5. Conjugate functions

- closed functions
- conjugate function
- duality

Closed set

a set C is **closed** if it contains its boundary:

$$x_k \in C, \quad x_k \to \bar{x} \qquad \Longrightarrow \qquad \bar{x} \in C$$

Operations that preserve closedness

- the intersection of (finitely or infinitely many) closed sets is closed
- the union of a finite number of closed sets is closed
- inverse under linear mapping: $\{x \mid Ax \in C\}$ is closed if C is closed

Image under linear mapping

the image of a closed set under a linear mapping is not necessarily closed

Example

$$C = \{(x_1, x_2) \in \mathbf{R}^2_+ \mid x_1 x_2 \geq 1\}, \qquad A = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad AC = \mathbf{R}_{++}$$

Sufficient condition: AC is closed if

- C is closed and convex
- and C does not have a recession direction in the nullspace of A, *i.e.*,

$$Ay = 0, \quad \hat{x} \in C, \quad \hat{x} + \alpha y \in C \text{ for all } \alpha \geq 0 \qquad \Longrightarrow \qquad y = 0$$

in particular, this holds for any matrix A if C is bounded

Closed function

Definition: a function is closed if its epigraph is a closed set

Examples

- $f(x) = -\log(1-x^2)$ with $\mathbf{dom}\, f = \{x \mid |x| < 1\}$
- $f(x) = x \log x$ with $\mathbf{dom}\, f = \mathbf{R}_+$ and $f(0) = 0$
- indicator function of a closed set C:

$$\delta_C(x) = \begin{cases} 0 & x \in C \\ +\infty & \text{otherwise} \end{cases}$$

Not closed

- $f(x) = x \log x$ with $\mathbf{dom}\, f = \mathbf{R}_{++}$, or with $\mathbf{dom}\, f = \mathbf{R}_+$ and $f(0) = 1$
- indicator function of a set C if C is not closed

Properties

Sublevel sets: f is closed if and only if all its sublevel sets are closed

Minimum: if f is closed with bounded sublevel sets then it has a minimizer

Common operations on convex functions that preserve closedness

- *sum:* $f = f_1 + f_2$ is closed if f_1 and f_2 are closed
- *composition with affine mapping:* $f = g(Ax + b)$ is closed if g is closed
- *supremum:* $f(x) = \sup_\alpha f_\alpha(x)$ is closed if each function f_α is closed

in each case, we assume $\mathrm{dom}\, f \neq \emptyset$

Outline

- closed functions
- **conjugate function**
- duality

Conjugate function

the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \operatorname{dom} f} (y^T x - f(x))$$

f^* is closed and convex (even when f is not)

Fenchel's inequality: the definition implies that

$$f(x) + f^*(y) \geq x^T y \quad \text{for all } x, y$$

this is an extension to non-quadratic convex f of the inequality

$$\frac{1}{2} x^T x + \frac{1}{2} y^T y \geq x^T y$$

Quadratic function

$$f(x) = \frac{1}{2}x^T Ax + b^T x + c$$

Strictly convex case ($A \succ 0$)

$$f^*(y) = \frac{1}{2}(y - b)^T A^{-1}(y - b) - c$$

General convex case ($A \succeq 0$)

$$f^*(y) = \frac{1}{2}(y - b)^T A^{\dagger}(y - b) - c, \qquad \mathrm{dom}\, f^* = \mathrm{range}(A) + b$$

Negative entropy and negative logarithm

Negative entropy

$$f(x) = \sum_{i=1}^{n} x_i \log x_i \qquad\qquad f^*(y) = \sum_{i=1}^{n} e^{y_i - 1}$$

Negative logarithm

$$f(x) = -\sum_{i=1}^{n} \log x_i \qquad\qquad f^*(y) = -\sum_{i=1}^{n} \log(-y_i) - n$$

Matrix logarithm

$$f(X) = -\log\det X \quad (\mathrm{dom}\, f = \mathbf{S}^n_{++}) \qquad f^*(Y) = -\log\det(-Y) - n$$

Indicator function and norm

Indicator of convex set C: conjugate is the *support function* of C

$$\delta_C(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases} \qquad \delta_C^*(y) = \sup_{x \in C} y^T x$$

