ON THE CLASSICAL EQUATIONS OF MOTION OF RADIATING ELECTRONS

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The classical equations of motion of point-electrons in an electromagnetic field as derived by Dirac (2), called Lorentz-Dirac equations, are exact in the sense that they satisfy strictly the laws of conservation of energy and momentum, and thus include the effect of radiation reaction on the motion of the electrons. The physical interpretation of these equations, however, is not free from difficulties. The equations are of the third order in the derivatives of the coordinates, so that the initial position and velocity of an electron no longer suffice to determine its motion uniquely as in usual mechanics, and extra constants of integration appear in the solutions of these equations. The complete solutions with the extra constants are found in general to represent motions with unfamiliar features, which do not seem plausible from the physical point of view and which have not yet been observed. The simplest example of this type is the self-accelerating motion of an electron in the absence of any incident field. In some cases, motions of a familiar type are obtained from special solutions.

It is usually assumed that general extra boundary conditions must be introduced explicitly into the theory to pick out the physically plausible solutions from all the mathematically possible ones, and several proposals have been made in this direction. (However, it is not yet clear how the physical solutions can be picked out in the quantum theory.) On the other hand, the question of whether the non-physical solutions should be definitely ruled out deserves further consideration. Therefore it is of great interest to examine more closely the complete solutions obtained when no extra boundary condition is imposed.

An attempt has been made in a previous paper (1) to form a 'physical picture' of the self-accelerating electron when it has nearly attained the velocity of light, by plotting surfaces of constant scalar potential. From the shape of these surfaces, it has been suggested there that the field energy-density has an appreciable value only inside a thin disk-like region of space close to the electron and perpendicular to its direction of motion, in a suitable frame where the motion is straightline. The total field energy calculated from the Maxwell energy-momentum tensor, which forms the energy of the disk, consists of the Coulomb energy (depending on the velocity of the electron), the acceleration energy and the radiated energy. The first of these contains an infinity which, however, can be pictured as cancelled by an infinite negative energy at the centre of the electron. The kinetic energy of the electron is then regarded as the difference of the Coulomb energy and an infinite energy of non-electromagnetic nature (2).

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The other two parts of the total field energy are finite. As is well known, the energy emitted by the electron, as calculated from the Poynting flux through a large fixed sphere, is mostly concentrated in a direction that approaches the forward direction of motion for relativistic velocities of the electron. When the self-accelerating electron follows with a velocity which is almost that of light, the energy emitted by it may not be able to leave the neighbourhood of the electron. Hence the energy which is radiated by the electron and succeeds in actually getting away from it is not necessarily that obtained from the usual formula $\frac{2}{3}e^2\ddot{z}^2\dot{z}_0$ for the rate of emission. It is therefore important to get a clear picture of the field-energy distribution when the electron has built up a large velocity. For this purpose, in the next section of the present work, the picture of the energy-disk suggested previously will be examined in greater detail and corrected, using numerical and graphical methods.

Another simple special case of the equations of motion of radiating electrons is the case of an electron disturbed by a pulse of radiation. The solution of these equations for the velocity of the electron has been worked out by Dirac (2) when relativistic effects are negligible. The physical solution has been discussed there and compared with the results of the elementary theory which ignores the radiation damping completely. Since the effect of radiation damping becomes more important for larger velocities, one may expect more marked departures from the results of the elementary theory when the electron has a relativistic velocity. In the last section of this work both the physical and non-physical solutions for the velocity of the electron will be worked out and discussed, without neglecting any relativistic effects.

Physical picture of the self-accelerating electron

We consider now the case of the motion of an electron when there is no incident field. As in (1), we use the customary notation. The units of length and time are so chosen that the velocity of light equals 1 and also $a \equiv 3m/2e^2 = 1$. In a suitable frame of reference, the solution of the equations of motion corresponding to the self-accelerating motion of the electron can be written in the form

$$\dot{z}_0 = \cosh\phi, \quad \dot{z}_1 = \sinh\phi, \quad \dot{z}_2 = \dot{z}_3 = 0, \tag{1}$$

where $\phi = Ae^s$, $A > 0$, A being the extra constant of integration. The equations

$$z_0 = \frac{\sinh\phi}{\phi}, \quad z_1 = \frac{\cosh\phi}{\phi}, \quad z_2 = z_3 = 0 \tag{2}$$

describe the world-line to a first approximation valid only for $\phi \gg 1$, i.e. for correspondingly large positive values of the proper-times (see (1)). It is convenient to take ϕ as the independent variable instead of s, as the constant of integration A does not then appear explicitly in the equations.

We shall examine the field-energy density produced by the electron in its neighbourhood for a large positive value of the time x_0, when the electron has already built up a velocity close to the velocity of light. It will be sufficient to confine our attention to field-points $\mathbf{x}$ such that the corresponding retarded points on the world-line fall into the region where equations (2) hold. The retarded value of ϕ corresponding to the field-point $\mathbf{x}$ then satisfies the equation, obtained in (1)

$$r^2\phi^2 = (1-\xi\phi e^{\phi})[(2x_0+\xi)\phi e^{-\phi}-1], \tag{3}$$

with

$$2x_0\phi - e^{\phi} + e^{-\phi} > 0, \tag{4}$$

where we have put $\xi = x_1 - x_0$ and $r^2 = x_2^2 + x_3^2$.

