Question 1:

a) $$E[f(x)] = \sum_{x=a,b,c} f(x)\,p(x)$$

$$= 10 \times 0.1 + 5 \times 0.2 + \tfrac{10}{7} \times 0.7$$

$$= 3$$

b) $$E\left(\tfrac{1}{p(x)}\right) = \sum_{x=a,b,c} \tfrac{1}{p(x)}\,p(x)$$

$$= \frac{1}{0.1} \times 0.1 + \frac{1}{0.2} \times 0.2 + \frac{1}{0.7} \times 0.7$$

$$= 3$$

c) For an arbitrary pmf, $E\left[\tfrac{1}{p(x)}\right] = \sum_{x=x_1,x_2 \dots x_n} \frac{1}{p(x)} \cdot p(x)$

$$= \sum_{x=x_1,x_2\dots x_n} 1$$

$$= 1 + 1 + \dots 1$$

$$= n$$

d) $$E[f(x)^2] = \sum_{x=a,b,c} f(x)^2\,p(x)$$

$$= 10^2 \times 0.1 + 5^2 \times 0.2 + \left(\tfrac{10}{7}\right)^2 \times 0.7$$

$$= 16.43$$

$$E[f(X)]^2 = \left\{ \sum_{x=a,b,c} f(x)\, p(x) \right\}^2$$

$$= \left(10 \times 0.1 + 5 \times 0.2 + \tfrac{10}{7} \times 0.7\right)^2$$

$$= 3^2$$

$$= 9$$

Question 2:

a) The outcomes of flipping three biased coins at once are = {TTT, THT, TTH, HTT, THH, HHT, HTH, HHH}

let X be the number of heads.

∴ Probability of no coins being head is $P(X=0)$

$= TTT$

$= (1-0.75) \times (1-0.5) \times (1-0.25)$

$= 0.09375$

∴ Probability of one coin being head is $P(X=1)$

$= THT + TTH + HTT$

$= (1-0.75) \times (0.5) \times (1-0.25) + (1-0.75) \times (1-0.5) \times 0.25 + (0.75) \times (1-0.5) \times (1-0.25)$

$= 0.09375 + 0.03125 + 0.28125$

$= 0.40625$

$\therefore$ Probability of two coins being head is $P(X=2)$

$= THH + HHT + HTH$

$= (1-0.75)\times 0.5\times 0.25 + 0.75\times 0.5\times(1-0.25) +$
$0.75\times(1-0.5)\times 0.25$

$= 0.03125 + 0.28125 + 0.09375$

$= 0.40625$

$\therefore$ Probability of all coins being head is $P(X=3)$

$= HHH$

$= 0.75\times 0.5\times 0.25$

$= 0.09375$

$\therefore E[X] = 0\times P(X=0) + 1\times P(X=1) + 2\times P(X=2) + 3\times P(X=3)$

$= 0\times 0.09375 + 1\times 0.40625 + 2\times 0.40625 +$
$3\times 0.09375$

$= 1.5$ (Ans)

b) There are 3 biased coins. A, B, C.

Probability of coin A is selected, $P(S_1) = \frac{1}{3}$

Probability of coin B is selected, $P(S_2) = \frac{1}{3}$

Probability of coin C is selected, $P(S_3) = \frac{1}{3}$

Total probability $= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$

Suppose one coin is selected randomly and flipping that coin 5 times resulting in 3 heads and 2 tails

let X = 3 heads and 2 tails

$P(X \mid S_1) = (0.75)^3 \times (1-0.75)^2 = 0.02636$

$P(X \mid S_2) = (0.5)^3 \times (1-0.5)^2 = 0.03125$

$P(X \mid S_3) = (0.25)^3 \times (1-0.25)^2 = 0.00879$

P(3 heads and 2 tails in 1st coin) $= (0.75)^3 \times (1-0.75)^2 = 0.02636$

P(3 heads and 2 tails in 2nd coin) $= (0.5)^3 \times (1-0.5)^2 = 0.03125$

P(3 heads and 2 tails in 3rd coin) $= (0.25)^3 \times (1-0.25)^2 = 0.00879$

$$\therefore P(S_3 \mid X) = \frac{P(S_3) \times P(X \mid S_3)}{P(S_1) \times P(X \mid S_1) + P(S_2) \times P(X \mid S_2) + P(S_3) \times P(X \mid S_3)}$$