Indicator of convex cone C: conjugate is indicator of polar (negative dual) cone

$$\delta_C^*(y) = \delta_{-C^*}(y) = \delta_{C^*}(-y) = \begin{cases} 0 & y^T x \leq 0 \;\; \forall x \in C \\ +\infty & \text{otherwise} \end{cases}$$

Norm: conjugate is indicator of unit ball for dual norm

$$f(x) = \|x\| \qquad f^*(y) = \begin{cases} 0 & \|y\|_* \leq 1 \\ +\infty & \|y\|_* > 1 \end{cases}$$

(see next page)

Proof: recall the definition of dual norm

$$\|y\|_* = \sup_{\|x\| \leq 1} x^T y$$

to evaluate $f^*(y) = \sup_x (y^T x - \|x\|)$ we distinguish two cases

- if $\|y\|_* \leq 1$, then (by definition of dual norm)

$$y^T x \leq \|x\| \quad \text{for all } x$$

  and equality holds if $x = 0$; therefore $\sup_x (y^T x - \|x\|) = 0$

- if $\|y\|_* > 1$, there exists an x with $\|x\| \leq 1$, $x^T y > 1$; then

$$f^*(y) \geq y^T (tx) - \|tx\| = t(y^T x - \|x\|)$$

  and right-hand side goes to infinity if $t \to \infty$

Calculus rules

Separable sum

$$f(x_1, x_2) = g(x_1) + h(x_2) \qquad\qquad f^*(y_1, y_2) = g^*(y_1) + h^*(y_2)$$

Scalar multiplication ($\alpha > 0$)

$$f(x) = \alpha g(x) \qquad\qquad f^*(y) = \alpha g^*(y/\alpha)$$

$$f(x) = \alpha g(x/\alpha) \qquad\qquad f^*(y) = \alpha g^*(y)$$

- the operation $f(x) = \alpha g(x/\alpha)$ is sometimes called "right scalar multiplication"
- a convenient notation is $f = g\alpha$ for the function $(g\alpha)(x) = \alpha g(x/\alpha)$
- conjugates can be written concisely as $(g\alpha)^* = \alpha g^*$ and $(\alpha g)^* = g^*\alpha$

Calculus rules

Addition to affine function

$$f(x) = g(x) + a^T x + b \qquad f^*(y) = g^*(y - a) - b$$

Translation of argument

$$f(x) = g(x - b) \qquad f^*(y) = b^T y + g^*(y)$$

Composition with invertible linear mapping (A square and nonsingular)

$$f(x) = g(Ax) \qquad f^*(y) = g^*(A^{-T} y)$$

Infimal convolution

$$f(x) = \inf_{u+v=x} (g(u) + h(v)) \qquad f^*(y) = g^*(y) + h^*(y)$$

The second conjugate

$$f^{**}(x) = \sup_{y \in \operatorname{dom} f^*} (x^T y - f^*(y))$$

- f^{**} is closed and convex
- from Fenchel's inequality, $x^T y - f^*(y) \le f(x)$ for all y and x; therefore

$$f^{**}(x) \le f(x) \quad \text{for all } x$$

  equivalently, $\operatorname{epi} f \subseteq \operatorname{epi} f^{**}$ (for any f)

- if f is closed and convex, then

$$f^{**}(x) = f(x) \quad \text{for all } x$$

  equivalently, $\operatorname{epi} f = \operatorname{epi} f^{**}$ (if f is closed and convex); proof on next page

Proof (by contradiction): assume f is closed and convex, and $\mathrm{epi}\, f^{**} \neq \mathrm{epi}\, f$

suppose $(x, f^{**}(x)) \notin \mathrm{epi}\, f$; then there is a strict separating hyperplane:

$$\begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} z - x \\ s - f^{**}(x) \end{bmatrix} \leq c < 0 \qquad \text{for all } (z, s) \in \mathrm{epi}\, f$$

for some a, b, c with $b \leq 0$ ($b > 0$ gives a contradiction as $s \to \infty$)

