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Accurate Coarse-Scale AMG-Based Finite Volume Reservoir Simulations in Highly Heterogeneous Media

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Abstract

To make repeated large-scale simulations feasible very often some dimension reduction technique is employed. We present a dimension reduction approach for reservoir simulations exploiting a finite element based algebraic multigrid (or AMGe) technique which provides coarse models with guaranteed accuracy. We refer to these dimension reduction techniques as numerical upscaling via AMGe coarsening, or for short as AMGe-upscaling. Our reservoir simulation exploits a conservative finite volume formulation of the Darcy equations of porous media flow. For the upscaling purpose we also use an auxiliary mixed finite element discretization of the problem, which is constructed by embedding each finite volume into a refined set of tetrahedral or hexahedral elements. Given a possibly unstructured mesh and the permeability coefficient of a highly heterogeneous medium, the AMGe-upscaled (coarse) discretization is generated from a spectral element agglomeration algebraic multigrid procedure developed at LLNL[3,1,5,6]. While AMG approaches are traditionally used to construct preconditioners for the fine grid problem, here we use this methodology to systematically generate coarse spaces that accurately represent the original operator with high-contrast coefficients at coarse scales.

The spectral construction of the coarse basis allows us to capture complicated porous media coefficients with few degrees of freedom (dof), and to do so systematically and locally using only information from the geometry of the reservoir and the coefficients. The AMG foundation also enables us to create an entire hierarchy of coarse spaces, allowing more flexibility and the possibility of selecting the appropriate tradeoff between coarse level error and computational cost. The well-developed underlying theory of our AMGe-upscaling[8] gives a strong theoretical foundation from which we expect this procedure to deliver good approximation properties, and the local construction of the coarse spaces leads to a potentially scalable method in parallel.

We demonstrate the effectiveness of this method in terms of error behavior and computational cost for a variety of model problems in our numerical experiments, considering both standard benchmark problems and more realistic reservoir simulations.

In contrast to more traditional multiscale and upscaling approaches that rely on an ad hoc approach to representing heterogeneous coefficients with few degrees of freedom, our AMGe-upscaling procedure is

systematic and theoretically well supported. This provides greater confidence in the accuracy of the simulation, and provides better understanding and control of the tradeoff between accuracy and cost.

Introduction

A fundamental task in petroleum reservoir simulation is solution of the pressure equation

$$\nabla \cdot (\lambda \nabla p) = \phi \frac{\partial p}{\partial t} + f, \tag{1}$$

where λ, φ, f represent some spatial coefficients. In general these coefficients may be dependent on p, that is (1) may be nonlinear, but in what follows we will assume we are solving a linear version of (1), as in a Newton-type iteration. This equation arises for example in the implicit pressure-explicit saturation (IMPES) algorithm or the sequential implicit algorithm, both of which are widely used in reservoir simulation[10]. In practice, (1) is solved at a discrete level, so that at each time step we solve

$$-K(\lambda)p_{n+1} = \frac{1}{\Delta t}M(\phi)(p_{n+} \tag{2}$$

for the unknown p_{n+1} given the pressure p_n at the previous time step. Here K and M are now discrete matrices that arise from a finite volume discretization of (1). Repeated numerical solution of the problem (2) can be very computationally expensive. A very successful approach to solving the discrete pressure equation efficiently has been the class of algebraic multigrid algorithms, which form the basis for our approach here[9]. Our goal, however, is not to solve the problem (2) itself, but instead to use the AMG methodology to substitute an easier problem

$$-K^H(\lambda)p^H_{n+1} = \frac{1}{\Delta t}M^H(\phi)(p^H_{n+1} - p^H_n) + M^H(\phi)f^H_{n+1} \tag{3}$$

where the dimensions of K^H, $M^H p^H$ are much smaller than the corresponding fine-grid problem, and in addition the number of nonzeros in the matrices K^H, M^H is much smaller than that in K, M. As a result, the upscaled problem (3) can be solved much more efficiently than the original problem. The hope is that the upscaled model can be used repeatedly, so that any cost in constructing the coarser discretization can be overcome by the cost savings in solving the smaller problem in place of the original. The goal of this paper is to describe how to systematically construct an upscaled system in such a way that we can preserve the important properties of the original system, controlling the error of the coarsening process, and understanding the tradeoff between accuracy and computational cost.

