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Numerical Experiments with AMG Solver in Reservoir Simulation

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Abstract

AMG (Algebraic Multi-Grid) solver proved to be one of the most efficient solvers handling linear systems from discretization of the pressure equation in IMPES and Sequential Implicit formulations. We share our experience in using it in production simulations. We also discuss how AMG solver can be used as a component in composite solvers for other formulations or in parallel algorithms.

Classical AMG (Ruge-Stuben) approach is very well suited for monotone matrices from finite volume discretizations, but cannot solve efficiently linear systems with the facility unknowns included because of the different nature of these equations. We have chosen to alleviate this problem by using the multiplicative Schwarz framework. One application of this preconditioner consists of solving the facility system to a relatively loose tolerance followed by one AMG V-cycle on the pressure system. Our experiments confirm there is no loss of efficiency due to the facility unknowns. Moreover, the same approach can be extended naturally to Adaptive Implicit (AIM) formulation with different parts of the matrix handled by different linear solvers. AMG is also very efficient when used as a sub-domain solver in the overlapping additive Schwarz domain decomposition algorithm. The straightforward generalization of the serial algorithm produced unexpected results with respect to the robustness of the algorithm. The theoretical analysis provided valuable insight on the interplay of the multiplicative and additive Schwarz methods and showed us how to reorganize the algorithm to correct the problems. We also report our new findings on the influence of the overlap size between the sub-domains on the convergence of additive Schwarz method.

Introduction

One of the most time consuming task during reservoir simulation is the solution of the linear systems resulting from the discretization and linearization of the partial differential equations describing the fluid flow in porous media. Traditionally SSOR, Nested Factorization, and ILU [Watts (1971), Appleyard et al (1983), Behie and Forsyth (1984)] preconditioners have been the most popular serial preconditioners in the oil industry because of their robustness and ease of use. We have extensive experience using ILU with threshold type of preconditioners (we call the particular implementation Fill-in ILU (FILU) Diyankov et al (2007)). FILU is very robust and efficient preconditioner for not very big problems, but its performance deteriorates with the increase of the dimension of the linear systems. Algebraic Multi-Grid (AMG) preconditioner has been successfully used for solving very large linear systems arising from discretization of elliptic or parabolic problems. It scales almost linearly and is very robust.

Classical AMG works very well on M-matrices (diagonally dominant, positive diagonal, entries, and negative off diagonal entries). We use AMG to solve the linear systems from the pressure equation in the IMPES and the Sequential Implicit formulation. Since the well terms in general can be very different from the reservoir equations the direct use of AMG is not always efficient and even the linear solver can diverge for more complicated wells. We fixed this problem by using AMG in the framework of the multiplicative Schwarz method. We first solve the well equations with another solver, for example FILU, to a relatively loose tolerance, followed by an AMG V-cycle on the reservoir equations.

A typical multigrid algorithm consists of several components: coarsening, interpolation, smoothing, coarsest level solution, cycling type and so on [Stuben (2001)]. Among these components, the most important ones, in the sense of performance impact and optimization difficulty are coarsening and interpolation. Coarsening is to determine the coarse level node set and the interpolation needs to define the way to transfer information between neighboring levels. A typical AMG algorithm can be separated into two phases: setup phase which includes coarsening and interpolation operator setup and solution phase. The computation cost for the setup phase is usually equivalent to the cost for 3-4 iterations in the solution

phase. Several researchers reported [Cao et al (2001), Stuben et al (2007), Stuben (2007)] also good results for using AMG in reservoir simulation applications. The parallel AMG methods [Henson and Yang (2002)] are becoming more mature and are successfully applied.

Domain decomposition methods are one of the most efficient algorithms for solving linear system on parallel computers. They are based on the divide-and-conquer strategy; the problem is split into several smaller problems, each problem is solved and the results are used to find a solution to the initial global problem. We are particularly interested in the multiplicative and additive versions of the overlapping Schwarz algorithms [[Smith et al (1996), Toselli and Widlund (2005)]. There exists a rigourous theory for the Schwarz methods applied to simple model problems. Unfortunately, many real problems are not covered. For example, there are still gaps in the theory for parabolic problems. The class of nonsymmetric problems is also not well understood. The behavior of the methods for problems with highly heterogeneous and discontinuous coefficients is still not fully explained and development of new more efficient methods is necessary to these types of problems. We share our knowledge from experiments and a give a simple theoretical argument that can be used to explain the behavior of some Schwarz methods and eventually lead to their more efficient use.

The rest of the paper is organized as follows. We first introduce the standard notations and present a simple theoretical argument. The next section contains the description and numerical experiment with AMG. We explore the same approach for AIM formulation. The last section is concerned with the use of different sub-domain solvers and different number of overlapping layers in additive and multiplicative Schwarz methods. We finish with a short conclusion section.

