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Studies of Robust Two Stage Preconditioners for the Solution of Fully Implicit Multiphase Flow Problems

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Abstract

The solution of the linear system of equations for a large scale reservoir simulation has several challenges. Preconditioners are used to speed up the convergence rate of the solution of such systems. In theory, a preconditioner defines a matrix $\boldsymbol{M}$ that can be inexpensively inverted and represents a good approximation of a given matrix $\boldsymbol{A}$. In this work, two-stage preconditioners consisting of the approximated inverses $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ are investigated for multiphase flow in porous media. The first-stage preconditioner, $\boldsymbol{M_1}$, is approximated from $\boldsymbol{A}$ using four different solution methods: (1) constrained pressure residuals (CPR), (2) lower block Gauss-Seidel, (3) upper block Gauss-Seidel, and (4) one iteration of block Gauss-Seidel. The pressure block solution in each of these different schemes is calculated using the Algebraic Multi Grid (AMG) method. The inverse of the saturation (or more generally, the nonpressure) blocks are approximated using Line Successive Over Relaxation (LSOR). The second stage preconditioner, $\boldsymbol{M_2}$, is a global preconditioner based on LSOR iterations for the matrix $\boldsymbol{A}$ that captures part of the original interaction of different coefficient blocks. Several techniques are also employed to weaken the coupling between the pressure block and the nonpressure blocks. Effective decoupling is achieved by: (1) an IMPES-like approach designed to preserve the integrity of pressure coefficients, (2) Householder transformations, (3) the alternate block factorization (ABF), and (4) the balanced decoupling strategy (BDS) based on least squares. The fourth method is a new technique developed in this work. The aforementioned preconditioning techniques were implemented in a parallel reservoir simulation environment, and tested for large-scale two-phase and three-phase black oil simulation models. This study demonstrates that a two-stage preconditioner based on balanced decoupling strategy (BDS) or ABF combined with Gauss-Seidel sweeps, that also incorporate nonpressure solutions for $\boldsymbol{M}$, delivers both the fastest convergence rate and the most robust option overall without compromising parallel scalability.

Introduction

Discretization of the governing partial differential equations for reservoir simulation in a fully implicit fashion generates a system of linear equations, Eq. 1, that is highly sparse, nonsymmetric and indefinite. The solution of Eq. 1 can be reached by either a direct solution such as Gaussian elimination or an iterative solution[1].

$$AX = b \tag{1}$$

For a large sparse matrix $\boldsymbol{A}$, iterative solution is employed[1]. There are many iterative methods discussed in detail in the literature[1]. Gauss-Seidel, Generalized Conjugate Residual (GCR) and Generalized Minimal Residual (GMRES) are examples of those methods. Depending on the iterative method, the convergence rate[1] of the iterative method is, in general, a function of the matrix condition number that is defined in Eq. 2. Where $\|\bullet\|$ is a matrix norm, the larger the condition number, the slower the convergence of a given iterative method. Reader may refer to the vast literature[1, 11, 16, 17, 18] for more details.

$$cond(A) = \|A\| \, \|A^{-1}\| \tag{2}$$

One way to reduce the condition number of the matrix ***A*** is to employ a preconditioner. The objective of a preconditioner is to define a matrix ***M*** that can be inexpensively inverted and represents a good approximation for a given matrix ***A***. For example, Aksoylu et al.[16], illustrates that the preconditioner helps to cluster the Eigen values of a given matrix, and therefore it improves the convergence rate of a given iterative method. There are two ways to apply a preconditioner, namely, from the left and/or from the right side of Eq. 1[1]. Therefore, the left preconditioner (M_L) is defined by Eq. 3, and the right preconditioner M_R is defined by Eq. 4.

$$M_L^{-1} AX = M_L^{-1} b \tag{3}$$

$$AM_R^{-1} M_R X = b \tag{4}$$

The preconditioned matrix must satisfy the following conditions defined by Eq. 5 and Eq. 6[6, 11, 16]. The closer M_L^{-1} or M_R^{-1} to A^{-1}, the smaller the condition number of $M_L^{-1}A$ or AM_R^{-1}.

$$cond(AM_R^{-1}) < cond(A) \tag{5}$$

$$cond(M_L^{-1}A) < cond(A) \tag{6}$$

The right preconditioner M_R is preferred, because it does not change the right hand side or the nonlinear residuals b[1, 13] generated in a nonlinear iteration (e.g., Newton's method). To minimize the impact of the left preconditioner on the residual b, the norm of M_L is kept as close as possible to 1.

Depending on the solution method, both preconditioners may also be used together as in Eq. 7. Both the left and the right preconditioners are discussed in details in the literature[1, 4, 5, 8, 10, 11] and are not repeated here:

$$M_L^{-1} AM_R^{-1} M_R X = M_L^{-1} b\,. \tag{7}$$

In this paper, the mathematical modeling used to generate the linear system, Eq. 1, will be briefly discussed. The left preconditioner $M_L^{-1} = G$ will be utilized as a decoupling process. Different decoupling matrix ***G*** and right preconditioners M_R^{-1} will be investigated in the area of reservoir simulation. Finally, numerical examples will be discussed.

