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A Fast Auxiliary Space Preconditioner for Numerical Reservoir Simulations

Xiaozhe Hu, Penn State Univ; Wei Liu, Monix Energy Solutions; Guan Qin, Univ of Wyoming; Jinchao Xu, Penn State Univ & BiCMR; Yan Yan, Penn State Univ; Chen-Song Zhang, BiCMR & Peking Univ.

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Abstract

The Fast Auxiliary Space Preconditioning (FASP) method is a general framework for constructing effective preconditioners for solving discretized partial differential equations (PDEs). By constructing appropriate auxiliary spaces, FASP transforms a complicated problem into a sequence of simpler solver-friendly systems. In this paper, we present a new FASP preconditioner for petroleum reservoir simulations. According to the analytic characteristics for the pressure and saturation variables in the black oil model, we choose appropriate auxiliary spaces for different parts of the Jacobian systems arising from the fully implicit method (FIM) with coupled implicit wells. By combining the new preconditioner with Krylov subspace methods (such as the GMRes method), we construct an efficient and robust solver, which can be easily generalized to more complicated models like the modified black oil model for simulating polymer flooding. Preliminary numerical experiments demonstrate the advantages of this new preconditioner.

Introduction

The solution of the linear system of equations arising from fully implicit method for a large scale reservoir simulation has many challenges. Very often more than 80% of CPU time in fully implicit reservoir simulation is usually spent on solving Jacobian systems resulting from the Newton linearization. These linear systems are large, sparse, highly nonsymmetric and ill-conditioned. The Krylov subspace methods[1], such as BiCGstab and GMRes, are efficient iterative methods for these Jacobian systems. Since these iterative methods do not converge very well by themselves, many preconditioners have been proposed over the years. They mainly fall in two categories: (i) purely algebraic preconditioners and (ii) preconditioners based on the different properties of the variables. In the category (i), there are, for example, point-wise incomplete lower-upper factorization (ILU) methods[2, 3, 4, 5], block ILU methods[6, 7], nest factorization methods[8, 9], and SVD-reduction methods[10]. In the category (ii), on the other hand, the methods are based on the understanding that pressure variables and saturation variables have different PDE properties; the representative examples are the Combinative method[11], Constrained Pressure Residual (CPR) method[12, 13], and multi-stage methods[14, 15, 16].

In order to solve a linear algebraic system of equations efficiently, a preconditioner is often used to accelerate a Krylov subspace iterative method. A preconditioner is an approximation to the inverse of A, the coefficient matrix of the linear system. Moreover, its action on a vector should be easy to compute. One approach to construct efficient and robust preconditioners is via the method of subspace correction (MSC)[17, 18]. This type of method aims to precondition a linear system by solving some appropriately chosen subspace problems. The domain decomposition (DD) and the multigrid (MG) method, which are popular preconditioners for a vast variety of problems, can be casted as MSC. The performance of MSC depends on the choice of subspaces of the original solution space. When the underlying geometry is relatively simple and the grid is uniform, subspaces can be constructed easily. However, for many practical problems, the geometry is usually complicated and the grids are unstructured, constructing "good" hierarchical subspaces can be very challenging. When there is lack of good subspaces, we can use a more general method, the Fast Auxiliary Space Preconditioning (FASP) method[19, 20]. FASP does not restrict auxiliary spaces to be subspaces of the original space. FASP provides a theoretical framework of design and analysis of efficient solvers for many different practical problems on highly unstructured grids. By carefully making use of special features that each auxiliary space or auxiliary problem possesses, FASP can be efficiently applied to a

large class of commonly used partial differential equations including diffusion-convection-reaction equation, linear elasticity, Stokes problem, Brikman model, Navier-Stokes equation, complex fluids models, and magneto hydrodynamics problem, see the work[21] for a summary of work in this direction. In this paper, we will adopt the idea of FASP framework to petroleum reservoir simulation, and develop a fast solver for multiphase flow in porous media problems.

The proposed FASP method for reservoir simulation has been implemented and intergrated with the Monix simulator[29], which is a reservoir simulator developed by Monix Energy Solutions, Inc. Monix simulator can efficiently solve reservoir problems involving up to five components (three pseudo-hydrocarbon components, inert gas and water) flowing in one, two or three dimensions in a single and/or dual porosity mode by fully implicit method. At each Newton's iteration, Krylov subspace method with the FASP preconditoner is applied to solve the resulting Jacobian matrix.

