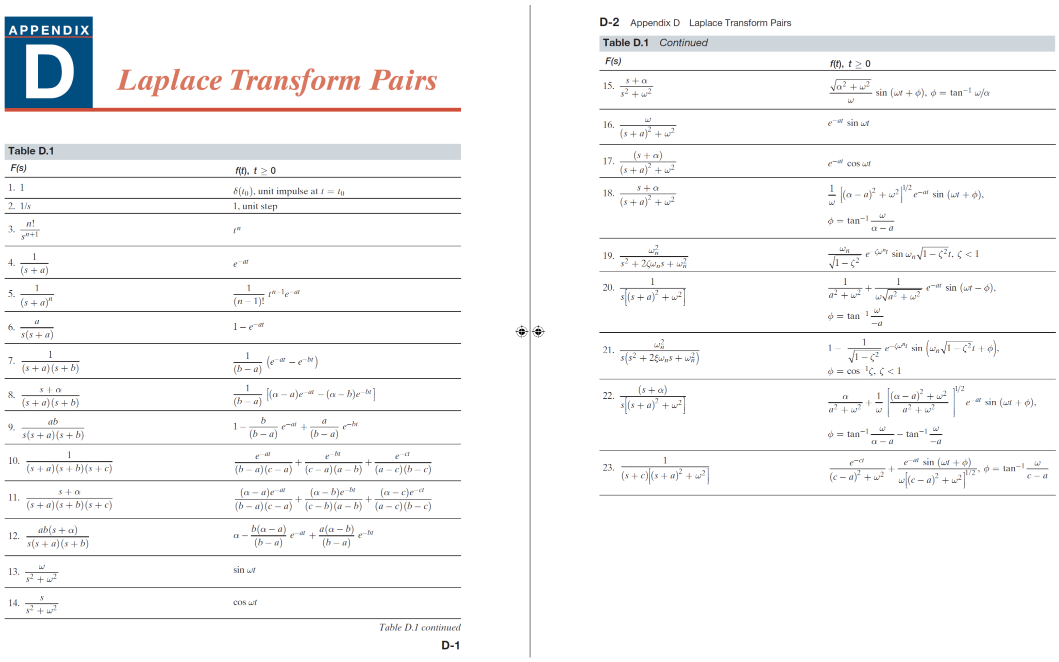
LTI results in sinusoidal fidelity, meaning any sinusoidal signal passing through a system results in a new sinusoidal signal of the same frequency (with a gain in amplitude and a shift in phase). To prove the sinusoidal fidelity, we need to learn to define the frequency response of a transfer function, or in other words, learn solving ODE using Laplace transform, see [1, Chapter 8]

3. Laplace Transform and Transfer Function

Watch Brian Douglas "What are Transfer Functions?"[6]

In Section 2.2, those derived ODEs are described using LTI operations shown in (4). We are now ready to have a different representation of those LTI systems, which is the transfer function. Transfer function is the output-input ratio after the Laplace transform is applied to an ODE with zero initial conditions.

3.1. Laplace Transform



APPENDIX D

Laplace Transform Pairs

Table D.1

F(s)	*f(t), t ≥ 0*
1. 1	$\delta(t_0)$, unit impulse at $t = t_0$
2. $1/s$	1, unit step
3. $\frac{n!}{s^{n+1}}$	t^n
4. $\frac{1}{(s+a)}$	e^{-at}
5. $\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!} t^{n-1}e^{-at}$
6. $\frac{a}{s(s+a)}$	$1 - e^{-at}$
7. $\frac{1}{(s+a)(s+b)}$	$\frac{1}{(b-a)}\left(e^{-at} - e^{-bt}\right)$
8. $\frac{s+\alpha}{(s+a)(s+b)}$	$\frac{1}{(b-a)}\left[(\alpha-a)e^{-at} - (\alpha-b)e^{-bt}\right]$
9. $\frac{ab}{s(s+a)(s+b)}$	$1 - \frac{b}{(b-a)}e^{-at} + \frac{a}{(b-a)}e^{-bt}$
10. $\frac{1}{(s+a)(s+b)(s+c)}$	$\frac{e^{-at}}{(b-a)(c-a)} + \frac{e^{-bt}}{(c-a)(a-b)} + \frac{e^{-ct}}{(a-c)(b-c)}$
11. $\frac{s+\alpha}{(s+a)(s+b)(s+c)}$	$\frac{(\alpha-a)e^{-at}}{(b-a)(c-a)} + \frac{(\alpha-b)e^{-bt}}{(c-b)(a-b)} + \frac{(\alpha-c)e^{-ct}}{(a-c)(b-c)}$
12. $\frac{ab(s+\alpha)}{s(s+a)(s+b)}$	$\alpha - \frac{b(\alpha-a)}{(b-a)}e^{-at} + \frac{a(\alpha-b)}{(b-a)}e^{-bt}$
13. $\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$
14. $\frac{s}{s^2+\omega^2}$	$\cos \omega t$

Table D.1 continued

D-1

D-2 Appendix D Laplace Transform Pairs

Table D.1 *Continued*

F(s)	*f(t), t ≥ 0*
15. $\frac{s+\alpha}{s^2+\omega^2}$	$\frac{\sqrt{\alpha^2+\omega^2}}{\omega}\sin(\omega t+\phi),\ \phi = \tan^{-1}\omega/\alpha$
16. $\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin\omega t$
17. $\frac{(s+\alpha)}{(s+a)^2+\omega^2}$	$e^{-at}\cos\omega t$
18. $\frac{s+\alpha}{(s+a)^2+\omega^2}$	$\frac{1}{\omega}\left[(\alpha-a)^2+\omega^2\right]^{1/2}e^{-at}\sin(\omega t+\phi)$, $\phi = \tan^{-1}\frac{\omega}{\alpha-a}$
19. $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega^n t}\sin\omega_n\sqrt{1-\zeta^2}t,\ \zeta<1$
20. $\frac{1}{s\left[(s+a)^2+\omega^2\right]}$	$\frac{1}{a^2+\omega^2} + \frac{1}{\omega\sqrt{a^2+\omega^2}}e^{-at}\sin(\omega t-\phi)$, $\phi = \tan^{-1}\frac{\omega}{-a}$
21. $\frac{\omega_n^2}{s\left(s^2+2\xi\omega_n s+\omega_n^2\right)}$	$1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega^n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)$, $\phi = \cos^{-1}\zeta,\ \zeta<1$
22. $\frac{(s+\alpha)}{s\left[(s+a)^2+\omega^2\right]}$	$\frac{\alpha}{a^2+\omega^2} + \frac{1}{\omega}\left[\frac{(\alpha-a)^2+\omega^2}{a^2+\omega^2}\right]^{1/2}e^{-at}\sin(\omega t+\phi)$, $\phi = \tan^{-1}\frac{\omega}{\alpha-a} - \tan^{-1}\frac{\omega}{-a}$
23. $\frac{1}{(s+c)\left[(s+a)^2+\omega^2\right]}$	$\frac{e^{-ct}}{(c-a)^2+\omega^2} + \frac{e^{-at}\sin(\omega t+\phi)}{\omega\left[(c-a)^2+\omega^2\right]^{1/2}},\ \phi = \tan^{-1}\frac{\omega}{c-a}$

Figure 11. Screenshot of the Laplace transform pairs from Appendix of [1].

In practice, Laplace transform is as simple as a look-up table, see the screenshot in Fig. 11. The minimum requirement is to remember the Laplace transform of

$$\delta(t),\ \mathbf{1}(t),\ t,\ t^k, e^{-at},\ \ \sin\omega t,\ \ \cos\omega t,\ \mathrm{e}^{-at}\sin\omega t,\ \mathrm{e}^{-at}\cos\omega t$$

where k is integer and $a, \omega \in \mathbb{R}$.

Laplace transform can be used to transform an ODE into an algebraic equation with ODE's initial conditions. For example, the damper-spring-mass system is transformed into:

$$\begin{aligned} &\mathcal{L}\left[M\frac{\mathrm{d}^2}{\mathrm{d}t^2}y + b\frac{\mathrm{d}}{\mathrm{d}t}y + ky\right] = \mathcal{L}\left[F(t)\right] \\ &M\left(s^2Y(s) - sy\left(0^-\right) - \frac{\mathrm{d}y}{\mathrm{d}t}\left(0^-\right)\right) + b\left(sY(s) - y\left(0^-\right)\right) + kY(s) = F(s) \end{aligned} \tag{15}$$

[6] https://ww2.mathworks.cn/en/videos/what-are-transfer-functions-1661846920974.html

where $y\left(0^{-}\right)$ and $\frac{\mathrm{d}y}{\mathrm{d}t}\left(0^{-}\right)$ are called the initial conditions of this second-order ODE. From (15), assuming $y\left(0^{-}\right)=y_0$ and $\frac{\mathrm{d}y}{\mathrm{d}t}\left(0^{-}\right)=0$ and solving for $Y(s)$ yield

$$Y(s)=\frac{(Ms+b)y_0+F\left(s\right)}{Ms^2+bs+k}=\frac{N(s)}{D(s)} \tag{16}$$

which can be transformed into time-domain via inverse Laplace transform. When $F(s)=0$ and $y_0\neq 0$, one possible *(depending on values of M, b, k)* impulse response is:

$$y\left(t\right)=\mathcal{L}^{-1}\left[Y\left(s\right)\right]=K_1\mathrm{e}^{-\alpha_1 t}\sin\left(\beta_1 t+\phi_1\right)$$

where α_1 and β_1 are constants associated with the parameters of the system. When $y_0=0$ and $F(t)=\delta(t)$, the solution shares a similar form but has a different initial phase angle than ϕ_1. Having more than one excitation channels makes the analysis of system response sometimes confusing, and we should stick with one input channel, preferably the input signal $F(t)$.