- if $b < 0$, define $y = a/(-b)$ and maximize left-hand side over $(z, s) \in \mathrm{epi}\, f$:

  $$f^*(y) - y^T x + f^{**}(x) \leq c/(-b) < 0$$

  this contradicts Fenchel's inequality

- if $b = 0$, choose $\hat{y} \in \mathrm{dom}\, f^*$ and add small multiple of $(\hat{y}, -1)$ to (a, b):

  $$\begin{bmatrix} a + \epsilon \hat{y} \\ -\epsilon \end{bmatrix}^T \begin{bmatrix} z - x \\ s - f^{**}(x) \end{bmatrix} \leq c + \epsilon \left(f^*(\hat{y}) - x^T \hat{y} + f^{**}(x) \right) < 0$$

  now apply the argument for $b < 0$

Conjugates and subgradients

if f is closed and convex, then

$$y \in \partial f(x) \quad \Longleftrightarrow \quad x \in \partial f^*(y) \quad \Longleftrightarrow \quad x^T y = f(x) + f^*(y)$$

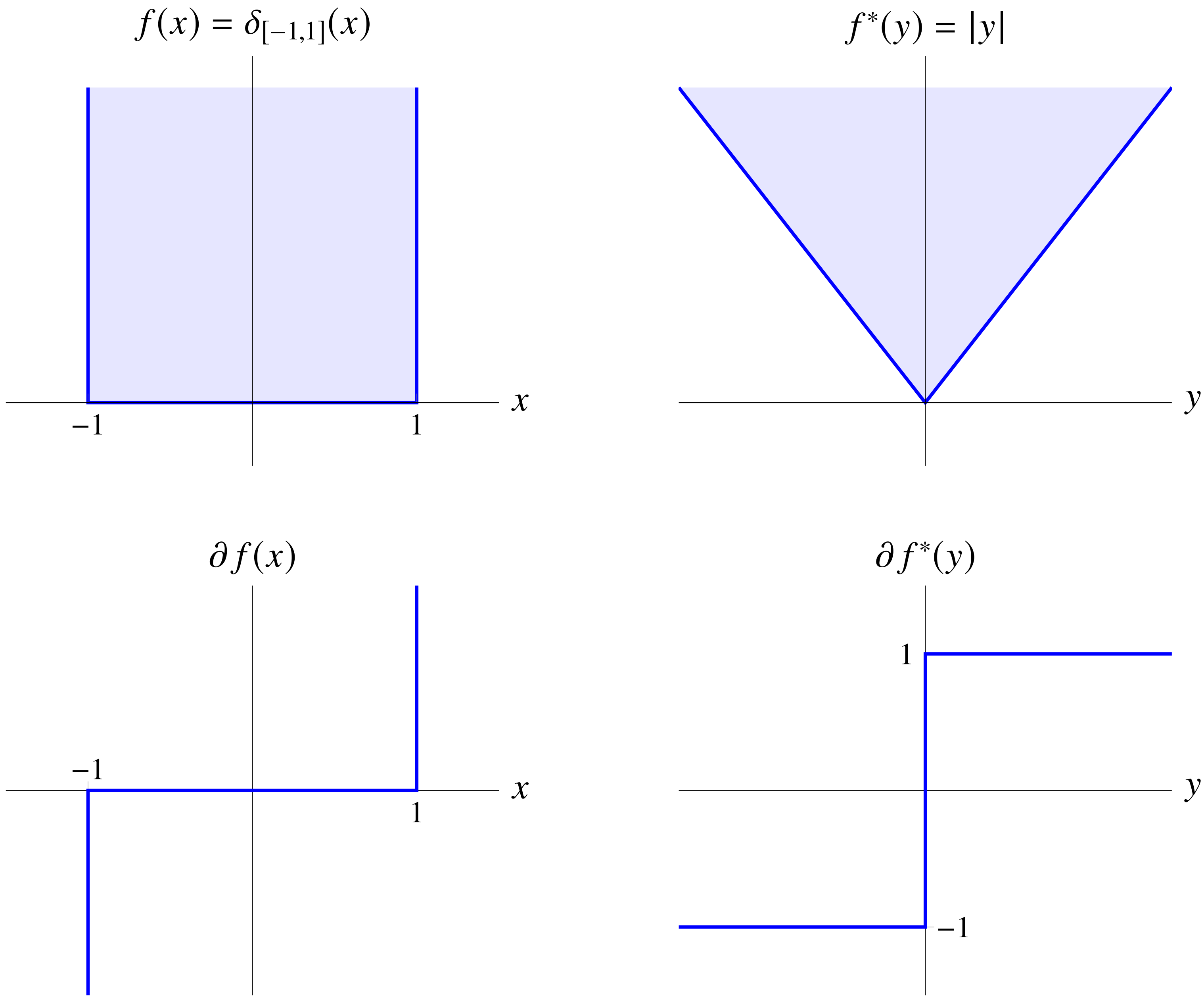
Proof. if $y \in \partial f(x)$, then $f^*(y) = \sup_u (y^T u - f(u)) = y^T x - f(x)$; hence