Fasp Auxiliary Space Preconditioning

The Fast Auxiliary Space Preconditioning (FASP) method is a general framework to develop mathematically optimal or near-optimal solvers that are robust and user-friendly for a variety of problems in practice. FASP represents a class of methods that

1) transform a complicated system into a sequence of simpler systems by using auxiliary spaces or problems;
2) produce an effective and robust preconditioner (to be used together with a Krylov subspace method).

FASP was first proposed for the following symmetric positive definite (SPD) system:

$$Au = f$$ Eq. E

on a vector space V equipped with an inner product $a(\cdot,\cdot) = (A\,\cdot,\cdot\,)$. In order to solve f
1q. 1, we consider

$$\bar{V} = V \times W_1 \times W_2 \times \cdots \times W_J,,$$, ,,,,,,,,,,,,,,,,,,,,,,,,,,,,Eq. E

where $W_1, \dots, W_J$ ($J \in \mathbb{N}$) are auxiliary (Hilbert) spaces with inner products $\bar{a}_j(\cdot,\cdot) = (\bar{A}_j\,\cdot,\cdot), j = 1, \dots, J$.

A distinctive feature of the auxiliary space method is the presence of V in Eq. E. The space V is equipped with an inner product $d(\cdot,\cdot)$ that is different from $a(\cdot,\cdot)$. The operator $D: V \to V$ induced by $d(\cdot,\cdot)$ on V leads to a smoother, $S = D^{-1}$. For each W_j, we can give an operator $\Pi_j: W_j \to V$. Then, we define an operator $\bar{V} \to V$ by

$$\Pi := I \times \Pi_1 \times \cdots \times \Pi_J.$$ q.

We assume that the operators Π_j and D satisfy the following properties:

$$\left\|\Pi_j w_j\right\|_A \le c_j \bar{a}\left(w_j, w_j\right)^{\frac{1}{2}}, \qquad \forall\, w_j \in W_j, j = 1, \dots, J.$$ . ..

$$\|v\|_A \le c_s d(v, v)^{\frac{1}{2}}, \qquad \forall\, v \in V,,,$$ q.

and, for every $v \in V$, there exist $v_0 \in V$ and $w_j \in W_j$ such that $v = v_0 + \sum_j \Pi_j w_j$ and

$$d(v_0, v_0)^{1/2} + \sum_j \bar{a}_j\left(w_j, w_j\right)^{\frac{1}{2}} \le c_0\, \|v\|_A$$ .... q... .

Then the auxiliary space preconditioner is given by

$$B = S + \sum_j \Pi_j \bar{A}_j^{-1} \Pi_j^*.$$ q.

It can be proved that the preconditioner Eq. 7 admits the following estimate[20]

$$\mathcal{K}(BA) \le c_0^2\left(c_s^2 + c_1^2 + \cdots + c_J^2\right), \qquad \text{Eq. 7}$$

where $\mathcal{K}(BA)$ denotes the condition number of the preconditioned linear system BA.

An important special case of FASP method is the Method of Subspace Correction (MSC) mentioned earlier, in which W_j is a subspace of V for every j. Another one is the Ficticious space lemma[30] developed by Nepomnyaschikh, where smoother is ommited. Eq. 7 may be called as the parallel or additive FASP method and we can naturally have a successive or multiplicative version of FASP, in which all the auxiliary spaces are used one after another.

FASP for Reservoir Simulations

In this section, we discuss a FASP method for fully implicit reservoir simulations. To demonstrate how the FASP framework can be applied, we first consider the classical black oil model:

$$\frac{\partial}{\partial t}\left[\phi\left(\frac{S_o}{B_o} + \frac{R_v S_g}{B_g}\right)\right] = -\nabla\cdot\left(\frac{1}{B_o}u_o + \frac{R_v}{B_g}u_g\right) + q_O, \qquad \text{Eq. 9}$$

$$\frac{\partial}{\partial t}\left[\phi\left(\frac{S_g}{B_g} + \frac{R_s S_o}{B_o}\right)\right] = -\nabla\cdot\left(\frac{1}{B_g}u_g + \frac{R_s}{B_o}u_o\right) + q_G, \qquad \text{Eq. 10}$$

$$\frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right) = -\nabla\cdot\left(\frac{1}{B_w}u_w\right) + q_W, \qquad \text{Eq. 11}$$

$$u_\alpha = -\frac{k k_{r\alpha}}{\mu_\alpha}(\nabla P_\alpha - \rho_\alpha g \nabla z), \quad \alpha = o, g, w, \qquad \text{Eq. 12}$$

$$S_o + S_g + S_w = 1, \qquad \text{Eq. 13}$$

$$P_o - P_w = P_{cow},\ P_g - P_o = P_{cog}. \qquad \text{Eq. 14}$$

As a widely accepted approach, the isothermal black oil model solves the three-dimensional and three-phase equations of conservation of mass (volume) in porous media at standard surface condition, subject to appropriate initial and boundary conditions.