Using the well-known formula which gives the retarded field $F_{\mu\nu}^{\text{ret.}}$ at $\mathbf{x}$ due to an electron of known world-line (2), and equations (1) and (2), one finds by a straightforward calculation

$$\left.\begin{aligned} F_{10}^{\text{ret.}} &= \frac{8(2x_0+\xi)\xi\phi - 4[(2x_0+\xi)e^{-\phi}+\xi e^{\phi}]}{[(2x_0+\xi)e^{-\phi}-\xi e^{\phi}]^3}e = E_1, \quad F_{32}^{\text{ret.}} = H_1 = 0,\\ F_{20}^{\text{ret.}} &= \frac{8x_1x_2\phi e}{[(2x_0+\xi)e^{-\phi}-\xi e^{\phi}]^3} = E_2, \quad F_{30}^{\text{ret.}} = \frac{8x_1x_3\phi e}{[(2x_0+\xi)e^{-\phi}-\xi e^{\phi}]^3} = E_3,\\ F_{13}^{\text{ret.}} &= \frac{-8x_0x_3\phi e}{[(2x_0+\xi)e^{-\phi}-\xi e^{\phi}]^3} = H_2, \quad F_{21}^{\text{ret.}} = \frac{8x_0x_2\phi e}{[(2x_0+\xi)e^{-\phi}-\xi e^{\phi}]^3} = H_3, \end{aligned}\right\} \tag{5}$$

where the retarded value of ϕ is to be used, and $\mathfrak{E} \equiv (E_1, E_2, E_3)$ and $\mathfrak{H} \equiv (H_1, H_2, H_3)$ are the three-vectors of the electric and magnetic intensities, respectively.

The field energy-density is given by the Maxwellian expression

$$\frac{1}{8\pi}(\mathfrak{E}^2+\mathfrak{H}^2) = \frac{8e^2}{\pi}w, \tag{6}$$

say. For convenience we shall in future refer to w as the energy-density. We evaluate w by substituting the expressions (5) in (6). For large x_0 and field-points in the neighbourhood of the electron, we can put $2x_0+\xi \sim 2x_0$. The result after making this approximation is

$$w = \frac{[2x_0\xi\phi - \frac{1}{2}(2x_0e^{-\phi}+\xi e^{\phi})]^2 + 2x_0^2r^2\phi^2}{(2x_0e^{-\phi}-\xi e^{\phi})^6}, \tag{7}$$

where ϕ is that root of the equation

$$r^2\phi^2 = (1-\xi\phi e^{\phi})(2x_0\phi e^{-\phi}-1) \tag{8}$$

which satisfies the inequality (4). The study of the energy-density given by (7) and (8) is awkward owing to the fact that ϕ cannot be eliminated directly between these equations. But we can derive some interesting results using numerical and graphical methods, these methods being suitable for our purpose of forming a picture of the field-energy distribution.

We can assign arbitrary fixed value to x_0 and r, and varying values to ϕ, then solve equation (8) for ξ and work out w using (7). By this method, the energy-density was calculated numerically for about 200 field-points, with the help of five-figure logarithms. The values assigned to x_0 were: (*a*) $x_0 = 10^{20}$, (*b*) $x_0 = 5\times 10^{20}$, (*c*) $x_0 = 10^{40}$. (For the first two of these values, surfaces of constant scalar potential, including the range of r considered below, have been plotted in (1).) The corresponding values of ϕ which determine the instantaneous position of the electron, i.e. the values of ϕ_E satisfying $z_0(\phi_E) = x_0$, are: (*a*) $\phi_E = 50\cdot6707$, (*b*) $\phi_E = 52\cdot3120$, (*c*) $\phi_E = 97\cdot3761$. The distances on the x_1-axis between the instantaneous position of the electron and its asymptote given by $\xi = 0, r = 0$, namely, the distances $\xi_E = z_1(\phi_E) - z_0(\phi_E)$, are: (*a*) $\xi_E = 1\cdot9475\times10^{-24}$, (*b*) $\xi_E = 3\cdot6546\times10^{-25}$, (*c*) $\xi_E = 5\cdot2731\times10^{-45}$. The values chosen for r are: $4\cdot472\times10^{-3}$,

$6{\cdot}325\times10^{-3}$, 10^{-2} and $2{\cdot}236\times10^{-2}$. For these values of r, the expression in square brackets in the numerator of (7) is negligible as compared with the rest of the numerator. These square brackets arise from the contribution of E_1 to w. This contribution becomes important only for much smaller values of r, namely, for r of the order of magnitude of ξ_E.

The values of w calculated in the manner indicated above have been plotted in twelve graphs, I–XII say, which are not reproduced here so as to save space. The essential information derived from these graphs, however, is shown in Figs. 1–4 as explained below. The general features of graphs I–XII are briefly as follows. Each of the graphs represents w as a function of ξ for fixed x_0 and r. All the curves I–XII are similar in so far as each has a maximum $w_{\text{max.}}$ occurring for the value $\xi = \xi_{\text{max.}}$ say. Each curve is almost symmetrical about the line $\xi = \xi_{\text{max.}}$, and falls off rapidly to zero as $|\xi - \xi_{\text{max.}}|$ increases. No asymmetry is detectable in the graphs, though the function w is not symmetrical analytically. For a fixed value of x_0, the maximum $w_{\text{max.}}$ becomes higher and steeper as r decreases; w is of course singular at the position of the electron ($r = 0$). For a fixed value of r, the maximum becomes higher and steeper as x_0 increases.

The graphs I–XII thus provide the twelve values $\xi_{\text{max.}}$ of ξ at which the maxima of the energy-density for constant x_0 and r occur. For each of the three values of x_0 considered, the four values of $\xi_{\text{max.}}$ are plotted against r in the three Figs. 1–3, and are joined there by smooth curves. In these three graphs the vertical scale for r is the same. The horizontal scale for ξ is the same in Figs. 1 and 2, but differs from this scale by a factor 10^{20} in Fig. 3. The scale taken for r and that for ξ in Fig. 1 as also in Fig. 2 differ by a factor $\frac{1}{3}\times10^{22}$, and in Fig. 3 by a factor $\frac{1}{3}\times10^{42}$. To give a measure of the steepness of the maxima of the graphs I–XII, a 'thickness' d is defined as being the distance between the two symmetrical points in the curves I–XII at which the energy-density has half its maximum value. These distances d, as read off the graphs I–XII, are plotted (in horizontal lines) in Figs. 1–3 about the points $\xi_{\text{max.}}$.