Remarks on some Schwarz methods

Here we give a very short description of methods used in the paper. The purpose is to provide some understanding about the factors controlling the convergence. We give a simple argument to support the discussion, although we did not provide a rigorous theory. In depth discussion can be found in the monographs [Smith et al (1996), Toselli and Widlund (2005)].

We would like to solve a problem defined in a domain Ω. Let A be the matrix corresponding to the global problem in Ω and b be the right hand side. The linear system is $Ax = b$. We suppose the domain Ω is spitted into two overlapping domains Ω_1 and Ω_2, i.e., $\Omega_1 \bigcap \Omega_2 \neq \phi$. Two examples are shown of Figure 1. The left graph is the conventional partition; the right is partition for the well. Ω_1 contains everything but the well.

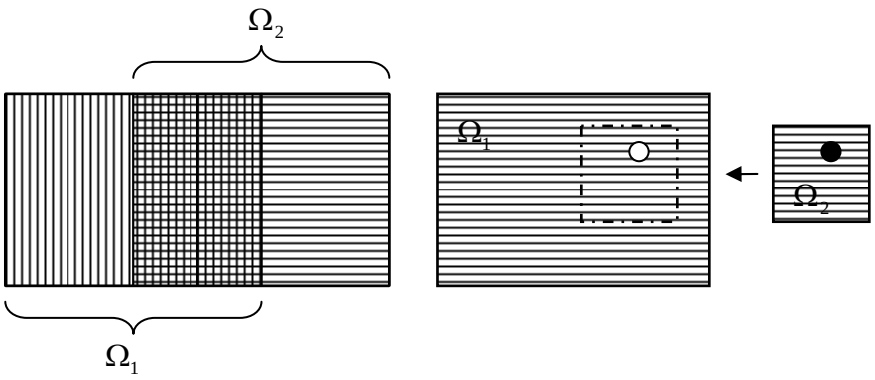

Figure 1: Example partitions of the domain Ω into Ω_1 and Ω_2.

We denote with R_i the restriction operator from Ω to Ω_i, $i = 1, 2$, and with R_i^T the extension (prolongation operator from Ω_i to Ω. Denote $A_i = R_i A R_i^T$. Then the additive Schwarz preconditioner is defined as

$$M_{AS}^{-1} = R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2 = B_1 + B_2$$

and the preconditioned matrix is $M_A^{-1} A = P_1 + P_2$. The action of the preconditioner is to solve the system with the matrix A_i in each sub-domain. This can be interpret as a Richardson iterative method

$$u^{k+1} = u^k + \tau(B_1 + B_2)(b - Au^k) = u^k(\bar{b} - (P_1 + P_2)u^k), \quad \bar{b} = (P_1 + P_2)b.$$

The multiplicative version is defined as:

$$u^{k+1/2} = u^k + B_1(b - Au^k),$$

$$u^{k+1} = u^{k+1/2} + B_2(b - Au^{k+1/2}).$$

We can eliminate $u^{k+1/2}$ to get

$$u^{k+1} = u^k + (B_1 + B_2 - B_2 A B_1)(b - Au^k).$$

Therefore the multiplicative version is defined as:
$M_{MS}^{-1} = B_1 + B_2 - B_2 A B_1$ and the preconditioned matrix is $M_{MS}^{-1} A = P_1 + P_2 - P_2 P_1$. Again we can interpret the method as a Richardson method
$u^{k+1} = u^k + \tau\left(B_1 + B_2 - B_2 A B_1\right)\left(b - Au^k\right) = u^k\left(\bar{b} - \left(P_1 + P_2 - P_2 P_1\right)u^k\right), \quad \bar{b} = \left(P_1 + P_2 - P_2 P_1\right)b.$

We also will consider composite methods: multiplicative additive Schwarz
$M_{MAS}^{-1} = B_1 + \left(B_2^1 + B_2^2\right) - \left(B_2^1 + B_2^2\right)AB_1$, i.e., the second component of the multiplicative Schwarz is the additive Schwarz. Alternatively we can consider additive multiplicative Schwarz:
$M_{AMS}^{-1} = \left(B_1^1 + B_2^1 - B_2^1 A B_1^1\right) + \left(B_1^2 + B_2^2 - B_2^2 A B_1^2\right)$, i.e., additive Schwarz with each component multiplicative Schwarz.

We are interested in the case when the operators P_i are approximated with $\tilde{P}_i$. Consider the case for one operator and the corresponding Richardson method $u^{k+1} = u^k + \tau\left(f - Mu^k\right)$. We assume the Richardson method converges with a rate $q < 1$.We perturb the operator M, that is replace it with approximate operator $\tilde{M}$ then we have a modified Richardson method $\tilde{u}^{k+1} = \tilde{u}^k + \tau\left(f - \tilde{M}u^k\right)$, We use the following result [Nepomnyschikh (2090].