AMGe-based coarsening

Given a linear matrix-vector system

$$Ax = b$$

where in the notation of the previous section we might have $A=K(\lambda)+(1/\Delta t)M(\varphi)$,$b=(1/\Delta t)M(\varphi)$,$p_n$-$M(\varphi)f_n+1=p_n+1$ In order to save computational effort, we do not solve the given linear problem $Ax = b$, but instead construct a projection matrix P and solve

$$(P^T AP)x_c = P^T b, \tag{4}$$

and then take Px_c as our approximation to the fine-scale solution x, where the operator $P^T AP$ has dimension much smaller than A (related to the notion of *arithmetic complexity*), and also has fewer nonzero entries than A (giving rise to the notion of *operator complexity*). The arithmetic and operator complexities are measures traditionally used in the AMG literature.

Our upscaling approach is based on the element-based algebraic multigrid (AMGe) setting, and specifically the spectral AMGe algorithm of[2],[4],[1] which takes as its input a connectivity graph and local element matrices. Although AMG methods are traditionally used to generate coarse meshes to aid in the

solution process of a fine scale model, here we use the coarse grids from AMG as upscaled representations of the system. Given a fine finite element mesh, we partition it into agglomerates, using for example a graph partitioner. Then on each agglomerate τ, we look for the local "near nulspace" of the operator A by approximately solving the eigenvalue problem

$$A_\tau y = \lambda R_\tau y \tag{5}$$

where A_τ is the finite element matrix assembled only on the aggregate τ and R_τ is a diagonal matrix related to the smoother we choose in the multigrid algorithm. In (5) we need only compute the eigenvectors corresponding to eigenvalues λ that are close to zero, which makes these local problems relatively cheap. In practice we set a tolerance θ and keep all the eigenvectors corresponding to eigenvalues less than θ. These eigenvectors then become columns in the projection matrix P, so that the coarse space contains all the components of the near nulspace of A. Next we discuss how we apply these finite element techniques to the finite volume problem at hand.

Direct application to the finite volume setting

Our first approach to upscaling (2) is to essentially interpret the finite volume system matrix as if it were a finite element matrix and apply the AMGe approach above. We interpret the matrix as a graph that gives us connectivity between cells, applying a graph partitioner to combine degrees of freedom into agglomerates without considering the actual physical geometry. Then we extract the principal submatrices of the global (finite volume) matrix corresponding to these degrees of freedom, and for an agglomerate τ we consider these submatrices as a kind of local agglomerate matrix $\tilde{A}_\tau$.

The AMGe method expects the local matrices for the eigenvalue problem to be assembled from local finite element matrices, which the above procedure does not give us. So we modify the diagonal of $\tilde{A}_\tau$ to make it an M-matrix that we call $\tilde{\tilde{A}}_\tau$, and then we solve local eigenvalue problems (5) and continue with the AMGe upscaling approach to construct P and therefore an upscaled problem. We emphasize that the modification of the diagonal is only used in constructing P, and the problem we solve in the end is still based on the unmodified matrix A.

This direct finite volume approach is purely algebraic, only relying on the matrix A. Whatever coefficients or physics are included in A are used in the construction of P, so that the upscaled system should be well adapted to the original system. On the other hand to make the spectral method work in this setting we need to make some modifications to local agglomerate matrices, and the theory behind AMGe methods does not directly apply.

Application to finite volumes through an auxiliary mixed finite element formulation

The pressure equation (1) can also be written in a mixed formulation

$$\lambda^{-1} u = \nabla p$$
$$-\nabla \cdot u = \phi \frac{\partial p}{\partial t} + f.$$

The mixed formulation is attractive because all of our upscaling theory is based on finite element discretizations, and the mixed finite element discretization is mass conservative.