The examination of Figs. 1–3 leads to the following conclusions. When the electron has built up a large velocity, the bulk of the energy of the field produced by it is concentrated in a thin region of space, forming a field-energy disk. This disk is symmetrical about the x_1-axis and is centred at the instantaneous position of the electron. The profile of the disk, as seen orthogonally to the x_1-axis, is given at the three different times x_0 considered by the three curves in Figs. 1–3, in the following sense. Consider one of these three curves. For a fixed value of r, the energy-density is then the same at any two points, one on each side of this curve, and at equal distances from it as measured parallel to the x_1-axis (due to the symmetry of the graphs I–XII about $\xi_{\text{max.}}$ mentioned earlier). The term 'disk' as used here to describe the region of greatest concentration of field energy is not to be taken in the strict sense of meaning a plane disk, whose profile would be a straight line perpendicular to the x_1-axis, but as meaning a slightly concave disk, with the concavity turned towards the asymptote $\xi = 0$, $r = 0$. The concavity of the two curves joining the points of maximum r in the surfaces of constant scalar potential for $x_0 = 10^{20}$ and for $x_0 = 5\times10^{20}$, given in (1), can be seen to be greater than that of the curves plotted in Figs. 1 and 2, respectively. The thickness d increases with r (but the energy-density decreases rapidly when r increases). The comparison of Figs. 1, 2 and 3 brings out how the disk changes as the electron advances. Its

Points ξ of maximum energy-density as functions of r.

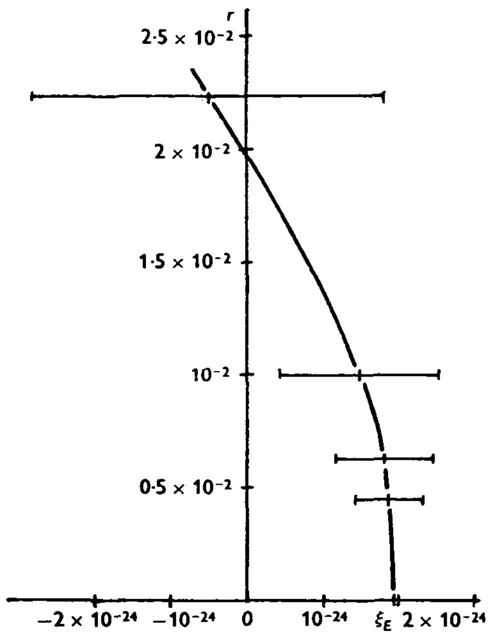

Fig. 1. $x_0 = 10^{20}$.

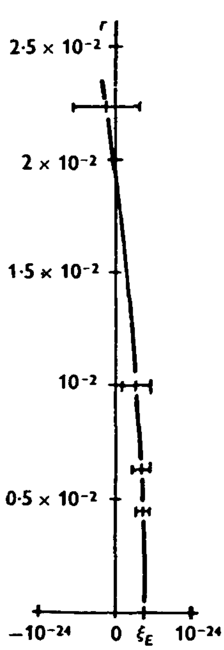

Fig. 2. $x_0 = 5 \times 10^{20}$.

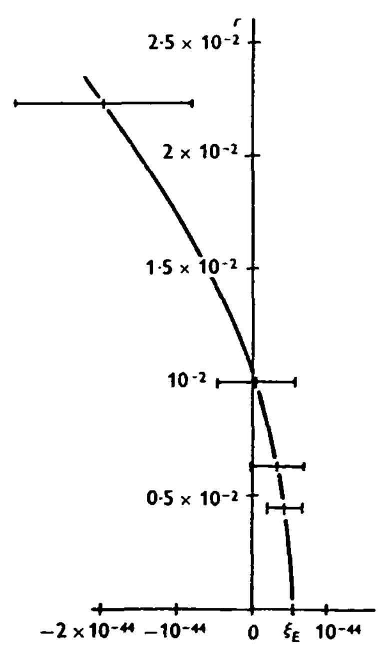

Fig. 3. $x_0 = 10^{40}$.

concavity gets flattened out as time goes on, so that the energy-disk tends to a true plane disk. It can be seen that the thickness d of the energy-disk at the various values of r and x_0 considered decreases somewhat more rapidly than x_0^{-1} as x_0 increases.

Next we must investigate the radial energy distribution in the disk, and the way this varies as the electron advances. For this purpose we consider the quantity $W(x_0, r)$ obtained by integrating the field-energy density from $+\infty$ to $-\infty$ along a direction parallel to the x_1-axis at a distance r from it,

$$W(x_0, r) = \int_{-\infty}^{+\infty} w(x_0, r, x_1)\, dx_1. \tag{9}$$

This W forms what may be called the energy per unit disk-area.

