Differential Equation

Hoseung Kang

This document is based on the contents of the **Differential Equation**.

I started this work to establish a solid foundation for my research activities.

I believe that theoretical understanding is essential in any field, so I studied to build background knowledge alongside basic research.

Since studying alone and leaving it behind leads to nothing remaining, I decided to organize my learning into this document.

This work was done based on the learning attitude taught to me by **Jung Byung-ho** and **Jung Byung-hoon**.

Although the contents may differ, the approach to studying concepts, proofs, and logical reasoning reflects the mindset and methods I learned from them.

I am truly grateful for all they have taught me.

Textbook
Elementary Differential Equations and Boundary Value Problems
Ninth Edition
William E. Boyce, Richard C. DiPrima

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I. First Order Linear Equation

A. The Integrating Factor Method

Consider the general first order linear equation

$$\frac{dy}{dt} + p(t)y = g(t)$$

where $p(t)$ and $g(t)$ are given functions of t.

To solve this equation, we use the method of integrating factors. The key idea is to multiply the entire equation by a carefully chosen function $\mu(t)$ that will allow us to integrate both sides.

Step 1: Multiply by the Integrating Factor

Multiply the differential equation by an unknown function $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

Step 2: Choose $\mu(t)$ to Create a Perfect Derivative

We want the left side to be the derivative of a product. For this to work, we need:

$$\frac{d\mu(t)}{dt} = p(t)\mu(t)$$

This is a separable differential equation for $\mu(t)$. Assuming $\mu(t) > 0$, we can write:

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t)$$

Integrating both sides:

$$\ln \mu(t) = \int p(t)\, dt + k$$

Taking the exponential of both sides:

$$\mu(t) = e^k \cdot \exp\left(\int p(t)\, dt\right)$$

Step 3: Choose the Simplest Form

By choosing the arbitrary constant $k = 0$, we obtain the simplest integrating factor:

$$\mu(t) = \exp\left(\int p(t)\, dt\right)$$

Note that $\mu(t) > 0$ for all t, which validates our assumption.

Step 4: Verify the Product Rule

With this choice of $\mu(t)$, the left side of our multiplied equation becomes:

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y = \frac{d}{dt}[\mu(t)y]$$

This follows from the product rule for differentiation.

Step 5: Integrate Both Sides

Our equation now becomes:

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t)$$

Integrating both sides:

$$\mu(t)y = \int \mu(t)g(t)\, dt + C$$

where C is an arbitrary constant of integration.

Step 6: Solve for y

The general solution is:

$$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\, dt + C\right]$$

conclusion

Sometimes it is convenient to express the solution using definite integrals. If we choose a specific lower limit t_0, we can write:

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)\, ds + C\right]$$

where s is used as the integration variable to distinguish it from the independent variable t.

B. Separable Differential Equations

The general first-order equation is

$$\frac{dy}{dx} = f(x, y)$$

To identify this class of equations, we first rewrite the general equation in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

If it happens that M is a function of x only and N is a function of y only, then the equation becomes

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Such an equation is said to be *separable*, because if it is written in the differential form

$$M(x)\,dx + N(y)\,dy = 0$$

then terms involving each variable may be placed on opposite sides of the equation. The differential form is also more symmetric and tends to suppress the distinction between independent and dependent variables.

A separable equation can be solved by integrating the functions M and N. We illustrate the process by examples and then discuss it in general.

Parametric Approach

The investigation of a first-order nonlinear equation can sometimes be facilitated by regarding both x and y as functions of a third variable t. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

If the differential equation is

$$\frac{dy}{dx} = \frac{F(x, y)}{G(x, y)}$$

then comparing numerators and denominators gives the system

$$\frac{dx}{dt} = G(x, y), \qquad \frac{dy}{dt} = F(x, y)$$

C. Differences Between Linear and Nonlinear Equations

Existence and Uniqueness of Solutions: Linear Equations

Theorem 1: If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0$$

where y_0 is an arbitrary prescribed initial value.

Observe that this theorem states that the given initial value problem has a solution and also that the problem has only one solution. In other words, the theorem asserts both the existence and uniqueness of the solution of the initial value problem.

In addition, it states that the solution exists throughout any interval I containing the initial point t_0 in which the coefficients p and g are continuous. That is, the solution can be discontinuous or fail to exist only at points where at least one of p and g is discontinuous. Such points can often be identified at a glance.

