ON THE SELF-ACCELERATING ELECTRON

By S. ASHAUER

Communicated by P. A. M. Dirac

Received 19 August 1946

In Dirac's (1) classical theory of radiating electrons, the relativistic equations of motion of a point-electron in an electromagnetic field involve third-order derivatives, whereas only second-order derivatives appear in the equations of motion of ordinary mechanics. Thus there are extra arbitrary constants of integration in Dirac's theory. Extra boundary conditions picking out the physically permissible solutions from all the mathematically possible ones have been discussed in a few cases by Dirac (1) and by others (2, 3), who find that in some cases even the physical motions display unexpected features. A closer examination of the mathematical solutions which have so far been rejected as non-physical is made in the following in the simplest case, namely, when there is no external field present (self-accelerating electron) and the electron has nearly attained the velocity of light, in order to get some sort of physical picture of it.

Equations of motion of the self-accelerating electron

Dirac's equations of motion for an electron when there is no external field are as follows (written in a suitable frame of reference):

$$a\ddot{z}_1 - \dddot{z}_1 - \ddot{\mathbf{z}}^2\dot{z}_1 = 0, \quad \dot{z}_0^2 - \dot{z}_1^2 = 1, \quad z_2 = z_3 = 0, \tag{1}$$

where $a = 3m/2e^2$, $(z_0, z_1, z_2, z_3) \equiv \mathbf{z}$ are the coordinates of the electron in flat space-time ($g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$; velocity of light $= 1$), and dots denote differentiation with respect to the proper-time s.

The equations (1) are easily solved for the velocity (1, 3), and give

$$\dot{z}_1 = \sinh\phi, \quad \dot{z}_0 = \cosh\phi, \tag{2}$$

where
$$\phi = Ae^{as} + B.$$

If the extra constant of integration A is taken to be zero, we get the physical solution, in which the electron moves in a straight line with constant velocity equal to its initial velocity $dz_1/dz_0 = \tanh B$.

We shall now consider the case $A > 0$. For simplicity we take a Lorentz frame in which the velocity of the electron at $s = -\infty$ is zero, so that $B = 0$. This case gives the self-accelerating motion; the velocity of the electron increases with s and tends to the velocity of light as $s \to \infty$, and the rate of radiation of energy (irreversible part), given by

$$-\tfrac{2}{3}e^2\ddot{\mathbf{z}}^2\dot{z}_0 = \tfrac{2}{3}e^2A^2a^2e^{2as}\cosh(Ae^{as}),$$

tends to infinity as $s \to \infty$.

The equations (2) give

$$z_1 = \frac{1}{a}\int \frac{\sinh\phi}{\phi}\,d\phi = \frac{1}{2a}\left[\int \frac{e^{\phi}}{\phi}\,d\phi - \int \frac{e^{-\phi}}{\phi}\,d\phi\right],$$

and with the help of the exponential integral defined as

$$Ei(\phi) = \int_{-\infty}^{\phi} \frac{e^t}{t}\,dt,$$

we get

$$z_1 = \frac{1}{2a}\left[Ei(\phi) - Ei(-\phi)\right] + \text{const.},$$

and similarly

$$z_0 = \frac{1}{2a}\left[Ei(\phi) + Ei(-\phi)\right] + \text{const.}$$

Let us choose the frame of reference so that both constants are zero. Then the solution of (2) is

$$\left.\begin{aligned} z_1 &= \frac{1}{2a}\left[Ei(Ae^{as}) - Ei(-Ae^{as})\right], \\ z_0 &= \frac{1}{2a}\left[Ei(Ae^{as}) + Ei(-Ae^{as})\right]. \end{aligned}\right\} \tag{3}$$

The exponential integral can be expressed in power-series form as follows (4):

$$Ei(\phi) = C + \log|\phi| + \phi + \frac{1}{2}\frac{\phi^2}{2!} + \frac{1}{3}\frac{\phi^3}{3!} + \dots,$$

where C = Euler's constant $= 0{\cdot}577215\dots$. If ϕ is small, we can take the first three terms only as a first approximation; then the equations of the world-line for correspondingly large negative s are

$$z_1 \approx \frac{Ae^{as}}{a}, \quad z_0 \approx \frac{1}{a}(C + \log A + as). \tag{4}$$

The asymptote of this world-line for $s \to -\infty$ is the z_1-axis. In order to examine the asymptotic behaviour of the electron as its velocity tends to the velocity of light, it is convenient to make use of the semi-convergent series (4)

$$Ei(\phi) = \frac{e^{\phi}}{\phi}\left(1 + \frac{1!}{\phi} + \frac{2!}{\phi^2} + \frac{3!}{\phi^3} + \dots\right)$$

which is valid for $|\phi| \gg 1$. For large ϕ, we can take the first term only, and get, as a first approximation for the equations of the world-line for correspondingly large positive s,

$$z_1 \approx \frac{\cosh(Ae^{as})}{Aae^{as}}, \quad z_0 \approx \frac{\sinh(Ae^{as})}{Aae^{as}}. \tag{5}$$

The asymptote for $s \to \infty$ is a null-line through the origin.