The mentioned graphs I–XII are used to carry out the integration (9) graphically, by measuring in each of those graphs the area bounded by the curve and the ξ-axis. The twelve values of W so obtained are given in Table 1, with an estimated error of a few per cent. For these values, the dependence of W on r may be expressed analytically in the form

$$W = f(x_0)\, r^{-3}, \tag{10}$$

where $f(x_0)$ is a factor depending on x_0 but varying very little (if at all) with r. This is seen from the fourth column of Table 1, which gives the values of $f(x_0)$ calculated on the assumption (10) from the values of W and r given in the third and second columns. (We observe that for the physical solution of the equations of motion (1), with the electron permanently at rest, the form of the energy per unit area, obtained by integrating the Coulomb energy-density along a straight line as in (9), is given by $W = cr^{-3}$, where c is a constant independent of the time x_0. This last form for W also gives an approximation in the case of the self-accelerating electron for sufficiently large values of r, for which the relevant retarded points fall into a region where the world-line of the electron approaches the x_0-axis.) Further, the values of $f(x_0)$ given are seen to increase a little more rapidly than x_0 (see also Fig. 4). As a rough approximation we have

$$f(x_0) = \alpha x_0 \log(x_0 \log x_0), \tag{11}$$

Table 1

x_0	r	W	$f(x_0)$	α
10^{20}	$4{\cdot}472 \times 10^{-3}$	$2{\cdot}08 \times 10^{27}$	$1{\cdot}87 \times 10^{20}$	$3{\cdot}75 \times 10^{-2}$
	$6{\cdot}325 \times 10^{-3}$	$7{\cdot}47 \times 10^{26}$	$1{\cdot}89 \times 10^{20}$	$3{\cdot}79 \times 10^{-2}$
	10^{-2}	$1{\cdot}87 \times 10^{26}$	$1{\cdot}87 \times 10^{20}$	$3{\cdot}75 \times 10^{-2}$
	$2{\cdot}236 \times 10^{-2}$	$1{\cdot}63 \times 10^{25}$	$1{\cdot}82 \times 10^{20}$	$3{\cdot}65 \times 10^{-2}$
5×10^{20}	$4{\cdot}472 \times 10^{-3}$	$1{\cdot}09 \times 10^{28}$	$9{\cdot}78 \times 10^{20}$	$3{\cdot}80 \times 10^{-2}$
	$6{\cdot}325 \times 10^{-3}$	$3{\cdot}81 \times 10^{27}$	$9{\cdot}64 \times 10^{20}$	$3{\cdot}74 \times 10^{-2}$
	10^{-2}	$9{\cdot}56 \times 10^{26}$	$9{\cdot}56 \times 10^{20}$	$3{\cdot}71 \times 10^{-2}$
	$2{\cdot}236 \times 10^{-2}$	$8{\cdot}40 \times 10^{25}$	$9{\cdot}38 \times 10^{20}$	$3{\cdot}64 \times 10^{-2}$
10^{40}	$4{\cdot}472 \times 10^{-3}$	$4{\cdot}11 \times 10^{47}$	$3{\cdot}69 \times 10^{40}$	$3{\cdot}82 \times 10^{-2}$
	$6{\cdot}325 \times 10^{-3}$	$1{\cdot}41 \times 10^{47}$	$3{\cdot}57 \times 10^{40}$	$3{\cdot}69 \times 10^{-2}$
	10^{-2}	$3{\cdot}55 \times 10^{46}$	$3{\cdot}55 \times 10^{40}$	$3{\cdot}67 \times 10^{-2}$
	$2{\cdot}236 \times 10^{-2}$	$3{\cdot}11 \times 10^{45}$	$3{\cdot}48 \times 10^{40}$	$3{\cdot}60 \times 10^{-2}$

with constant α. The values of α calculated on the assumption (11) are given in the fifth column of Table 1. Their average is $\alpha = 3{\cdot}717 \times 10^{-2}$.

To show the form of the energy per unit disk-area, the values of W given in Table 1 are plotted against r in Fig. 4, those for $x_0 = 10^{20}$ and those for $x_0 = 10^{40}$ being joined by smooth curves. It should be noted that the scale taken for r is the same for all the three values of x_0, but the scale for W is taken proportional to x_0 in order to bring out the way in which W increases with x_0.

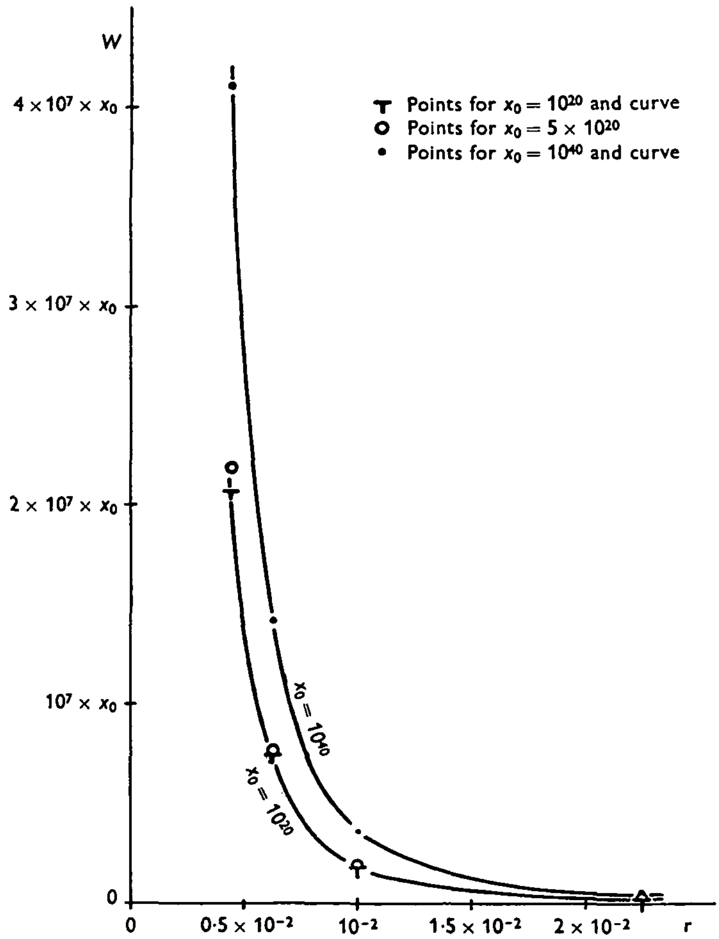

Fig. 4. Energy per unit area $W = W(x_0, r)$.