3.2. Transfer Function

Assuming zero initial conditions, we can derive the ratio between system output and system input for the damper-spring-mass system

$$T(s)=\frac{Y(s)}{F\left(s\right)}=\frac{1}{Ms^2+bs+k}=\frac{N(s)}{D(s)} \tag{17}$$

where $T(s)$ is the transfer function from input $F(s)$ to output $Y(s)$. Neglecting initial conditions, the differential operator s can be defined as follows

$$\begin{aligned}\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}y\left(t\right)\right]&=sY\left(s\right)-y\left(0^{-}\right)\\ \Rightarrow s&\triangleq\frac{\mathrm{d}}{\mathrm{d}t}\end{aligned}$$

In the sequel, I will always use operator s instead of taking time-derivative $\frac{\mathrm{d}}{\mathrm{d}t}$. The time-domain and s-domain functions are indicated by its variable, e.g., $y(t)$ and $Y(s)$, $\mathit{\Omega}(t)$ and $\mathit{\Omega}(s)$.

3.2.1. Relation between Impulse Function and Transfer Function

Note the Laplace transform of the impulse function is $1=\mathcal{L}[\delta(t)]$. Therefore, in s-domain, transfer function is the same as the system's impulse response. In other words, signal and system become the same concept in s-domain.

3.2.2. Pole, Zero, and Gain.

Pole is defined as the s-value that makes a transfer function to become infinity or that makes the denominator polynomial $D(s)=0$.

Zero is defined as the s-value that makes a transfer function to become zero or that makes the numerator polynomial $N(s)=0$.

Gain is defined as the transfer function value when $s = 0$ is substituted.

3.2.3. Strictly Proper

Consider a transfer function $T(s) = \frac{\text{Num}(s)}{\text{Den}(s)}$. If the order of the polynomial $\text{Den}(s)$ is equal or higher than that of $\text{Num}(s)$, we say the transfer function is **proper** [4]. In other words, define

$$T() = \lim_{s\to\infty} T(s)$$

we have $0 < |T()| < \infty$ for a proper system, and we have $0 = T()$ for a strictly proper system.

The strictly proper function can be defined as [4]

$$T_{\text{sp}}(s) = T(s) - T(\infty)$$

3.3. Block Diagram in s-Domain

We have mentioned in Section 1.2 that the block in a block diagram is often an ODE, and the arrow in a block diagram is convolution.

The block is a transfer function in s-domain, and the arrow between two connected blocks are multiplication in s-domain.

A number of blocks in a block diagram can be reduced by applying the [1, Table 2.5].

In my opinion, block diagram has one key advantage over the O.D.E.. In a block diagram, it is not necessary to give a name to all state variables, and it becomes quite easy to pay attention to those state variables that matter.

*3.4. Signal Flow Diagram and Mason's Signal-Flow Gain Formula**

Signal flow diagram is only meaningful when the block diagram has too many nodes. In that case, Mason's signal-flow gain formula can be applied to derive the linear dependence between two independent variables in the signal flow graph. While in practical scenarios as far as this course concerns, signal flow graph is the same as block diagram, so it is safe to skip it in this course.

4. Feedback Control System Characteristics

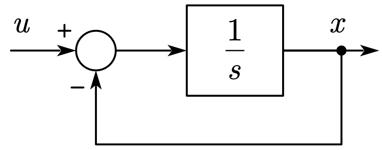


Figure 12. Motivation: the converging response system $sx = -x$ forms a loop.

By making $\frac{1}{s}$ a block in a block diagram, we realize the converging response system $sx = -x$ forms a loop, as shown in Fig. 12. This motivates us that a closed loop might be what we desire for designing a control system that does not have diverging response, which, in most cases, is true.

This section is going to answer why feedback control system is better than a system having no feedback path.

4.1. Open Loop and Closed Loop

Our goal is to make state $x(t)$ follow reference signal $r(t)$.

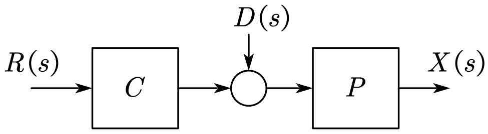


Figure 13. Open loop system.

For an open loop control system like the one in Fig. 13, the full transfer function of the control system is $\frac{X}{R} = CP$. Putting $R(s) = X(s)$ requires $CP = 1$ or $C(s) = P(s)^{-1}$. This kind of controller is known as the inverse system controller, which often is not realizable in practice.

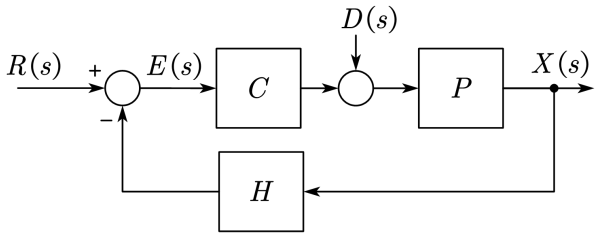


Figure 14. Closed loop system.

As shown in Fig. 14, a closed loop system, on the other hand, gives $\frac{X}{R} = \frac{CP}{1+CPH}$. Note CP is a complex number in nature. As long as $|CP|$ is large enough such that $|CP| \gg 1$, we have $X \approx R$. One realizes that the closed loop control has non-zero error $E(s) \triangleq R(s) - X(s) = \frac{1}{1+CP}$ in nature, unless $|CP| = \infty$.

4.2. Model of DC Motor

Example open loop and closed loop system with a motor can be found in FIGURE 4.12 in [1].

All motor is AC. Even though the voltage applied to the motor can be DC at its terminals, the conductors along the air gap of the motor must carrying an alternating current to maintain a steady torque. In order to provide an alternating current to the conductors, carbon brushes or power electronic devices are necessary to a motor, which realize mechanical and electronic commutation for the current-carrying conductors, respectively.

Assuming perfect conductor commutation, the dc motor consists of a first-order electrical subsystem and a first-order mechanical subsystem. Recall the analogy between the two subsystems is called force-voltage analogy.

4.2.1. Second Order Model of a DC Motor

See FIGURE 4.28 and 4.29 in [1] for a full model of a dc motor when the coil inertia is not neglectable as compared with the rotor inertia. In other words, the disk read head is light in weight. We will address second order plant later.

4.2.2. Simplified Model of a DC Motor

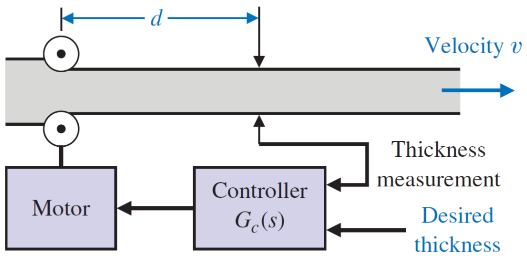


Figure 15. Steel rolling mill control system.

For this chapter, let's consider an simple example of industrial application, and compare the differences when open loop control and closed loop control. The steel rolling mill shown in Fig. 15 [1, FIGURE 4.7] has a heavy rotor, such that the pole of the coil dynamics is far away from the pole of the rotor dynamics, implying that the former can be neglected with limited influence on accuracy. In other words, we can say the mechanical pole dominates the electrical pole. As a consequence, in s-domain, the dc motor with heavy rotor can be modelled as a first-order transfer function $P(s)$:

$$P(s) = \frac{\Omega(s)}{V(s)} = \frac{g_m}{\tau_m s + 1}$$

where $\Omega(s)$ is the angular speed, $V(s)$ is the voltage applied to the motor armature terminal, g_m is the gain and τ_m is the time constant of the mechanical system.

The block diagram of a simplified DC motor model is shown in [1, FIGURE 4.8], which is a closed loop and the loop reduction $T(s) = X(s)/R(s) = CP/(1 + CPH)$ still works, but it is by definition not a closed loop control system. For a system to be feedback controlled, it has to equip some kind of sensor hardwares.

4.3. Transient Response Comparison

(todo: missing block diagram)

When the expression of a response $x(t)$ has nonzero exponential terms, it is then called a transient response. Closed loop control is able to modify the pole of the closed loop transfer function, so its transient response can be modified to have a larger exponent. To see this, let's compare between the open loop controlled dc motor and the closed loop controlled one.

- The transfer function from reference signal to angular speed using an open loop control is:

$$T(s) = \frac{\Omega(s)}{R(s)} = C(s)P(s) = K_P \frac{g_m}{\tau_m s + 1}$$

which has a real-valued pole $\lambda_1 = -\frac{1}{\tau_m}$.

- The transfer function from reference signal to angular speed using an closed loop control is:

$$T(s) = \frac{\Omega(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{K_P \frac{g_m}{\tau_m s+1}}{1 + K_P \frac{g_m}{\tau_m s+1}} = \frac{K_P g_m}{\tau_m s + 1 + K_P g_m}$$

which has a real-valued pole $\lambda_1 = -\frac{1+K_P g_m}{\tau_m}$.