Mathematical Model

The mathematical formulation for multiphase flow in porous media is discussed in detail by Wheeler[12], and it will be briefly stated here. The material balance equations for porous medium flow can be described as follows:

The mass balance for oil and gas:

$$\frac{V\phi_m}{\Delta t}\Delta\left[\rho_o x_i S_o + \rho_g y_i S_g\right] = \sum_j \left[T\rho_o x_i \frac{k_{ro}}{\mu_o}\left(\Delta P - \rho_o g \Delta z\right)\right] + \sum_j \left[T\rho_g y_i \frac{k_{rg}}{\mu_g}\left(\Delta P - \rho_g g \Delta z + P_{cog}\right)\right]_m - q_{mi} \tag{8}$$

The mass balance for water:

$$\frac{V\phi_m}{\Delta t}\Delta\left[\rho_w S_w\right] = \sum_j \left[T\rho_w \frac{k_{rw}}{\mu_w}\left(\Delta P - \rho_w g \Delta z - P_{cwo}\right)\right] - q_{wm} \tag{9}$$

$i = 1, 2, ..., N_c$

$j \in n_p$ (number of phases)

The nomenclature used in Eq. 8 and Eq. 9 is presented at the end of the paper. Here it is assumed that water is immiscible with oil and gas. The number of component, N_c, can be either water, oil, and gas for black oil simulation or chemical components for compositional modeling. The formulation presented in Eq. 8 and Eq. 9 is similar to the compositional formulation described in Coats et al.[2] Per each of this formulations, the input of the pressure, volume and temperature (PVT) data such as the formation volume factors and solution gas-oil ratios (GOR) are converted to oil and gas phase molar density (ρ) and compositions as single-valued functions (x_i and y_i) of pressure. The PVT data can be either input as tabulated values, or calculated from equation of state (EOS) in compositional modeling.

Additional constraint equations in each grid block consist of phase saturation balances, phase equilibrium equations and capillary function relationships. These equations are available in the literature[3, 12] and are not repeated here.

Finite volume is used for space discretization, and the fully implicit Euler scheme with Newton's method are used for solving the equations at each timestep. Discretizations of Eq. 8 and Eq. 9 lead to a sequence of nonsymmetric and indefinite algebraic systems of linear equations[12] to be solved in each Newton iteration, Eq. 10.

$$\begin{bmatrix} A_{pp} & A_{ps} \\ A_{sp} & A_{ss} \end{bmatrix} \begin{bmatrix} P \\ S \end{bmatrix} = \begin{bmatrix} R_p \\ R_s \end{bmatrix} \tag{10}$$

Where $A_{pp} \in \Re^{ng \times ng}$ is a pressure block matrix for pressure, $A_{ss} \in \Re^{ns \times ns}$ is a block matrix for saturation/concentration or other primary unknowns, $A_{ps} \in \Re^{ng \times ns}$ and $A_{ps} \in \Re^{ng \times np}$ are the block matrixes coupling pressure with the other primary unknowns. The dimension ng is the number of active grid blocks, and $ns = nv \times np$, where nv is the number of nonpressure variables.

The solution of block matrix, Eq. 10, has been also investigated in detail in the literature[4, 5, 8, 10, 11, 13-16]. The generated linear system of equations is highly sparse, nonsymmetric and indefinite. As a result, an efficient and robust iterative method is needed to seek for the solution of the system. In this study, preconditioned GMRES[1, 10, 11] is employed. GMRES is discussed in detailed by Saad[1], and the preconditioners as presented in Eq. 7 will be discussed in the following sections.

Left Preconditioned (Decoupling Operator)

The left preconditioner $M_L^{-1} = G$ is used as a preprocessing step to weaken the coupling between the pressure block matrix A_{pp} and A_{ps}, and between A_{ss} and A_{sp}. This is needed to achieve overall effectiveness of the preconditioning procedure. The preprocessed matrix $\overline{A}$, Eq. 11, should satisfy the conditions given in Eq. 12. The second norm (L_2), for example, can be used in Eq. 12. To minimize the impact of G on the original residual b, the norm of G is kept close to 1. Although, some of the decoupling operators that will be discussed later may fail to satisfy such conditions.

$$\overline{A} = G \cdot A = \begin{bmatrix} \overline{A}_{pp} & \overline{A}_{ps} \\ \overline{A}_{sp} & \overline{A}_{ss} \end{bmatrix} \tag{11}$$

$$\frac{\left\|\overline{A}_{ps}\right\|}{\left\|\overline{A}_{pp}\right\|} << 1 \quad , \quad \frac{\left\|\overline{A}_{sp}\right\|}{\left\|\overline{A}_{ss}\right\|} << 1 \tag{12}$$

Klie[11] discussed that the decoupling operation is effective in clustering the Eigen values of the preprocessed matrix $\overline{A}$. This step may improve the overall convergence rate of Krylov-based iterative schemes such as GMRES. A successful decoupling is critical to make the right two-stage preconditioner (given by Eqs. 21-23) to work effectively since the strength of the off-diagonal is reduced.

There are many options that can be used to reach the bounds stated in Eq. 12. The first method is using Householder transformations[12, 13], where saturations/concentrations are projected into the pressure space[11-13]. This method satisfies[13] $\left\|G\right\| \approx 1$ and it will decrease the ratio $\left\|\overline{A}_{ps}\right\| / \left\|\overline{A}_{pp}\right\|$ but not $\left\|\overline{A}_{sp}\right\| / \left\|\overline{A}_{ss}\right\|$. The second method is to use IMPES like or Quasi-IMPES[13] method, where the total mass balance equation is used for solving the pressure equations (first block

equation in Eq. 10). Scheichl et al.[13], compares between Householder and Quasi-IMPES and they found that the Quasi-IMPES approach works well for most of their cases. Additionally, Masson et al.[14] asserts that the Quasi-IMPES works well for dealing with reservoir cases with high heterogeneity and relative fast flow velocities.

