Calculus

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This document is based on the contents of the **calculus**.

I started this work to establish a solid foundation for my research activities.

I believe that theoretical understanding is essential in any field, so I studied to build background knowledge alongside basic research.

Since studying alone and leaving it behind leads to nothing remaining, I decided to organize my learning into this document.

This work was done based on the learning attitude taught to me by **Jung Byung-ho** and **Jung Byung-hoon**.

Although the contents may differ, the approach to studying concepts, proofs, and logical reasoning reflects the mindset and methods I learned from them.

I am truly grateful for all they have taught me.

Textbook

Calculus: Early Transcendentals for Scientists and Engineers
Metric Edition
James Stewart

Page 68: Problem 69
Prove that:

$$\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$$

Given:

$$\text{Domain of } x \in [-1,1],$$
$$\text{Range of } \sin^{-1}(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Therefore, since cosine is non-negative on this interval:

$$\begin{aligned}\cos(\sin^{-1}(x)) &= \sqrt{1-\sin^2(\sin^{-1}(x))}\\ &= \sqrt{1-x^2}\end{aligned}$$

Alternative geometric approach (triangle):

$$\sin\theta = x, \quad \cos\theta = y$$
$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow x^2 + y^2 = 1 \Rightarrow \cos\theta = \sqrt{1-x^2}$$
$$\therefore \cos(\sin^{-1}(x)) = \sqrt{1-x^2}$$

Page 68: Problem 70

$$\begin{aligned}\tan(\sin^{-1}(x)) &= \sqrt{\sec^2(\sin^{-1}(x)) - 1}\\ &= \sqrt{\frac{1}{1-\sin^2(\sin^{-1}(x))} - 1}\\ &= \sqrt{\frac{1}{1-x^2} - 1}\\ &= \sqrt{\frac{x^2}{1-x^2}}\\ &= \frac{x}{\sqrt{1-x^2}}\end{aligned}$$

Page 68: Problem 71

$$\begin{aligned}\sin(\tan^{-1}(x)) &= \sqrt{1-\cos^2(\tan^{-1}(x))}\\ &= \sqrt{1-\frac{1}{\sec^2(\tan^{-1}(x))}}\\ &= \sqrt{1-\frac{1}{1+\tan^2(\tan^{-1}(x))}}\\ &= \sqrt{1-\frac{1}{1+x^2}}\\ &= \frac{x}{\sqrt{1+x^2}}\end{aligned}$$

Page 75 : Example 3
Given:

$$f_0(x) = \frac{x}{x+1}, \quad f_{n+1} = f_0(f_n)$$

(1) First Approach:

$$f_n = \frac{x}{nx+1}$$
$$f_{n+1} = f_0(f_n) = \frac{f_n}{f_n+1} = \frac{\frac{x}{nx+1}}{\frac{x}{nx+1}+1} = \frac{x}{(n+1)x+1}$$

(2) Inductive Approach:

$$f_n = \frac{x}{nx+1}$$

Base case $n = 1$:

$$f_1 = f_0(f_0) = \frac{f_0}{f_0+1} = \frac{x}{x+2}$$

Inductive step: assume $f_k = \frac{x}{kx+1}$, then

$$f_{k+1} = f_0(f_k) = \frac{f_k}{f_k+1} = \frac{\frac{x}{kx+1}}{\frac{x}{kx+1}+1} = \frac{x}{(k+1)x+1}$$

$$\therefore \text{ by induction,} \quad f_n = \frac{x}{nx+1}$$

II. Limits and Theorems

Definition of Limit
Given $f(x)$ defined near $x = a$,

$$\lim_{x \to a} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta > 0 :\ 0 < |x - a| < \delta \ \Rightarrow\ |f(x) - L| < \varepsilon$$

Alternative notation:

$$f(x) \to L \quad \text{as} \quad x \to a$$

One-Sided Limits
Left-hand limit:

$$\lim_{x \to a^-} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta > 0 :\ a - \delta < x < a \ \Rightarrow\ |f(x) - L| < \varepsilon$$