The proof is partly contained in the discussion leading to the formula

$$\mu(t)y = \int \mu(t)g(t)\,dt + c$$

where

$$\mu(t) = \exp\left(\int p(t)\,dt\right)$$

Since p is continuous for $\alpha < t < \beta$, it follows that μ is defined in this interval and is a nonzero differentiable function. Upon multiplying the equation by $\mu(t)$, we obtain

$$[\mu(t)y]' = \mu(t)g(t)$$

Since both μ and g are continuous, the function μg is integrable, and the formula follows. The initial condition determines the constant c uniquely, so there is only one solution of the initial value problem.

The solution of the initial value problem can be written as:

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)\,ds + y_0\right]$$

where $\mu(t) = \exp\left(\int_{t_0}^{t} p(s)\,ds\right)$.

Existence and Uniqueness of Solutions: Nonlinear Equations

Turning now to nonlinear differential equations, we must replace the previous theorem by a more general theorem.

Theorem 2: Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

The proof of the linear theorem was comparatively simple because it could be based on the explicit expression that gives the solution. There is no corresponding expression for the solution of the nonlinear differential equation, so the proof of this theorem is much more difficult.

Here we note that the conditions stated in this theorem are sufficient to guarantee the existence of a unique solution in some interval $t_0 - h < t < t_0 + h$, but they are not necessary. The existence of a solution (but not its uniqueness) can be established on the basis of the continuity of f alone.

An important geometrical consequence of the uniqueness parts of both theorems is that the graphs of two solutions cannot intersect each other. Otherwise, there would be two solutions that satisfy the initial condition corresponding to the point of intersection, in contradiction to the uniqueness theorem.

D. Exact Differential Equations

Consider a first-order differential equation of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where $M(x,y)$ and $N(x,y)$ are given functions of x and y.

Integrating Factors

When $M_y \neq N_x$, the equation may be made exact by multiplying by an integrating factor $\mu(x,y)$.

Step 1: General Condition For $\mu M dx + \mu N dy = 0$ to be exact:

$$(\mu M)_y = (\mu N)_x$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

Case 1: Integrating Factor $\mu(x)$

If μ depends only on x, then $\mu_y = 0$ and:

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

Condition: This works when $\frac{M_y - N_x}{N}$ is a function of x only.

Solution:

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N} dx\right)$$

Case 2: Integrating Factor $\mu(y)$

If μ depends only on y, then $\mu_x = 0$ and:

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu$$

Condition: This works when $\frac{N_x - M_y}{M}$ is a function of y only.

Solution:

$$\mu(y) = \exp\left(\int \frac{N_x - M_y}{M} dy\right)$$

Definition: The differential equation is called *exact* if there exists a function $\psi(x,y)$ such that:

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y)$$

When exact, solutions are given implicitly by $\psi(x,y) = C$.

Step 1: Necessary Condition for Exactness

If the equation is exact, then there exists $\psi(x,y)$ with $\psi_x = M$ and $\psi_y = N$.

Computing mixed partial derivatives:

$$M_y = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = \psi_{xy}$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \psi_{yx}$$

Since mixed partials are equal for continuous functions:

$$M_y = N_x$$

Step 2: Sufficient Condition - Constructing $\psi(x,y)$

Suppose $M_y = N_x$. We construct $\psi(x,y)$ satisfying both conditions.

Substep 2a: Partial Integration Integrate the first condition $\psi_x = M$ with respect to x:

$$\psi(x,y) = \int M(x,y)\,dx + h(y)$$

where $h(y)$ is an unknown function of y only.

Substep 2b: Determine $h(y)$ Differentiate with respect to y and use the second condition $\psi_y = N$:

$$\frac{\partial}{\partial y}\int M(x,y)\,dx + h'(y) = N(x,y)$$

Therefore:

$$h'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

Substep 2c: Verify Consistency For $h(y)$ to exist, the right side must be independent of x. Check by differentiating with respect to x:

$$\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx\right] = N_x - M_y$$

Using our assumption $M_y = N_x$, this equals zero. Hence $h'(y)$ depends only on y.

Substep 2d: Complete the Solution Integrate to find $h(y)$, and the function $\psi(x,y)$ is constructed.

Conclusion

$$\boxed{M_y = N_x \iff \text{equation is exact}}$$