The third method is to use the alternate block factorization (ABF) strategy, where Eq. 10 is multiplied by the inverse of a block matrix given by the diagonal blocks of matrix ***A*** (Eq. 13). This decoupling is also studied by Cao[10] and Klie[11]. Klie[11] argued that $\left(BLOCK\_D_A^{-1}\cdot A\right)$ is preferred over $\left(A\cdot BLOCK\_D_A^{-1}\right)$, because the latter one may spoil the diagonal dominance of *A*.

$$G = BLOCK\_D_A^{-1} = \begin{bmatrix} diag(A_{pp}) & diag(A_{ps}) \\ diag(A_{sp}) & diag(A_{ss}) \end{bmatrix}^{-1} = \left(L_{DA}D_{DA}U_{DA}\right)^{-1}. \quad (13)$$

In this study, Gaussian elimination is used to define the lower matrix (L_{DA}), the diagonal matrix (D_{DA}) and the upper matrix (U_{DA}) in Eq. 13. To get a decoupled matrix with unit diagonal as will be discussed later, the block matrix in Eq. 10 is multiplied by *G:*

$$BLOCK\_D_A^{-1}\begin{bmatrix} A_{pp} & A_{ps} \\ A_{sp} & A_{ss} \end{bmatrix}\begin{bmatrix} P \\ S \end{bmatrix} = BLOCK\_D_A^{-1}\begin{bmatrix} R_p \\ R_s \end{bmatrix}. \quad (14)$$

On the other hand, to get a decoupled matrix with a non-unit entries in A_{pp} and A_{ss} , Eq. 15 is used.

$$L_{DA}^{-1}\begin{bmatrix} A_{pp} & A_{ps} \\ A_{sp} & A_{ss} \end{bmatrix}U_{DA}^{-1}U_{DA}\begin{bmatrix} P \\ S \end{bmatrix} = L_{DA}^{-1}\begin{bmatrix} R_p \\ R_s \end{bmatrix}. \quad (15)$$

The third method will work well if the diagonal matrix $BLOCK\_D_A$ is relatively dominant with respect to the off-diagonal blocks. In reservoir simulation, the upstream weighting method is used for computing the right mobility[12]. Heterogeneity and upstream weighting may result in the off-diagonal blocks being more dominant than $BLOCK\_D_A$.

The fourth method is the BDS based on least squares. This method is a new technique developed in this work. The idea behind it is to multiply Eq. 10 by matrix $\overline{C}$ (Eq. 16) to produce a new matrix $\overline{A}$ (Eq. 17).

$$\overline{C} = \begin{bmatrix} I & 0 \\ C_1 & C_2 \end{bmatrix}, \quad (16)$$

$$\overline{A} = \overline{C}\cdot A = \begin{bmatrix} A_{pp} & A_{ps} \\ C_1A_{pp} + C_2A_{sp} & C_1A_{ps} + C_2A_{ss} \end{bmatrix} = \begin{bmatrix} A_{pp} & A_{ps} \\ \overline{A}_{sp} & \overline{A}_{ss} \end{bmatrix}. \quad (17)$$

The constant matrixes C_1 and C_2 are calculated using a least squares method to minimize $\left\|\overline{A}_{sp}\right\| / \left\|\overline{A}_{ss}\right\|$. Then the matrix $\overline{A}$ in Eq. 17 is multiplied by another matrix $\overline{\overline{C}}$ (Eq. 18) to generate the matrix $\overline{\overline{A}}$ (Eq. 19) that satisfies the criteria given in Eq. 12.

$$\overline{\overline{C}} = \begin{bmatrix} C_3 & C_4 \\ 0 & I \end{bmatrix} \quad (18)$$

$$\overline{\overline{A}} = \overline{\overline{C}}\cdot\overline{A} = \begin{bmatrix} C_3A_{pp} + C_4\overline{A}_{sp} & C_3A_{ps} + C_4\overline{A}_{ss} \\ \overline{A}_{sp} & \overline{A}_{ss} \end{bmatrix} = \begin{bmatrix} \overline{\overline{A}}_{pp} & \overline{\overline{A}}_{ps} \\ \overline{A}_{sp} & \overline{A}_{ss} \end{bmatrix} \quad (19)$$

The constant matrixes C_3 and C_4 are also calculated using the least squares method to minimize $\left\|\overline{\overline{A}}_{ps}\right\| / \left\|\overline{\overline{A}}_{pp}\right\|$. The values of C_1, C_2, C_3 and C_4 are given in Appendix A. The fourth method will work well if upstream weighting leads to significant loss of diagonal dominance.

Right Preconditioner

One way to improve the condition number of the matrix $\overline{A}$ (Eq. 11) is to use the right preconditioner, Eq. 4. The closer M_R to the matrix $\overline{A}$, the smaller the condition number is. Klie[11] presented that the right preconditioner, M_R, helps to cluster the Eigen values of $\overline{A}M_R^{-1}$, and therefore improves the convergence of GMRES.