Right-hand limit:

$$\lim_{x \to a^+} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta > 0 :\ a < x < a + \delta \ \Rightarrow\ |f(x) - L| < \varepsilon$$

Infinite Limit

$$\lim_{x \to a} f(x) = \infty \iff \forall M > 0,\ \exists \delta > 0 :\ 0 < |x - a| < \delta \ \Rightarrow\ f(x) > M$$

Alternative notation:

$$f(x) \to \infty \quad \text{as} \quad x \to a$$

Limit Laws
Suppose

$$\lim_{x \to a} f(x) = L, \quad \lim_{x \to a} g(x) = M$$

Then

$$\begin{aligned}
\lim_{x \to a} [f(x) + g(x)] &= L + M \\
\lim_{x \to a} [f(x) - g(x)] &= L - M \\
\lim_{x \to a} [c \cdot f(x)] &= cL \\
\lim_{x \to a} [f(x) g(x)] &= LM \\
\lim_{x \to a} \frac{f(x)}{g(x)} &= \frac{L}{M},\ M \neq 0
\end{aligned}$$

Proof of Limit Laws
Assume

$$\lim_{x \to a} f(x) = L, \quad \lim_{x \to a} g(x) = M.$$

For all $\varepsilon > 0$, there exist $\delta_f > 0$ and $\delta_g > 0$ such that

$$0 < |x - a| < \delta_f \ \Rightarrow\ |f(x) - L| < \varepsilon, \quad 0 < |x - a| < \delta_g \ \Rightarrow\ |g(x) - M| < \varepsilon.$$

(1) Sum Rule
For $\varepsilon > 0$, choose $\varepsilon/2$ for each function and set

$$\delta = \min\{\delta_f, \delta_g\}.$$

Then,

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(2) Difference Rule
Same proof as the sum, since

$$|f(x) - g(x) - (L - M)| \le |f(x) - L| + |g(x) - M| < \varepsilon.$$

(3) Constant Multiple Rule
For constant c,

$$|c \cdot f(x) - cL| = |c| \cdot |f(x) - L| < \varepsilon \quad \text{if} \quad |f(x) - L| < \frac{\varepsilon}{|c|}.$$

If $c = 0$, both sides are zero.
(4) Product Rule
Use

$$|f(x)g(x) - LM| \le |f(x) - L||g(x)| + |L||g(x) - M|.$$

Since $g(x) \to M$, pick δ_1 so that $|g(x) - M| < 1$ implies

$$|g(x)| \le |M| + 1.$$

Choose δ_2 so that

$$|f(x) - L| < \frac{\varepsilon}{2(|M| + 1)}, \quad |g(x) - M| < \frac{\varepsilon}{2|L| + 2}.$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$|f(x)g(x) - LM| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Law 5 (Quotient Rule)
Assume $M \neq 0$. We want to show

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Step 1: Reciprocal Limit
First show

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}.$$

Given $\varepsilon > 0$, choose $\delta_1 > 0$ so that

$$0 < |x - a| < \delta_1 \;\Rightarrow\; |g(x) - M| < \frac{|M|}{2}.$$

Then,

$$|g(x)| = |M + (g(x) - M)| \ge |M| - |g(x) - M| > |M| - \frac{|M|}{2} = \frac{|M|}{2}.$$

So,

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \frac{|M - g(x)|}{|M| \cdot \frac{|M|}{2}} = \frac{2}{M^2}|M - g(x)|.$$

Pick $\delta_2 > 0$ so that

$$0 < |x - a| < \delta_2 \;\Rightarrow\; |g(x) - M| < \frac{M^2}{2}\varepsilon.$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \frac{2}{M^2} \cdot \frac{M^2}{2}\varepsilon = \varepsilon.$$

