1)

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |A| = 1 \neq 0$

Therefore the given vectors are linearly independent basis of $\mathbb{R}^3$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ The given vectors are linearly dependent

c $\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$ linearly independent vectors that span $\mathbb{R}3$

d $\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix}$ linearly independent, not basis of $\mathbb{R}^3$

e) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & -1 & 4 & -2 \end{bmatrix}$ vectors not a basis, is not a set of linearly independent vectors, but spans R^3

2) B is row echelon form of A, we see that the first, third and fifth columns of A are its pivot columns.

Basis of Col A is:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$$

$x_1 = -2x_2 - 4x_4$, $x_3 = (7/5)x_4$, $x_5 = 0$ with x_2 and x_4 free.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix}$$

basis for ~~Nul A~~ $N(A)$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

3)

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix} \quad v_5 = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

col A, $A = [v_1, v_2, v_3, v_4, v_5]$

$$A = \begin{matrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{matrix}$$

reduce matrix:

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \xrightarrow{R4-R_1} \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 0 & 3 & -7 & -9 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 0 & 3 & -7 & 9 & 1 \end{bmatrix} \xrightarrow{R_3+R_2,\ R4-3R_2} \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & 0 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 3 & 42 \\ 0 & 0 & -4 & -4 & -8 \end{bmatrix} \xrightarrow{R_4+4R_3} \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix A is: $A = \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$b = b_1 b_2 b_3 b_4 b_5$ $\quad$ $\text{col}(A) = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}; v_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix} \right\}$

vector $\{\{v_1, v_2, v_3\}\}$

4)

a) For each B in R^m, the equation has solution $Ax=b$ is equivalent to the columns of A span R^m. The matrix A has pivot position in each row. each b in R^m is a linear combo of collumn A.

b) $(v_1, \ldots v_p\}$ in R^n is linearly dependent if $p>n$.
the only difference is p instead of F.
therefore $\{v_1, \ldots, v_k\}$ is linearly dependent, and cant be a basis.

5) $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix}$

$$x_B = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

$$x = 8(b_1) + -5(b_2)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} 4 \\ 5 \end{bmatrix} + -5\begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 32 \\ 40 \end{bmatrix} + \begin{bmatrix} -30 \\ -35 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

5) $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix}$

a) $x_B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$x = 8(b_1) + -5(b_2)$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} 4 \\ 5 \end{bmatrix} + -5\begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 32 \\ 40 \end{bmatrix} + \begin{bmatrix} -30 \\ -35 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

5)=

$B = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}, x_B = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (-4)\, b_1 + 8(b_2) + -7(b_3)$$

$$x = (-4)\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} + -7\begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 + 24 - 28 \\ -8 + (-40) + 49 \\ 0 + 16 - 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$$

6)

a)

b) $b = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$, $x = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 2 & -1 & -5 \end{bmatrix}$$

so $x_1 + 2x_2 + x_3 = 3$

$x_2 - x_3 = -5$

$x_3 = 5$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

so $x_2 = 0$, $x_1 = 8$

coordinate vector $= (8, 0, 5)$

7)

the change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^2$ is $P_B = \begin{matrix} 4 & 6 \\ 5 & 7 \end{matrix}$.

$$x_B = \begin{matrix} 4 & 6 \\ 5 & 7 \end{matrix}^{-1} \begin{matrix} 2 \\ 0 \end{matrix} = \begin{matrix} -3.5 & 3 \\ 2.5 & -2 \end{matrix} \begin{matrix} 2 \\ 0 \end{matrix} = \begin{matrix} -7 \\ 5 \end{matrix}$$

8) $x_1(1-t^2) + x_2(t-t^2) + x_3(2-2t+t^2) = 3+t-6t^2$

$\hookrightarrow (x_1 + 2x_3) + (x_2 - 2x_3)t + (-x_1 - x_2 + x_3)t^2 = 3+t-6t^2$

$x_1 + 2x_3 = 3$

$x_2 - 2x_3 = 1$

$-x_1 - x_2 + x_3 = -6$

augmented matrix $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$x_1 = 7,\ x_2 = -3,\ x_3 = -2$

$7(1-t^2) + (-3)(t-t^2) + (-2)(2-2t+t^2) = 3+t-6t^2$

$[3+t-6t^2]_B = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix} \in \mathbb{R}^3$

9)

$1t + 0t - 2t^2 - t^3$ = $v_1 = (1, 0, -2, -1$ | $0t + t + 0t^2 + 2t^3$ $v_2 = (0, 1, 0, 2$ | $1t + t - 2t^2 + 0t^3$ $v_3 = (1, 1, -2, 0)$

$$[v_1, v_2, v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$; $R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$R_3 \to R_4 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 \to 2R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \frac{R_3}{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

v_1, v_2, v_3 are linearly ~~of~~ independent

10)

a) for any polynomial in P_2 is of the form $a_0 + a_1 t + a_2 t^2$ then the coordinate vectors are $(1,0,1), (2,-1,3) (1,2,-4)$

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{vmatrix} = -2+4+1 = 3 (\neq 0)$$

the polynomials $\{P_1(t), P_2(t), P_3(t)\}$ form a basis for P_2

b) $\{P_1(t), P_2(t), P_3(t)\}$

$= x(1+t^2) + y(2-t+3t^2) + z(1+2t-4t^2)$

$= (x+2y+z) + t(-y+2z) + t^2(x+3y-4z)$

$= x+2y+z = a_0, -y+2z = a_1$ and $x+3y-4z = a_2$

$x = \frac{1}{3}(-2a_0 + 11a_1 + 5a_2), y = \frac{1}{3}(2a_0 - 5a_1 - 2a_2$ and $z = (\frac{1}{3})(a_0 - a_1 - a_2)$

$a_0 + a_1 + a_2 = \frac{1}{3}(-2a_0 + 11a_1 + 5a_2)(1+t^2) + \frac{1}{3}(2a_0 - 5a_1 - 2a_2)(2-t+3t^2)$
$+ \frac{1}{3}(a_0 - a_1 - a_2)(1+2t-4t^2$

coordinate vector $[q]_b = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

$\frac{1}{3}(-2a_0 + 11a_1 + 5a_2) = -3, \frac{1}{3}(2a_0 - 5a_1 - 2a_2) = 1$

$\frac{1}{3}(a_0 - a_1 - a_2) = 2$

$a_0 = 1, a_1 - 3, a_2 = -8 \quad q = 1 + 3t - 8t^2$