LECTURE NOTES

EE160: Introduction to Control (Fall 2023)

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ARTICLE HISTORY
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ABSTRACT
The 14th edition of a textbook is not making it a better material for "newbies". When I realized this, I decided to draft my own lecture notes for you, my dear friends.

KEYWORDS
Control theory, linear systems.

Contents

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1. Introduction to Control

In my first lecture on Sept. 26th, 2023, I was suggesting there should be a second episode for the Youtube video entitled "Animation vs. Math" featuring TSC, the sticker man.[1] But this time, we are going to further allow him to bring a new set of tools, including:

$$u, \text{ and } \frac{\mathrm{d}}{\mathrm{d}t} \tag{1}$$

where u grants TSC the ability to control, and $\frac{\mathrm{d}}{\mathrm{d}t}$ is the magic operator that brings some random variable $x \in \mathbb{R}$ to life and begins to evolve with time t.

1.1. What is Time?

Before we move on to discuss what is control, let's define time. The time is measured in terms of periodic events. The sun rise and sun set make a day. The SI unit second is defined in terms of the unperturbed transition frequency of the caesium 133 atom.[2]. The positive direction of time elapse is defined in the second law of thermodynamics: *"The entropy of the universe tends to a maximum, or in loose terms, energy spreads out over time."* From a macroscopic point of view, entropy (dented by S) changes whenever there is a transfer of heat:

$$\Delta S = \int_{t_0}^{t_1} \frac{-\mathrm{d}Q}{T(t)} \tag{2}$$

where $T(t)$ is the temperature when the dissipated *(note the negative sign)* heat $\mathrm{d}Q$ is made, and the differential change of heat energy $\mathrm{d}Q$ is the work done by the friction force:

$$\begin{aligned} \mathrm{d}Q &= \boldsymbol{F}_{\text{friction}} \mathrm{d}\boldsymbol{x} \ [\mathrm{J}] \\ Q &= \int_{\boldsymbol{x}_0}^{\boldsymbol{x}_1} \boldsymbol{F}_{\text{friction}} \mathrm{d}\boldsymbol{x} \ [\mathrm{J}] \end{aligned} \tag{3}$$

where $\boldsymbol{F}_{\text{friction}}$ is the friction force and $\boldsymbol{x}(t) \in \mathbb{R}^3$ is a trajectory in space.

Think, if there is no longer transfer of heat in a universe, does this mean its time stops evolving?

1.2. What is Control?

We have been playing with those math toys to study how $x(t)$ evolves when its dynamics are one of the followings

$$\frac{\mathrm{d}}{\mathrm{d}t}x = 1,\ \frac{\mathrm{d}}{\mathrm{d}t}x = -1,\ \frac{\mathrm{d}}{\mathrm{d}t}x = x,\ \frac{\mathrm{d}}{\mathrm{d}t}x = -x,\ \frac{\mathrm{d}}{\mathrm{d}t}x = x^2,\ \frac{\mathrm{d}}{\mathrm{d}t}x = -x^2$$

When learning math, we tend to want to avoid diverging to infinity. The only system

[1](TSC stands for The Second Coming, the fourth stick figure that was created by Alan Becker.
[2]`https://en.wikipedia.org/wiki/Second`

above that gives a non-diverging response $x(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}x = -x \;\Rightarrow\; x(t) = x(0)\mathrm{e}^{-t}$$

This system evolves and it gives a response $x(t)$ that converges towards 0.

1.2.1. Feedforward Control

When TSC decides to further append the tool u to any of the system, it yields, e.g.,

$$\frac{\mathrm{d}}{\mathrm{d}t}x = x^2 + u \tag{4}$$

Assume x is known, the control is simply realized by setting $u = -x^2 + v$, leading to

$$\frac{\mathrm{d}}{\mathrm{d}t}x = v \tag{5}$$

This means we are essentially treating the term x^2 as a disturbance to the system, and u is able to cancel the effect of such disturbance.

Therefore, control is subjective: y^2 can be dynamics and it can also be treated as (internal) disturbance.

1.2.2. Book Recommendations

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1.2.3. Model Predictive Control

Apply control effort in a period of time just enough to exactly reach the goal.

1.2.4. Proportional Control

Proportional control is the basic form of negative feedback control.

1.2.5. Sliding Mode Control

From our view, SMC is more like bang-bang control.

1.2.6. Adaptive / Integral / Dynamic Control

The control input u itself behaves like a dynamical system.