We may point out at this stage a special feature of the advanced field variables of the self-accelerating electron, arising from the fact that the asymptote of the world-line of the electron in the distant future is a null-line.

Consider the past part of the light-cone from a point $\mathbf{z}$ on the world-line of the electron for which $\phi \gg 1$, so that equations (2) hold. Its equation is given by (3), where x_0 must satisfy the reverse of inequality (4). Let the point $\mathbf{z}$ move along the world-line in such a way that $\phi \rightarrow \infty$, and consider those points of the past light-cone whose coordinates always remain finite. For these points the expression in square brackets in (3) does not vanish, and we can put (3) into the form

$$\xi \equiv x_1 - x_0 = \frac{r^2}{e^{\phi}/\phi - (x_1 + x_0)} + \frac{e^{-\phi}}{\phi}, \tag{12}$$

which shows that these points form the hyperplane of three dimensions

$$x_1 - x_0 = 0. \tag{13}$$

Any point x for which $x_1 - x_0 < 0$ is outside the past light-cone from all points on the entire world-line of the self-accelerating electron. Therefore all advanced field variables are zero for the region of space-time where $x_0 - x_1 < 0$.

To examine possible discontinuities of the advanced field variables on the hyperplane (13), we consider a field-point x close to it and let $x_1 - x_0$ tend to zero by positive values: $\xi \to +0$. Then $\phi_{\text{adv.}} \to \infty$. Using (1) and (2) in the well-known formula (1) for the advanced four-vector potential, one gets for its non-vanishing components

$$\left.\begin{aligned} \frac{1}{e} A_0^{\text{adv.}} &= \frac{1 + e^{-2\phi}}{x_1 - x_0 - (x_1 + x_0)\, e^{-2\phi}} \to +\infty, \\ \frac{1}{e} A_1^{\text{adv.}} &= \frac{1 - e^{-2\phi}}{x_1 - x_0 - (x_1 + x_0)\, e^{-2\phi}} \to +\infty, \quad (\phi = \phi_{\text{adv.}}), \end{aligned}\right\} \tag{14}$$

remembering that the inequality (4) is reversed for the advanced point. The potential having no physical interpretation in itself, it is more interesting to consider the advanced field quantities $F_{\mu\nu}^{\text{adv.}}$. In their formation some of the infinities of the potential (14) cancel out. The $F_{\mu\nu}^{\text{adv.}}$ are obtained from equations (5) by changing the sign of each and using the advanced value for ϕ. Suppose first that $r \neq 0$. Then, using (12), we find $(x_1 + x_0)\, e^{-\phi} - (x_1 - x_0)\, e^{\phi} \to -\infty$, and from (5) all $F_{\mu\nu}^{\text{adv.}} \to 0$. If $r = 0$, all $F_{\mu\nu}^{\text{adv.}}$ vanish from (5) except $F_{10}^{\text{adv.}}$. We have in this case from (12) $(x_1 + x_0) e^{-\phi} - (x_1 - x_0) e^{\phi} \to -\dfrac{1}{\phi} \to -0$, and from (5) $e^{-1} F_{10}^{\text{adv.}} \to -4\phi^2 \to -\infty$. Thus the only discontinuity in the advanced field quantities $F_{\mu\nu}^{\text{adv.}}$ occurs in $F_{10}^{\text{adv.}}$ for field-points on the null-line $x_1 - x_0 = 0$, $x_2 = x_3 = 0$.

MOTION OF AN ELECTRON DISTURBED BY A PULSE

In this section we consider the motion of an electron disturbed by a pulse of electromagnetic radiation of infinitely short duration passing over it. Suppose that the electric and magnetic intensities of the pulse are represented by δ-functions in the form

$$\mathfrak{E} = \mathfrak{l} k \delta(z_0 - \mathfrak{z} . \mathfrak{n}), \quad \mathfrak{H} = \mathfrak{m} k \delta(z_0 - \mathfrak{z} . \mathfrak{n}), \tag{15}$$

where $\mathfrak{l}$, $\mathfrak{m}$ and $\mathfrak{n}$ form a set of unit mutually orthogonal three-vectors, k is a positive constant and $\mathfrak{z}.\mathfrak{n}$ denotes the scalar product $z_1 n_1 + z_2 n_2 + z_3 n_3$. The equations (15) describe a pulse polarized in the direction $\mathfrak{l}$ and moving in the direction $\mathfrak{n}$. With the units mentioned at the beginning of the last section and the usual notation, the Lorentz-Dirac equations of motion of the electron under the influence of the force (15) are

$$\dot{v}_\mu - \ddot{v}_\mu - (\dot{\mathbf{v}}, \dot{\mathbf{v}})\, v_\mu = F_\mu \quad (\mu = 0, 1, 2, 3), \tag{16}$$

with $v_\mu = \dot{z}_\mu$ and
$$F_\mu = v_\nu F^\nu_{\mu\text{in.}} = \frac{3}{2e}\{\mathfrak{v}\,.\,\mathfrak{E}, v_0\mathfrak{E} + \mathfrak{v}\wedge\mathfrak{H}\}, \tag{17}$$
where $\wedge$ denotes the vector product of two three-vectors.