$$\begin{aligned} f^*(v) &= \sup_u (v^T u - f(u)) \\ &\geq v^T x - f(x) \\ &= x^T(v - y) - f(x) + y^T x \\ &= f^*(y) + x^T(v - y) \end{aligned}$$

this holds for all v; therefore, $x \in \partial f^*(y)$

reverse implication $x \in \partial f^*(y) \Longrightarrow y \in \partial f(x)$ follows from $f^{**} = f$

Example



Strongly convex function

Definition (page 1.18) f is μ-strongly convex (for $\|\cdot\|$) if $\operatorname{dom} f$ is convex and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y) - \frac{\mu}{2}\theta(1-\theta)\|x-y\|^2$$

for all $x, y \in \operatorname{dom} f$ and $\theta \in [0,1]$

First-order condition

- if f is μ-strongly convex, then

$$f(y) \geq f(x) + g^T(y-x) + \frac{\mu}{2}\|y-x\|^2 \qquad \text{for all } x, y \in \operatorname{dom} f,\ g \in \partial f(x)$$

- for differentiable f this is the inequality (4) on page 1.19

Proof

- recall the definition of directional derivative (page 2.28 and 2.29):

$$f'(x, y - x) = \inf_{\theta > 0} \frac{f(x + \theta(y - x)) - f(x)}{\theta}$$

and the infimum is approached as $\theta \to 0$

- if f is μ-strongly convex and subdifferentiable at x, then for all $y \in \operatorname{dom} f$,

$$\begin{aligned} f'(x, y - x) &\leq \inf_{\theta \in (0,1]} \frac{(1 - \theta)f(x) + \theta f(y) - (\mu/2)\theta(1 - \theta)\|y - x\|^2 - f(x)}{\theta} \\ &= f(y) - f(x) - \frac{\mu}{2}\|y - x\|^2 \end{aligned}$$

- from page 2.31, the directional derivative is the support function of $\partial f(x)$:

$$\begin{aligned} g^T(y - x) &\leq \sup_{\tilde{g} \in \partial f(x)} \tilde{g}^T(y - x) \\ &= f'(x; y - x) \\ &\leq f(y) - f(x) - \frac{\mu}{2}\|y - x\|^2 \end{aligned}$$

Conjugate of strongly convex function

assume f is closed and strongly convex with parameter $\mu > 0$ for the norm $\|\cdot\|$

- f^* is defined for all y (*i.e.*, $\mathrm{dom}\, f^* = \mathbf{R}^n$)
- f^* is differentiable everywhere, with gradient

$$\nabla f^*(y) = \underset{x}{\operatorname{argmax}}\, (y^T x - f(x))$$

- ∇f^* is Lipschitz continuous with constant $1/\mu$ for the dual norm $\|\cdot\|_*$:

$$\|\nabla f^*(y) - \nabla f^*(y')\| \le \frac{1}{\mu}\|y - y'\|_* \quad \text{for all } y \text{ and } y'$$