1.2.7. Linear Quadratic Regulator (LQR)

For a linear system: $\dot{x} = ax$, the LQR is $u = -(a + \sqrt{a^2 + \psi})x$ with tuning button ψ. The objective of optimum is

$$J_{\mathrm{LQR}} = \int_0^{t_f} (\psi x(t)^2 + u^2)\mathrm{d}t \tag{6}$$

Solving it will lead to something called CTDRE.

1.2.8. Using States as Control Input

Using states as control input is one simple principle that is at the center of control, e.g., (integral) back-stepping and input-output linearizing control.

Nested loop control wants to use the same principle, but it really assumes the control transients of the inner loop are short enough.

2. Mathematic Model of Linear Systems

This course, however, does not treat control like what we have done (in time domain) in Chapter 1, as our controlled dynamics can only be solved using numerical integrations. In order to have a closed form as solution, we need to study a class of simple systems originated from physics laws.

2.1. Analogue Systems and Variables

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2.1.1. Force-current analogy

Kirchhoff's current law states that all currents flowing into a node must be equal to the current flowing out of it (as a consequence of charge conservation):

$$\frac{v}{R} + C\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{1}{L}\int_0^t v\mathrm{d}t = i\left(t\right) \tag{7}$$

Newtonian mechanics state that the change of momentum equals to the sum of forces applied to the particle with mass M:

$$M\frac{\mathrm{d}^2}{\mathrm{d}t^2}y + b\frac{\mathrm{d}}{\mathrm{d}t}y + ky = F\left(t\right) \tag{8}$$

Think, where defines $y = 0$? Hint: where the spring force $ky = 0$.

2.1.2. Force-voltage analogy

Newton 2nd law === Faraday's law of induction

2.2. Across Variable and Through Variable

Concepts for helping to take abstract of various different physics systems.

2.3. Excitation and Response

The excitations are respectively $i(t) = \text{Const.}$ and $y(0) = y(0)$.

The response shows the

2.4. Dirac Delta Function

The excitations to the LCR system and the mass-spring-damper system are distinct

Heaviside step function is the integral of Dirac delta function (impulse function)

2.5. Linear System

Two necessary conditions for a linear system are homogeneity and superposition.

Further requirement is time-invariance, which results in sinusoidal fidelity. To prove the sinusoidal fidelity, we need to learn to define the frequency response of a transfer function, or in other words, learn solving ODE using Laplace Transform, see Chapter 8 of textbook.[3]

2.6. Laplace Transform

Transform an ODE into an algebraic equation, with initial conditions.

Define: Poles, Zeros, and Gain.

2.7. Transfer Function

Zero initial conditions. We have the differential operator $s \triangleq \frac{\mathrm{d}}{\mathrm{d}t}$.

In s-domain, transfer function is the same as the system's impulse response. In other words, signal and system become the same concept in s-domain.

2.8. Block Diagram

The operation between two connected blocks are multiplication in s-domain. In other words, block diagram is multiplication of transfer functions.

*2.9. Signal Flow Diagram**

This diagram is only meaningful when the block diagram has too many nodes. In most scenarios, it is the same as block diagram, so it is not required in my course.

[3] Another interpretation is to change the Kernel of the integral transform to get steady state results.

3. Feedback Control System Characteristics

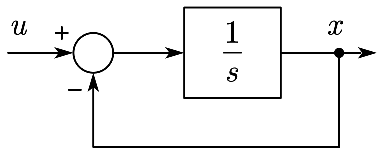

Figure 1. Motivation: the converging response system $sx = -x$ forms a loop.

By making $\frac{1}{s}$ a block in a block diagram, we realize the converging response system $sx = -x$ forms a loop, as shown in Fig. 1. This motivates us that a closed loop might be what we desire for designing a control system.

This section is going to answer why feedback control system is better than a system having no feedback path.

3.1. Open Loop and Closed Loop

Our goal is make state $x(t)$ follow reference signal $r(t)$.

For an open loop control system like the one in Fig. 2, the full transfer function of the control system is $\frac{X}{R} = CP$. Let $R(s) = X(s)$, we have $CP = 1$ or $C = P^{-1}$. This kind of controller is known as the inverse system controller, which often has difficulty for realization in reality.

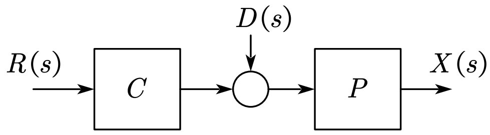

Figure 2. Open loop system.

A closed loop system, on the other hand, gives $\frac{X}{R} = \frac{CP}{1+CP}$. Note CP is a complex number in nature. As long as $|CP|$ is large enough such that $|CP| \gg 1$, we have $X \approx R$.