Let us choose the origin from which s is measured in such a way that the pulse strikes the electron at $s = 0$, so that $z_0 - \mathfrak{z}\,.\,\mathfrak{n} = 0$ for $s = 0$. Using the property (2) of δ-functions that, for $\dot{g}(s) > 0$, $\delta\{g(s)\}$ is equivalent to $[\dot{g}(s)]^{-1}\delta(s - s_0)$, where s_0 satisfies $g(s_0) = 0$, we can put the four components of F_μ given by (17) and (15) into the form
$$F_\mu = K f_\mu \delta(s), \tag{18}$$
where $K = 3k/2e$ and
$$\left.\begin{aligned} f_0 &= \frac{\mathfrak{v}\,.\,\mathfrak{l}}{v_0 - \mathfrak{v}\,.\,\mathfrak{n}}, & f_1 &= \frac{v_0 l_1 + v_2 m_3 - v_3 m_2}{v_0 - \mathfrak{v}\,.\,\mathfrak{n}}, \\ f_2 &= \frac{v_0 l_2 - v_1 m_3 + v_3 m_1}{v_0 - \mathfrak{v}\,.\,\mathfrak{n}}, & f_3 &= \frac{v_0 l_3 + v_1 m_2 - v_2 m_1}{v_0 - \mathfrak{v}\,.\,\mathfrak{n}}. \end{aligned}\right\} \tag{19}$$

Equations (16) and (18) show that before and after $s = 0$ the equations of motion of the electron are the same as the well-known equations with no incident field, namely,
$$\dot{v}_\mu - \ddot{v}_\mu - (\dot{\mathrm{v}}, \dot{\mathrm{v}})\, v_\mu = 0. \tag{20}$$
Let the velocity of the electron before $s = 0$ be denoted by v_μ^- and that after $s = 0$ by v_μ^+. They are given by
$$v_\mu^- = \alpha_{\mu 0}\cosh A e^s + \alpha_{\mu 1}\sinh A e^s \quad (s < 0), \tag{21}$$
$$v_\mu^+ = \beta_{\mu 0}\cosh B e^s + \beta_{\mu 1}\sinh B e^s \quad (s > 0), \tag{22}$$
where A, B are positive constants and the α's and β's satisfy
$$g^{\mu\nu}\alpha_{\mu r}\alpha_{\nu s} = g_{rs}, \quad g^{\mu\nu}\beta_{\mu r}\beta_{\nu s} = g_{rs} \quad (r, s = 0, 1), \tag{23}$$
$g_{\mu\nu}$ being the metric tensor. (This is the result of transforming the solution (1) of (20) to a general Lorentz frame of reference.) The six independent constants in v_μ^+ of (22) are to be determined in terms of the six independent constants in v_μ^- of (21) (or vice versa) through the equations of motion of the electron in an infinitesimal neighbourhood of $s = 0$, which we proceed to examine.

Equations (16) and (18) show that at $s = 0$, $\ddot{v}_\mu$ must contain a δ-function, so that the acceleration $\dot{v}_\mu$ is discontinuous there. To express the electron variables in a compact form, we make use of the discontinuous factor $\epsilon(s)$ defined by
$$\epsilon(s) = 0 \quad \text{for} \quad s < 0, \qquad \epsilon(s) = 1 \quad \text{for} \quad s > 0.$$
We write the velocity for all s as
$$v_\mu = v_\mu^- \epsilon(-s) + v_\mu^+ \epsilon(s), \tag{24}$$
with the continuity condition that
$$v_\mu = v_\mu^- = v_\mu^+ \quad \text{for} \quad s = 0. \tag{25}$$
Since $\dot{\epsilon}(s) = \delta(s)$, we have, from (24) and (25),
$$\left.\begin{aligned} \dot{v}_\mu &= \dot{v}_\mu^- \epsilon(-s) + \dot{v}_\mu^+ \epsilon(s), \\ \ddot{v}_\mu &= \ddot{v}_\mu^- \epsilon(-s) + \ddot{v}_\mu^+ \epsilon(s) - \dot{v}_\mu^- \delta(s) + \dot{v}_\mu^+ \delta(s), \\ (\dot{\mathrm{v}}, \dot{\mathrm{v}})\, v_\mu &= (\dot{\mathrm{v}}^-, \dot{\mathrm{v}}^-)\, v_\mu^- \epsilon(-s) + (\dot{\mathrm{v}}^+, \dot{\mathrm{v}}^+)\, v_\mu^+ \epsilon(s) + \dots \end{aligned}\right\} \tag{26}$$

In the third equation of (26) we have used the definition of $\epsilon(s)$, and terms containing products of $\epsilon(-s)$ by $\epsilon(s)$ with a coefficient that remains finite at $s=0$ have not been written out. They are different from zero only at $s=0$, and there they are finite, so that they make no contribution in the equations of motion since these contain a factor $\delta(s)$. Remembering that v_μ^- and v_μ^+ satisfy the equation (20), we have, from (26),

$$\dot{v}_\mu - \ddot{v}_\mu - (\dot{\mathbf{v}}, \dot{\mathbf{v}})\, v_\mu = (\dot{v}_\mu^- - \dot{v}_\mu^+)\,\delta(s). \tag{27}$$

Thus, from (16) and (18),

$$\dot{v}_\mu^- - \dot{v}_\mu^+ = Kf_\mu \quad \text{for} \quad s=0. \tag{28}$$

The equations (28), together with (25), determine the relation between the constants which appear in v_μ^- of (21) and those in v_μ^+ of (22). With the help of (21) and (22), (25) and (28) give

$$\alpha_{\mu 0}\cosh A + \alpha_{\mu 1}\sinh A - (\beta_{\mu 0}\cosh B + \beta_{\mu 1}\sinh B) = 0, \tag{29}$$

$$A(\alpha_{\mu 0}\sinh A + \alpha_{\mu 1}\cosh A) - B(\beta_{\mu 0}\sinh B + \beta_{\mu 1}\cosh B) = Kf'_\mu, \tag{30}$$

where f'_μ is f_μ taken at $s=0$. The value of f'_μ in (30) can be expressed in two forms. One is in terms of the α's and A, by replacing the value of v_μ^- at $s=0$ obtained from (21) in (19). The other form is in terms of the β's and B, by replacing the value of v_μ^+ at $s=0$ obtained from (22) in (19). In the presence of a pulse ($K \neq 0$), A and B cannot both vanish as is seen from (30). We therefore consider two cases.