The Fully Implicit Method (FIM)[22] is applied to solve the black oil model. At each Newton's iteration, we need to solve the following linear system (the Jacobian system)

$$Au = f \qquad i.e.\ \begin{pmatrix} A_{ResRes} & A_{ResWel} \\ A_{WelRes} & A_{WelWel} \end{pmatrix}\begin{pmatrix} u_{Res} \\ u_{Wel} \end{pmatrix} = \begin{pmatrix} f_{Res} \\ f_{Wel} \end{pmatrix} \qquad \text{Eq. 15}$$

in a finite-dimensional vector space V. Here, the sub-indices, Res and Wel, stand for the reservoir and well parts of the unknowns, respectively. Usually, A_{ResRes} is a large-size sparse matrix and the whole matrix A is called the bordered matrix because of its shape. Moreover, we can further write the reservoir block A_{ResRes} in the following two-by-two block form

$$A_{ResRes} = \begin{pmatrix} A_{PP} & A_{PS} \\ A_{SP} & A_{SS} \end{pmatrix}, \qquad \text{Eq. 16}$$

where P denotes the pressure variables and S denotes the saturation variables.

It is straightforward to see that, in the coefficient matrix A, different parts have different properties due to the analytic nature of their corresponding continuous differential equations. The equation corresponding to the pressure unknowns is elliptic, the equation corresponding to the saturation unknowns is hyperbolic, and the equation corresponding to the well unknowns is algebraic. To adopt the FASP framework to the black oil model, we should take full advantage of these analytic properties. The basic idea is as follows:

1) First, transform the complicated Jacobian system into three simpler auxiliary problems: an elliptic problem for the pressure variables, a hyperbolic problem for the saturation variables, and a purely algebraic problem for the well bottom-hole pressure variables;
2) Then, design efficient and robust smoothers or preconditioners for each auxiliary problem.

Therefore, following the FASP framework, for the black oil model, we consider the following auxiliary spaces,

$$\bar{V} = V \times W_P \times W_S \times W_{Wel}, \quad \text{Eq. 171}$$

where W_P, W_S and W_{Wel} are auxiliary spaces for the pressure unknowns, the saturation unknowns, and the well unknowns, respectively. By introducing the inner products $a_P(\cdot,\cdot) = (A_P \cdot,\cdot)$, $a_S(\cdot,\cdot) = (A_S \cdot,\cdot)$, $a_{Wel}(\cdot,\cdot) = (A_{Wel} \cdot,\cdot)$, as well as a smoother R, an additive version of auxiliary space preconditioner can be defined as following

$$B = R + \Pi_P A_P^{-1} \Pi_P^* + \Pi_S A_S^{-1} \Pi_S^* + \Pi_{Wel} A_{Wel}^{-1} \Pi_{Wel}^*, \quad \text{Eq. 181}$$

where $\Pi_P: W_P \to V$, $\Pi_S: W_S \to V$ and $\Pi_{Wel}: W_{Wel} \to V$ are the transfer operators.

The additive version FASP defined above gives the idea about how to apply the FASP framework to reservoir simulations. In practice, to achieve better performance and robustness, we apply the following multiplicative FASP algorithm:

Algorithm FASP: Given an initial guess u_0,

1) $u_1 = u_0 + \Pi_{Wel} A_{Wel}^{-1} \Pi_{Wel}^* (f - A u_0)$
2) $u_2 = u_1 + \Pi_S A_S^{-1} \Pi_S^* (f - A u_1)$
3) $u_3 = u_2 + \Pi_P A_P^{-1} \Pi_P^* (f - A u_2)$
4) $u_4 = u_3 + R(f - A u_3)$
5) $B u_0 = u_4$

We will discuss the details on the choice of auxiliary spaces, corresponding solvers, and other details of the FASP method for reservoir simulations.