To improve the estimation of the preconditioner M_R, a two-stage preconditioner is employed[9-11].

$$M_R = M_{R,2}\left(I - \overline{A}M_{R,1}\right) + M_{R,1} \tag{20}$$

Where $M_{R,1}$ is the first-stage preconditioner and $M_{R,2}$ is the second-stage preconditioner, the preconditioner $M_{R,1}$ decouples pressure from other unknowns such as saturations/concentrations, and relies on a strong solver for the pressure block equation that have an elliptic behavior. The second preconditioner $M_{R,2}$ resolves the original coupling between pressure and the other unknowns. More detail about $M_{R,1}$ will be discussed later. The inverse of $M_{R,2}$ is calculated using LSOR for the matrix $\overline{A}$[12]. Other methods, such as Incomplete LU factorization (ILU), can be used to approximate the inverse of $M_{R,2}$. Masson et al.[14], discussed that ILU(0) deteriorates as the mesh size increases in a heterogeneous environment. The algorithm for performing Eq. 20 is discussed in details in other literature[9, 10, 11].

To achieve the first preconditioner $M_{R,1}$, the matrix $\overline{A}$ is written in the form of two block equations, Eq. 10[5, 11, 13]. Many approaches are studied to approximate $M_{R,1}$ from $\overline{A}$. These approaches are:

1. **Approach 1.** Use only the pressure block matrix, Eq. 21. This leads to the popular constrained pressure residual method (CPR) discussed in detail in the literature[10, 11].

$$M_{R,1} = \begin{bmatrix} \overline{A}_{pp} & 0 \\ 0 & 0 \end{bmatrix} \tag{21}$$

2. **Approach 2.** Lower block Gauss-Seidel, Eq. 22, for solving the pressure followed by the solution of the other unknowns.

$$M_{R,1} = \begin{bmatrix} \overline{A}_{pp} & 0 \\ \overline{A}_{sp} & \overline{A}_{ss} \end{bmatrix} \tag{22}$$

3. **Approach 3.** Upper block Gauss-Seidel, Eq. 23, for solving the nonpressure, e.g., saturation, block first followed by the pressure solution:

$$M_{R,1} = \begin{bmatrix} \overline{A}_{pp} & \overline{A}_{ps} \\ 0 & \overline{A}_{ss} \end{bmatrix}. \tag{23}$$

4. **Approach 4.** Lower block Gauss-Seidel, Eq. 22, followed by a pressure solve, Eq. 21.

5. **Approach 5.** Two iterations of block Gauss-Seidel. First iteration uses lower block Gauss-Seidel, Eq. 22, and the second iteration uses upper block Gauss-Seidel, Eq. 23.

Block preconditioners proposed by Bai[4, 5] were also investigated but they yielded similar results to Approaches 2 and 3.

The nonpressure block $\overline{A}_{ss}$ (Eq. 21 to Eq. 23) presents a hyperbolic character[11, 13, 14]. Therefore, the approximate inverse for $\overline{A}_{ss}$ is calculated using one iteration of LSOR. On the other hand, the pressure matrix $\overline{A}_{pp}$ (Eq. 21 to Eq. 23) in general is closer to elliptic[11, 13, 14]. Therefore, an algebraic multigrid method (AMG)[15] is used to approximate the inversion of $\overline{A}_{pp}$. It has been proven in literature that the AMG is robust and fast for equations of an elliptic type[13-15]. In this study, it has been found that one V-cycle of AMG is enough to approximate the inversion of $\overline{A}_{pp}$. Scheichl et al.[13] also perform comparisons between one V-cycle of AMG and the exact inversion of $\overline{A}_{pp}$ that corroborates results of the present work.

The preconditioner $M_{R,1}$ better approximates the matrix $\overline{A}$ by increasing the number of iterations of the block Gauss-Seidel. One notices from the five approaches that the approximation of $M_{R,1}$ is improved by going from Approach 1 (CPR) to Approach 5 (Two iterations of block Gauss-Seidel). The lower block Gauss-Seidel, Eq. 22, and upper block Gauss-Seidel, Eq. 23, approaches can have the same accuracy as shown in the numerical examples.

Numerical Experiments

The different methods for the decoupling matrix *G* and the different approaches for the right preconditioners are implemented in the Integrated Parallel Accurate Reservoir Simulator (IPARS)[12]. The GMRES option is used as an iterative linear solver.

Some of the models in the database of IPARS are employed to study the robustness of the preconditioners. These models are BLACK OIL, SPE 9[6], and SPE 10[7]. The BLACK OIL model consists of 4,000 active cells and it is a three-phase flow. The permeability and the porosity are homogenous, and it has a fault in the middle of x-direction. The permeability is set to $K_z = 20$ md and $K_x = K_y = 200$ md. There is one injector and one producer located at opposite corners.

The model SPE 9[6] is another three phase model. It consists of 9,000 active cells and it dips in the x-direction at 9,000 ft. The porosity and the layer's thickness is constant within a given layer but they vary from one layer to another. The permeability is heterogeneous in each direction. The oil-water capillary pressure is discontinuous at water saturation $S_w = 0.35$. This can cause convergence problems for Newton's method to converge. SPE 9 has 25 producers and one injector.