Therefore,

$$\lim_{x\to a}\frac{1}{g(x)}=\frac{1}{M}.$$

Step 2: Use Product Rule
Write

$$\frac{f(x)}{g(x)}=f(x)\cdot\frac{1}{g(x)}.$$

Then by the product law,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}f(x)\cdot\lim_{x\to a}\frac{1}{g(x)}=L\cdot\frac{1}{M}=\frac{L}{M}.$$

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{L}{M}.}$$

Example:

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Solution 1 (Direct Inequality)
Use

$$\cos^2 x\ <\ \frac{\cos x\cdot\sin x}{x}\ <\ 1\qquad (x\neq 0,\ |x|<\pi/2).$$

Divide both sides by $\cos x>0$:

$$\cos x\ <\ \frac{\sin x}{x}\ <\ \frac{1}{\cos x}.$$

Take limits as $x\to 0$:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1.}$$

Solution 2 (Geometric Squeeze)
From the unit circle:

$$\sin x<x<\tan x.$$

Divide by $\sin x>0$:

$$1<\frac{x}{\sin x}<\frac{1}{\cos x}.$$

Take reciprocals:

$$\cos x<\frac{\sin x}{x}<1.$$

Take the limit $x\to 0$:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1.}$$

Solution 3 (Integral Bound)
Use

$$\sin x-x=\int_0^x(\cos t-1)\ dt=-\int_0^x(1-\cos t)\ dt.$$

Since

$$1 - \cos t \geq \frac{t^2}{2} - \frac{t^4}{24},$$

integrating gives

$$\int_0^x (1 - \cos t)\ dt \geq \frac{x^3}{6} - \frac{x^5}{120}.$$

Thus,

$$\sin x - x \leq -\left(\frac{x^3}{6} - \frac{x^5}{120}\right).$$

Simplify:

$$\frac{\sin x}{x} - 1 \leq -\frac{x^2}{6} + \frac{x^4}{120}.$$

Take the limit $x \to 0$:

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1.}$$

Comparison Theorem

If $f(x) \leq g(x)$ for all x in an open interval containing a (except possibly at $x = a$), and

$$\lim_{x\to a} f(x) = L, \quad \lim_{x\to a} g(x) = M,$$

then

$$\boxed{L \leq M.}$$

Proof

We use the method of **proof by contradiction**.
Suppose, for the sake of contradiction, that

$$L > M.$$

Consider

$$\lim_{x\to a} \big(g(x) - f(x)\big) = M - L.$$

Since $L > M$, we have $M - L < 0$.
By the definition of limit, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad \big|\big(g(x) - f(x)\big) - (M - L)\big| < \varepsilon.$$

Take $\varepsilon = L - M$ (note that $L - M > 0$). Then, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad \big|\big(g(x) - f(x)\big) - (M - L)\big| < L - M.$$

Since $|b| \geq b$ for any real number b, this implies

$$\big(g(x) - f(x)\big) - (M - L) < L - M,$$

which simplifies to

$$g(x) - f(x) < (M - L) + (L - M) = 0.$$

That is,

$$g(x) < f(x).$$

But this contradicts the assumption that $f(x) \le g(x)$ for all x near a.
Therefore, the assumption $L > M$ must be false. Hence,

$$\boxed{L \le M.}$$

Squeeze Theorem

If $f(x) \le g(x) \le h(x)$ for all x in an open interval containing a (except possibly at $x = a$), and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then

$$\boxed{\lim_{x \to a} g(x) = L.}$$

Proof

Let $\varepsilon > 0$ be given.
Since $\lim_{x \to a} f(x) = L$, there exists $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

That is,

$$L - \varepsilon < f(x) < L + \varepsilon.$$

Similarly, since $\lim_{x \to a} h(x) = L$, there exists $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \quad \Rightarrow \quad |h(x) - L| < \varepsilon.$$