Case (i). Suppose that $A \neq 0$. The system of linear equations formed by (29) and (30), with the f'_μ expressed in terms of the β's and B, may now be solved for the α's. The equation $g^{\mu\nu}\alpha_{\mu 0}\alpha_{\nu 1} = 0$ of (23) then leads by a straightforward but lengthy calculation to an equation giving A in terms of the β's and B. The result is that in this case the velocity of the electron is given explicitly in terms of the β's and B by the equation (22) for $s \geqslant 0$, and by

$$\begin{aligned} v_\mu^- = -\frac{1}{A}\{Kf'_\mu + B(\beta_{\mu 0}\sinh B + \beta_{\mu 1}\cosh B)\}\sinh[A(1-e^s)] \\ + (\beta_{\mu 0}\cosh B + \beta_{\mu 1}\sinh B)\cosh[A(1-e^s)] \end{aligned} \tag{31}$$

for $s \leqslant 0$, where A is the positive root of

$$A^2 = B^2 - 2BKf'^\mu(\beta_{\mu 0}\sinh B + \beta_{\mu 1}\cosh B) - K^2 f'^\mu f'_\mu. \tag{32}$$

Case (ii). Suppose that $B \neq 0$. The system of linear equations formed by (29) and (30), with the f'_μ expressed in terms of the α's and A, may now be solved for the β's. The equation $g^{\mu\nu}\beta_{\mu 0}\beta_{\nu 1} = 0$ of (23) then leads to an equation giving B in terms of the α's and A. The result is that in this case the velocity of the electron is given explicitly in terms of the α's and A by the equation (21) for $s \leqslant 0$, and by

$$\begin{aligned} v_\mu^+ = \frac{1}{B}\{-Kf'_\mu + A(\alpha_{\mu 0}\sinh A + \alpha_{\mu 1}\cosh A)\}\sinh[B(e^s-1)] \\ + (\alpha_{\mu 0}\cosh A + \alpha_{\mu 1}\sinh A)\cosh[B(e^s-1)] \end{aligned} \tag{33}$$

for $s \geqslant 0$, where B is the positive root of

$$B^2 = A^2 + 2AKf'^\mu(\alpha_{\mu 0}\sinh A + \alpha_{\mu 1}\cosh A) - K^2 f'^\mu f'_\mu. \tag{34}$$

We discuss the general solution of our problem worked out above in the next two subsections.

THE PHYSICAL SOLUTION

The physical solution is obtained from the general solution by adding the extra condition that $B = 0$. For its discussion it is convenient to take a frame of reference in which the electromagnetic pulse is polarized in the direction of the x_1-axis and moves in the direction of the x_3-axis. Then $\mathfrak{l} \equiv (1,0,0)$, $\mathfrak{m} \equiv (0,1,0)$ and $\mathfrak{n} \equiv (0,0,1)$. The results of Case (i) above give the velocity of the electron as

$$v_\mu^- = -f'_\mu \sinh[K(1-e^s)] + \beta_{\mu 0}\cosh[K(1-e^s)] \quad (s \leqslant 0), \qquad v_\mu^+ = \beta_{\mu 0} \quad (s \geqslant 0), \tag{35}$$

where

$$f'_0 = f'_3 = \frac{\beta_{10}}{\beta_{00}-\beta_{30}}, \quad f'_1 = 1, \quad f'_2 = 0. \tag{36}$$

The expressions (35) and (36) give the physical solution for a prescribed final velocity $\beta_{\mu 0}$. In practice it is the velocity at $s = -\infty$, w_μ say, that is given. From (35) and (36) we find that the constants are given in terms of w_μ by

$$\left.\begin{aligned} &\beta_{00} = \frac{(w_0-w_3)\,w_0 + (w_1+\sinh K)\sinh K}{(w_0-w_3)\cosh K}, \quad \beta_{10} = \frac{w_1+\sinh K}{\cosh K}, \\ &\beta_{20} = \frac{w_2}{\cosh K}, \quad \beta_{30} = \frac{(w_0-w_3)\,w_3 + (w_1+\sinh K)\sinh K}{(w_0-w_3)\cosh K}, \quad f'_0 = f'_3 = \frac{w_1+\sinh K}{w_0-w_3}. \end{aligned}\right\} \tag{37}$$

Passing to three-space, we note that in general the motion given by (35) is three-dimensional. Up to the time $s = 0$ the path of the electron is a plane curve, and after that it is a straight line, not necessarily coplanar with the curve. If we suppose that $w_0 \neq 1$ and $\beta_{00} \neq 1$, the angle ϕ between the direction of the velocity at $s = -\infty$ and the direction of the velocity after $s = 0$ is found to be given by

$$\cos\phi = \frac{(w_0-w_3)(w_0^2-1) + w_0 w_1 \sinh K + w_3 \sinh^2 K}{(w_0^2-1)^{\frac{1}{2}}\{[(w_0-w_3)\,w_0 + (w_1+\sinh K)\sinh K]^2 - (w_0-w_3)^2\cosh^2 K\}^{\frac{1}{2}}}. \tag{38}$$

If the intensity of the pulse is small so that $|K| \ll 1$, we find from (38) by expansion including terms up to the order of K^2

$$\cos\phi = 1 - \frac{K^2}{2}\frac{(w_0-w_3)^2(w_0^2-1) - w_1^2}{(w_0-w_3)^2(w_0^2-1)^2} + \dots. \tag{39}$$