In order to construct an upscaled discretization in the finite volume case one way is to go through an auxiliary finite element method. In contrast to the previous (direct finite volume) upscaling approach, to derive the auxiliary finite element discretization, we need additional information (not only the fine-grid finite volume matrix). For the given finite volume mesh, we construct a finite element mesh by subdividing volumes that are not hexahedra or tetrahedra. Since pressures are represented as piecewise constants in both cases, interpolating pressures between finite element and finite volume meshes is straightforward. We use the permeability field of the original problem to construct a finite element system, and then use a spectral version of the mixed AMGe based upscaling on this finite element system. This

results in a projection matrix P that, after combining with the map between finite elements and finite volumes, can be used to upscale the original finite volume matrix A.

The mixed approach is well grounded in element-based algebraic multigrid theory. However, it relies on a finite element mesh, which is not always easy to construct, and the construction of P includes only whatever coefficients can be expressed on this mesh—for example, at present we do not include face-based mobilities in the construction of P (although the upscaled system does include them through A).

Numerical examples

Below we present numerical results for our pressure equation upscaling applied to linear systems arising in a petroleum reservoir simulation. Sequential implicit formulation[10] is used in the simulation, where the pressure equation is first solved implicitly followed by an implicit solution of the saturation equations as well. We consider two reservoir simulations, the Tarbert portion of the SPE[10] testcase and also a test case from the Saigup project.

The key questions at this stage of our research are about error and cost. To measure error, we compare the upscaled pressure solution to the fine scale pressure solution. In practice the simulator is solving a linear system $Axcor = b$ for a pressure correction $xcor$ and then adding this correction to a base pressure, so that the total pressure is $x = x_{\text{base}} + x_{\text{cor}}$. We report

$$\frac{||x_{\text{upscaled}} - x||}{||x||}$$

as a relative error (where the norm can be either l_2 or a discrete energy norm) and report

$$\frac{||x_{\text{upscaled}} - x||}{||x_{\text{cor}}||}$$

as "delta error". We also report the (absolute) error in the max norm.

To get some idea of cost, we report the number of coarse degrees of freedom (dof) in each case, and also a multigrid measure known as *operator complexity*, which is defined to be the ratio between the total number of nonzero entries in both fine and upscaled system matrices divided by the number of nonzeros in the original fine scale system matrix. That is, an operator complexity of 1.25 in our context can be interpreted as saying the coarse system has one fourth the number of nonzeros as the fine system.

Tarbert The Tarbert formation is a piece of the standard SPE10 testcase. It has 462000 cells in a structured mesh with high-constrast anisotropic coefficients. We run a reservoir simulation with two wells (one injector and one producer), doing two phase oil-gas flow, and analyze the quality of the pressure equation upscaling (as compared to the solution on the fine mesh) at selected timesteps. Plots of a typical pressure solution and the upscaling error are shown in Figure 1. Results are reported in Table 1. We note here that both methods are producing quite accurate results, but the mixed finite element upscaling provides lower error and simpler, easier to solve upscaled linear systems. The actual process of upscaling is much more complicated and expensive for the mixed upscaling, however.

One of the strengths of our upscaling approach is the flexibility we have in choosing both the size of our unstructured agglomerates and also the tolerance for the local eigenvectors we add to the upscaled representation. In Tables 2 and 3 we illustrate the tradeoff between cost and accuracy, showing operator complexity for a variety of agglomerate sizes and spectral tolerances in Table 2 and showing the corresponding relative l_2 error in Table 3, here for the case of direct finite volume upscaling. In these tables, an eigenvalue tolerance $\theta = 0$ corresponds to taking only one eigenvector per agglomerate in the coarse basis. These results show that we can tune the properties of our upscaling method so that the user can select a desired tradeoff between accuracy and cost.

