1)

$$x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$Ax = -2x$, $A = -2$ is an eign value of matrix A

2

$$A - 3\lambda = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$[(A-3I)\ 0]$

$$\begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(A-3I)x = 0$ has a non trival solution so 3 is an eigenvalue. If $x_3 = 1$ then $x = (3, 2, 1)$

3

a) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$ $\lambda = 2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 5 & 0 \\ -4 & 13 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$E_1: x_1$

b) $\begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix}$ $\lambda = 4$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

4

$$2\times 2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A - \lambda I = 0$$

$$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$a = d$ $bc = 0$, $\begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$ has one distinct Eigen value

5

a)

$$\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} = \lambda^2 - 10\lambda + 16$$

$\lambda = 8 \quad \lambda = 2$

b $$\begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 5-\lambda & -3 \\ -4 & 3-\lambda \end{bmatrix} = \lambda^2 - 8\lambda + 3$$

$\lambda = 4+\sqrt{13}, \ \lambda\ 4-\sqrt{13}$

c $$\begin{bmatrix} 3 & -4 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 3-\lambda & -4 \\ 4 & 8-\lambda \end{bmatrix} = \lambda^2 - 11 + 46$$

$\lambda\ 11/2$

d

6

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Let λ be the eigen value of A

$$= \begin{vmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (-1-\lambda) \begin{vmatrix} 4-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} -3 & -4-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= (-1-\lambda)(4-\lambda)(2-\lambda)$$

$$= (1-\lambda)\ (8-6\lambda+\lambda^2)$$

$$= (-\lambda^3 + 5\lambda^2 - 2\lambda - 8$$

7

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$$

eigenvalues of matrix A are obtained by solving $|A - \lambda I| = 0$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 8 & -4-\lambda & 0 & 0 \\ 0 & 7 & 1-\lambda & 0 \\ 1 & -5 & 2 & 1-\lambda \end{vmatrix}$$

$$(5-\lambda)\begin{vmatrix} -4-\lambda & 0 & 0 \\ 7 & 1-\lambda & 0 \\ -5 & 2 & 1-\lambda \end{vmatrix} - 0 + 0 - 0 = 0$$

$(\because a_{12} = a_{13} = a_{14} = 0$

$(5-\lambda)[(4-\lambda)(1-\lambda)(1-\lambda) - 0) - 0 + 0] = 0$

$(5-\lambda)[(-4-\lambda)(1-\lambda)(1-\lambda) = 0$

8

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & -2 & 6 & -1 & 0 \\ 0 & -2 & h & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & -2 & 6 & -1 & 0 \\ 0 & 0 & h-6 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & h-6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

system above needs two free variables
this happens iff $h=6$

9

$P = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ $A = PDP^{-1}$

$P^{-1} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $D^4 \begin{bmatrix} 1 & 0 \\ 0 & 1/16 \end{bmatrix}$, $A^4 = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} =$

$1/16 \begin{bmatrix} 151 & 90 \\ -225 & -134 \end{bmatrix} = \begin{bmatrix} 151/16 & 45/8 \\ -255/16 & -67/8 \end{bmatrix}$

10

$$\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$$

$A = PDP^{-1}$

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}, P = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}, P^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$$

eigenvalues of Matrix A = 4, 5

bases for the eigspaces to the eigenvalues 5 and 4 are $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

11

a)

$$\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$$

$$\det\left(\left(\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right)$$

$$\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5-\lambda & 1 \\ 0 & 5-\lambda \end{bmatrix}$$

$$= \det \begin{bmatrix} 5-\lambda & 1 \\ 0 & 5-\lambda \end{bmatrix}$$

$(5-\lambda)(5-\lambda) - 1 \cdot 0$

$1 \cdot 0 = 0$

$= (-\lambda + 5^2) - 0$ $\quad$ $\lambda = 5$ with multiplicity of 2

$$\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(A - 5I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$y = 0$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ = Not diagonalizabile b)

$$\det\left(\left(\begin{bmatrix}2 & 3\\4 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\right)$$

$$\begin{bmatrix}2 & 3\\4 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \det\begin{bmatrix}2-\lambda & 3\\4 & 1-\lambda\end{bmatrix}$$

$$\det\begin{bmatrix}2-\lambda & 3\\4 & 1-\lambda\end{bmatrix} = (2-\lambda(1-\lambda) - 3\cdot 4$$

$$= \lambda^2 - 3\lambda - 10$$

$$\lambda = 5, \lambda = -2$$

$$D = \begin{bmatrix}5 & 0\\0 & -2\end{bmatrix} \qquad P = \begin{bmatrix}1 & -3\\1 & 4\end{bmatrix}$$

$$P^{-1} = \begin{bmatrix}4/7 & 3/7\\-1/7 & 1/7\end{bmatrix}$$

$$P = \begin{bmatrix}1 & -3\\1 & 4\end{bmatrix}, D = \begin{bmatrix}5 & 0\\0 & -2\end{bmatrix}, P^{-1}\begin{bmatrix}4/7 & 3/7\\-1/7 & 1/7\end{bmatrix}$$

c

$$\det\left(\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) =$$

$$= \lambda^4 - 12\lambda^3 + 52\lambda^2 - 96\lambda + 64$$

$$(\lambda - 2)^2 \cdot (\lambda - 4)^2) = 0$$

$\lambda = 2$ $\lambda = 4$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = P^{-1}$$

$$PDP^{-1} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

12.

Matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has 2 linearly independend eigenvalues

but ading 1, 1 $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has one linearly independent eigven value.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible but not diagnonalizable

13.

a 2x2 triangular matrix whose diogonal entries are 0 and a nonzero realnumber.

Matrix $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ satifies this