Proof: if f is strongly convex and closed

- $y^T x - f(x)$ has a unique maximizer x for every y
- x maximizes $y^T x - f(x)$ if and only if $y \in \partial f(x)$; from page 5.15

$$y \in \partial f(x) \qquad \Longleftrightarrow \qquad x \in \partial f^*(y) = \{\nabla f^*(y)\}$$

  hence $\nabla f^*(y) = \operatorname{argmax}_x (y^T x - f(x))$

- from first-order condition on page 5.17: if $y \in \partial f(x)$, $y' \in \partial f(x')$:

$$\begin{aligned} f(x') &\geq f(x) + y^T(x' - x) + \frac{\mu}{2}\|x' - x\|^2 \\ f(x) &\geq f(x') + (y')^T(x - x') + \frac{\mu}{2}\|x' - x\|^2 \end{aligned}$$

  combining these inequalities shows

$$\mu\|x - x'\|^2 \leq (y - y')^T(x - x') \leq \|y - y'\|_* \|x - x'\|$$

- now substitute $x = \nabla f^*(y)$ and $x' = \nabla f^*(y')$

Outline

- closed functions
- conjugate function
- **duality**

Duality

$$\begin{array}{lll} \text{primal:} & \text{minimize} & f(x) + g(Ax) \\ \text{dual:} & \text{maximize} & -g^*(z) - f^*(-A^T z) \end{array}$$

- follows from Lagrange duality applied to reformulated primal

$$\begin{array}{ll} \text{minimize} & f(x) + g(y) \\ \text{subject to} & Ax = y \end{array}$$

dual function for the formulated problem is:

$$\inf_{x,y} \left(f(x) + z^T Ax + g(y) - z^T y\right) = -f^*(-A^T z) - g^*(z)$$

- Slater's condition (for convex f, g): strong duality holds if there exists an $\hat{x}$ with

$$\hat{x} \in \operatorname{int} \operatorname{dom} f, \qquad A\hat{x} \in \operatorname{int} \operatorname{dom} g$$

this also guarantees that the dual optimum is attained, if optimal value is finite

Set constraint

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax - b \in C \end{array}$$

Primal and dual problem

$$\begin{array}{lll} \text{primal:} & \text{minimize} & f(x) + \delta_C(Ax - b) \\ \text{dual:} & \text{maximize} & -b^T z - \delta_C^*(z) - f^*(-A^T z) \end{array}$$

Examples

	constraint	set C	support function $\delta_C^*(z)$
equality	$Ax = b$	$\{0\}$	0
norm inequality	$\|Ax - b\| \leq 1$	unit $\|\cdot\|$-ball	$\|z\|_*$
conic inequality	$Ax \preceq_K b$	$-K$	$\delta_{K^*}(z)$

Norm regularization

$$\text{minimize} \quad f(x) + \|Ax - b\|$$

- take $g(y) = \|y - b\|$ in general problem

$$\text{minimize} \quad f(x) + g(Ax)$$

- conjugate of $\|\cdot\|$ is indicator of unit ball for dual norm

$$g^*(z) = b^T z + \delta_B(z) \quad \text{where } B = \{z \mid \|z\|_* \leq 1\}$$

- hence, dual problem can be written as

$$\begin{array}{ll} \text{maximize} & -b^T z - f^*(-A^T z) \\ \text{subject to} & \|z\|_* \leq 1 \end{array}$$

Optimality conditions

$$\begin{array}{ll} \text{minimize} & f(x) + g(y) \\ \text{subject to} & Ax = y \end{array}$$

assume f, g are convex and Slater's condition holds

Optimality conditions: x is optimal if and only if there exists a z such that

1. primal feasibility: $x \in \operatorname{dom} f$ and $y = Ax \in \operatorname{dom} g$
2. x and $y = Ax$ are minimizers of the Lagrangian $f(x) + z^T Ax + g(y) - z^T y$:

$$-A^T z \in \partial f(x), \qquad z \in \partial g(Ax)$$

if g is closed, this can be written symmetrically as

$$-A^T z \in \partial f(x), \qquad Ax \in \partial g^*(z)$$

References

- J.-B. Hiriart-Urruty, C. Lemaréchal, *Convex Analysis and Minimization Algoritms* (1993), chapter X.
- D.P. Bertsekas, A. Nedić, A.E. Ozdaglar, *Convex Analysis and Optimization* (2003), chapter 7.
- R. T. Rockafellar, *Convex Analysis* (1970).