Decoupling

As suggested by many researchers[14, 15, 16, 23], in order to weaken the strong coupling between the pressure and saturation unknows, a decoupling step needs to be applied on the reservoir block A_{ResRes}. This decoupling procedure should be computationally cheap and weaken the coupling between the pressure and saturation unknowns. Here, we choose the so-called alternative block factorization (ABF) strategy[24]. This strategy is basically the block diagonal preconditioning, i.e.

$$\widetilde{A}_{ResRes} = D_{ResRes}^{-1} A_{ResRes}, \quad \text{Eq. 199}$$

where

$$D_{ResRes} = \begin{pmatrix} diag(A_{PP}) & diag(A_{PS}) \\ diag(A_{SP}) & diag(A_{SS}) \end{pmatrix} \quad \text{Eq. 202}$$

Here the notation $diag(M)$ stands for a diagonal matrix whose diagonal entries are just the diagonal entries of M.

Auxiliary Space for Pressure Variables

A simple choice of W_P is the vector space of all the pressure unknowns. In this case, $A_P = A_{PP}$. It is well known that the equations describing the mass balance in terms of pressure unknowns are mainly elliptic[25]. Therefore, we can use algebraic multigrid (AMG) methods to solve the pressure block. However, in practice, the performance and efficiency of AMG may degenerate when the physical and geometric properties of the problems become more complicated, for example, the heterogeneous media and anisotropic permeability. Moreover, the decoupling step may destroy the structure of the pressure block to certain extent, which will in turn affect multigrid solvers.

In order to improve the performance of the AMG solver, we can use an approach to combine an iterative method (AMG method) with some other preconditioner (like ILU methods) to obtain a new solver for the pressure block[26]. We note this approach can be viewed as a FASP method for the pressure block by itself.

Remark. There are other choices of W_P, for example, on can choose the vector space of all the pressure unknowns together with the well unknowns. In this case, A_P is a two-by-two block matrix with the following structure:

$$A_P = \begin{pmatrix} A_{PP} & A_{PW} \\ A_{WP} & A_{WW} \end{pmatrix}. \qquad \text{Eqq1q}$$

AMG methods can be used to approximate the inverse of A_P above. Clearly, this choice of W_P helps decoupling the well equations, which has non-PDE characters. Similarly, we can also include some saturation unknowns, which are strongly coupled to pressure unknowns, into W_P.

Auxiliary Space for Saturation Variables

Similar to pressure unknowns, one simple choice of W_S is the vector space of all saturation unknowns. In this case, $A_S = A_{SS}$. The saturation block A_{SS} has hyperbolic characters. Due to this fact, we solve the saturation block by one sweep of block Gauss-Seidel with downwind ordering and crosswind blocks. We order the gridblocks according to the direction of the multiphase flow. Since the fluid flows from gridblocks of higher pressure to gridblocks of lower pressure, we can order the girdblocks according to the pressure unknowns, and form the crosswind blocks. This idea has been shown to be efficient for convection-dominated problems; see the liturature[27] for details. Theoretically, after downwind ordering, the saturation block is close to a lower triangular matrix, which can be solved efficiently by the block Gauss-Seidel method. In addition, well bottom-hole-pressure unknowns and some pressure unknowns, which are strongly coupled to saturation unknowns, can also be included in W_S.

Auxiliary Sapce for Implicit Wells

In practice, we have found that many iterative solvers may fail to converge or converge very slowly for practical problems when the simulation involves complicated well control, espeically when simulating enhanced oil recovery (EOR) using polymer flooding together with water flooding. The strong doupleing between the implicit wells and the reservoir contributes to some of these difficulties. Generally speaking, the non-PDE character of the well constraints is very different than the rest of the Jacobian system. This suggests that we may treat the reservoir and well equations separately. However, the coupling between the reservoir and well equations might be strong (especially when simulating complex wells.) This means the well equations and the reservoir part should be solved together.

Our choice of W_{Wel} takes these two facts into account. We choose W_{Wel} in a way that the reservoir and well equations are treated separately, yet W_{Wel} is still large enough to be able to handle the coupling between the reservoir part and the well part effectively. To be more specific, W_{Wel} is the vector space of all the well unknowns and all the reservoir unknowns that are on the grid-blocks of perforation, since the coupling between reservoir and wells mainly exists on those perforated grids. Due to the non-PDE character of the well equations and the number of wells and perforated grid-blocks are relatively small with respect to the total number of grid-blocks, we solve A_{Wel} by the direct method (Gaussian Elimination).