If $\left\|Mu - \tilde{M}u\right\| \le \varepsilon_M \left\|u\right\|$ and if $\tilde{q} = q + \varepsilon_M \dfrac{2}{\left\|M\right\|} < 1$ then the method converges and

$$\left\|u^k - \tilde{u}^k\right\| \le q^k \left\|u^0 - \tilde{u}^0\right\| + \varepsilon_M \frac{2}{\left\|M\right\|\left(1-q\right)}\left(\left\|\tilde{u}^0\right\| + \frac{2\left\|f\right\|}{\left\|M\right\|\left(1-\tilde{q}\right)}\right).$$

Note that the convergence of the perturbed method is controlled by how accurate the approximation is (ε_M) and how ill-conditioned the operator M is.

We use the simple result above to get some insight about the methods considered above. We interested in replacing the exact solves A_i^{-1} with approximate ones. It is relatively easy to control the error of the additive Schwarz method. Recall the operator is $M_{AS}^{-1} A = P_1 + P_2$ and the approximate operator is $\tilde{M}_{AS}^{-1} A = \tilde{P}_1 + \tilde{P}_2$. It is enough to make $\varepsilon_{\tilde{P}_1} + \varepsilon_{\tilde{P}_2} < 1$. $\left\|\tilde{M}_{AS}^{-1} A\right\|$ can be control by increased overlap and / or coarse grid correction. In reservoir simulation we also can make the matrix A easier by reducing the time step size. There are several complications with the additive Schwarz method. The first one is that not always the Richardson method for the unperturbed method converges even with exact solves. This method requires an iterative method (CG, GMRES, BiCGStab, etc.) as accelerator to guarantee the convergence. If $q_{AS} = 1$ then $\tilde{q}_{AS} > 1$. Nevertheless, knowing the factors that can improve the convergence is always helpful. With more elaborate analysis it is possible to show that if $\varepsilon_{\tilde{P}_1} < 1, \quad \varepsilon_{\tilde{P}_2} < 1$ then the method converges. Even with not very accurate sub-domain solves the method is robust. The second complication is that we use the more efficient restricted additive Schwarz [Cai and Sarkis (1999)] $M_{AR}^{-1} = \bar{R}_1^T A_1^{-1} R_1 + \bar{R}_2^T A_2^{-1} R_2$ where $\bar{R}_i$ only extracts the unknowns in the no overlapping part of Ω_i. There is still no comprehensive theory for the restricted additive Schwarz method.

Next we consider the multiplicative Schwarz method. It is known that the corresponding Richardson method converges, i.e., $q_{MS} < 1$. It is clear that if we make the approximate solves in the sub-domains accurate enough then $\tilde{q}_{MS} < 1$.Our experience is that $\varepsilon_{\tilde{P}_1} \approx 0.1, \quad \varepsilon_{\tilde{P}_2} \approx 0.1$ work well.

The analysis of the multiplicative additive Schwarz method is still not complete. In general we cannot guarantee that it will always converge because of the facts discussed in the analysis of the additive method. Nevertheless, we can troubleshoot using the insights. The additive multiplicative Schwarz method is easier to analyze. We need each sub-domain solve to be relatively accurate (see the recommendation in the discussion for the additive method). Therefore it is enough the two sub-domain solves to have some accuracy as discussed in the analysis of the multiplicative method.

AMG for Reservoir simulation problems

Our experiments are focusing on solving pressure systems. We use AMG as a preconditioner with an acceleration method FGMRES. The matrix we are dealing with consists of two parts: reservoir part and well part. The reservoir block represents pressure equation coefficients which usually have desirable algebraic properties although they can be affected by factors such

as time step size, pore volume size, etc. The well block represents the well equation coefficients which usually does not have the same algebraic properties as the reservoir block has.

Directly applying AMG to the whole system usually causes slow convergence and sometimes makes AMG fail to converge. This is because the well block does not have elliptic property required by AMG method. One way to alleviate this problem is to separate the well block from the reservoir block and solve them separately by using the multiplicative Schwarz method. We method is rewritten in the expanded matrix form below. Suppose the matrix allows the following block decomposition:

$$Ax = \begin{bmatrix} A_r & A_{ro} & \\ A_{br} & A_o & A_{ow} \\ & A_{wo} & A_w \end{bmatrix} \begin{bmatrix} x_r \\ x_o \\ x_w \end{bmatrix} = \begin{bmatrix} b_r \\ b_o \\ b_w \end{bmatrix} = b,$$

with the subscripts r, o, w representing reservoir, overlapping part, and well respectively (see Figure 1, right graph). Usually the overlap is one layer.