On the other hand, when $|K| \to \infty$, (38) gives $\cos\phi \to w_3(w_0^2-1)^{-\frac{1}{2}}$, and the final straight-line motion tends to be in the direction of motion of the pulse, as can be seen directly from (37). If the velocity of the electron has no component parallel to $\mathfrak{H}$ initially ($w_2 = 0$), then this remains so always and the complete motion of the electron is confined to a plane. Further, if the electron is at rest initially ($w_0 = 1$), its complete motion is along a straight line, whose direction cosines are

$$\frac{K}{|K|}\left(\frac{1}{\cosh K}, 0, \tanh K\right).$$

(In the approximate treatment of an electron initially at rest, when $|K|$ is small and when the magnetic effects of the pulse are neglected, it is assumed that the electron moves along the x_1-axis (2).) The motion is also in a straight line if the velocity of the electron at $s = -\infty$ is such that the electron comes to rest at $s = 0$ ($w_1 = \sinh K$, $w_2 = w_3 = 0$, so that $\beta_{00} = 1$).

The total energy and momentum $\Delta p_\mu = m(v_\mu^+ - w_\mu)$ acquired by the electron are given by

$$\left.\begin{aligned}
\Delta p_0 &= m\frac{(w_0-w_3)(1-\cosh K)w_0+(w_1+\sinh K)\sinh K}{(w_0-w_3)\cosh K} \sim mK\frac{w_1}{w_0-w_3} \\
&\qquad + \frac{mK^2}{2}\frac{2-(w_0-w_3)w_0}{w_0-w_3}+\dots, \\
\Delta p_1 &= m\frac{(1-\cosh K)w_1+\sinh K}{\cosh K} \sim mK - \tfrac{1}{2}mK^2w_1+\dots, \\
\Delta p_2 &= m\frac{(1-\cosh K)w_2}{\cosh K} \sim -\tfrac{1}{2}mK^2w_2+\dots, \\
\Delta p_3 &= m\frac{(w_0-w_3)(1-\cosh K)w_3+(w_1+\sinh K)\sinh K}{(w_0-w_3)\cosh K} \sim mK\frac{w_1}{w_0-w_3} \\
&\qquad + \frac{mK^2}{2}\frac{2-(w_0-w_3)w_3}{w_0-w_3}+\dots,
\end{aligned}\right\} \tag{40}$$

where the expressions to the right of $\sim$ are expansions valid for $|K| \ll 1$. If only terms up to the first order in K are included, these expansions give the value for Δp_μ obtained from the elementary theory which ignores radiation damping completely (2), but for a pulse of great intensity the effect of radiation damping becomes important. In particular, when $|K| \to \infty$, $\Delta p_1 \to m(K/|K| - w_1)$ and $\Delta p_2 \to -mw_2$, whereas in the elementary theory Δp_1 increases indefinitely with $|K|$ and $\Delta p_2 = 0$.

The non-physical solutions

We consider now the general solution of our equations of motion (16) without imposing any extra condition on it, and suppose therefore that $B \neq 0$. When sufficient data are known to determine the six independent constants which enter into (21), the velocity of the electron is given in a general Lorentz frame by the results of Case (ii) above. In this motion the velocity of the electron tends to the velocity of light as $s \to \infty$. Furthermore, the rate of emission of energy $-\frac{2}{3}e^2(\dot{\mathrm{v}}, \dot{\mathrm{v}})\,v_0$ leads to an infinite energy being emitted by the electron per unit frequency range. However, a result of physical interest that can still be considered is the 'angle of deflexion' θ in three-space due to the pulse, which may be taken to be the angle between the direction of the actual final three-velocity at $s = \infty$, and the direction which the velocity would have at $s = \infty$ if the electron had started off in the same state of motion for $s < 0$ but without meeting the pulse at $s = 0$. The latter velocity is given by (21) at $s = \infty$.

In order to get a simple expression for θ, we take a Lorentz frame in which the electron is at rest at $s = -\infty$ and moves along the x_1-axis for $s < 0$. Then in (21) we have $\alpha_{\mu r} = \delta_{\mu r}$ $(r = 0, 1)$, where $\delta_{\mu r}$ is the Kronecker symbol. From (21) and (33)

$$\cos\theta = \frac{(A\cosh A + B\sinh A)(\cosh A - n_1\sinh A) - Kl_1\cosh A}{|(A\sinh A + B\cosh A)(\cosh A - n_1\sinh A) - Kl_1\sinh A|}, \tag{41}$$

where B is the positive root of

$$B^2 = A^2 - \frac{2AKl_1}{\cosh A - n_1\sinh A} + K^2. \tag{42}$$

In particular, if the electron starts off in a straight-line uniform motion with $A = 0$, the direction cosines of its velocity at $s = \infty$ are found to be $-(K/|K|)(l_1, l_2, l_3)$, from (33). In this case therefore an electron of negative charge ($K < 0$) tends asymptotically to move in the direction of the electric field $\mathfrak{E}$, whereas a particle of positive charge ($K > 0$) would tend to move in the opposite direction $-\mathfrak{E}$. This non-physical solution thus leads to results which are contrary to what one expects to happen on elementary considerations.

Summary

This paper deals with some details of the application of the Lorentz-Dirac equations of motion of an electron to two simple cases, (*a*) with no incident field, and (*b*) with an incident pulse of radiation. In case (*a*), the field-energy distribution in the self-accelerating motion of the electron when the electron has built up a velocity close to the velocity of light is considered. Numerical and graphical methods are used to form a picture showing how the field energy tends to be concentrated in a disk for this state of motion. A discontinuity of the advanced field variables of the self-accelerating electron is also examined. In case (*b*), the general solution of the equations of motion for the velocity of the electron is worked out accurately. In the physical solution, this brings out the effects of radiation damping when the pulse has great intensity. Some aspects of the non-physical solutions are pointed out.

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