The third model SPE 10[7] is a two-phase oil and water system. In this work, only the top 60 layers are utilized, so the total number of cells is 660,000. There are four producers at the corners and one injector in the middle of the reservoir. SPE 10 is widely studied in the literature and it is a challenging problem for traditional linear solvers due to the high heterogeneity in permeability, and due to the nearly incompressible fluid. Aksoylu et al.[16], illustrates how heterogeneous permeability with a small correlation length causes a wide spectrum of Eigen values.

For the models BLACK-OIL and SPE 9, the relative residual tolerance for the linear solver is set to 10^{-8}. For SPE 10, it is set to 10^{-3}. For the right preconditioner, one V-cycle is employed for AMG; and one iteration is set for LSOR.

Results and Discussions

Different combinations for using the decoupling matrices and the different approaches for $M_{R,1}$ are investigated for the three aforementioned reservoir models. For the decoupling matrix *G*, a comparison of Householder, Quasi-IMPES and least squares (Eq. 17 and Eq. 19) will be analyzed first. Also, the impact of forcing a unit diagonal in these combinations will be studied. The average number of iterations for the studied examples are provided in Tables 1, 2 and 3. The average number of the iterations is defined as the accumulated number of the GMRES linear iteration divided by the accumulated number of Newton during the simulation. In these Tables, two iterations of block Gauss-Seidel, Approach 5, are employed as the default first-stage right preconditioner $M_{R,1}$. As will be discussed later, Approach 5 yields the best result among all approaches for $M_{R,1}$.

One can notice from the Tables that the Householder transformation method combined with either Eq. 17 or Eq. 19 takes more iterations than when the given combination is normalized using Eq. 13. In fact, the normalized Householder combined with Eq. 17 takes less iterations than combined with Eq. 19. In addition, it delivers either the minimum number of iterations (e.g., BLACK-OIL) or close to the minimum number of iterations (e.g., SPE 9 and SPE 10). The non-normalized option to Householder transformation does not show differences when combined with either Eq. 17 or Eq. 19.

Unlike Householder transformation, Quasi-IMPES takes less iteration combined with Eq. 17 than when combined with Eq. 19. It does not converge for SPE 9 and SPE 10 when combined with Eq. 17 and normalized using Eq. 13. On the other hand, when

the Quasi-IMPES method is combined with Eq. 19 and normalized with Eq. 13, it converges for all models, and it also yields the minimum number of iterations for the SPE 10 case. For SPE 9, the normalization of the Quasi-IMPES combined with Eq. 19 requires more iterations. A more in-depth investigation is required to understand the performance of the Quasi-IMPES method in this case. Nevertheless, as a concluding remark, the combination of Quasi-IMPES with Eq. 17 should not be normalized to guarantee convergence of GMRES.

The different decoupling matrices (i.e., G) given in Table 4 are compared against the different approaches for the first-stage right preconditioner $M_{R,1}$. The average number of iterations of GMRES for BLACK-OIL, SPE 9 and SPE 10, are plotted in Figure 1, Figure 2 and Figure 3, respectively.

As discussed earlier, the approximation of $M_{R,1}$ is improved by moving from Approach 1 (CPR) to Approach 5 (Two iterations of block Gauss-Seidel). It is expected that the GMRES number of iterations decreases by improving the approximation of $M_{R,1}$. For BLACK-OIL and SPE 10, the GMRES number of iterations using Householder (Case 1) as the decoupling operator does not decrease by improving $M_{R,1}$. It arbitrary depends on the approaches for $M_{R,1}$. For SPE 10, Approach 1 takes less iteration than both Approach 2 (Lower block Gauss-Seidel) and Approach 3 (Upper block Gauss-Seidel). The SPE 10 model causes difficulties to Approach 3 since GMRES takes the highest number of iterations to converge. Similarly for BLACK-OIL, using Approach 1 for $M_{R,1}$, GMRES takes less iterations than Approach 3 and Approach 5. SPE 9 is the only studied model where the GMRES number of iterations using Case 1 decreases by improving the approximation of $M_{R,1}$.

When the decoupling operator (G) is calculated using Case 2, the GMRES takes less numbers of iterations than without normalization (Case 1) for SPE 10 and BLACK-OIL. Same as Case 1, the improvement in the approximation of $M_{R,1}$ has no impact in decreasing the number of iterations. For BLACK-OIL, the normalization of the diagonal (Case 2) reduces the number of iterations of GMRES in a significant way with respect to Approach 5. For SPE 9, on the other hand, Case 2 increases the number of iterations for some of the approaches (e.g., Approaches 3 and 5). Unlike Case 1 for SPE 9, Case 2 causes the GMRES iterations to not necessarily decrease by improving the approximation of $M_{R,1}$.

Householder transformations combined with Eq. 17 and normalized with Eq. 13 (Case 3) reduces the number of iterations for all the models considered. With no exceptions, the number of iteration decreases with improving the approximation of $M_{R,1}$. Approach 5 gives the minimum number of iteration among all approaches. For SPE 10, Case 3 with Approach 4 reduces the number of iterations by a considerable factor when compared to Case 1 and Case 2.

Using Quasi-IMPES (Case 4), the GMRES converges for BLACK-OIL, and it gives roughly the same number of iterations as Householder (Case 1). For SPE 9 and SPE 10, the GMRES either takes many iterations or does not converge for any of the Approaches considered for $M_{R,1}$. Matters get worse when the main diagonal is normalized with Eq. 13.