The preconditioner process is defined as follows:

Algorithm 1.

1. Solve the well system:

$$A_W x_W = \begin{bmatrix} A_o & A_{ow} \\ A_{wo} & A_w \end{bmatrix} \begin{bmatrix} x_{ow} \\ x_w \end{bmatrix} = \begin{bmatrix} b_o \\ b_w \end{bmatrix} = b_W ,$$

2. Compute the new residual for the reservoir system:

$$\hat{r}_R = \begin{bmatrix} b_r \\ b_o \end{bmatrix} - \begin{bmatrix} A_{ro} & \\ A_o & A_{ow} \end{bmatrix} \begin{bmatrix} x_{ow} \\ x_w \end{bmatrix},$$

3. Solve the reservoir system with AMG:

$$A_R x_R = \begin{bmatrix} A_r & A_{ro} \\ A_{or} & A_r \end{bmatrix} \begin{bmatrix} x_r \\ x_{or} \end{bmatrix} = \begin{bmatrix} \hat{r}_r \\ \hat{r}_o \end{bmatrix} = \hat{r}_R$$

4. Form the final solution:

$$x = \begin{bmatrix} x_r \\ x_{ow} + x_{or} \\ x_w \end{bmatrix}.$$

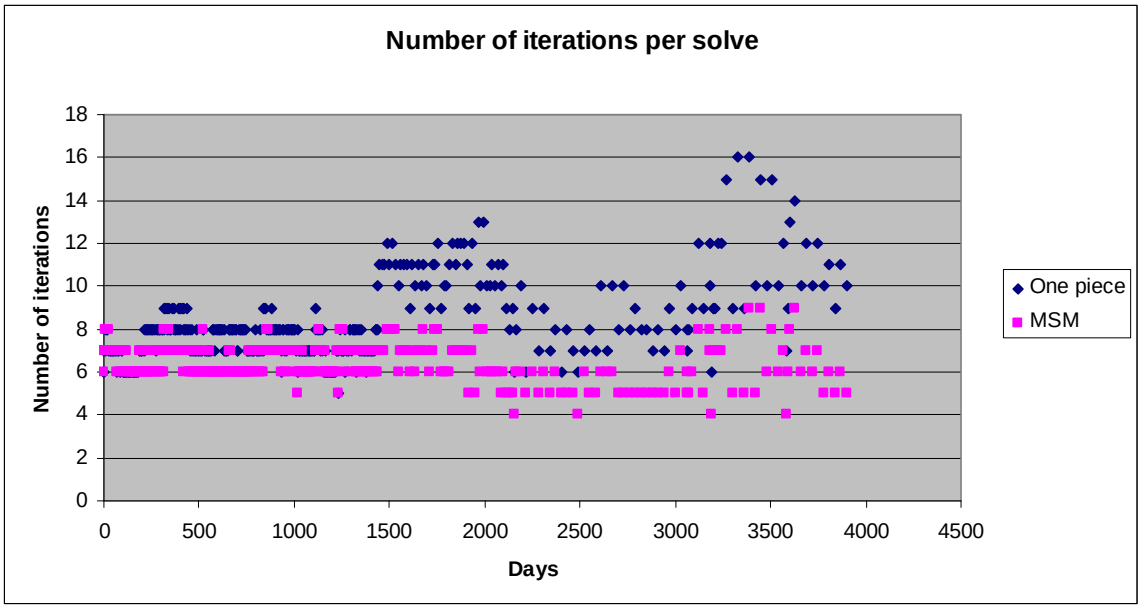

Figure 2: Number of iterations per solve for Multiplicative Schwarz and AMG for the whole system

If the calculations in the process are accurate, the residual components r_r and r_b will be zero and $r_w = -A_{wo} r_{or}$. Several comments are in order. We use the insights form the analysis of the multiplicative Schwarz method in the previous section. We need each solve to have some accuracy. The well block is usually much smaller than the reservoir block and the cost for solving it is negligible compared to the cost of solving the reservoir block. An iterative or direct solver can be used to solve it. In our case, we use iterative method BiCGStab with FILU preconditioner as the well block solver and set the tolerance to 1.e-2. We use only one V-cycle with one smoothing iteration on each level up and down for the reservoir block. AMG usually exhibits uniform convergence and one V-cycle provides enough accuracy. We do not recommend performing more than one multiplicative Schwarz iteration as a preconditioner. We did not see any performance benefit from applying more than one iteration in our experiments. Specifically, the second well block solve after the reservoir solve does not reduce the number of iterations.