Their time-domain solutions of impulse excitation share the form of

$$\Omega(t) = \mathcal{L}^{-1}\left\{T(s) \times \mathcal{L}[\delta(t)]\right\} = K_P g_m \mathrm{e}^{\lambda_1 t}$$

Who has a larger λ_1 has a faster transient response.

Their s-domain step responses can be derived by substituting $R(s) = 1/s$:

$$\Omega(s) = T(s) \times \mathcal{L}[\mathbf{1}(t)] = \begin{cases} \frac{K_P g_m}{\tau_m s+1}\frac{1}{s}, & \text{open loop} \\ \frac{K_P g_m}{\tau_m s+1+K_P g_m}\frac{1}{s}, & \text{closed loop} \end{cases} = \frac{g_1}{s-\lambda_1}\frac{1}{s}$$

$$g_1 = \frac{K_P g_m}{\tau_m}, \quad \lambda_1 = \begin{cases} \frac{-1}{\tau_m}, & \text{open loop} \\ -\frac{1+K_P g_m}{\tau_m}, & \text{closed loop} \end{cases}$$

For open loop control, we have speed response

$$\Omega(s) = T(s) \times \mathcal{L}[\mathbf{1}(t)] = \frac{g_1}{s-\lambda_1}\frac{1}{s} = \frac{A}{s} - \frac{B}{s-\lambda_1} = \frac{As - A\lambda_1 - sB}{s(s-\lambda_1)} = \frac{g_1}{-\lambda_1}\left(\frac{1}{s} - \frac{1}{s-\lambda_1}\right)$$

$$\Rightarrow \left\{ \begin{array}{c} A = B \\ -A\lambda_1 = g_1 \end{array} \right. \Rightarrow A = B = \frac{g_1}{-\lambda_1}$$

which gives a time-domain response as

$$\Omega(t) = \frac{g_1}{-\lambda_1}\left(1 - \mathrm{e}^{\lambda_1 t}\right)$$

See FIGURE 4.13 to have a visualization of the transient response comparison.

4.4. Steady State Error Comparison

With the step response available as $\Omega(t) = \frac{g_1}{-\lambda_1}\left(1 - \mathrm{e}^{\lambda_1 t}\right)$, we can get steady state value by setting $t = \infty$ to get

$$\Omega(\infty) = \frac{g_1}{-\lambda_1}\left(1 - \mathrm{e}^{\lambda_1 \infty}\right) = \frac{g_1}{-\lambda_1}$$

which is equivalent to applying final value theorem to the s-domain solution:

$$\mathit{\Omega}\left(t\right)|_{t=\infty} = \lim_{s\to 0} s\mathit{\Omega}\left(s\right) = \frac{g_1}{-\lambda_1} s\left(\frac{1}{s} - \frac{1}{s-\lambda_1}\right) = \frac{g_1}{-\lambda_1}$$

4.4.1. Steady State Error of a Step Response

Recall our goal is to make state $x(t) = \mathit{\Omega}(t)$ follow reference signal $r(t)$. It is convenient to evaluate the error signal $e(t) = r(t) - x(t)$ instead. Its s-domain step response is:

$$E\left(s\right) = R\left(s\right) - \mathit{\Omega}\left(s\right) = \frac{1}{s} - \frac{g_1}{-\lambda_1}\left(\frac{1}{s} - \frac{1}{s-\lambda_1}\right) \tag{18}$$

The steady state value of the error signal is

$$e\left(\infty\right) = \lim_{s\to 0} s\left[R\left(s\right) - \mathit{\Omega}\left(s\right)\right] = 1 - \frac{g_1}{-\lambda_1} \tag{19}$$

In order to have zero steady state error, such that $x = \mathit{\Omega}$ coincides with $r(t)$ when t approaches infinity, we need to make $g_1/\lambda_1 = 1$. Think what should the controller $C(s)$ be to make this happen.

For open loop controller, the gain K_P must be tuned to ensure $g_1/\lambda_1 = 1$, assuming the parameters of the system are not time-varying.

For closed loop control, a simple trick to have zero steady state (step) error is to use a infinity loop gain $L(0) = C(0)P(0) = \infty$. This results in an proportional-integral (PI) controller $C(s) = K_P + K_I/s$ with $C(0) = \infty$.

4.4.2. Steady State Error of a Ramp Response

Using the same proportional controller $C(s) = K_P$, the steady state error of a ramp excitation $r(t) = t$ or $R(s) = 1/s^2$ is

$$e\left(\infty\right) = \lim_{s\to 0} sE\left(s\right) = s\left[\frac{1}{s^2} - \frac{g_1}{-\lambda_1}\left(\frac{1}{s} - \frac{1}{s-\lambda_1}\right)\right] = \infty \tag{20}$$

which means the proportional control cannot follow a ramping reference signal and its tracking error grows with time.

4.5. Frequency Response

Let's further extend the concept of steady state evaluation of the response to arbitrary sinusoidal inputs $R(s) = \frac{\omega}{s^2+\omega^2}$.

Frequency response is the system's steady state response to sinusoidal inputs, in which the transients are not important thus shall be neglected. To this end, replacing $s = \sigma + j\omega$ with $j\omega$ in $T(s)R(s)$ provides steady state frequency response. Alternatively, we can prove above "$\sigma + j\omega \to j\omega$" trick by considering the following example. Let the excitation be $r(t) = \sin\omega t$ or $R\left(s\right) = \frac{\omega}{s^2+\omega^2}$, a second order system's response is

$$Y\left(s\right) = T\left(s\right)R\left(s\right) = \frac{N\left(s\right)}{\left(s-\lambda_1\right)\left(s-\lambda_2\right)}\frac{\omega}{s^2+\omega^2} = \frac{g_1}{s-\lambda_1} + \frac{g_2}{s-\lambda_2} + \frac{\alpha s + \beta\omega}{s^2+\omega^2} \tag{21}$$

where α and β are coefficients to be determined from the partial fraction expansion and are related to coefficients of $N(s)$ and poles λ_1, λ_2. The time-domain response is

$$\begin{aligned} y(t) &= g_1 e^{\lambda_1 t} + g_2 e^{\lambda_2 t} + \mathcal{L}^{-1}\left[\frac{\alpha s + \beta\omega}{s^2+\omega^2}\right] \\ &= g_1 e^{\lambda_1 t} + g_2 e^{\lambda_2 t} + \alpha\cos\omega t + \beta\sin\omega t \end{aligned}$$

In the limit $t \to \infty$, the first two exponential terms vanish, and we have[7]

$$\begin{aligned} y(t)\,|_{t\to\infty} &= \mathcal{L}^{-1}\left[\frac{\alpha s + \beta\omega}{s^2+\omega^2}\right] = \alpha\cos\omega t + \beta\sin\omega t \\ &= \operatorname{sgn}(\alpha)\sqrt{\alpha^2+\beta^2}\cos\left[\omega t + \arctan\frac{-\beta}{\alpha}\right] \\ &= |T(j\omega)|\sin\left[\omega t + \angle T(j\omega)\right] \\ &\Leftarrow \begin{cases} |T(j\omega)| = \operatorname{sgn}(\alpha)\sqrt{\alpha^2+\beta^2} \\ \angle T(j\omega) = \arctan\frac{-\beta}{\alpha} \end{cases} \end{aligned}$$

where $\alpha \neq 0$. Final value theorem cannot be used to attain $y(t)$ as $t \to \infty$, because final value theorem can only be applied to a response $Y(s)$ when $Y(j\omega)$ exists when $j\omega \neq 0$. A non-rigorous proof to show that the last row of equation holds is as follows.

$$\begin{aligned} & Y(s) = T(s)R(s) = T(s)\frac{\omega}{s^2+\omega^2} = 0 + 0 + \frac{\alpha s + \beta\omega}{s^2+\omega^2} \\ & \Rightarrow T(s) = \frac{\frac{\alpha s+\beta\omega}{s^2+\omega^2}}{\frac{\omega}{s^2+\omega^2}} = \frac{\alpha s + \beta\omega}{\omega} \\ & \Rightarrow T(j\omega) = \frac{\alpha j\omega + \beta\omega}{\omega} = \frac{\alpha j + \beta}{1} \\ & \Rightarrow |T(j\omega)|^2 = \alpha^2 + \beta^2 \end{aligned}$$

4.6. Foes

So far, the sole input to our system is the reference signal. In a practice, however, there are at least three input channels to a closed loop control system.

There are undesired phenomena present in a control system, including external disturbance *[measurement noise $n(t)$ and unknown input $d(t)$]* and internal disturbance *[parameter uncertainty ΔP]*, leading to degrade in control performance, e.g., causing a remarkable steady state error.

The ultimate goal of the control system design is to keep the reference tracking ability while rejecting all those disturbances to the system. To this end, we need to first introduce the idea of sensitivity function, in order to describe how sensitive to disturbance is our control system.