That is,

$$L - \varepsilon < h(x) < L + \varepsilon.$$

Now set

$$\delta = \min\{\delta_1, \delta_2\}.$$

Then, for $0 < |x - a| < \delta$, both conditions hold:

$$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon.$$

In particular,

$$L - \varepsilon < g(x) < L + \varepsilon.$$

This means

$$|g(x) - L| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude

$$\boxed{\lim_{x \to a} g(x) = L.}$$

Continuity

A function f is continuous at a point a if

$$\lim_{x \to a} f(x) = f(a)$$

That is,

$$\forall \varepsilon > 0,\ \exists \delta > 0 :\ 0 < |x - a| < \delta \ \Rightarrow\ |f(x) - f(a)| < \varepsilon$$

Definition of Continuity on an Interval
A function f is continuous on an interval if it is continuous at every number in the interval.
If f is defined only on one side of an endpoint of the interval, then:

- **Right-continuous at** a if
$$\lim_{x \to a^+} f(x) = f(a)$$
- **Left-continuous at** b if
$$\lim_{x \to b^-} f(x) = f(b)$$

Theorem (Algebra of Continuous Functions)
If f and g are continuous at a, and c is a constant, then the following functions are also continuous at a:

1. $f(x) + g(x)$
2. $f(x) - g(x)$
3. $c \cdot f(x)$
4. $f(x) \cdot g(x)$
5. $\dfrac{f(x)}{g(x)}$, provided that $g(a) \neq 0$

Theorem (Limit of a Composite Function)
If f is continuous at b and

$$\lim_{x \to a} g(x) = b,$$

then

$$\lim_{x \to a} f(g(x)) = f(b).$$

In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

Proof
Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad |f(g(x)) - f(b)| < \varepsilon.$$

Since f is continuous at b, we have

$$\lim_{y \to b} f(y) = f(b).$$

So there exists $\delta_1 > 0$ such that

$$0 < |y - b| < \delta_1 \quad \Longrightarrow \quad |f(y) - f(b)| < \varepsilon.$$

Also, since

$$\lim_{x \to a} g(x) = b,$$

there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad |g(x) - b| < \delta_1.$$

Combining these two statements:

If $0 < |x - a| < \delta$, then

$$|g(x) - b| < \delta_1 \quad \Longrightarrow \quad |f(g(x)) - f(b)| < \varepsilon.$$

Therefore, by the definition of limit,

$$\lim_{x \to a} f(g(x)) = f(b).$$

Theorem (Intermediate Value Theorem)
Suppose that f is continuous on the closed interval $[a, b]$.
Let N be any number between $f(a)$ and $f(b)$, where

$$f(a) \neq f(b).$$

Then there exists a number c in (a, b) such that

$$f(c) = N.$$

In other words:
If f is continuous on $[a, b]$ and

$$f(a) < N < f(b) \quad \text{or} \quad f(b) < N < f(a),$$

then there exists $c \in (a, b)$ such that

$$f(c) = N.$$

Definition of Derivative
The. derivative of a function f at a number a, denoted by $f'(a)$, is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

III. New Derivatives from Old

When new functions are formed from old functions by addition, subtraction, or multiplication by a constant, their derivatives can be calculated in terms of derivatives of the old functions. In particular, the following formula says that *the derivative of a constant times a function is the constant times the derivative of the function*.

A. The Constant Multiple Rule

If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

PROOF Let $g(x) = cf(x)$. Then

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} && (1)\\
&= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] && (2)\\
&= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{(by Limit Law 3)} && (3)\\
&= cf'(x) && (4)
\end{aligned}$$

B. The Sum Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

PROOF Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} && (5)\\
&= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} && (6)\\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] && (7)\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \quad \text{(by Limit Law 1)} && (8)\\
&= f'(x) + g'(x) && (9)
\end{aligned}$$

The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(f+g+h)' = [(f+g)+h]' = (f+g)' + h' = f' + g' + h'$$

By writing $f - g$ as $f + (-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get the following formula.