Surfaces of constant scalar potential

For an electron of given world-line $\mathbf{z} = \mathbf{z}(s)$ the retarded (advanced) potentials at the point (x_0, x_1, x_2, x_3) are (1)

$$A_{\mu,\,\text{ret.(adv.)}} = \overset{+}{(-)}\frac{e\dot{z}_\mu}{(\dot{\mathbf{z}}, \mathbf{x} - \mathbf{z})} \tag{6}$$

taken at the retarded (advanced) proper-time, which is the value of s satisfying

$$(\mathbf{x} - \mathbf{z})^2 = 0 \tag{7}$$

with $x_0 - z_0$ positive (negative).

In order to get a physical picture of the self-accelerating electron when its velocity is close to the velocity of light, we shall consider the surfaces (in three-dimensional space) of constant retarded (advanced) scalar potential $A_{0,\mathrm{ret.(adv.)}} = \mathrm{const.}$, at a fixed time x_0, taking x_0 to be positive and large.

For points at sufficiently large distances from the position of the electron at time x_0 the corresponding retarded proper-times belong to the region where equations (4) hold. It can then be checked that the surfaces $A_{0,\mathrm{ret.}} = \mathrm{const.}$ tend to spheres (with centre in the origin) as the distance from the electron becomes larger and larger.

For points sufficiently close to the position of the electron at time x_0, the corresponding retarded proper-times belong to a region where equations (5) hold (x_0 being sufficiently large). For all points (x_0 being sufficiently large) we can take (5) for the equations of the world-line in order to find the corresponding advanced proper-times.

Taking (5) for $\mathbf{z} = \mathbf{z}(s)$, we get from (7)

$$\left(x_0 - \frac{\sinh\phi}{a\phi}\right)^2 - \left(x_1 - \frac{\cosh\phi}{a\phi}\right)^2 - x_2^2 - x_3^2 = 0,$$

which can be written as

$$\left(x_0 - x_1 + \frac{e^{-\phi}}{a\phi}\right)\left(x_0 + x_1 - \frac{e^{\phi}}{a\phi}\right) - (x_2^2 + x_3^2) = 0. \tag{8}$$

From (6) we see that the surfaces $A_{0,\mathrm{ret.(adv.)}} = \mathrm{const.}$ will be given by

$$K = \frac{\cosh\phi}{x_0\cosh\phi - x_1\sinh\phi}$$

at the retarded (advanced) ϕ corresponding to (x_0, x_1, x_2, x_3). This gives

$$K = \frac{e^{2\phi} + 1}{(x_0 - x_1)\,e^{2\phi} + x_0 + x_1}. \tag{9}$$

Eliminating $\phi_{\mathrm{ret.(adv.)}}$ between (8) and (9), we get

$$r^2 = \left\{x_1 - x_0 - \frac{1}{a}\left(\frac{K(x_1+x_0)-1}{K(x_1-x_0)+1}\right)^{-\frac{1}{2}}\left[\log\left(\frac{K(x_1+x_0)-1}{K(x_1-x_0)+1}\right)^{\frac{1}{2}}\right]^{-1}\right\}$$
$$\times\left\{\frac{1}{a}\left(\frac{K(x_1+x_0)-1}{K(x_1-x_0)+1}\right)^{\frac{1}{2}}\left[\log\left(\frac{K(x_1+x_0)-1}{K(x_1-x_0)+1}\right)^{\frac{1}{2}}\right]^{-1} - x_1 - x_0\right\}, \tag{10}$$

where we have used cylindrical coordinates with $r^2 = x_2^2 + x_3^2$.

Equation (10) gives the double locus of points for which either $A_{0,\mathrm{ret.}} = eK$ or $A_{0,\mathrm{adv.}} = -eK$, for fixed large positive x_0. In three-dimensional space the locus is a surface of rotation whose axis is the line of motion of the electron.

PHYSICAL PICTURE

We consider now in particular the surfaces of constant *retarded* scalar potential, as the Coulomb energy depends only on the retarded field quantities.

In order to form a physical picture of the self-accelerating electron when its velocity is close to the velocity of light, we have plotted in Figs. 1 and 2 the values of the radius r of the surfaces $A_{0,\mathrm{ret.}} = \mathrm{const.}$ against x_1, at two different times, namely, $x_0 = 10^{20}$ when the velocity of the electron is $dz_1/dz_0 = \tanh 50{\cdot}7$, and $x_0 = 5\times 10^{20}$ when the velocity is $dz_1/dz_0 = \tanh 52{\cdot}3$. It should be noted that the scale taken for r and that for x_1 differ by a factor 10^{21}.