Choice of Smoothers

One of the distinctive features of the FASP method is the presence of the smoother R, for the whole space V, which resolves the coupling between pressure unknowns and saturation unknowns, as well as the coupling between reservoir unknowns and well unknowns. In the literature, LSOR method and Block ILU methods have been successfully applied and are often used in practice. However, both LSOR and ILU are noticed to deteriorate when the size of the problems becomes larger, or the porous media become more heterogeneous. Moreover, ILU becomes too expensive (in terms of memory usage) in practice when more and more fill-in is allowed. Therefore, to reduce computational cost (both CPU time cost and memory cost), we use block Gauss-Seidel method as the smoother here. Moreover, downwind ordering and crosswind blocks are used to improve the overall robustness.

Remark. We would like to point out that, although we derive the FASP for the black oil model, the proposed method can be

easily extended to the modified black oil model for EOR[31]. In the modified black oil model, equations, which describe the mass conservation of concentrations of polymer and sodium chloride, are also considered besides the original black oil model. Similar as the saturations, concentrations also present transport properties in general, and have hyperbolic characters. Therefore, in the FASP method for EOR, concentrations are treated with saturations simultaneously. In another word, for EOR, notation S stands for non-pressure unknowns, including both saturations and concentrations. Then Algorithm FASP still works for the EOR simulation. The FASP method for EOR proceeds exactly the same as the FASP method for black oil model, except the auxiliary problem for saturations becomes the auxiliary problem for non-pressure unknowns, which is still solved by block Gauss-Seidel with downwind ordering. Numerical results of the FASP method for EOR will be presented in the next section to demonstrate the efficiency of the FASP method for EOR.

Numerical Experiments

The proposed FASP method is implemented in the MONIX. It is used as a preconditioner of GMRES method. We applied the FASP method to several different test problems to study the computational optimality, efficiency and robustness of the FASP method.

Test 1 (A black oil model problem). The first test problem is based on the SPE1 comparative test[28]. SPE1 is a standard three-phase black oil model. The permeability and the porosity are homogenous. The porosity is 0.3 and the permeability is set to $k_z = 100$ and $k_x = k_y = 500$. There is one injector and one producer located at two opposite corners. The total simulation time is 10 years. We use this test problem to study the computational optimality of the FASP method for black oil model. Therefore, four different structured grids are employed---their sizes are 10x10x3, 50x50x10, 100x100x20 and 200x200x30. From Figure 1, we can see that the CPU time for solving one Jacobian matrix is proportional to the size of the test problem. It shows the optimal computational complexity of the FASP method for this black oil problem. This shows the potential scalability of the FASP method for the black oil model.

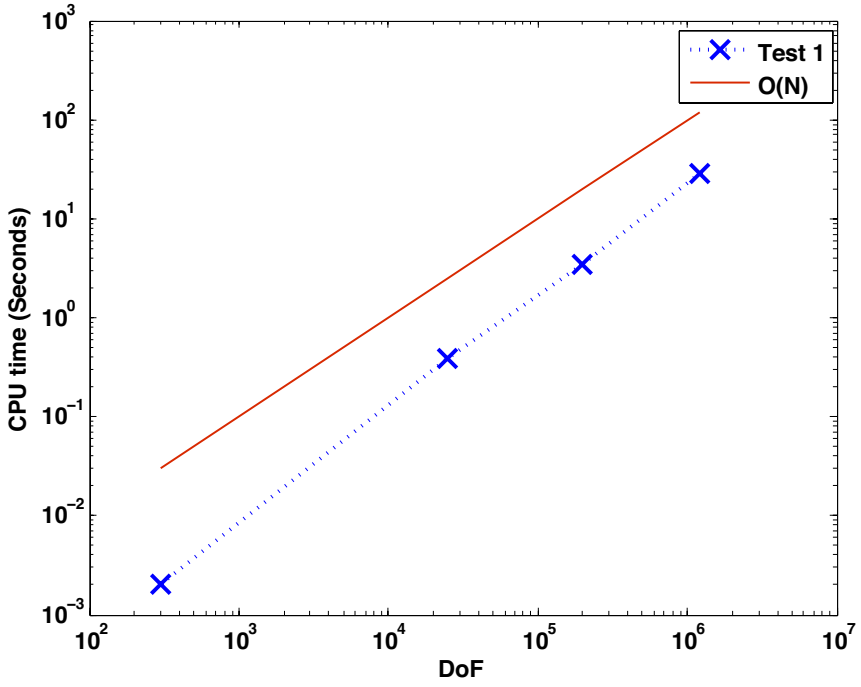

Figure 1: Optimal computational complexity of the FASP method for the black oil model (Test 1)