C. The Difference Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

D. The Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

PROOF Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} && (10)\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} && (11)
\end{aligned}$$

We add and subtract $f(x+h)g(x)$ in the numerator:

$$F'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \tag{12}$$

$$= \lim_{h \to 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h} \tag{13}$$

$$= \lim_{h \to 0} \left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right] \tag{14}$$

Since f is differentiable at x, it is continuous at x, so $\lim_{h \to 0} f(x+h) = f(x)$. Therefore:

$$F'(x) = \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + g(x) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{15}$$

$$= f(x) \cdot g'(x) + g(x) \cdot f'(x) \tag{16}$$

$$= f(x)g'(x) + g(x)f'(x) \tag{17}$$

E. The Quotient Rule

If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

PROOF Let $F(x) = \frac{f(x)}{g(x)}$. Then

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \tag{18}$$

$$= \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \tag{19}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h \cdot g(x+h)g(x)} \tag{20}$$

We add and subtract $f(x)g(x)$ in the numerator:

$$F'(x) = \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h \cdot g(x+h)g(x)} \tag{21}$$

$$= \lim_{h \to 0} \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{h \cdot g(x+h)g(x)} \tag{22}$$

$$= \lim_{h \to 0} \frac{g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)} \tag{23}$$

Since g is differentiable at x, it is continuous at x, so $\lim_{h \to 0} g(x+h) = g(x)$. Therefore:

$$F'(x) = \frac{g(x)\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} - f(x)\lim_{h \to 0} \frac{g(x+h)-g(x)}{h}}{g(x) \cdot g(x)} \tag{24}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \tag{25}$$

F. The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

PROOF Let $F(x) = f(g(x))$. If we let $u = g(x)$ and $y = f(u)$, then $y = F(x) = f(g(x))$.
Let $\Delta u = g(x+h) - g(x)$ and $\Delta y = f(u + \Delta u) - f(u) = f(g(x+h)) - f(g(x)) = F(x+h) - F(x)$.
Then

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\Delta y}{h} \tag{26}$$

If $\Delta u \neq 0$, we can write

$$\frac{\Delta y}{h} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{h}$$

Since g is differentiable at x, we have $\lim_{h \to 0} \frac{\Delta u}{h} = g'(x)$.
Since g is continuous at x (as it's differentiable there), we have $\lim_{h \to 0} \Delta u = 0$.
Since f is differentiable at $g(x)$, we have $\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = f'(g(x))$.
Therefore:

$$F'(x) = \lim_{h \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{h \to 0} \frac{\Delta u}{h} \tag{27}$$

$$= f'(g(x)) \cdot g'(x) \tag{28}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

G. Definition of the Number e

e is the number such that $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h} = 1$

IV. Extrema and Critical Points

A. Definition (Absolute Extrema)

Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

B. Definition (Local Extrema)

The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c.

C. The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The Extreme Value Theorem is illustrated in Figure 8. Note that an extreme value can be taken on more than once. Although the Extreme Value Theorem is intuitively very clear, its proof is beyond the scope of this book.

D. Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

PROOF Suppose, for the sake of definiteness, that f has a local maximum at c. Then, according to Definition 2, $f(c) \geq f(x)$ if x is sufficiently close to c. This implies that if h is sufficiently close to 0, with h being positive or negative, then

$$f(c) \geq f(c+h)$$

and therefore

$$f(c+h) - f(c) \leq 0$$

We can divide both sides of an inequality by a positive number. Thus, if $h > 0$ and h is sufficiently small, we have

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

Taking the right-hand limit of both sides of this inequality (using Theorem 2.3.2), we get

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0^+} 0 = 0$$

But since $f'(c)$ exists, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

and so we have shown that $f'(c) \leq 0$.

If $h < 0$, then the direction of the inequality is reversed when we divide by h:

$$\frac{f(c+h) - f(c)}{h} \geq 0 \quad h < 0$$

So, taking the left-hand limit, we have

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

Since we have shown that $f'(c) \leq 0$ and $f'(c) \geq 0$, it follows that $f'(c) = 0$.