Saigup Since the agglomerates in our upscaling method can in general be any shape, this approach is well suited to upscaling unstructured reservoirs. Here we look at a smaller but more unstructured mesh,

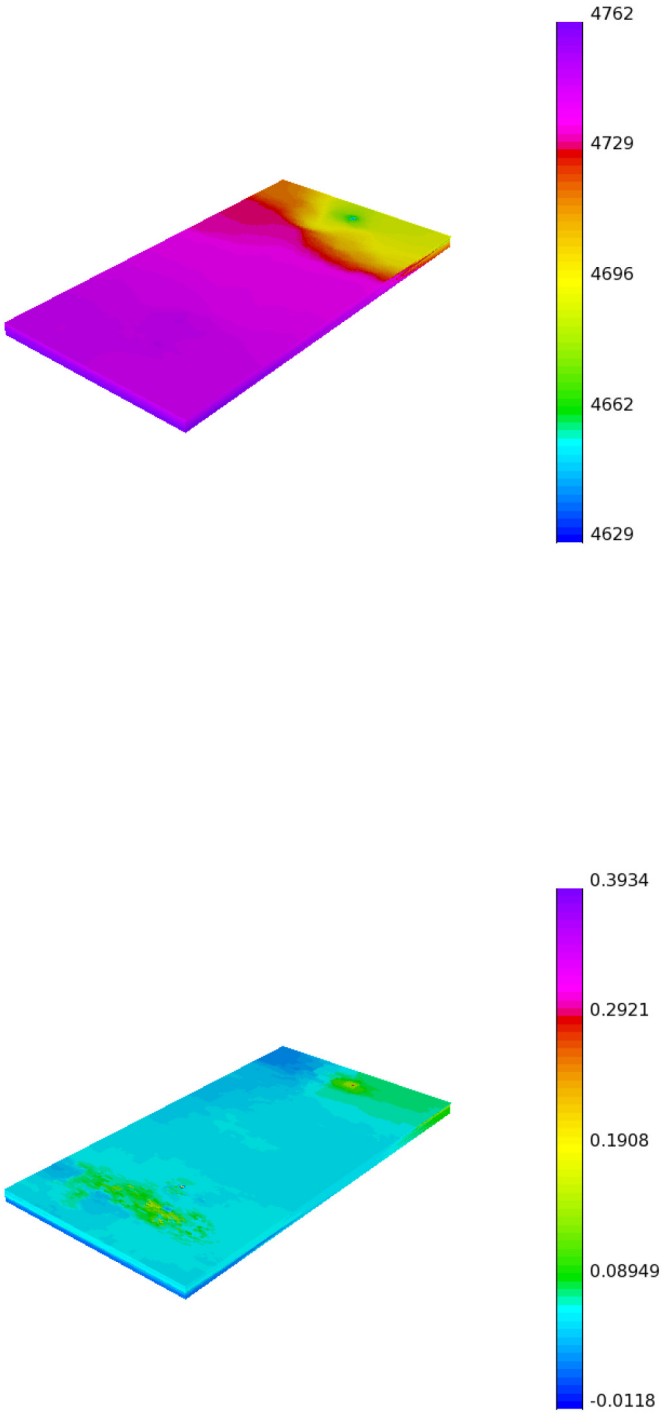

Figure 1—Typical pressure solution for the Tarbert mesh (top) and the upscaling error (bottom). These results come from the direct finite volume upscaling at timestep with 128 volumes per aggregate and a spectral tolerance of 0.001.

the test case from the Saigup project[7] which has about 75000 degrees of freedom. The geometry and x-direction permeability field are visualized in Figure 2. The simulation we run here has 10 wells, five injectors and five producers, and the problem is two phase oil-water flow.

We demonstrate some typical agglomerated elements for this unstructured model in Figure 3, while a comparison of the simulator solution and the upscaled solution using the mixed method at the 300th timestep are shown in Figure 4.

1—Results from the Tarbert test case at various timesteps. The direct method is using 128 volumes per agglomerate and $\theta = 0.001$, while the mixed method has 64 volumes per agglomerate and $\theta = 10^{-8}$.

type	timestep	l_2 error	energy error	abs max error	delta error	complexity	coarse dofs
direct	0	3.31e-05	1.21e-04	8.59e+00	6.33e-01	1.40	24111
direct	100	1.46e-05	1.52e-05	3.93e-01	3.12e-01	1.18	11006
direct	200	3.86e-05	2.31e-05	6.22e-01	4.19e-01	1.22	12900
direct	300	1.85e-05	1.30e-05	3.54e-01	3.74e-01	1.25	14520
direct	400	2.34e-05	1.32e-05	3.30e-01	4.48e-01	1.26	15369
mixed	0	4.23e-05	1.75e-04	1.11e+01	9.17e-01	1.05	8272
mixed	100	9.18e-07	3.83e-06	4.43e-02	7.84e-02	1.05	8272
mixed	200	1.25e-06	3.76e-06	2.03e-01	6.83e-02	1.05	8272
mixed	300	7.08e-07	4.00e-06	7.57e-02	1.15e-01	1.05	8272
mixed	400	5.40e-07	2.63e-06	3.01e-02	8.93e-02	1.05	8272

2—Operator complexity of Tarbert simulation at timestep 200 with direct finite volume upscaling for various agglomerate sizes and spectral tolerances.

