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Algebraic Multilevel Method AMG: Comparison with the Method BICGSTAB + ILU and its Use in the Method CPR

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Abstract—The algebraic multilevel method AMG is compared with the method BICGSTAB+ILU on a model problem with a strong anisotropy. The method AMG was used as a part of the method CPR for solution of the filtration problems of a viscous compressible fluid flow in porous media. The algorithm CPR+AMG is compared with CPR+ILU on the base of this problem. The work of the CPR+AMG is analyzed by solving the problem for a model oil field.

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Introduction. The algebraic multilevel method AMG (Algebraic Multigrid) is one of the most popular solvers for system of linear equations obtained in approximation of differential equations by finite elements or finite difference methods [1, 2]. This method and its modifications were considered in detail in [3, 4]. The authors of those papers also implemented the algorithm in FORTRAN (AMG1R1 code).

In this paper we describe the method AMG and indicate its modifications for the use within the method CPR. The algorithm AMG was implemented in C++ and compared with the method of biconjugate gradients with the incomplete LU preconditioner on the example of a problem simulating the heat distribution in a strongly anisotropic medium. Moreover, the qualitative behavior of the algorithm was studied depending on the rate of the anisotropy.

The method AMG is used within the method CPR (Constrained Pressure Residual [5]) for solution of the problem of filtration of a viscous compressible fluid in a porous medium [6]. The combination of the methods CPR + AMG is the preconditioner for the system of linear equations obtained at each Newton iteration step in solution of the filtration scheme by an implicit scheme. The method CPR + AMG is compared with the method CPR + ILU on this problem taken as an example. The analysis of the work of CPR + AMG is performed in solution of the problem for a model oil field for different values of the parameters of the model.

Description of the algorithm AMG. The detailed description of the algorithm was given in [3, 4]. We consider the system of linear equations $A * u = f$. In order to apply the algebraic multilevel method, the matrix A must satisfy the following conditions:

A is a positive definite square matrix;

A has an "essentially" positive type, i.e.,

A is a sparse matrix;

all its diagonal elements are positive;

the most of off-diagonal elements are less than or equal to zero;

the row sums of elements are greater than or equal to zero (weak diagonal dominance).

Consider the two-level algebraic algorithm [3, 4] for solution of the system of linear equations $A * u = f$ (the multilevel variant is constructed similarly).

The terminology of fictitious grids is used in the algebraic method (h and H are the fine and coarse grids, respectively). Let the ith point be each element $i \in \Omega^h$, $\Omega^h = \{1, 2, \ldots, n\}$, where i is the row (column) number in the original matrix. The matrix A is represented as a point connectivity graph. A point $i \in \Omega^h$ is connected to a point $j \in \Omega^h$ if $a^h_{ij} \neq 0$. The system of linear equations can be written as the following grid equation on the fictitious fine grid Ω^h: $A_h u^h = f^h$.

On the coarse grid we have $A_H u^H = f^H$, where $A_H = I^H_h A_h I^h_H$.

The two-level algorithm has the form $u^h_{\text{new}} = u^h_{\text{old}} + I^h_H e^H$, where $A_H e^H = r^H = I^H_h(r^h_{\text{old}}) = I^H_h(f^h - A_h u^h_{\text{old}})$.

The method AMG may be divided into two phases: initialization and solution.

The initialization phase includes the construction of the F/C-partitioning and the operators I_H^h and I_h^H for each grid. In this case the partitioning and the operators must be such that the operator A_H remains sparse and is of considerably lesser dimension than A_h, and such that a sufficient convergence of the algorithm would be attained. The initialization phase can be implemented once in the case when AMG is used as a preconditioner in the iterative algorithm.

The solution phase includes the iterative process whose each iteration is a V-, W-, or F-cycle with the smoothing Gauss–Seidel procedure and the solution of the system on the most coarse level.

Comparison of the methods AMG and BICGSTAB+ILU on a problem with a strong anisotropy. The algorithm described here was implemented in C++, numerical experiments on the example of a problem with a strong anisotropy were performed.

Tables 1 and 2 present the results of computer simulation compared to the method BICGSTAB+ILU [7]. The tests were performed on the computer AMD Athlon, 1833 MHz, 512 Kb.

Consider the problem $-\mathrm{div}(k\nabla u) + \lambda u = f$ with zero boundary conditions in the domain $[0,1] \times [0,1]$. We use a uniform grid with 4096 nodes in x and y directions.

Example 1. Take the matrix k of the following form: $k = \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix}$.

Table 1

Method	Parameter w					
	3	100	500	2000	10000	40000
AMG	2(10)	2(8)	1(7)	1(7)	1(7)	1(7)
BICGSTAB+ILU	8(53)	13(87)	23(148)	36(228)	61(373)	80(471)

We compare the computation speed of the algorithms for different values of the parameter w. AMG uses the V-cycle. Table 1 presents the comparison of the computation time and the number of iterations (the latter is presented in brackets). The table shows that with the growth of the parameter w the algorithm AMG more and more surpasses the method BICGSTAB + ILU in computation time.

Example 2. Now consider the case of the W-cycle, k is the following matrix: $k = \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix}$.

Table 2

Method	Parameter ε					
	0.001	0.01	0.03	0.035	0.038	0.042
AMG	4(10)	5(10)	7(16)	11(24)	20(40)	Diverges
BICGSTAB+ILU	10(69)	10(68)	12(71)	12(81)	13(85)	21(138)

For small ε the method AMG surpasses BICGSTAB + ILU in speed, however, when the parameter ε increases the method loses its advantage (Table 2). Further the algorithm AMG begins to diverge, whereas BICGSTAB + ILU still converges.

Application of AMG within the method CPR for solution of a filtration problem. The problem of filtration of a viscous compressible mixture in a porous medium (the detailed statement was given in [6]) is written in the following form:

$$\begin{aligned} &\frac{\partial}{\partial t}(\phi N_c) = \mathrm{div} \sum_{P=O,W,G} x_{c,P}\xi_P\beta\Big(\mathbf{k}\frac{k_{r,P}}{\mu_P}(\nabla p_P - \rho_P g \nabla D)\Big) + q_c, \quad c = 1,\dots,n_c, \quad n_c = 3, \\ &p_O - p_G = P_{cOG}, \quad p_O - p_W = P_{cOW}, \\ &S_W + S_O + S_G = 1, \quad N_1 = N_W = \xi_{W,SC}\frac{S_W}{B_W}, \\ &N_2 = N_O = \xi_{O,SC}\Big(\frac{S_O}{B_O} + R_{O,G}\frac{S_G}{B_G}\Big), \quad N_3 = N_G = \xi_{G,SC}\Big(\frac{S_G}{B_G} + R_{G,O}\frac{S_O}{B_O}\Big). \end{aligned} \tag{1}$$

Here the following values are to be determined:

1) the molar densities $N_c = N_c(t,x,y,z)$;

2) the pressure $p_P = p_P(t,x,y,z)$ ($P = O, W, G$ are the oil, water, and gas phases, respectively);

3) the saturations of phases $S_P = S_P(t,x,y,z)$.

The elements ϕ, $x_{c,P}$, ξ_P, $\mathbf{k}$, $k_{r,P}$, μ_P, ρ_P, q_c, P_{cOG}, P_{cOW}, B_P, $R_{G,O}$, $R_{O,G}$ are given functions nonlinearly depending on the required values and defined in (1).