Test 2 (A polymer flooding model problem). The second test problem is a three-phase modified black oil model for polymer flooding (five components: oil, gas, water, polymer, brine). Since we choose this problem to study the computational complexity of the FASP method for modified black oil model, the porosity and permeability are homogeneous. There are one injector located at the center of the reservoir and four producers located at the four corners. The total simulation time is 20 years---in the first 18 years, only water flooding is used and, in the last two years, polymer flooding is empolyed on the field. Four different structured grids, 40x40x4, 80x80x4, 40x40x20, and 40x40x40, are used in our simulations. The computational time for solving one Jacobian system is shown in Figure 2. We can see that the CPU time for solving one Jacobian matrix is

proportional to the problem size, which means the FASP method has nearly optimal complexity for this test problem.

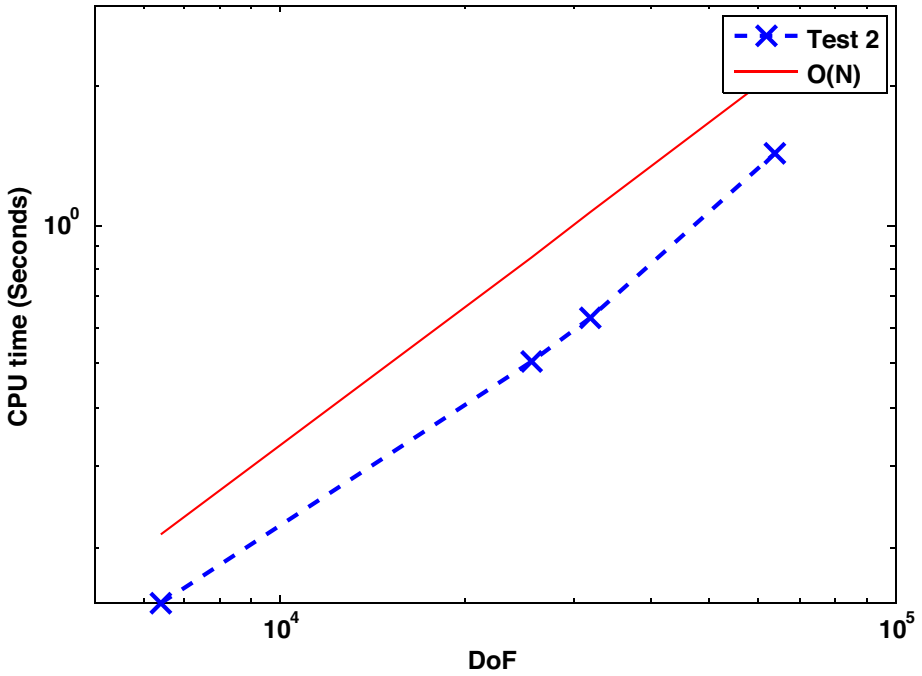

Figure 2: Optimal computational complexity of the FASP method for the polymer flooding (Test 2)

Test 3 (A black oil field test). The field test is from an oil field in the United States. We did the simulation with the classical balck oil model with water flooding. Figure 3 shows the computational domain and the initial oil saturation distribution in Test 3. A corner-point grid with 83592 active grid blocks is used. There are 169 active wells in total and two faults with 994 non-local connections. The porosity and permeability are very heterogeneous. There properties make this real-world problem very challenge for classical solvers.