We compared the performance of the multiplicative Schwarz method (MSM) using AMG for the reservoir block to that using AMG for solving the whole system (One Piece). One-piece approach often caused the solver to diverge if the model contains hydraulic wells. Even for models with simple wells one-piece approach usually takes more iterations. Figure 2 and 3 show the comparison on the number of solver iterations per solve and cumulative iterations of the whole run respectively for a representative real simulation model with more than 100K nodes and about 80 simple well nodes.

In our implementation, the AMG configuration and parameter settings are selected based on robustness, user-friendly and performance so that the users have to do very little parameter adjustment to achieve a consistent good performance on a broad range of models. We tested our AMG solver on 29 models of a variety of model types, sizes and features. Results show that AMG perform very well regardless of model type or features. Figure 4 shows the performance comparison between FGMRES-AMG and BiCGStab-FILU in terms of percentage difference on pressure solver CPU time defined as $\left(\left(T_{AMG} - T_{FILU}\right)/T_{FILU}\right)\times 100\%$. For relatively easy problems (defined as problems that require less that forty iterations with a solver using FILU as a preconditioner) FILU is faster. This is shown on Figure 4 as points with positive y coordinate. AMG is faster for more difficult problems (more than forty iterations with FILU). The explanation for the existence such a threshold is the fixed initialization cost of AMG and the number of iterations is relatively insensitive to the difficulty of the problem. It turned out that the size of the problem is not always a good indicator for the difficulty of the problem. We encountered several small (~20-30K node), but extremely heterogeneous models for which AMG was considerably faster than FILU.

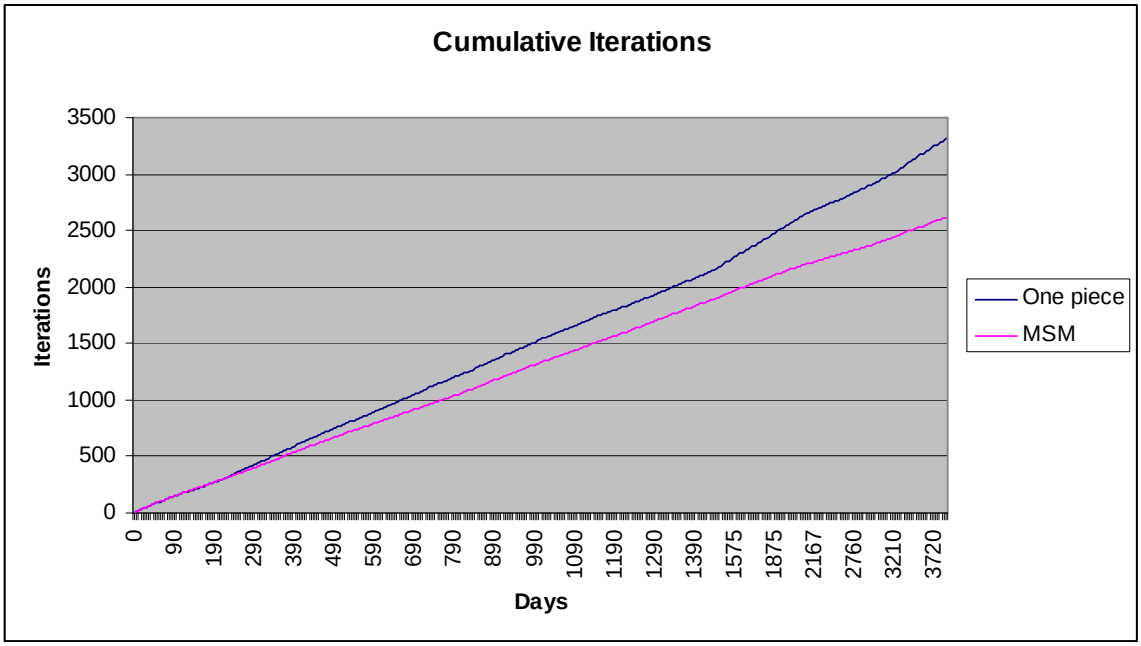

Figure 3: Cumulative number of iterations per solve for Multiplicative Schwarz and AMG for the whole system