[7] Trigonometry identity used here can be found at `https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Sine_and_cosine`

4.7. Sensitivity Function

The internal disturbance ΔP (which is often a parameter uncertainty) causes a deviation ΔT from T. A metric that evaluates how much perturbation it causes to our system is the sensitivity function, defined by

$$S = \frac{\Delta T(s)/T(s)}{\Delta P(s)/P(s)} \tag{22}$$

where the deviation can be calculated as per definition:

$$\Delta T(s) = \frac{C(P+\Delta P)}{1+C(P+\Delta P)} - \frac{CP}{1+CP}$$

In the limit, small incremental changes leads to following de
nition:

$$S_P^T = \frac{\partial T(s)/T(s)}{\partial P(s)/P(s)} = \frac{\partial \ln T(s)}{\partial \ln P(s)} \tag{23}$$

where the following calculus relation has been substituted:

$$\frac{\mathrm{d}x}{x} = \mathrm{d}\ln x \;\Leftrightarrow\; \int \frac{\mathrm{d}x}{x} = \ln x$$

When control system transfer function is $T(s) = \frac{CP}{1+CP}$, the sensitivity function is

$$S_P^T = \frac{1}{1+CP} \tag{24}$$

When control system transfer function is $T(s) = CP$, the sensitivity function is

$$S_P^T = 1 \tag{25}$$

This is the second advantage of using a closed loop control system. The amplitude of the sensitivity function is subject to a factor that is less than 1. Also, it is important to use a **negative** feedback loop, otherwise the denominator in (24) becomes $1 - CP$, making $|S_P^T| > 1$.

In most cases, the transfer function $T(s)$ is a rational fraction:

$$T(s;\alpha) = \frac{N(s;\alpha)}{D(s;\alpha)}$$

where α is a parameter that experiences variation, and N and D are numerator and denominator polynomials in s. As a result, $T(s)$'s sensitivity with respect to parameter α becomes

$$S_\alpha^T = \frac{\partial \ln \mathrm{T}}{\partial \ln \alpha} = \left.\frac{\partial \ln \mathrm{N}}{\partial \ln \alpha}\right|_{\alpha=\alpha_0} - \left.\frac{\partial \ln \mathrm{D}}{\partial \ln \alpha}\right|_{\alpha=\alpha_0} = S_\alpha^N - S_\alpha^D$$

where α_0 is the nominal value of α.

4.8. Gang of Six

Watch video of Douglas "Gang of Six".[8]

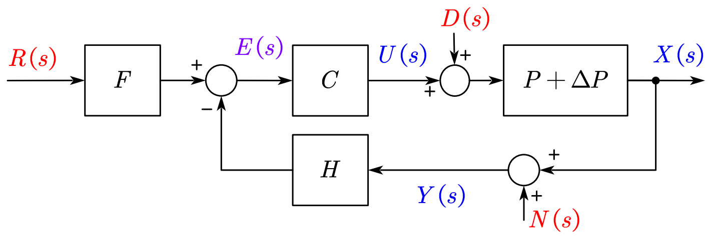


Figure 16. Closed loop system with three input channels.

The closed-loop control system shown in Fig. 16 has considered all three different foes that perturb the control performance. Assuming $H(s) = 1$ in Fig. 16, we can derive the following relationships among the input signals and state/output/input/error:

$$X = \frac{CP}{1+CP}FR + \frac{P}{1+CP}D - \frac{CP}{1+CP}N \tag{26a}$$

$$Y = \frac{CP}{1+CP}FR + \frac{P}{1+CP}D + \frac{1}{1+CP}N \tag{26b}$$

$$U = \frac{C}{1+CP}FR - \frac{CP}{1+CP}D - \frac{C}{1+CP}N \tag{26c}$$

$$E = \frac{1}{1+CP}FR + \frac{P}{1+CP}D - \frac{CP}{1+CP}N \tag{26d}$$

When $F(s) = 1$, the gang of six is reduced as gang of four. We define loop gain as $L \triangleq CP$, and the definitions of the four gang members are now in order:

- Sensitivity function $S = 1/(1 + L)$.
- Complementary sensitivity function is $1 - S$.
- Disturbance sensitivity function is PS.
- Noise sensitivity function is CS.

See also Fig. 17.

4.9. Error Signal Analysis

Assuming feedforward block $F = 1$, (26d) is rewritten in terms of sensitivity function S as follows:

$$\begin{aligned} E &= \frac{1}{1+L}R - \frac{P}{1+L}D + \frac{L}{1+L}N \\ &= S \times R - PS \times D + (1 - S) \times N \end{aligned} \tag{27}$$

[8] https://ww2.mathworks.cn/en/videos/control-systems-in-practice-part-8-the-gang-of-six-in-control-theor html

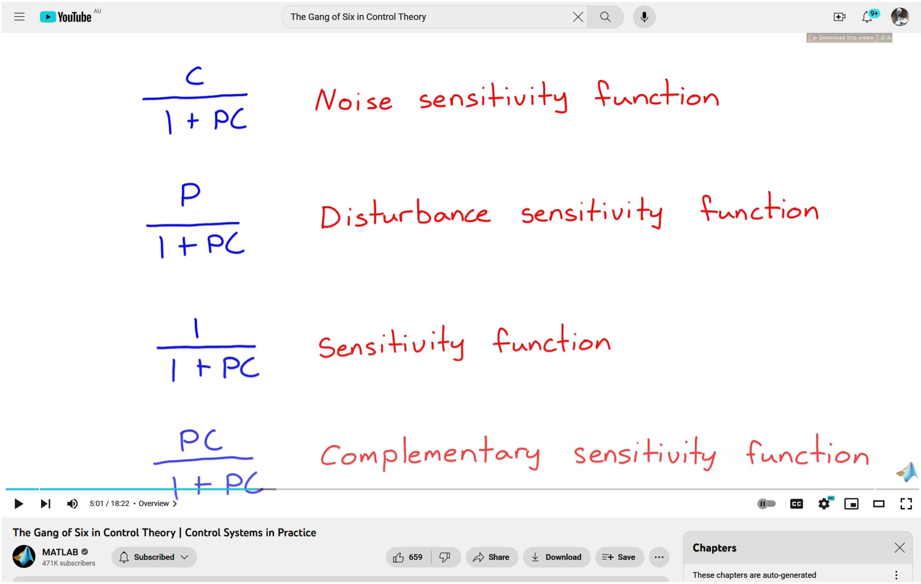


Figure 17. Gang of four by Brian Douglas (Youtube: `b8v8scghh8`)

4.10. Disturbance Rejection

Using the principle of superposition, let's analyze the effect of external disturbance input D by putting $R = N = 0$:

$$E(s) = -\frac{P}{1+CP}D = -\frac{P}{1+L}D = -PS \times D \tag{28}$$

The disturbance will be rejected if we use a "large" loop gain. Or in rigorous terms, disturbance rejection occurs whenever s is making the gain $|S(s)P(s)|$ small enough.

4.11. Reference Tracking

The error due to change in reference is

$$E(s) = S \times R = \frac{1}{1+L} \times R \tag{29}$$

which suggests that "large" loop gain also minimizes the tracking error.

4.12. Noise Attenuation

The complementary sensitivity function $1 - S$ shows how noise is attenuated in error. Unfortunately, the error excited by noise $N(s)$

$$E(s) = (1 - S) \times N(s) = \frac{L}{1+L} \times N(s) \tag{30}$$

is less attenuated when a "large" loop gain $|L|$ is used. We conclude that there is a compromise between the reference tracking and noise attenuation., because

$$S(s) + (1 - S(s)) \equiv 1 \tag{31}$$

4.13. Frequency Response of The Gang Members

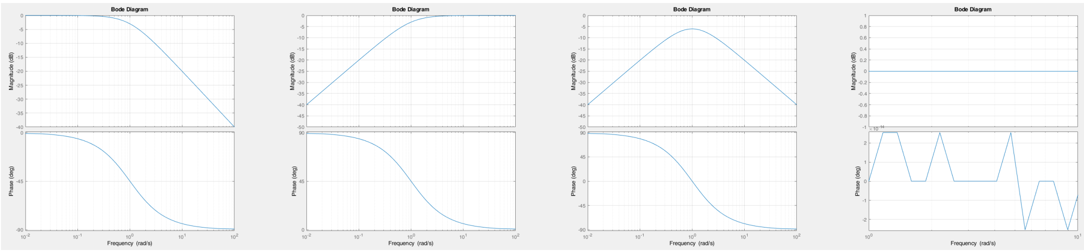


Figure 18. Sensitivity function frequency response. From left to right are noise to error $1 - S$, reference to error S, disturbance to error PS, and CS (which equals to 1).

The gang members' frequency responses are important for practical control system design. The frequency response in logarithm plot is known as Bode plot. Let's consider a simple example with the aid of Matlab.

```
P = tf([1], [1, 1])
C = tf([1, 1], [1 0])
subplot(141); bode(C*P/(1+C*P)); grid
subplot(142); bode(1/(1+C*P)); grid
subplot(143); bode(P/(1+C*P)); grid
subplot(144); bode(C/(1+C*P)); grid
```

where the controller $C(s) = (s+1)/s$ is a PI regulator.

4.14. Sensitivity to Parameter Variation

Uncertainty ΔP affects all three channels of the input. We will take reference input for illustration. Assume $D = N = 0$, and substitute $P + \Delta P$ for P in error analysis (27) yields

$$\begin{aligned} E + \Delta E &= \frac{1}{1 + C\left(P + \Delta P\right)} R \\ \Rightarrow \Delta E &= \left(\frac{1}{1 + C\left(P + \Delta P\right)} - \frac{1}{1 + CP} \right) R \\ &\approx \frac{1}{1 + CP} \frac{\Delta P}{P} R \\ &= S \frac{\Delta P}{P} R \end{aligned} \tag{32}$$

which reveals the reason why S is called as sensitivity function.