On the other hand, when the Quasi-IMPES method is coupled with Eq. 17 (Case 5), the convergent rate of the GMRES is improved relative to the simulation Case 4. In addition, the GMRES takes less iterations when the approximation of $M_{R,1}$ is improved. The decoupling, Case 5, takes more iterations than Case 3 for BLACK-OIL, and it takes roughly the same number of iterations as Case 3 for SPE 9. For the model SPE 10, the decoupling method in Case 5 takes more iterations than Case 3 for Approach 4 and Approach 5, but it takes nearly the same number of iterations for the other approaches defining $M_{R,1}$.

The decoupling achieved by the block diagonal inverse (Case 6) takes more iterations than the decoupling achieved by Case 3 and Case 5 for all studied models. When the block diagonal inverse is combined with the normalization (Case 7), it takes roughly the same number of iterations as Case 5 for both SPE 9 and SPE 10. For BLACK-OIL, Case 7 marginally takes less iteration than Case 5. As in the case of decoupling via Case 3 and Case 5, the GMRES with the decoupling Case 6 and Case 7 takes less iterations by improving the approximation of $M_{R,1}$.

For a given decoupling method, Approach 1 is supposed to be faster in CPU time than the other approaches per one linear iteration of the solver, because it involves only pressure to solve using the AMG. Approach 2 and Approach 3 are slower, because they need LSOR to solve for $\overline{A}_{ss}$ in addition to the pressure solve. Approach 4 needs extra pressure to solve in addition to Approach 3. Approach 5 needs extra pressure and LSOR to solve in addition to Approach 3. The number of iterations below which Approach 2 to Approach 5 will be faster than Approach 1 is determined and given in Table 5. In the Table, n is the number of iterations for a given approach, and $n_{Approch1}$ is the number of iterations for Approach 1. The ratios in Table 5 are averaged among all, using the decoupled method. It can be seen in Table 5 that Approach 2 and Approach 3 requires roughly less than 77% of $n_{Approch1}$ to be faster than Approach 1. Approach 4 and 5 requires less than 58% and 53%, respectively, of $n_{Approch1}$ to be faster than Approach 1. The algorithm of the LSOR in Integrated Parallel Accurate Reservoir

Simulator (IPARS) requires optimization for multiple calls per Newton. If the LSOR algorithm is optimized, it is expected that these ratios will be higher, especially for Approach 5 in Table 5.

Conclusion

Different decoupling operators and different right preconditioners have been investigated for solving the linear system of equations arising from the fully implicit formulation. From the analysis of the numerical examples, it was found that Householder transformations integrated with Eq. 17 and normalized using Eq. 13 produces a matrix that will take a fewer number of GMRES iterations than the original matrix. On the other hand, integration of Quasi-IMPES with Eq. 17 seems to negatively affect convergence upon normalization of the diagonal. Decoupling the system of equations with the block diagonal inverse, Eq. 13, leads to a better conditioned matrix than the original matrix. For the mentioned decoupling cases, using Approach 5 (two iterations of block Gauss-Seidel) for $M_{R,1}$ provides the lowest number of iterations relative to all other approaches considered in this work.

Table 1. Number of iterations for BLACK-OIL

	Eq. 17	Eq. 19	Eq. 17 + Eq. 13	Eq. 19 + Eq. 13
Quasi-IMPES	8.4	9.1	8.7	7.6
Householder	9.1	9.2	7.6	7.6

Table 2. Number of iterations for SPE 9

	Eq. 17	Eq. 19	Eq. 17 + Eq. 13	Eq. 19 + Eq. 13
Quasi-IMPES	8.2	11.4	Does not converge	18.5
Householder	10.2	10.7	8.6	9.

Table 3. Number of iterations for SPE 10

	Eq. 17	Eq. 19	Eq. 17 + Eq. 13	Eq. 19 + Eq. 13
Quasi-IMPES	46.2	58.8	Does not converge	21.0
Householder	57.2	57.4	21.6	47.8

Table 4. Cases for the decoupling matrix G

Cases	Decoupling Matrix (*G*)
Case 1	Householder
Case 2	Householder + Eq. 13
Case 3	Householder + Eq. 17 + Eq. 13
Case 4	Quasi-IMPES
Case 5	Quasi-IMPES + Eq. 17
Case 6	Eq. 15 (Inverse Diagonal without Normalization)
Case 7	Eq. 14 (Inverse Diagonal with Normalization)

Table 5. The ratio of the number iterations below which the given approach will be faster than Approach 1

	$n/n_{Approch1} \times 100$
Approach 2	78
Approach 3	77
Approach 4	58
Approach 5	53