	θ				
ratio	0.0	0.0001	0.0005	0.001	0.002
32	1.34	1.46	1.69	1.93	2.49
64	1.10	1.15	1.23	1.29	1.53
128	1.07	1.11	1.17	1.22	1.39

3—Relative l_2 upscaling error corresponding to Table 2.

	θ				
ratio	0.0	0.0001	0.0005	0.001	0.002
32	1.454e-04	3.448e-05	3.408e-05	3.387e-05	3.355e-05
64	1.555e-04	3.772e-05	3.703e-05	3.682e-05	3.651e-05
128	1.710e-04	3.935e-05	3.885e-05	3.860e-05	3.823e-05

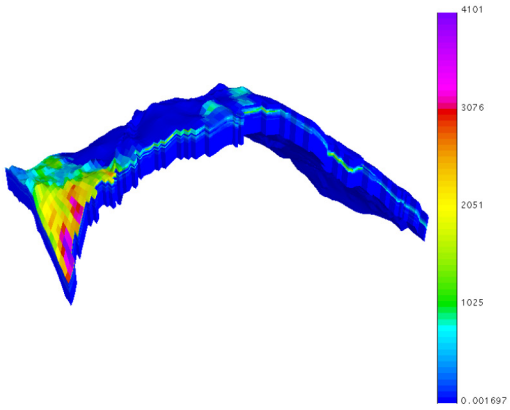

Figure 2—Permeability in the x-direction for the Saigup reservoir model.

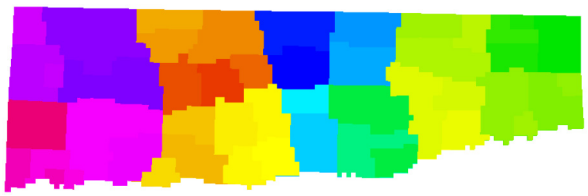

Figure 3—Typical unstructed agglomerated elements for the Saigup reservoir model.