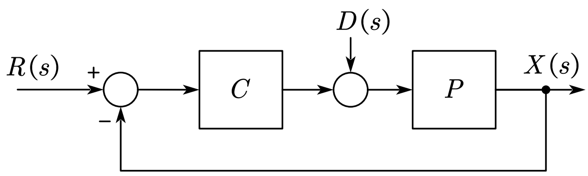

Figure 3. Closed loop system.

3.2. Steel Rolling Mill

Let's consider an example of industrial application.

3.2.1. Model of a DC Motor

All motor is AC. Even though the voltage applied to the motor can be DC at its terminals, the conductors along the air gap of the motor must carrying an alternating current. In order to provide an alternating current to the conductors, carbon brushes or power electronic devices are necessary to a motor.

3.3. Transient Response

Closed loop control is able to modify the pole of the transient response!

3.4. Steady State Response

3.4.1. Steady State Step Response

A simple trick to to have zero steady state (step) error is to use an infinity loop gain $L(0)$.

3.4.2. Frequency Response

Let's further extend the concept of steady state response to arbitrary sinusoidal inputs $R(s) = \frac{\omega^2}{s^2+\omega^2}$. In its extreme, $s = 0$, it becomes Steady State Step Response.

Frequency response is the system's steady state response to sinusoidal inputs, in which the transients are not important thus shall be neglected. To this end, replacing $s = \sigma + j\omega$ with $j\omega$ in a transfer function $T(s) \times 1$ or system response $T(s)R(s)$ provides steady state response.

For a second order transfer function to have nonzero frequency response, the damper b should be zero.

3.5. Foe

There are, of course, undesired phenomena present in a control system, including external disturbance (measurement noise N and unknown input D) and internal disturbance (parameter uncertainty ΔP).

3.6. Sensitivity Function

The internal disturbance ΔP (which is often a parameter uncertainty) causes a deviation ΔT from T.

A metric that evaluates how much perturbation it causes to our system is the sensitivity function, defined by

$$S = \frac{\Delta T(s)/T(s)}{\Delta P(s)/P(s)} \tag{9}$$

where the deviation can be calculated as per definition:

$$\Delta T(s) = \frac{C(P + \Delta P)}{1 + C(P + \Delta P)} - \frac{CP}{1 + CP} \tag{10}$$

In the limit, small incremental changes leads to following definition:

$$S_P^T = \frac{\partial T\left(s\right)/T\left(s\right)}{\partial P\left(s\right)/P\left(s\right)} = \frac{\partial \ln T\left(s\right)}{\partial \ln P\left(s\right)} \tag{11}$$

where the following calculus relation has been substituted:

$$\frac{\mathrm{d}x}{x} = \mathrm{d}\ln x \;\Leftrightarrow\; \int \frac{\mathrm{d}x}{x} = \ln x \tag{12}$$

When control system transfer function is $T\left(s\right) = \frac{CP}{1+CP}$, the sensitivity function is

$$S_P^T = \frac{1}{1+CP} \tag{13}$$

When control system transfer function is $T\left(s\right) = CP$, the sensitivity function is

$$S_P^T = 1 \tag{14}$$

This is the second advantage of using a closed loop control system. The amplitude of the sensitivity function is subject to a factor that is less than 1. Also, it is important to use a negative feedback loop, otherwise the denominator in (13) becomes $1 - CP$, making $|S_P^T| > 1$.

In most cases, the transfer function T is a rational fraction:

$$T\left(s;\alpha\right) = \frac{N\left(s;\alpha\right)}{D\left(s;\alpha\right)} \tag{15}$$

where α is a parameter that has variation, and N and D are numerator and denominator polynomials in s. As a result, T's sensitivity with respect to parameter α becomes

$$S_\alpha^T = \frac{\partial \ln \mathrm{T}}{\partial \ln \alpha} = \left.\frac{\partial \ln \mathrm{N}}{\partial \ln \alpha}\right|_{\alpha=\alpha_0} - \left.\frac{\partial \ln \mathrm{D}}{\partial \ln \alpha}\right|_{\alpha=\alpha_0} = S_\alpha^N - S_\alpha^D \tag{16}$$

where α_0 is the nominal value of α.

3.7. Gang of Six

Watch video of Douglas "Gang of Six".