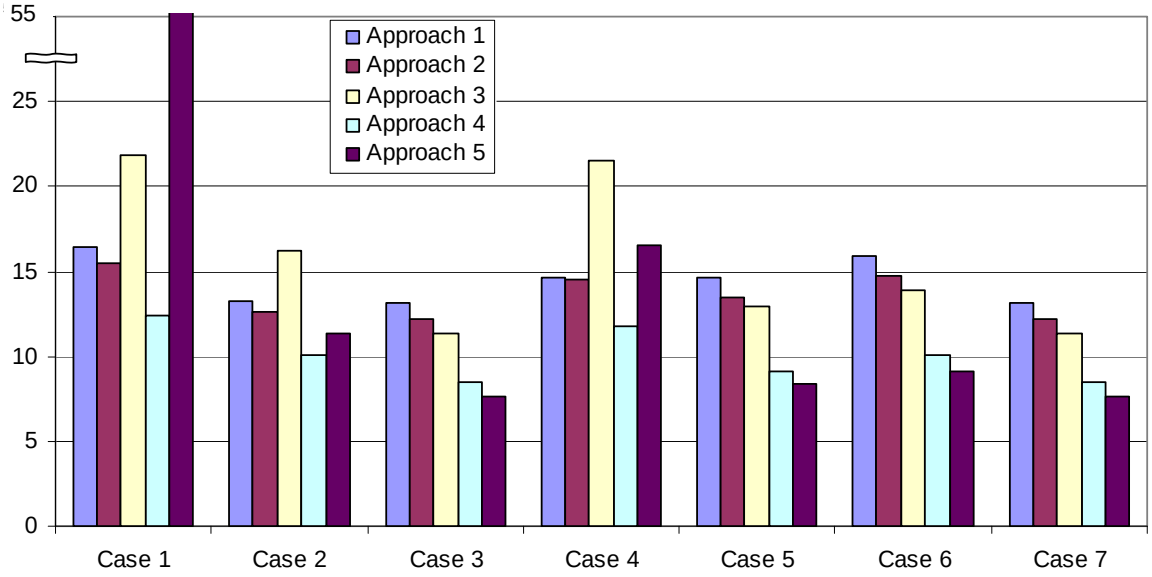

Figure 1. Number of iterations of GMRES for BLACK-OIL.

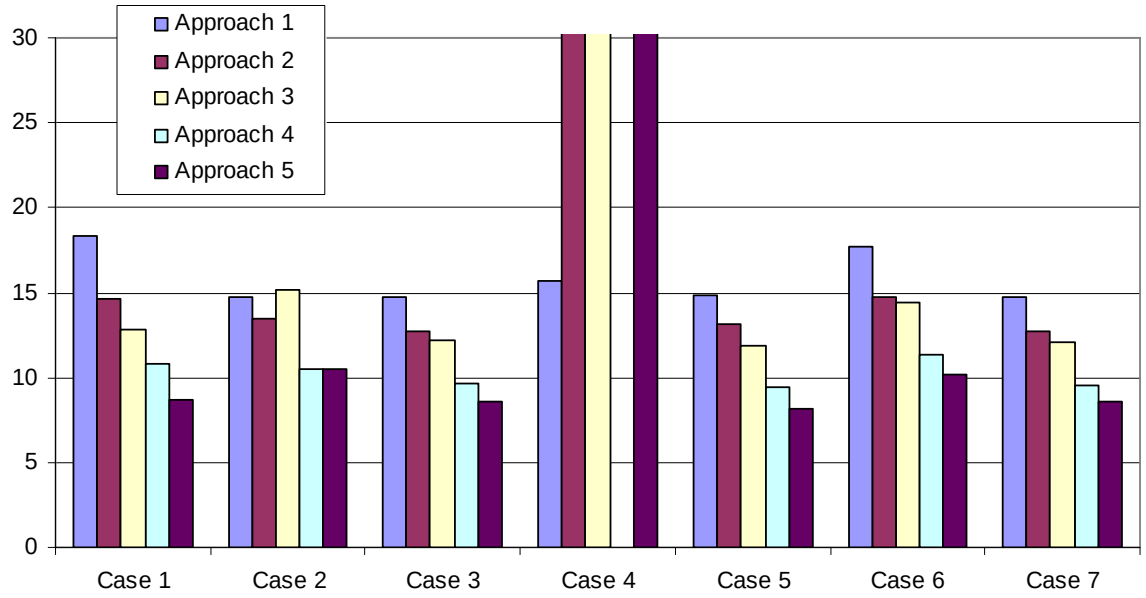

Figure 2. Number of iterations of GMRES for SPE 9.

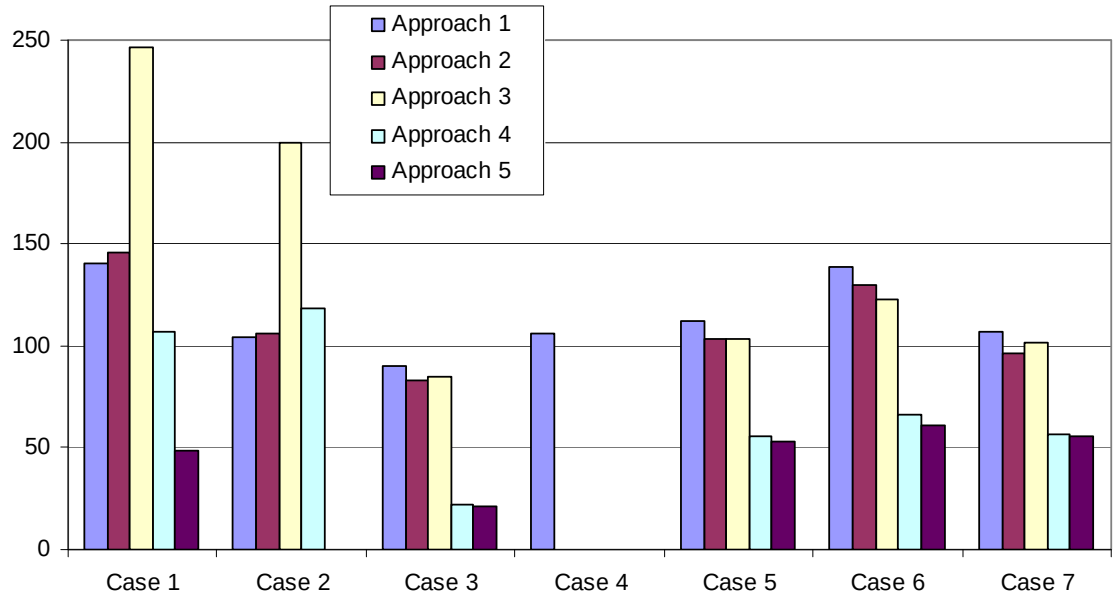

Figure 3. Number of iteration of GMRES for SPE 10.