A similar argument can be used to prove the result for a local minimum.

E. Definition (Critical Number)

A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

F. Theorem

If f has a local maximum or minimum at c, then c is a critical number of f.

G. Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1) f is continuous on the closed interval $[a,b]$.
2) f is differentiable on the open interval (a,b).
3) $f(a)=f(b)$

Then there is a number c in (a,b) such that $f'(c)=0$.

Before giving the proof let's take a look at the graphs of some typical functions that satisfy the three hypotheses. Figure 1 shows the graphs of four such functions. In each case it appears that there is at least one point $(c,f(c))$ on the graph where the tangent is horizontal and therefore $f'(c)=0$. Thus Rolle's Theorem is plausible.

PROOF There are three cases:

CASE I $f(x)=k$, a constant

Then $f'(x)=0$, so the number c can be taken to be any number in (a,b).

CASE II $f(x)>f(a)$ for some x in (a,b) [as in Figure 1(b) or (c)]

By the Extreme Value Theorem (which we can apply by hypothesis 1), f has a maximum value somewhere in $[a,b]$. Since $f(a)=f(b)$, it must attain this maximum value at a number c in the open interval (a,b). Then f has a local maximum at c and, by hypothesis 2, f is differentiable at c. Therefore $f'(c)=0$ by Fermat's Theorem.

CASE III $f(x)<f(a)$ for some x in (a,b) [as in Figure 1(c) or (d)]

By the Extreme Value Theorem, f has a minimum value in $[a,b]$ and, since $f(a)=f(b)$, it attains this minimum value at a number c in (a,b). Again $f'(c)=0$ by Fermat's Theorem.

H. The Mean Value Theorem

Let f be a function that satisfies the following hypotheses:

1) f is continuous on the closed interval $[a,b]$.
2) f is differentiable on the open interval (a,b).

Then there is a number c in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \tag{29}$$

or, equivalently,

$$f(b)-f(a)=f'(c)(b-a) \tag{30}$$

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points $A(a,f(a))$ and $B(b,f(b))$ on the graphs of two differentiable functions. The slope of the secant line AB is

$$m_{AB}=\frac{f(b)-f(a)}{b-a} \tag{31}$$

which is the same expression as on the right side of Equation 1. Since $f'(c)$ is the slope of the tangent line at the point $(c,f(c))$, the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point P where $f'(c)=m_{AB}$. In other words, there is a point P where the tangent line is parallel to the secant line AB. (Imagine a line that moves parallel to AB while moving toward AB until it touches the graph for the first time.)

PROOF We apply Rolle's Theorem to a new function h defined as the difference between f and the function whose graph is the secant line AB. Using the point-slope equation of a line, we see that the equation of the line AB can be written as

$$y-f(a)=\frac{f(b)-f(a)}{b-a}(x-a) \tag{32}$$

$$\text{or as}\quad y=f(a)+\frac{f(b)-f(a)}{b-a}(x-a) \tag{33}$$

So, as shown in Figure 5,

$$h(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a) \tag{34}$$

First we must verify that h satisfies the three hypotheses of Rolle's Theorem.

1) The function h is continuous on $[a,b]$ because it is the sum of f and a first-degree polynomial, both of which are continuous.
2) The function h is differentiable on (a,b) because both f and the first-degree polynomial are differentiable. In fact, we can compute h' directly from Equation 4:

$$h'(x)=f'(x)-\frac{f(b)-f(a)}{b-a} \tag{35}$$

(Note that $f(a)$ and $[f(b) - f(a)]/(b - a)$ are constants.)

3)

$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0 \tag{36}$$

$$h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) \tag{37}$$

$$= f(b) - f(a) - [f(b) - f(a)] = 0 \tag{38}$$

Therefore $h(a) = h(b)$.

Since h satisfies the hypotheses of Rolle's Theorem, that theorem says there is a number c in (a, b) such that $h'(c) = 0$. Therefore

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \tag{39}$$

and so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{40}$$