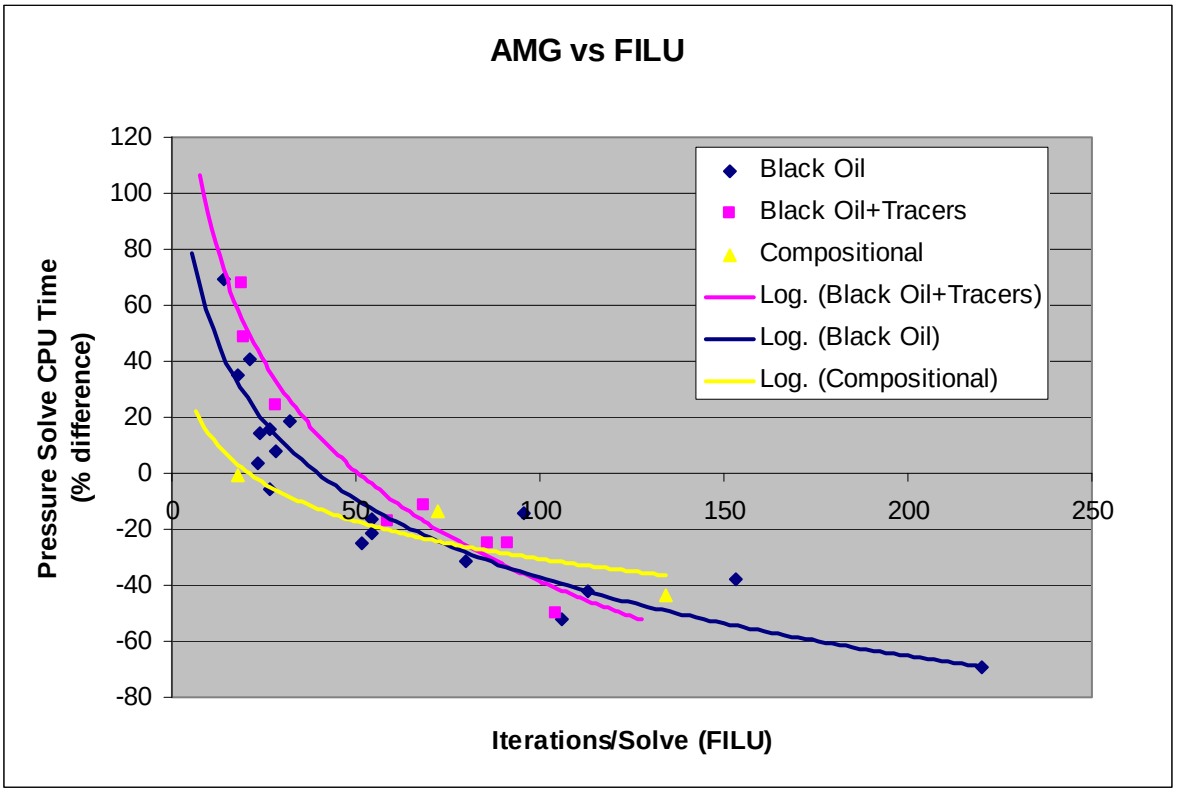

Figure 4: Performance comparison between AMG and FILU solvers.

Multiplicative Schwarz method for Adaptive Implicit (AIM) models

The success of MSM for the pressure equation (FILU for the well block and AMG for the reservoir block) encouraged us to try it for AIM formulation. The algorithm is the same as in the previous section, the well block replaced with the implicit block and the reservoir block replaced with the IMPES block. The wells are assumed to be in the implicit block. First we experimented with a small implicit region. We present the results for a model with about 20K cells, 1% of them implicit in the left graph of Figure 5. Four layers of overlap were used. The FILU solver performed very well with average 37 iterations per linear solve. MSM was about 10% faster than the FILU solver with about sixteen iterations per linear solve. We expect for more difficult for FILU problems the gain from MSM to be even bigger. The second example has considerably bigger implicit region (120K cell, 14K implicit). We used six layers of overlap. The FILU solver was more efficient for this example. The explanation is that the bigger size of the implicit domain required more global iterations (more than 20 on average) and the implicit domain with the overlap required many local iterations.