4.15. Q & A

Some questions and answers are listed as a brief summary.

Q1: What is a transfer function?

A1: A transfer function is

- a polynomial fraction with a complex variable $s \in \mathbb{C}$;
- an LTI system;
- the system's impulse response in s-domain.

5. Feedback Control System Performance

Watch Brian Douglas "The Step Response"[9]

This chapter develops performance metric as design requirements for feedback control system design.

5.1. Test signals

Standard test signals include $\delta(t)$, $\mathbf{1}(t)$, t, t^2, and $\sin\omega t$.

5.2. Standard Second Order System

Closed loop transfer function $T(s)$

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{33}$$

is called the standard second order system, where ω_n is **natural frequency**, and ζ is the **damping ratio**. "Standard" puts an emphasis on the absence of zeros. Can you guess what system's open loop transfer function is?[10]

The characteristic equation of transfer function $T(s)$ is its denominator polynomial:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{34}$$

whose roots are

$$\begin{aligned}(34) \Rightarrow s &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &\triangleq -\tau^{-1} \pm \omega_n\beta\end{aligned} \tag{35}$$

where $\beta \triangleq \sqrt{\zeta^2 - 1}$, and τ is the time constant defined as

$$\tau \triangleq \frac{1}{\zeta\omega_n} \tag{36}$$

which suggests the decaying exponential term in the transient response has an exponent of $-t/\tau$.

The impulse response of a second order system can be obtained using inverse Laplace transform:

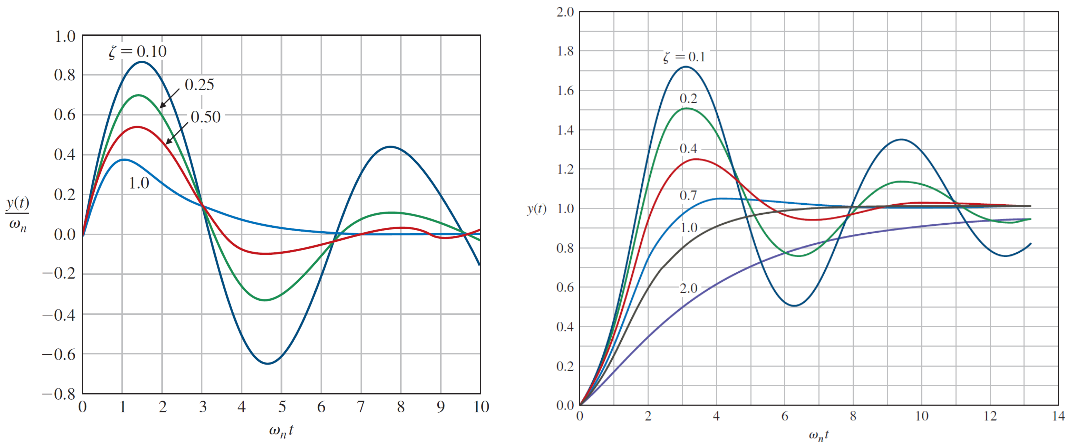
$$\mathcal{L}^{-1}\left[\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right] = \frac{\omega_n}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right), \text{ only when } \zeta < 1 \tag{37}$$

as shown in Fig. 19a. Note the y-axis is re-scaled as y/ω_n and t-axis is re-scaled as $\omega_n t$. Re-scaling the axes will keep the waveform invariant when ω_n changes.

---

[9] https://ww2.mathworks.cn/en/videos/control-systems-in-practice-part-9-the-step-response-1593067191882.html

[10] Hint: divide both numerator and denominator with $s^2 + 2\zeta\omega_n s$.



(a) Impulse responses. (b) Step responses.

Figure 19. Responses of a standard second order system [1].

The step response of a second order system is

$$\mathcal{L}^{-1}\left[\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\frac{1}{s}\right] = 1 - \frac{1}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta\right),\ \zeta<1 \tag{38}$$

as shown in Fig. 19b. Note t-axis is re-scaled as $\omega_n t$, implying that the waveform shape of the response is not dependent on ω_n. To see this, you can play with the GUI I made with two sliders for adjusting the values of ω_n and ζ, using python packages DearPyGUI and python-control.[11]

From Fig. 19, responses having higher values than final value $y(\infty)$ is said to have an overshoot thus **underdamped**. If the peak of the response is less than the final value $y(\infty)$, then there is no overshoot and the system is said to be **overdamped**. The system is said **critically damped** when the peak value is equal to the final value.

5.3. Performance Metrics

The swiftness of the response is measured by the rise time T_r and the peak time T_p [1]. The 0–100% rise time T_r is mainly used for underdamped system with an overshoot. The 10–90% rise time T_{r1} becomes useful for overdamped system because the time spent in 90–100% would be remarkably long for an overdamped system, which is unfair to be accounted for rising time.

The settling time, T_s, is defined as the time required for the system to settle within a certain percentage δ of the input amplitude (i.e., the command or reference) [1]. In [1], settling time is approximately estimated from

$$\mathrm{e}^{-\zeta\omega_n T_s} < 2\% \tag{39}$$

[11] `https://github.com/horychen/ee160/blob/master/step_response_visual.py`

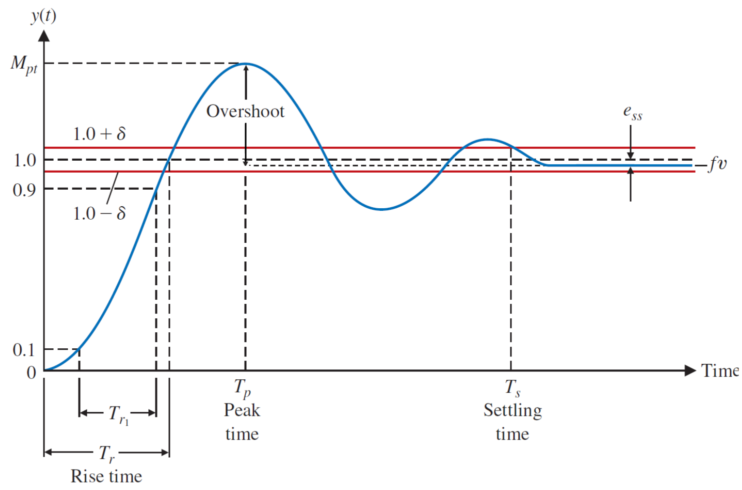


Figure 20. Graphical definitions of the performance metrics in a transient step response.

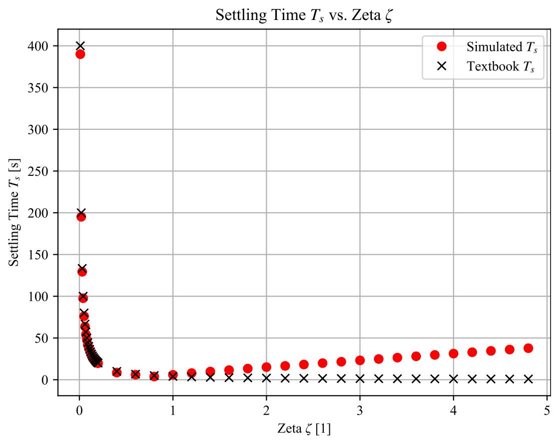


Figure 21. Settling time T_s versus damping ratio ζ.
Premise: step response and standard second order system.

which gives an estimated settling time that is only related to the product of ζ and ω_n:

$$T_s = \frac{4}{\zeta \omega_n} \tag{40}$$

when ζ is not too large. But how large is too large? To have a feel about this, I have used a python snippet to visualize the comparison between the settling time read from graph (results of simulation) and the estimated one $T_s = \frac{4}{\zeta \omega_n}$. The results are shown in Fig. 21. The code snippet is listed here.

```
import control
from pylab import np, plt, mpl
mpl.rc('font', family='Times New Roman', size=10.0)
mpl.rcParams['mathtext.fontset'] = 'stix'
mpl.rc('legend', fontsize=10)
omega_n = 1.0
def get_attr(zeta, key='SettlingTime'):
    T = control.TransferFunction([omega_n ** 2], [1, 2 * zeta *
        omega_n, omega_n ** 2])
```

```
    return control.step_info(T)['SettlingTime']
zeta_list = np.concatenate(( np.arange(0.0, 0.2, 0.01), np.arange(0.2, 5, 0.2) ))
approximated_settling_time_list = [4 / zeta / omega_n for zeta in zeta_list]
settling_time_list = [get_attr(zeta, key='SettlingTime') for zeta in zeta_list]
plt.plot(zeta_list, settling_time_list, 'o', color='red', label=r'True $T_s$')
plt.plot(zeta_list, approximated_settling_time_list, 'x', color='black', label=r'Estimated $T_s$')
plt.xlabel(r'Zeta $\zeta$ [1]')
plt.ylabel(r'Settling Time $T_s$ [s]')
plt.title(r'Settling Time $T_s$ vs. Zeta $\zeta$')
plt.grid(); plt.legend(); plt.show()
plt.savefig(r'D:\horyc\Desktop\SettlingTimeVsZeta.pdf', dpi=400, bbox_inches='tight', pad_inches=0)
```

From Fig. 21, we learn that the T_s estimate in (40) works quite well when $\zeta < 1$. Therefore, it is safe to say settling time is equal to four times system time constants $4\tau = 4/(\zeta\omega_n)$ [1].

