Differential Equation

Hoseung Kang

This document is based on the contents of the **Differential Equation**.

I started this work to establish a solid foundation for my research activities.

I believe that theoretical understanding is essential in any field, so I studied to build background knowledge alongside basic research.

Since studying alone and leaving it behind leads to nothing remaining, I decided to organize my learning into this document.

This work was done based on the learning attitude taught to me by **Jung Byung-ho** and **Jung Byung-hoon**.

Although the contents may differ, the approach to studying concepts, proofs, and logical reasoning reflects the mindset and methods I learned from them.

I am truly grateful for all they have taught me.

Textbook
Elementary Differential Equations and Boundary Value Problems
Ninth Edition
William E. Boyce, Richard C. DiPrima

I. First Order Differential Equations

General Form

$$\frac{dy}{dt} + P(t)y = Q(t)$$

Multiplying both sides by integrating factor $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + \mu(t)P(t)y = \mu(t)Q(t)$$

Left side is:

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)Q(t)$$

Integrate both sides:

$$\mu(t)y = \int \mu(t)Q(t)\,dt + C$$

General solution:

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)Q(t)\,dt + C\right)$$

Integrating Factor

Definition:

$$\mu(t) = e^{\int P(t)\,dt}$$

Example:

$$\mu(t) = e^{\int \frac{4t}{1+t^2}\,dt} = e^{2\ln(1+t^2)} = (1+t^2)^2$$

Example Problems

Problem: Solve the initial value problem

$$t\frac{dy}{dt} + 2y = 4t^2, \quad y(1) = 2$$

Solution:

$$\begin{aligned} t^2\frac{dy}{dt} + 2ty &= 4t^3 \\ \frac{d}{dt}(t^2y) &= t^4 \\ \int d(t^2y) &= \int t^4\,dt \\ t^2y &= t^4 + C \\ y &= t^2 + \frac{C}{t^2} \end{aligned}$$

Apply $y(1) = 2$:

$$2 = 1 + C \quad \Rightarrow \quad C = 1$$

Final answer:

$$y = t^2 + \frac{1}{t^2}$$

—

Page 39: Problem 8

Given:

$$(1+t^2)\frac{dy}{dt} + 4ty = (1+t^2)^{-2}$$

Solution:

$$\begin{aligned} (1+t^2)^2\frac{dy}{dt} + 4t(1+t^2)y &= (1+t^2)^{-1} \\ \frac{d}{dt}\left((1+t^2)^2y\right) &= \frac{1}{1+t^2} \\ \int d\left((1+t^2)^2y\right) &= \int \frac{1}{1+t^2}\,dt \\ (1+t^2)^2y &= \tan^{-1}(t) + C \\ y &= \frac{\tan^{-1}(t) + C}{(1+t^2)^2} \end{aligned}$$

—

Problem 21

Given:

$$y' - \frac{1}{2}y = \cos t, \quad y(0) = a$$

Solution:

Integrating factor:

$$\mu(t) = e^{\int -\frac{1}{2}\,dt} = e^{-\frac{1}{2}t}$$

Multiply both sides:

$$\frac{d}{dt}\left(e^{-\frac{1}{2}t}y\right) = e^{-\frac{1}{2}t}\cos t$$

Integrate:

$$e^{-\frac{1}{2}t}y = \int e^{-\frac{1}{2}t}\cos t\,dt$$

Standard formula:

$$\int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$$

Apply $a = -\frac{1}{2},\ b = 1$:

$$= \frac{e^{-\frac{1}{2}t}}{\frac{1}{4}+1}\left(-\frac{1}{2}\cos t + \sin t\right) = \frac{4}{5}e^{-\frac{1}{2}t}\left(-\frac{1}{2}\cos t + \sin t\right)$$

Back to solution:

$$e^{-\frac{1}{2}t}y = \frac{4}{5}e^{-\frac{1}{2}t}\left(-\frac{1}{2}\cos t + \sin t\right) + C$$

Simplify:

$$y = \frac{2}{5}\sin t - \frac{1}{5}\cos t + Ce^{\frac{1}{2}t}$$

Apply initial condition $y(0) = a$:

$$a = 0 - \frac{1}{5} + C \Rightarrow C = a + \frac{1}{5}$$

Final answer:

$$y = \frac{2}{5}\sin t - \frac{1}{5}\cos t + \left(a + \frac{1}{5}\right)e^{\frac{1}{2}t}$$

—

Problem 22

Given:

$$2y' - y = e^{-\frac{t}{6}}$$

Solution:

Integrating factor:

$$\mu(t) = e^{\int -\frac{1}{2}\,dt} = e^{-\frac{1}{2}t}$$

Multiply both sides:

$$\frac{d}{dt}\left(2e^{-\frac{1}{2}t}y\right) = e^{-\frac{1}{2}t}e^{-\frac{t}{b}} = e^{t\left(-\frac{1}{b}-\frac{1}{2}\right)}$$

Case 1: $b = 2$

$$e^{t\left(-\frac{1}{2}-\frac{1}{2}\right)} = e^{-t}$$

Integrate:

$$\int e^{-t}\,dt = -e^{-t} + C$$
$$2e^{-\frac{1}{2}t}y = -e^{-t} + C$$
$$y = -\frac{1}{2}e^{-\frac{1}{2}t} + Ce^{\frac{1}{2}t}$$

—

Problem 24

Given:

$$t\frac{dy}{dt} + (t+1)y = 2te^{-t}$$

Solution:

Integrating factor:

$$\mu(t) = e^{\int \frac{t+1}{t}\,dt} = e^{t+\ln t} = te^{t}$$

Multiply both sides:

$$\frac{d}{dt}\left(te^{t}y\right) = 2t$$

Integrate:

$$te^{t}y = t^2 + Cy = \frac{t^2 + C}{te^{t}}$$

—

Problem 33

Given:

$$y' + ay = be^{-\lambda t}$$

Case 1: $a \neq \lambda$

Integrating factor:

$$\mu(t) = e^{at}$$

Multiply both sides:

$$\frac{d}{dt}\left(e^{at}y\right) = be^{(a-\lambda)t}$$

Integrate:

$$e^{at}y = \frac{b}{a-\lambda}e^{(a-\lambda)t} + Cy = \frac{b}{a-\lambda}e^{-\lambda t} + Ce^{-at}$$

Case 2: $a = \lambda$

$$\frac{d}{dt}\left(e^{at}y\right) = b$$

Integrate:

$$e^{at}y = bt + Cy = \frac{bt + C}{e^{at}}$$

Behavior at infinity

$$\lim_{t\to\infty} y = 0 \quad \text{if} \quad a > 0$$