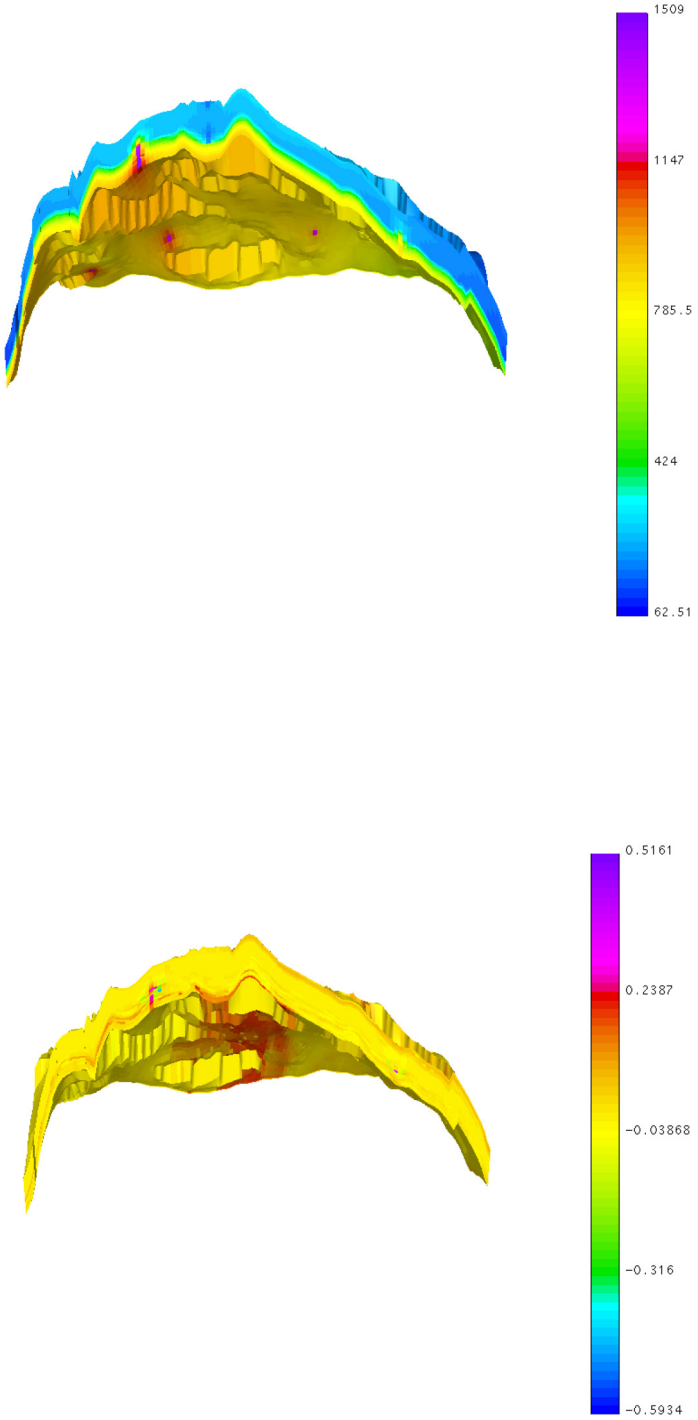

Figure 4—The pressure solution for the Saigup model (top) and the error for our upscaling approximation (bottom). Notice the error is at most 0.6 for a background pressure that ranges from 60 to 1500. Vertical scale is exaggerated in this picture

Results in terms of upscaling error and operator complexity for this model are shown in Table 4. Here, in constrast to the Tarbert test, the direct finite volume upscaling and the mixed finite element upscaling show similar behavior.

4—Relative errors in discrete l_2 and energy norms for the Saigup model.

type	timestep	l_2 error	energy error	abs max error	delta error	complexity	coarse dofs
direct	0	6.75e-07	5.78e-07	1.47e-02	1.49e-01	1.87	10754
direct	100	2.17e-03	2.40e-03	1.22e+01	3.74e-01	1.47	6341
direct	200	1.47e-03	1.71e-03	5.33e+01	3.39e-01	1.51	6797
direct	300	7.89e-05	8.34e-05	5.03e-01	4.69e-01	1.52	6903
direct	400	6.67e-04	7.47e-04	3.11e+00	4.45e-01	1.66	8447
mixed	0	1.38e-06	1.18e-06	2.69e-02	3.02e-01	1.30	5132
mixed	100	1.71e-03	2.50e-03	7.82e+00	3.90e-01	1.23	2709
mixed	200	1.09e-03	1.40e-03	1.16e+01	2.78e-01	1.24	3051
mixed	300	5.27e-05	6.30e-05	6.13e-01	3.55e-01	1.23	2786
mixed	400	8.08e-04	9.03e-04	1.58e+00	5.38e-01	1.27	3896

Conclusion

We have demonstrated an upscaling approach where the coarse model is derived from the coarse grids of an algebraic multigrid hierarchy. In this approach, the solution of small local eigenvalue problems provides a basis for the upscaled discretization that is tailored to whatever coefficients and physics are in the original linear system. Tests of the upscaling technique on a large structured reservoir simulation problem and also a smaller unstructured grid show that the upscaled discretization can be made much smaller than the original system both in terms of number of degrees of freedom and also the number of nonzeros in the system matrices. The upscaling method can be adjusted by the user to provide the desired tradeoff between accuracy and cost. Although the actual construction of the upscaled discretization is at present quite expensive (but local), it is believed that this cost can be spread out over many timesteps in a time-dependent simulation or many realizations in an uncertainty quantification setting, and that the method can provide significant speedup over the fine scale model. Also, since the main cost involves local (relatively) small eigenporoblems, the perfromance of the method can be speeded up substantially on a multicore system.

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