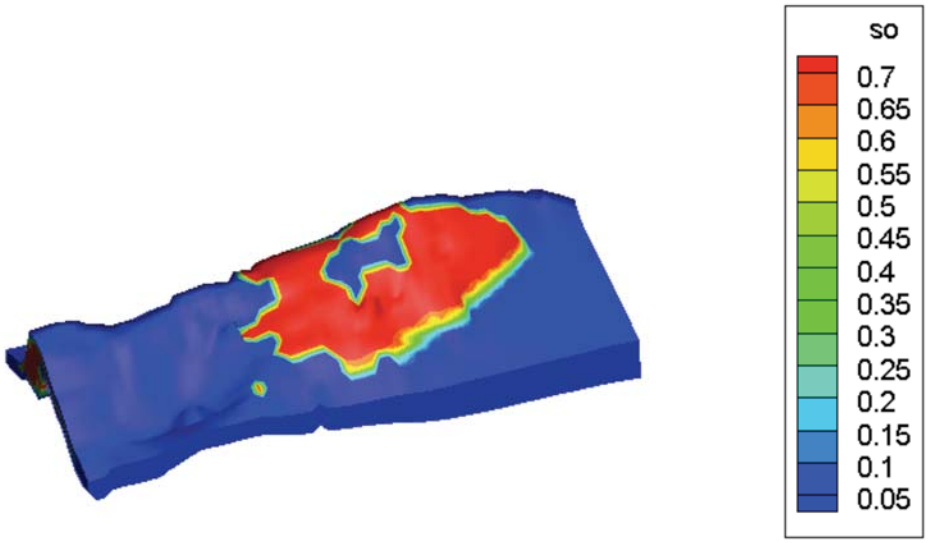

Figure 3: Computational domain and initial oil saturation of Test 3 problem

Total CPU time (seconds)	3879
Total time steps	671
Total Newton iterations	2298
Average Newton iterations per time step	3.42
Average CPU time per time step (seconds)	5.78
Average Number of iterations of GMRES per Newton iteration	9.36
Average CPU time of GMRES per Newton iteration	0.64
Average CPU time of GMRES per time step (seconds)	2.21

Table 1: Summary of numerical results for Test 3. .

Summary and Conclusions

In this work, we discuss practical and efficient solvers for large sparse linear systems arising from the black oil model discretized by the fully implicit method with coupled implicit wells. The general FASP framework is adopted and a new FASP method, which specially tailored for field-scale petroleum reservoir simulations, is proposed. By introducing auxiliary problems for different unknowns, the FASP transforms an original highly coupled Jacobian system into several auxiliary problems and use different methods to solve them according to their analytic characteristics. The new FASP method can be employed as a preconditioner of Krylov subspace iterative methods and provides an efficient and robust solver. This new FASP method can also be easily generalized to the modified black oil model for simulating polymer flooding. Primary numerical experiments are presented. For both standard and modified black oil model on structured grid and homogeneous porous media, numerical results show that FASP method is efficient and has optimal computational complexity. We also apply the new FASP method to two practical problems from the real world. One is the black oil model and another one is the modified black oil model. Both problems are on real heterogeneous porous media and the meshes are highly unstructured, which make the resulting Jacobian system very difficult to solve. From the numerical results, our FASP method is efficient and robust for these two real-world problems, which demonstrates the potential of our new method.

Nomencalture

S	saturation
P	pressure
u	velocity
ϕ	porosity
k	permeability
k_r	relative permeability
μ	viscosity
B	formation value factor
q	source/sink term (wells)
α	the oil (o), gas (g) and water (w) phases
ρ	fluid density
P_{cow}, P_{cog}	capillary pressures
R_s	gas solubility
R_v	oil volatility
g	gravitational acceleration
z	depth