The **steady state error** e_{ss} can be also read on Fig. 20. The response's magnitude at peak time is denoted as M_{pt}. The **percent overshoot** (denoted by P.O.) is defined as

$$\text{P.O.} = \frac{M_{pt} - fv}{fv} \times 100\%$$

where fv is the **final value** of the response. Final value can be calculated as $fv = r(\infty) - e_{ss} = y(\infty)$, where $r(t)$ is reference signal and $e_{ss} = e(\infty)$ is error signal.

For an overdamped system with $\zeta > 1$, theoretically speaking, its peak time can only be read when $t \to \infty$. When $\zeta < 1$, the peak time can be calculated by differentiating the response $y(t)$ with respect to time to get[12]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}y\left(t\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[1-\frac{1}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta\right)\right] \\ \Rightarrow\frac{\mathrm{d}}{\mathrm{d}t}y\left(t\right) &= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\end{aligned} \tag{41}$$

Putting (41) to zero and solving the equation yield an exact expression for peak time

$$\begin{aligned}0 &= \omega_n\mathrm{e}^{-\zeta\omega_n t}\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) \\ \Rightarrow T_p &= \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\end{aligned} \tag{42}$$

Since this is a theoretical solution, we can in turn use it to validate the accuracy of our numerical simulation. To this end, the previous python script for validating settling time accuracy can be modified to compare between the simulated peak time and the theoretical one in (42). The results are shown in Fig. 22a. From Fig. 22a, the simulated peak time becomes quite off when ζ becomes larger than 0.9. Furthermore, when $\zeta < 0.3$, the zoomed-in plot in Fig. 22b suggests there is remarkable error in simulated peak time—it becomes almost invariant to ζ. This simulation error is probably due to large simulation steps.

[12]The detailed derivation is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}y\left(t\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta\right)\right]$$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}t}y\left(t\right) = -\left(\frac{-\zeta\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\right)\sin\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta\right)-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\omega_n\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta\right)\right]$$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}t}y\left(t\right) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left\{\zeta\sin\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta\right)-\left[\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta\right)\right]\right\}$$

$$\Rightarrow\frac{\frac{\mathrm{d}}{\mathrm{d}t}y\left(t\right)}{\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}} = \zeta\sin\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta\right)-\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta\right)$$

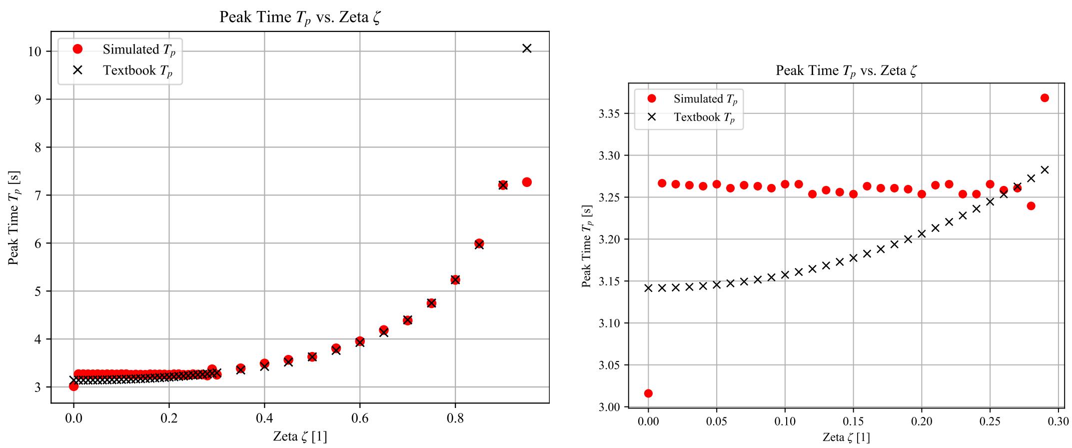
$$\Rightarrow\frac{\frac{\mathrm{d}}{\mathrm{d}t}y\left(t\right)}{\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}} = \sqrt{\zeta^2+1-\zeta^2}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta+\arctan\frac{-\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\Rightarrow\frac{\frac{\mathrm{d}}{\mathrm{d}t}y\left(t\right)}{\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}} = \sin\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta+\arctan\frac{-\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\text{note }\arctan\frac{-\sqrt{1-\zeta^2}}{\zeta} = -\arccos\frac{\zeta}{1}$$

$$\Rightarrow\frac{\frac{\mathrm{d}}{\mathrm{d}t}y\left(t\right)}{\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}} = \sin\left(\omega_n\sqrt{1-\zeta^2}t+\cos^{-1}\zeta-\arccos\frac{\zeta}{1}\right)$$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}t}y\left(t\right) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)$$



(a) Zoomed out. (b) Zoomed in.

Figure 22. Peak time T_p versus damping ratio ζ. Premise: step response and standard second order system.

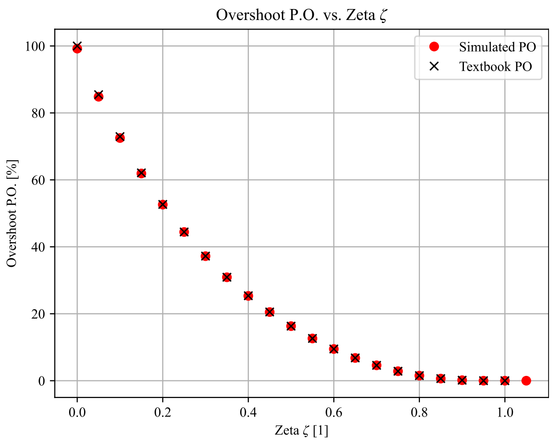


Figure 23. Percentage overshoot P.O. versus damping ratio ζ. Premise: step response and standard second order system.

The peak response at the estimated peak time (42) is

$$M_{pt} = 1 + \mathrm{e}^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}}$$

and the resulting estimated percentage overshoot is

$$\text{P.O.} = 100 \times \mathrm{e}^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}} \quad [\%]$$

which is measured from the command $r(\infty)$ to the peak magnitude M_{pt}, thus there is an minor approximation because the overshoot should be measured from final value $y(\infty)$ to M_{pt}. One can also validate accuracy of P.O.'s expression using the python script, and the results are shown in Fig. 23. Note the overshoot is independent on natural frequency ω_n.

Comparing between Fig. 22a and Fig. 23, one realizes that the textbook peak time T_p and overshoot P.O. are conflicting performance metrics.

Comparing between Fig. 21 and Fig. 23, one finds that settling time T_s and overshoot P.O. both decrease as damping ratio increases. In a physical system, damping ratio often increases when the value of the dissipator component (e.g., damper and resistor) increases.

As a general design guideline, we need to first pick a ζ value to meet the overshoot requirement. Then the swiftness of the response can be tuned by picking a reasonable ω_n value.

5.4. Dominant Poles

When a system has a large negative real pole, or a pair of conjugate complex poles with large negative real part, it is said those poles far away from the imaginary axis are dominated by other poles that are significantly closer to the imaginary axis. For example, If the far pole $|\lambda_3| \geq 10|\zeta\omega_n|$, then the following approximation is reasonable

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{1}{(s - \lambda_3)} \approx \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{43}$$

The roots of its characteristic equation $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ are called dominant roots of this third order system.

The concept of dominant poles are valid only when there is no zero near the dominant poles. This can be understood by considering an extreme case in which the zero-pole cancellation occurs. Generally speaking, The poles determine the particular **response modes** (i.e., terms of different exponent) that will be present, and the zeros establish the relative weightings of the individual mode functions. In other words, moving a zero closer to a specific pole will reduce the relative contribution to the output response [1].

5.5. Complex Plane Root Location and the Transient Response

Since the roots of the characteristic equation are complex number, we can mark them in the complex plane (a.k.a. s-plane).

A step response of a general transfer function can be converted into a partial fraction expansion [1, Equation (5.21)], which implies that the transient response consists of terms that have different modes, depending on the location of the characteristic equation roots.

The relation between complex plane root location and the transient response is revealed in [1, FIGURE 5.17]:

- Along the real axis of the s-plane from $-\infty$ to 0, the convergence rate of the response becomes slower and stops to converge when the root is located on the imaginary axis.
- Along the real axis of the s-plane from 0 to ∞, the diverging speed of the response becomes faster and faster. The response is not bounded by any finite number. Unbounded response is also said to be unstable.
- Along the imaginary axis of the s-plane from $j0$ to $j\infty$, the oscillating frequency of the response becomes higher and higher.

5.6. System Types in Terms of Steady State Error

See Table 5.2 in [1].

5.7. Discussions

For higher order systems more than two dominant poles, the performance indices developed in this chapter cannot be directly applied, and the **stability** becomes the highest priority.

6. Stability

A **stable** system is a dynamic system with a bounded response to a bounded input [1]. This input-output property is known as bounded input and bounded output (BIBO) stability [4, Theorem 3.1]. There are other definitions of stability.[13]

A stable response is, therefore, a broader concept that includes converging response and bounded response. A typical converging response is $x(t) = \mathrm{e}^{-t}$ or $x(t) = 1 - \mathrm{e}^{-t}$, and a typical bounded response is $x(t) = 0.707\sin(t+\pi/4)$. So far, we are only capable to evaluate the stability of a system by looking at its response with respect to a certain input, such as impulse, step or sinusoidal excitation.