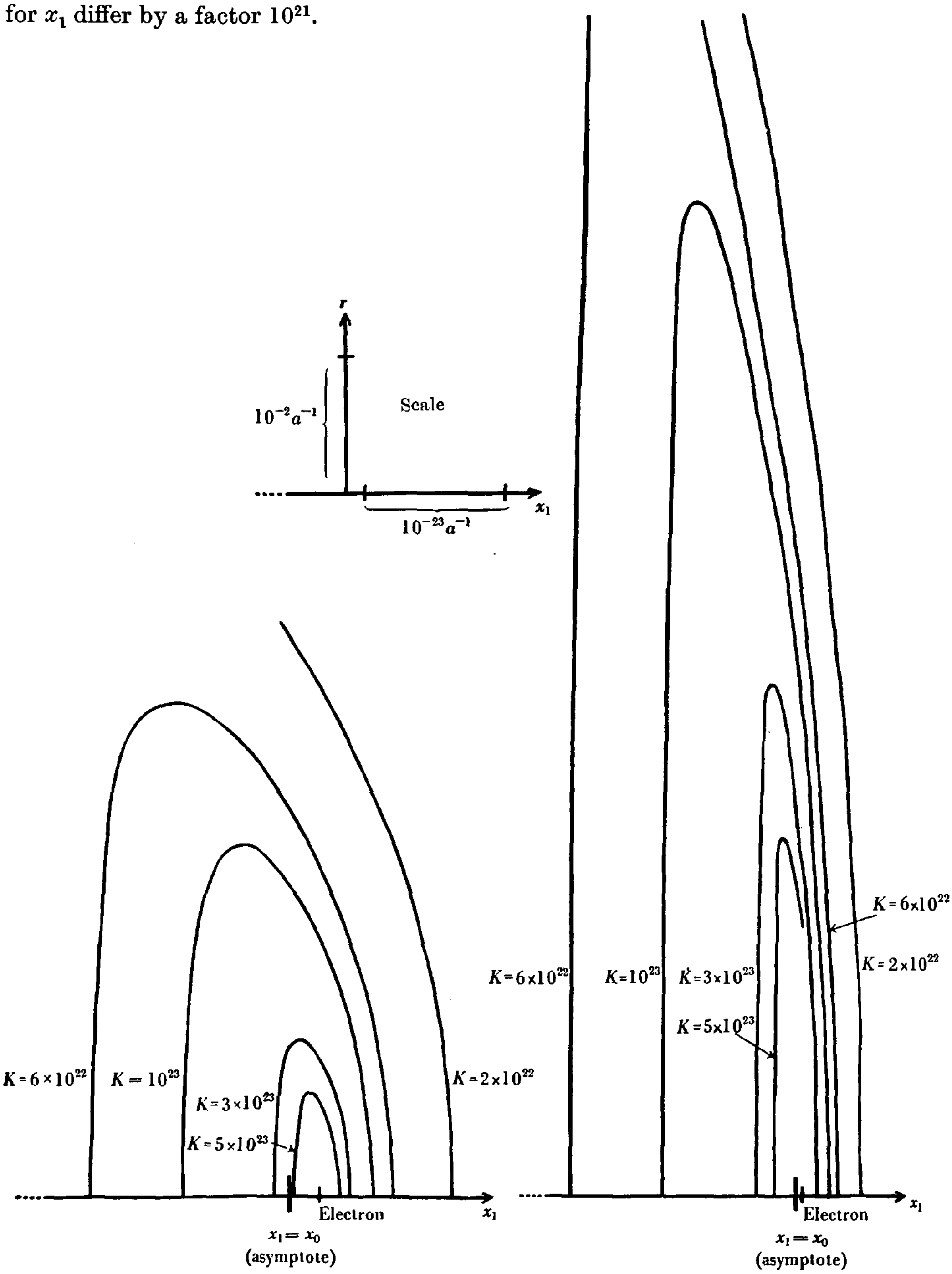

Fig. 1. $x_0 = 10^{20}$. Fig. 2. $x_0 = 5\times 10^{20}$.

Figs. 1, 2. Radius of the surfaces of constant retarded scalar potential $A_{0,\text{ret.}} = eK$. Units such that velocity of light $= 1$.

The exact plotting of the surfaces of constant energy-density is troublesome; however, the closeness of the lines drawn in the diagram in any region gives an indication of the energy-density in that region. It is seen that the surfaces $A_{0,\text{ret.}} = \text{const.}$ are closest together in the forward direction of motion, in a way that suggests that, when the velocity of the electron is close to the velocity of light, the energy-density has an appreciable value only inside a thin disk-like region, perpendicular to the direction of motion and just ahead of the electron, and falls off rapidly outside this region. The thickness of the disk diminishes, its radius increases and its distance from the electron diminishes as the velocity of the electron increases.

Summary

After solving the equations of motion of Dirac's self-accelerating electron, a physical picture of it is formed by plotting graphically the surfaces of constant scalar potential when the electron has built up a velocity close to the velocity of light.

I take this opportunity to express my gratitude to Professor Dirac for his helpful guidance, and to the British Council for the award of a Scholarship.

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Newnham College
Cambridge