Nomenclature

Ac	Accentric factor
k_r	Relative permeability
L	Matrix block size
MW	Molecular weight
N	Number of iterations
p	Pressure
Pc	Critical pressure
q	Production or injection rates
Δt	Time step
T	Transmissibility
V	Grid block volume
Vc	Critical volume
x	Mole fraction in the liquid phase
y	Mole fraction in the gas phase
S	Saturation
Tc	Critical temperature
λ	Phase mobility
σ	Shape factor
τ	Matrix-fracture transfer term
Φ	Phase potential
ϕ	Porosity
μ	Viscosity
ρ	Phase molar density

Subscripts

F	Fracture
g	Gas
i	Component number
j	Grid block neighbor
m	Matrix
o	Oil
p	Phase
w	Water or well
x, y, z	Coordinate directions
α	Upstream

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Appendix A

In this Appendix, the constants C_1, C_2, C_3 and C_4 for the balanced decoupling strategy (BDS), based on least squares (Eq. 17 and Eq. 19), will be calculated. The constants will be calculated for two unknowns such as pressure and saturation. The same procedures can be used for more than two unknowns. To find those constants, the discretized governing equations will be numbered in such a way that the unknowns go first. As a result, the discretized equations for a given grid block are

$$\sum_{j=1}^{ncoff} a_{1,j} P + \sum_{j=1}^{ncoff} b_{1,j} S = r_1 \qquad \text{(A-1)}$$

$$\sum_{j=1}^{ncoff} a_{2,j} P + \sum_{j=1}^{ncoff} b_{2,j} S = r_2 \qquad \text{(A-2)}$$

where a and b are the pressure and saturation coefficient, respectively. The subscript *1* and *2* are the equation number. The subscript *j* goes from 1 to the number of coefficients (*ncoff*) for a given equation and unknown. To get saturation equation with a minimum pressure coefficient, Eq. (A-1) and Eq. (A-2) are multiplied with constants C_1 and C_2, respectively, and added

$$\left(\sum_{j=1}^{ncoff} \left(C_1 a_{1,j} + C_2 a_{2,j} \right) P + \sum_{j=1}^{ncoff} \left(C_1 b_{1,j} + C_2 b_{2,j} \right) b_{1,j} S \right) = C_1 r_1 + C_2 r_2 . \quad \text{(A-3)}$$

The constants C_1 and C_2 will be calculated using the least square method to minimize pressure coefficients

$$\left(\sum_{j=1}^{ncoff} \left(a_{1,j} C_1 + a_{2,j} C_2 \right)^2 \right). \quad \text{(A-4)}$$

By differentiating Eq. (A-4) with respects to C_1 and C_2 and equate the results to 0,

$$\sum_{j=1}^{ncoff} \left(a_{1,j} C_1 + a_{2,j} C_2 \right) a_{1,j} = 0 \quad \text{(A-5)}$$

$$\sum_{j=1}^{ncoff} \left(a_{1,j} C_1 + a_{2,j} C_2 \right) a_{2,j} = 0 \quad \text{(A-6)}$$

Rearrange

$$C_1 \sum_{j=1}^{ncoff} a_{1,j}^2 + C_2 \sum_{j=1}^{ncoff} a_{1,j} a_{2,j} = 0 \quad \text{(A-7)}$$

$$C_1 \sum_{j=1}^{ncoff} a_{1,j} a_{2,j} + C_2 \sum_{j=1}^{ncoff} a_{2,j}^2 = 0 \quad \text{A-8}$$

One notices that the solution of Eq. (A-7) and Eq. (A-8) is trivial when C_1=C_2=0. This trivial solution is a global minimum. Therefore, we will look for the other local minimum. One of the local minimum solutions are:

$$C_1 = -\frac{C_2 \sum_{j=1}^{ncoff} a_{1,j} a_{2,j}}{\sum_{j=1}^{ncoff} a_{1,j}^2} \quad , \quad C_2 = \text{constant} = 1 . \quad \text{(A-9)}$$

Other local minimum solutions are:

$$C_2 = -\frac{C_1 \sum_{j=1}^{ncoff} a_{1,j} a_{2,j}}{\sum_{j=1}^{ncoff} a_{2,j}^2} \quad , \quad C_1 = \text{constant} = 1 . \quad \text{(A-10)}$$

The value of the constants C_2 in Eq. (A-9) and C_1 in Eq. (A-10) has no impact on the final solutions. Therefore, they are chosen to be 1. For a given grid block, two pairs of C_1 and C_2 is calculated from both Eq. (A-9) and Eq. (A-10). Then, we chose the solution of either equation that will give the minimum pressure coefficient in the saturation equation, Eq. (A-3).

The same procedure can be used to find C3 and C4 that will minimize saturation coefficient in pressure equations.