Since this chapter, we are going to learn a bunch of tools for analyzing the stability of a control system, including Routh-Hurwitz stability criterion, root locus, Bode plot, Nyquist plot, and Nichols plot. Those tools are especially useful for design of a controller that stabilizes an unstable open loop system. Examples of unstable open loop systems include air-plane, motorcycle, maglev iron ball, inverted pendulum, and bipedal robot.

6.1. Stability and Root Locations

There are three stability results depending on the pole locations. [1].

(1) A system is **stable** if all the poles of the transfer function are in the left hand s-plane (LHP).
(2) A system is said to have **marginal stability**, if the characteristic equation has **simple roots**[14] on the imaginary axis (e.g., $s = \pm j\omega_u$) with all other roots in the left half-plane. Its steady-state output will be sustained oscillations for a bounded input as long as the input does not contain a frequency that matches the roots on imaginary axis. However, if the input contains a sinusoid (which is bounded) whose frequency ($\omega_u/(2\pi)$) happens to be equal to the magnitude of the imaginary axis roots, then the system response will be unbounded and unstable.
(3) For an **unstable** system, the characteristic equation has at least one root in the right half of the s-plane or repeated $j\omega$ roots (i.e., non-simple imaginary roots); for this case, the output will become unbounded for any input.

For example,

6.2. Motivation of a Stability Criterion

When the characteristic equation, i.e., the denominator of a transfer function, is of low orders, we can resort to the quadratic formula, cubic formula, and quartic formula to get a closed form solution.[15] However, according to Abel–Ruffini theorem[16], there is no closed form solution to polynomial equations of degree five or higher, and an

[13]Except BIBO stability, we have, e.g., absolute and relative stability, internal stability, Lyapunov stability, L-2/L-infinity stability, input-to-state stability, asymptotical stability, exponential stability.

[14]A simple root is a root with a multiplicity of 1. As a counter-example, we can solve the following large order polynomial equation:

$$(s+5)^{100} = 0$$

where the root $s = -5$ has a multiplicity of 100, thus not a **simple root**.

[15]See, e.g., `https://en.wikipedia.org/wiki/Quartic_function#General_formula_for_roots`

[16]`https://en.wikipedia.org/wiki/Abel%E2%80%93Ruffini_theorem`

example of non-solvable equation is

$$x^5 - x - 1 = 0 \tag{44}$$

To determine the stability of a system, there is in fact no need to solve for the roots of polynomial, and we only need to determine the signs of the real parts of the roots. For example, for polynomial equation of degree two

$$as^2 + bs + c = 0$$

Vieta's formulas state that the roots λ_1 and λ_2 satisfies

$$\lambda_1 + \lambda_2 = -b/a$$
$$\lambda_1 \lambda_2 = c/a$$

As a result, the system must be stable if $-b/a < 0$ and $c/a > 0$, which is a sufficient and necessary condition of stability of this second order polynomial.

For large order polynomial, the Vieta's formulas become only a necessary condition for stability [1, Equation (6.5)] A system is unstable, if the polynomial coefficients do not share a same sign. For example, the following polynomial

$$s^5 + s^4 + s^3 - s^2 + s + 1 = 0$$

is not stable. This motivates that: "sign changing indicates instability".

6.3. The Routh-Hurwitz Stability Criterion

The generalization of using algebraic combinations of the polynomial coefficients to determining stability, is known as **Routh-Hurwitz stability criterion**.

The best way to learn how to apply this criterion is not to read the determinant based definition in the textbook [1]. Instead, watch the three videos on Routh-Hurwitz stability criterion by Brian Douglas. The screenshot of the first episode is shown in Fig. 24.[17] The table in Fig. 24 is called the Routh array.

The Routh-Hurwitz stability criterion states that

(1) the number of roots of characteristic equation with positive real parts is equal to the number of changes in sign of the first column of the Routh array;
(2) and there should be no changes in sign in the first column for a stable system, which is both necessary and sufficient.

6.4. Steps to Determine Stability from Polynomial Coefficients

The full procedure to determine stability from polynomial coefficients are as follows.

S1 Check if all coefficients have the same sign. If not, it is not stable.
S2 Write down Routh array, and if it is a regular case positive and negative numbers in first column, apply the criterion for determining stability.

[17] See also `https://www.youtube.com/watch?v=WBCZBOB3LCA`.

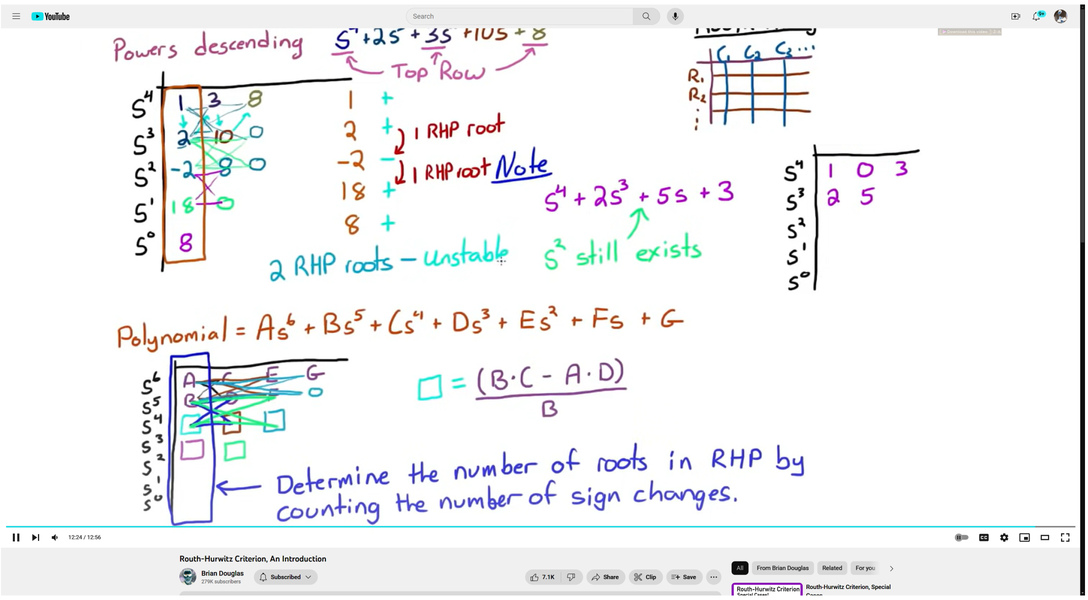


Figure 24. Routh-Hurwitz criterion for regular cases.

S3 If there is a zero in the first column of row s^k of Routh array, we stop. The system is not stable, but we can do more to learn more about the pole locations about the system.

S3.1 Special case one (unstable system): at row s^k, except the first column, there are at least one element being nonzero. We can replace the zeros with a small positive number $\epsilon = 0^+$ at row s^k and continuing to write down Routh array for further information about the number of unstable poles.

S3.2 Special case two (unstable system): at row s^k, including the first column, the rest of elements are all zero. For further information about unstable roots, an auxiliary polynomial is constructed using the row preceding the all-zero row. The order of the auxiliary polynomial is always even and indicates the number of symmetrical root pairs. To be more specific, the auxiliary polynomial contains roots that are symmetrically located about the origin of the s-plane, e.g.,

$$\begin{aligned}
&(s+\sigma)(s-\sigma),\\
&(s+j\omega)(s-j\omega),\\
\text{or }&(s+\sigma+j\omega)(s+\sigma-j\omega)(s-\sigma+j\omega)(s-\sigma-j\omega)
\end{aligned}$$

The all-zero row is replaced with the coefficients of the time derivative of the auxiliary polynomial. The auxiliary polynomial is a factor of the characteristic polynomial, which can be verified using polynomial division as exemplified in Fig. 25.

A step S2 example is

$$\text{Den}\,(\text{s}) = \left(s-1+j\sqrt{7}\right)\left(s-1-j\sqrt{7}\right)(s+3) = s^3+s^2+2s+24$$

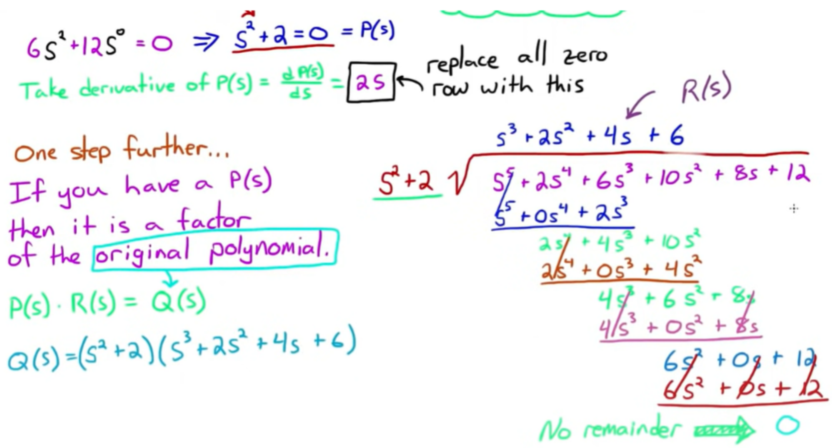


Figure 25. Special case two, auxiliary polynomial and polynomial division.