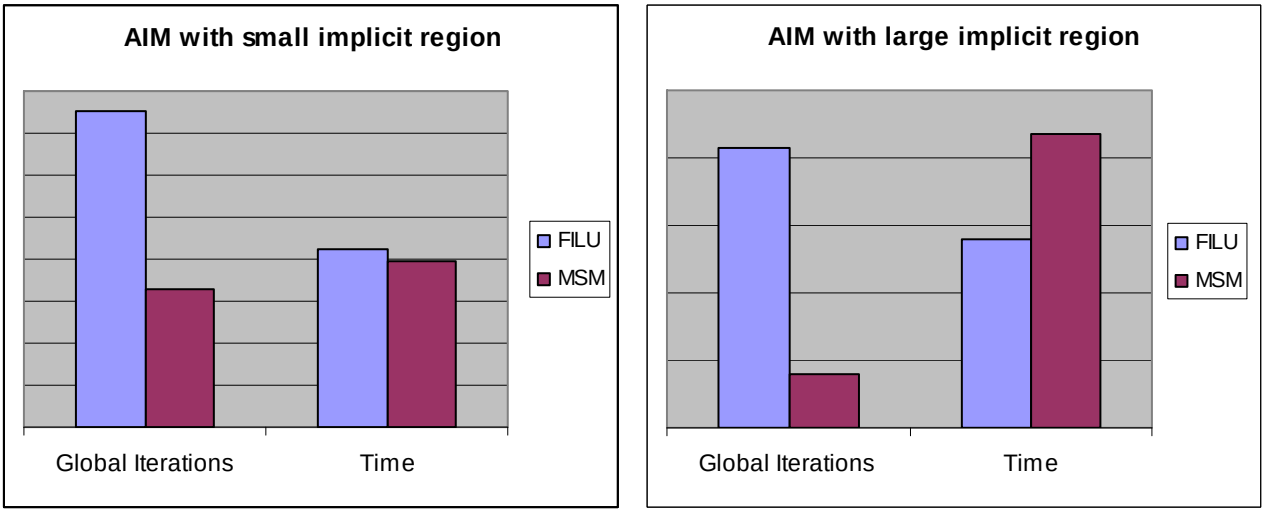

Figure 5: MSM for AIM problems. Left graph small implicit region, right graph large implicit region..

AMG as a subdomain solver for the overlapping additive Schwarz method

We have considerable experience using additive Schwarz method for parallel reservoir simulations [Usadi et al (2007)]. The results reported in the previous SPE paper were for additive Schwarz method with FILU as a sub-domain solver. In particular only one preconditioner solve was performed with FILU without the iterative method as accelerator. These not very accurate sub-domain solves led to high number of global iterations, but the total CPU time was much smaller compared with solvers with more accurate sub-domain solvers. We compared the two sub-domain solve on a representative model, black oil, sequential formulation, 350K cells, 2000 day simulation with 1300 linear solves. The results are presented in Figure 6 for FILU preconditioner solve and FILU solve with tolerance 0.1. It is clear that FILU preconditioner sub-domain solve provides considerably more efficient preconditioner.

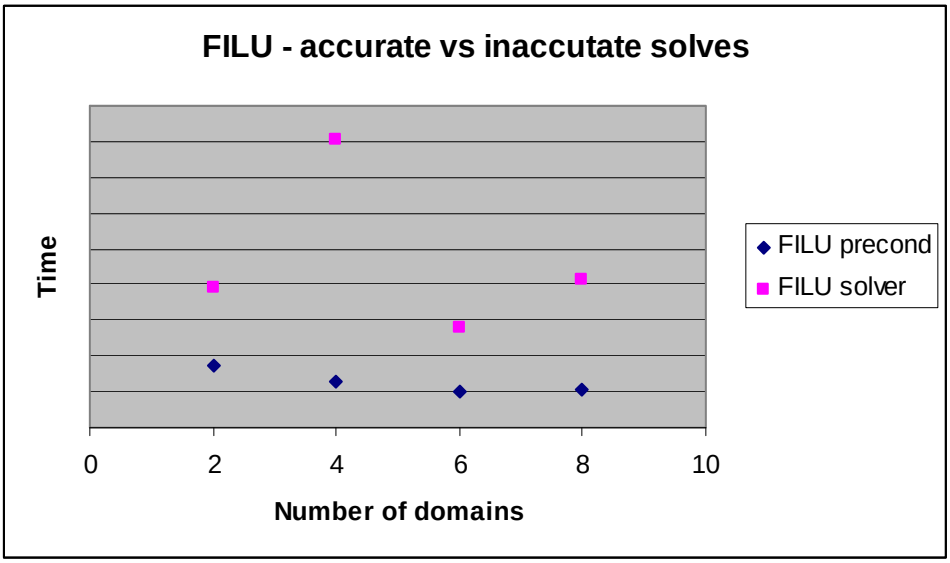

Figure 6: Inaccurate (FILU preconditioner solve) versus accurate (FILU solve with tolerance 0.1).

Next we investigate the use of AMG as a sub-domain solver. Usually each sub-domain contains wells and therefore we have to use a composite algorithm: multiplicative additive or additive multiplicative Schwarz method. We tested the multiplicative additive methods. It is essentially **Algorithm 1** with step 3 changed to "Solve the reservoir system with the restricted additive Schwarz [Cai and Sarkis (1999)] using for the sub-domain solves AMG". We apply only one V-cycle in each sub-domain. The general theory predicts that increase overlap leads to smaller number of iterations. On the other hand bigger overlap requires solving bigger problems. These two trends balance each other and there is an optimal number of overlapping layers for a given number of sub-domains and fixed partition that provide the best performance. The results in Figure 7 confirm that for AMG. Apparently one level of overlap does not provide fast convergence. Even there were some divergent solves for four domains. That can be explained with the theoretical argument discussed earlier. It seems that the bigger overlap is more important for the smaller number of domains.