The closed-loop control system shown in Fig. 4 has considered all three different foes that perturb the control performance. Assuming $H(s) = 1$ in Fig. 4, we can derive the following relationships among the input signals and state/output/input/error:

$$X = \frac{CP}{1+CP}FR + \frac{P}{1+CP}D - \frac{CP}{1+CP}N \tag{17a}$$
$$Y = \frac{CP}{1+CP}FR + \frac{P}{1+CP}D + \frac{1}{1+CP}N \tag{17b}$$
$$U = \frac{C}{1+CP}FR - \frac{CP}{1+CP}D - \frac{C}{1+CP}N \tag{17c}$$
$$E = \frac{1}{1+CP}FR - \frac{P}{1+CP}D + \frac{CP}{1+CP}N \tag{17d}$$

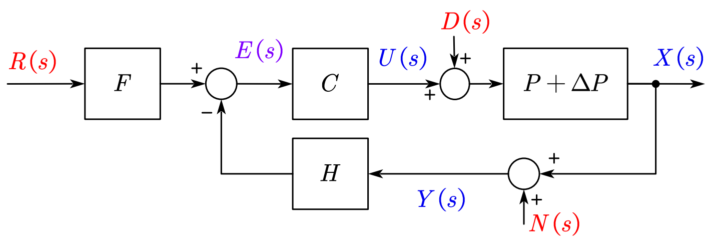

Figure 4. Closed loop system with three input channels.

We define loop gain as $L \triangleq CP$.
Sensitivity function S.
Complementary sensitivity function $1 - S$.
Disturbance sensitivity function PS.
Noise sensitivity function CS.

3.8. Error Signal Analysis

Assuming feedforward block $F = 1$, (17d) is rewritten in terms of sensitivity function S as follows:

$$\begin{aligned} E &= \frac{1}{1+L}R - \frac{P}{1+L}D + \frac{L}{1+L}N \\ &= SR + SPD - S^{'}N \end{aligned} \tag{18}$$

3.9. Sensitivity to Parameter Variation

Uncertainty ΔP affects all three channels of the input. We will take reference input for illustration. Assume $D = N = 0$, and substitute $P + \Delta P$ for P in error analysis

(18) yields

$$
\begin{aligned}
E + \Delta E &= \frac{1}{1 + C\left(P + \Delta P\right)} R \\
\Rightarrow \Delta E &= \left(\frac{1}{1 + C\left(P + \Delta P\right)} - \frac{1}{1 + CP}\right) R \\
&\approx \frac{1}{1 + CP} \frac{\Delta P}{P} R \\
&= S \frac{\Delta P}{P} R
\end{aligned}
\tag{19}
$$

3.10. Disturbance Rejection

Using the principle of superposition, let's analyze the effect of external disturbance input D:

$$
E = -\frac{P}{1 + CP} D = -\frac{P}{1 + L} D = -SPD \tag{20}
$$

in which $R = N = 0$ has been substituted. The disturbance will be rejected if we use a "large" loop gain. Or in rigorous terms, disturbance rejection occurs whenever s is making the gain $|S(s)P(s)|$ small enough.

3.11. Noise Attenuation

The complementary sensitivity function.

3.12. Frequency Response of The Gang Members

```
P = tf([1], [1, 1])
C = tf([1, 1], [1 0])
subplot(141); bode(C*P/(1+C*P)); grid
subplot(142); bode(1/(1+C*P)); grid
subplot(143); bode(P/(1+C*P)); grid
subplot(144); bode(C/(1+C*P)); grid
```

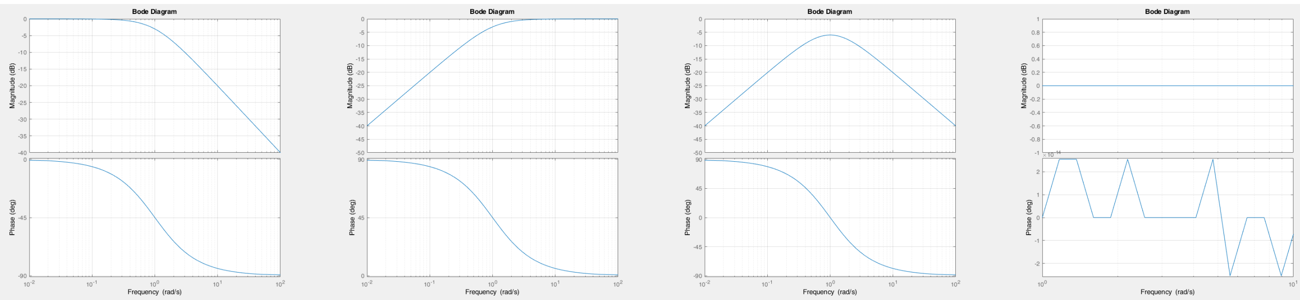

Figure 5. Sensitivity function

Appendix A. Review of Key Math Concepts: Two Kernels

A.1. Kernel in Integral Transform

Kernel as in integral transform.

A.2. Kernel in Linear Algebra

Kernel as null space in linear algebra

Appendix B. Zeros and Zero dynamics