A step S3 or S3.1 example is

$$\text{Den}\,(\text{s}) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

A step S3 or S3.2 example is (when $K = 8$)

$$\text{Den}\,(\text{s}) = s^3 + 2s^2 + 4s + K$$

The resulting auxiliary polynomial is $A(s) = 2s^2 + K$.

Another step S3 or S3.2 example is

$$\text{Den}(s) = (s+1)(s+j)(s-j)(s+j)(s-j) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1$$

in which special case two occurs twice. Writing down Routh array will end up with two auxiliary polynomials that are both factors of Den(s):

- one is $A_1(s) = (s+j)(s-j)(s+j)(s-j)$ at row s^4 of Routh array,
- and the other is $A_2(s) = (s+j)(s-j)$ at row s^2 of Routh array.

6.5. Discussions

Can you think of a counter-example to reject "the order of the auxiliary polynomial is always even and indicates the number of symmetrical root pairs"?

Initial condition as zeros to make zero-pole cancellation. Consider the following transfer funciton

$$\frac{1}{\left(s^2+16\right)^2} \tag{45}$$

6.6. Strength

RH criterion can be used for controller design. Find the range of value K such that the following characteristic equation has stable roots:

$$\mathrm{Den}\,(\mathrm{s}) = 1 + C\,(s)\,P\,(s) = \frac{K\,(s+a)}{s+1}\frac{1}{s\,(s+2)\,(s+3)} \tag{46}$$

with a a constant. This is the welding control example from textbook.

The advantages using a Routh–Hurwitz criterion is when the transfer function has a polynomial in its denominator instead of explicit pole form. For example, consider the following open loop system:

$$CP(s) = \frac{K}{s^4 + 10s^3 + 35s^2 + 50s + 24} \tag{47}$$

While other tools will need to know the poles of this open loop transfer function, RH criterion can be applied directly to determine the range of K that makes system stable.

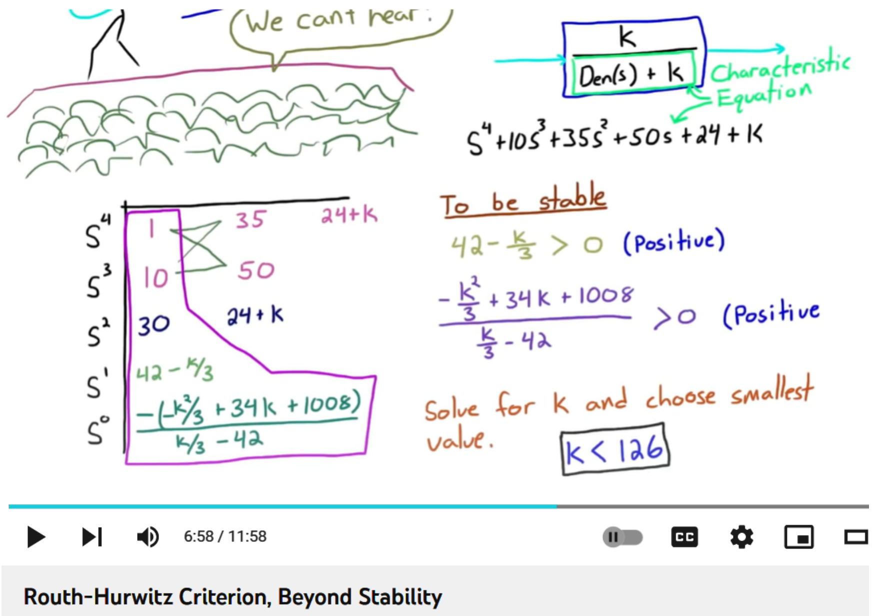


Figure 26. Routh-Hurwitz criterion for controller design of a transfer function having implicit poles.

6.7. Weakness

In order to have an estimation of how far our roots are away from imaginary axis, we need to substitute, e.g., $s' = s - \sigma$ into the polynomial with $\sigma \in \mathbb{R}$. If the resulting Routh array in terms of s' is stable, we then say that the roots are **relatively stable** with a margin of at least σ away from the imaginary axis. The issue of using RH criterion for relative stability, is that we cannot easily determine the maximal value of σ.

6.8. Discussions on Stability

Consider the following unstable system

$$Y(s)/U(s) = \frac{1}{(s^2+16)^2} \tag{48}$$

with repeated pole $4j$ on the imaginary axis. Since $4j$ is not a simple root, the system is unstable. We will now discuss from 4 different aspects as follows.

6.8.1. Signal and System are the Same Thing in s-Domain

Since signal and system are the same in s-domain, we can switch our perspective a bit. Recall the gain for frequency response to $\sin(4t)$ is infinity for the following marginally stable system with simple roots on imaginary axis:

$$\frac{1}{\left((4j)^2+16\right)} = \frac{1}{0} = \infty$$

So its frequency response to a sinusoidal input $\frac{1}{\left((4j)^2+16\right)}$ will be unbounded.

6.8.2. Inverse Laplace Transform

Consider the following inverse Laplace transform: (todo: need to double check)

$$\mathcal{L}^{-1}\left[\frac{4s}{(s^2+16)^2}\right] = t\sin(4t)$$

and we conclude the system's impulse response is unbounded, as $t \to \infty$.

6.8.3. Simulation, Modulation, and Envelope

The following MATLAB snippet numerically simulates the impulse, step and frequency responses of (48).

```
%%
close all; cla; clc
s = zpk(0, [], 1)
% s = zpk('s')

P = 10*16^2 / ((s+10)*(s^2+16)^2)
% P = 10*16^2 / ((s+10)*(s^2+16))
subplot(311); step(P)
subplot(312); impulse(P)

t = linspace(0, 10000, 10000);
u = sin(3*t);
y = lsim(P, u, t);
subplot(313); plot(t,y)
```

The frequency response to $\sin(3t)$ is a modulation of two sinusoidals of the natural frequency 4 rad/s and forcing frequency 3 rad/s, such that the envelop of the response oscillates at a frequency of 1 rad/s. When the exciting angular speed approaches 4 rad/s, the period of this envelop becomes infinite, hence the response is unbounded.

6.8.4. Steady State Frequency Response

The gain for frequency response to $\sin(4t)$ is

$$\frac{1}{\left((4j)^2+16\right)^2}=\frac{1}{0}=\infty$$

The gain for frequency response to $\sin(t)$ is

$$\frac{1}{\left((1j)^2+16\right)^2}=\frac{1}{15^2}$$

which is finite. In order to produce a steady state frequency response for this system (48), we need to carefully set the initial states so we can avoid any transient response. In other words, the transient response of (48) is always unstable. To understand how it is possible to have a bounded response to (48), let's convert it back to its ODE form

$$(48) \Rightarrow \left(s^2+16\right)^2 y = u \Rightarrow s^4 y + 32y^2 + 16^2 y = u$$

where s has been used as the differential operator $\mathrm{d}/\mathrm{d}t$. Recall that when we apply a Laplace transform to above ODE, we have:

$$\left(s^4 Y - \frac{\mathrm{d}^3 y}{\mathrm{d}t^3}\left(0^-\right) - s\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\left(0^-\right) - s^2\frac{\mathrm{d}y}{\mathrm{d}t}\left(0^-\right) - s^3 y\left(0^-\right)\right) + 32\left(s^2 Y - s\frac{\mathrm{d}y}{\mathrm{d}t}\left(0^-\right) - s^2 y\left(0^-\right)\right) + 16^2 Y = U$$

With proper initial conditions, it is possible to introduce zeros to $Y(s)/U(s)$ such that the pole zero cancellation occurs and a bounded response becomes possible.

7. Bode Plot

Sinusoidal input is the only function that keeps its shape while passing through an LTI system.

Let's consider an example system with the following transfer function:

$$T(s) = 1/s + 2$$

which consists of an ideal integrator and a proportional gain.

8. Root Locus Method

We will learn root locus method and see how it is useful in understanding the effect of lead-lag compensator on the system poles and zeros.

Appendix A. Review of Key Math Concepts: Two Kernels

A.1. Kernel in Integral Transform

The integral transform is a math operation that changes variable of interest, and it has a general form as follows

$$F(\alpha) = \int_a^b f(t)\, K(\alpha, t)\, \mathrm{d}t \tag{A1}$$

where $K(\alpha, t)$ is known as the kernel of the integral transform.

There are three integral transforms will be used in this course: Laplace transform, Fourier transform and convolution:

$$F(s) = \int_{0^-}^{\infty} f(t)\, \mathrm{e}^{-st} \mathrm{d}t \tag{A2a}$$

$$F(j\omega) = \int_{0^-}^{\infty} f(t)\, \mathrm{e}^{-j\omega t} \mathrm{d}t \tag{A2b}$$

$$F(t) = \int_{0^-}^{\infty} f(\tau)\, \delta(t-\tau)\, \mathrm{d}\tau \tag{A2c}$$

where δ is the impulse function and $j = \sqrt{-1}$.

A.2. Kernel in Linear Algebra

Kernel as null space in linear algebra.

Appendix B. Five Ways Solving Ordinary Differential Equations

See "Five Levels for Differential Equations in Physics" by Physics with Elliot `https://www.physicswithelliot.com/odes-help-room-notes`

Appendix C. Zeros and Zero dynamics

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