Subscripts

α	the oil (o), gas (g) and water (w) phases
Res	reservoir
Wel	well

P	Pressure variables
S	Saturation variables
x, y, z	Coordinate directions

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References

1. Y. Saad: "Iterative Methods for Sparse Linear Systems," Society for Industrial and Applied Mathematics, 2003.
2. T. Dupont, R.P. Kendall and H.H.Jr. Rachford: "An approximate factorization procedure for solving self-adjoint elliptic difference equations," SIAM J. Numer. Anal. (1968) 5, 559--573.
3. J.A. Meijerink and H.A. van der Vorst: "An iterative solution method for linear systems of which the coefficient matrix is a symmetric M-matrix, " Math. Comp. (1977) 31, 148--162.
4. J.R. Wallis: "Incomplete Gaussian elimination as a preconditioning for generalized conjugated gradient acceleration," SPE12265, presented at the seventh SPE symposium on reservoir simulation, San Francisco, 1983.
5. A. Behie and P.A. Forsyth Jr.: "Incomplete factorization methods for fully implicit simulation of enhanced oil recovery," SIAM J. Sci. Stat. Comp. 5 (1984) 543--561.
6. P. Concus, G.H. Golub and G. Meurant: "Block preconditioning for the conjugate gradient method," SIAM J. Sci. Stat. Comput (1985) 6, 220--252.
7. J.A. Meijerink: "Iterative methods for the solution of linear equations based on the incomplete block factorization of the Matrix," SPE12262 presented at the 1983 SPE Reservoir Simulation Symposium, Lubbock, TX. Nov. 14--15.
8. J. R. Appleyard, I.M. Cheshire and R.K. Pollard: "Special techniques for fully implicit simulators," Proc. European Symposium on enhanced oil recovery, Bournemouth, England (1981), 395--408.
9. J. R. Appleyard and I.M. Cheshire: "Nested factorization," paper SPE 12264 presented at Proc. 7th SPE Symposium on Reservoir Simulation, 1983.
10. J.W. Watts and J.S. Shaw: "A new method for solving the implicit reservoir simulation matrix equation," SPE 93068 presented at the SPE Reservoir Simulation Symposium, Texas, TX, Jan 31--Feb 2, 2005.
11. G. Behie and P. Vinsome: "Block iterative methods for fully implicit reservoir simulation," Soc. Pet. Eng. J., July, (1982) 658-668.
12. J.R. Wallis: "Incomplete Gaussian Elimination as a preconditioning for generalized conjugate gradient acceleration," SPE 12265 presented at the SPE Reservoir Simulation Symposium, San Francisco, California, Nov 15-18, 1983.
13. J.R. Wallis, R.P. Kendall, and T.E. Little: "Constrained residual acceleration of conjugate residual methods," SPE 13536 presented at the SPE Reservoir Simulation Symposium, Dallas, TX, February 10-13, 1985.
14. S. Lacroix, Y. Vassilevski and M. Wheeler: "Decoupling preconditioners in the implicit parallel accurate reservoir simulator (IPARS)," Numer. Linear Algebra with applications Vol. 8, No. 8, pp. 537-549, 2001.
15. K. Stuben, T. Clees, H. Klie and M. Wheeler: "Algebraic Multigrid Methods (AMG) for the Efficient Solution of Fully Implicit Formulations in Reservoir Simulation," paper SPE 105832 presented at 2007 SPE Reservoir Simulation Symposium, Houston, TX.
16. T. M. Al-Shaalan, H. Klie, A. H. Dogru and M. Wheeler: "Studies of Robust Two Stage Preconditioners for the Solution of Fully Implicit Multiphase Flow Problems," paper SPE 118722 presented at 2009 SPE Reservoir Simulation Symposium, Woodlands, Texas, USA, 2, February 2009.
17. J. Xu: "Iterative Methods by Space Decomposition and Subspace Correction," Siam Review, *34*(4) 1992, 581-613.
18. J. Xu: "The method of subspace corrections," Journal of Computational and Applied Mathematics, *128*(1-2) 2001, 335-
19. J. Xu: "The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids," Computing, *56*(3) 1996, 215-235.
20. R. Hiptmair and J. Xu: "Nodal auxiliary space preconditioning in H (curl) and H (div) spaces," SIAM Journal on Numerical Analysis, *45*(6) 2007, 2483-2509.
21. J. Xu: "Fast Poisson-Based Solvers for Linear and Nonlinear PDEs," In R. Bhatia (Ed.), Proceedings of the Internitonal Congress of Mathematicians 2010 (ICM 2010) (pp. 2886 – 2912). World Scientific.
22. J. Douglas, Jr., D.W. Peaceman, and H.H. Rachford, Jr,: "A method of calculating multi-dimentiosnal immiscible displacement," SPE AIME, 216 (1959), 297-396.

23. R.P. Hammersley and D.K. Ponting: “Solving Linear Equations In Reservoir Simulation Using Multigrid Methods,” SPE Russian Oil and Gas Technical, October 2008, 28-30. Society of Petroleum Engineers.
24. R.E. Bank, T.F. Chan, J.W.M. Coughran and R. K. Smith: “The alternate-block-factorization procedure for systems of partial differential equations,” BIT, 29(4), 938-954.
25. H. Klie: “Krylov-secant methods for solving large-scale systems of coupled nonlinear parabolic equations,” PhD thesis, 1997.
26. R. Falgout, X. Hu, X.-H. Wu, J. Xu, C.-S. Zhang, S. Zhang and L. Zikatanov: “A Combined Utilization of ILU and AMG with Applications in Reservoir Simulation”, 2011, in preparation.
27. F. Wang and J. Xu: “A Crosswind Block Iterative Method for Convection-Dominated Problems,” SIAM Journal on Scientific Computing, 21(2) 1999, 620.
28. A. Odeh: “Comparison of Solution to a 3D Black-Oil Reservoir Simulation Problem,” JPT, 1981, 13-25.
29. Monix Ebergy Solution. Inc. http://www.monixe.com/index.htm.
30. S. V. Nepomnyaschikh: “Decomposition and fictitious domains methods for elliptic boundary value problems,” In Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Norfolk, VA, 1991), pages 62–72. SIAM, Philadelphia, PA, 1992.
31.