$$= \frac{\frac{1}{3} \times 0.00879}{\left(\frac{1}{3} \times 0.02636\right) + \left(\frac{1}{3} \times 0.03125\right) + \left(\frac{1}{3} \times 0.00879\right)}$$

$$= \frac{\frac{293}{100000}}{\frac{659}{75000} + \frac{1}{96} + \frac{293}{100000}}$$

$$= \frac{293}{100000} \times \frac{3750}{83}$$

$$= \frac{879}{6640}$$

$$= 0.13238$$

Probability that I chose coin C, P(Probability of coin C is selected | 3 heads and 2 tails)

$$= 0.13238$$

Question 3:

$$p(x) = \begin{cases} 1/10 & \text{if } 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

The cost of an electrical breakdown of duration $x$ is $x^3$.

$\therefore f(x) = x^3$

$$E[f(x)] = \int_0^{10} f(x)\, p(x)\, dx$$

$$= \int_0^{10} x^3 \cdot \frac{1}{10}\, dx$$

$$= \frac{1}{10} \int_0^{10} x^3\, dx$$

$$= \frac{1}{10} \left[\frac{x^4}{4}\right]_0^{10}$$

$$= \frac{1}{10} \times \frac{10^4}{4} - 0$$

$$= \frac{10^4}{10 \times 4}$$

$$= 250 \quad \text{(Ans)}.$$

## Question 4:

a) Filled in the code, all checks passed.

b) $n = 10$

$\mu = 0$

$\sigma = 1$

$M = [\underbrace{0.0419}_{M_1}, \underbrace{-0.2432}_{M_2}, \underbrace{0.6882}_{M_3}, \underbrace{-0.1052}_{M_4}, \underbrace{0.0886}_{M_5}]$ (generated from Pluto Notebook)

Sample Variance $= 0.1273$ (generated from Pluto Notebook)

c) $n = 100$

$\mu = 0$

$\sigma = 1$

$M = [\underbrace{0.1354}_{M_1}, \underbrace{0.0966}_{M_2}, \underbrace{-0.0349}_{M_3}, \underbrace{0.1137}_{M_4}, \underbrace{0.1004}_{M_5}]$ (generated from Pluto Notebook)

Sample Variance $= 0.0045$ (generated from Pluto Notebook)

The sample variance is much smaller compared to the sample variance we got from 4b because of the difference in sample size.

d&e) $n = 30$

$\sigma = \sqrt{10}$

$M = 0.6380$

95% C.I (assuming Gaussian): $(-0.3018, 1.5778)$

95% C.I (without assuming Gaussian): $(-1.5063, 2.7823)$

Question 5:

$\bar{x} = 1/n \sum_{i=1}^{n} x_i$ is an unbiased estimator of the mean $\mu$

a) We know, $E[\bar{V}] = \sigma^2$

$\Rightarrow E\left[\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2\right] = \sigma^2 \quad \left[\bar{V} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2\right]$

$\Rightarrow E\left[n \cdot \frac{1}{n-1} \cdot \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2\right] = \sigma^2$

$\Rightarrow E\left[n \cdot \frac{1}{n-1} \cdot \bar{V}_b\right] = \sigma^2 \quad \left[\bar{V}_b = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2\right]$

$\Rightarrow n \cdot \frac{1}{n-1} \cdot E[\bar{V}_b] = \sigma^2$

$\Rightarrow E[\bar{V}_b] = \left(\frac{n-1}{n}\right) \sigma^2$

$\Rightarrow E[\bar{V}_b] = \left(1 - \frac{1}{n}\right) \sigma^2$ (Proved)

b)