The bigger overlap did not improve the performance when FILU was used as a sub-domain solver. The explanation is that the solve is not accurate enough and the bigger overlap did not provide better approximation, but increased the cost. In general we do not recommend use of bigger overlap for additive Schwarz with FILU sub-domain solves, but we observed that for models with very bad domain partition bigger overlap can improve performance. Better domain partition will improve it even more even with minimal overlap.

Conclusions

We considered the different ways AMG can be used as a component in efficient solvers for reservoir simulation problems. Our experiments confirmed that AMG is one of the best serial solvers for our applications. Further research is required to improve the performance for the AIM formulation models.

Acknowledgements

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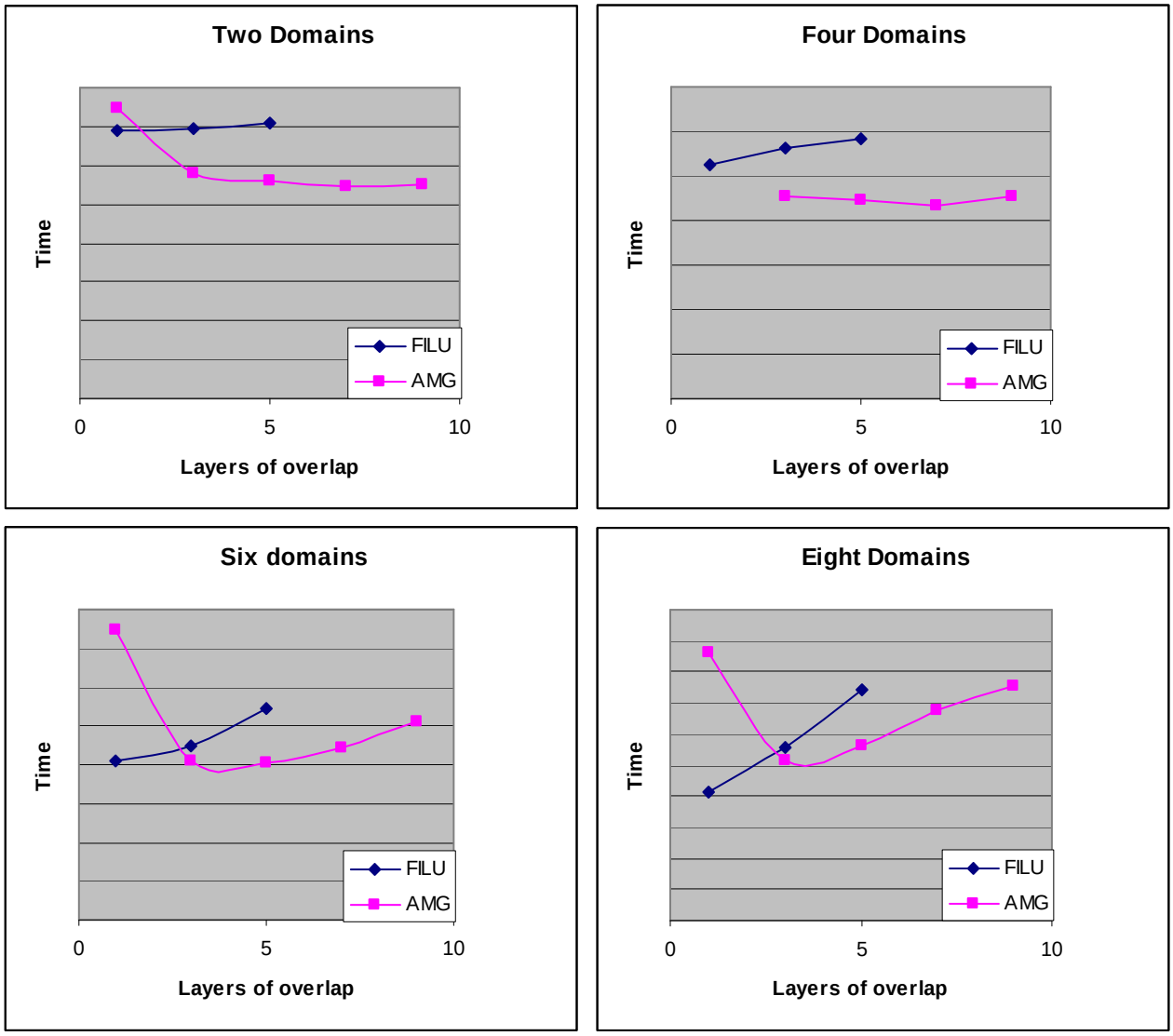

Figure 7: Influence f the number of overlapping layers on the performance of FILU and AMG sub-domain solvers.

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