CS605/CS635 - Modeling and Simulation
Summary of Important Mathematical Equations

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Contents

List of Mathematical Symbols

Notation	Description
$\max\limits_{a\leq x\leq b} f(x)$	maximum of a function $f(x)$ on the interval $a \leq x \leq b$
$\widehat{\theta}$	Point estimator of a parameter
e	A mathematical constant, the base of the natural logarithm. It is approximately equal to **2.71828**
$\exp x$	exponential (exponential function), denoted also by $\exp x = e^x$
$\ln x$	natural logarithm (logarithm to base e)
$\log_a x$	logarithm of x to base a
$\log x$	logarithm of x to base 10
$\int$	integral; $\int_a^b f(x)dx$ is the integral of a function $f(x)$ over the interval $[a, b]$
$\lim\limits_{x\to a} f(x)$	limit of a function $f(x)$ as the value of x approaches a
$\sum$	summation; $\sum\limits_{i=a}^{n} a_i$ is the summation from i equals a to n of a_i
$\binom{n}{r}$	n choose r, the number of r-combinations of a set of n elements, the number of r-element subsets of a set of n elements. It is also called binomial coefficient
$n!$	n factorial
λ	Greek letter lambda, in this course it is used as the parameter for exponential and poisson distribution, it also denotes **arrival rate** in queuing theory

Notation	Description
μ	Greek letter small Mu, used to denote mean of the normal distribution. It is also used to denote **service rate** in queuing theory
Φ	Greek letter capital phi, $\Phi(z)$ is the cumulative distribution function of the standard normal distribution
ϕ	Greek letter small Phi, $\phi(z)$ is the probability distribution function of the standard normal distribution
ρ	Greek letter small Rho, used to denote **server utilization** in queuing thoery
σ	Greek letter small Sigma. Used to denote **standard deviation**
θ	Greek Letter theta
$F(x)$	Cumulative Distribution Function(cdf), gives the probability of the random variable taking on a value up to and including the given value x
$E(X)$	The expected value of the random variable associated with a random experiment, also called the **expectation**, **long-run value**, or **mean**.
$f(x)$	Probability distribution function(pdf) for a given value x of the **continuous** random variable
$p(x)$	Probability mass function(pmf) for a given value x of the **discrete** random variable
$V(X)$	Variance of a Random Variable X
Z	Standard Normal Distribution
$\approx$	Approximately equal to
∞	infinity

Chapter 2

Simulation Examples

2.1 Simulation of Queuing Systems

Hand Simulation Table Template

Table 2.1: Simulation Table for Single-Channel Queueing Problem

Step	**Activity**	**Clock**	**Activity**	**Clock**	**Output**	**Clock**	**Output**	**Output**
Customer	**Inter-arrival Time**	**Arrival Time**	**Service Time**	**Time Service Begins**	**Waiting Time in Queue**	**Time Service Ends**	**Time Customer Spends in The System**	**Idle Time of Server**
1								
2								
3								
...								
n								
total	$\sum =$		$\sum =$		$\sum =$		$\sum =$	$\sum =$

See simulation table example in textbook [See Banks et al., 2009, p. 65]

Chapter 5

Probability Distributions

5.1 Discrete Probability Distributions

5.1.1 Binomial Distribution

Usage : Used to denote the number of successes in n Bernoulli trials.
(x successes in n trials with p probability of success and q probability of failure)
Notation : $X \sim Bin(n, p)$
Probability mass function(pmf):

$$p(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0, 1, 2, ..., n \\ 0, & \text{otherwise} \end{cases}$$

Where :
p : probability of success
q : probability of failure $= (1 - p)$
x : number of success
n : total number of trials

Cumulative distribution function :

$$F(x) = \sum_{i=0}^{x} \binom{n}{i} p^i q^{n-i}$$

$$\textbf{Mean} = E(X) = np$$
$$\textbf{Variance} = V(X) = npq$$

5.1.2 Geometric Distribution

Usage : Used to denote the number of trials to achieve the first success.
(probability of doing the expirment exactly x times to achieve the first success, with p probaility of success and q probability of failure)
Probability mass function(pmf):

$$p(x) = \begin{cases} q^{x-1} p, & x = 1, 2, ... \\ 0, & \text{otherwise} \end{cases}$$

Where :
p : probability of success.
q : probability of failure $= (1 - p)$
x : number of trials to achieve the first success (when there are $x - 1$ failures followed by a success).
Cumulative distribution function :

$$F(x) = \sum_{i=1}^{x} q^{i-1} p$$

$$\textbf{Mean} = E(X) = \frac{1}{p}$$
$$\textbf{Variance} = V(X) = \frac{q}{p^2}$$

5.1.3 Negative Binomial Distribution

Usage : Used to denote the number of trials until the kth success.

(probability of doing the expirment exactly x times to achieve the kth success, with p probaility of success and q probability of failure)

Probability mass function(pmf):

$$p(x) = \begin{cases} \binom{x-1}{k-1} q^{x-k} p^k, & x = k, k+1, k+2, ... \\ 0, & \text{otherwise} \end{cases}$$

Where :
p : probability of success.
q : probability of failure $= (1 - p)$
x : number of trials to achieve the kth success.
k : total number of successes.

Cumulative distribution function :

$$F(x) = \sum_{i=0}^{x} \binom{i-1}{k-1} q^{i-k} p^k$$

$$\textbf{Mean} = E(X) = \frac{k}{p}$$

$$\textbf{Variance} = V(X) = \frac{kq}{p^2}$$

5.1.4 Poisson Distribution

Usage : Used to denote the number of rare events occurring within a fixed period of time.
Rare Events : Two of such events are extremely unlikely to occur simultaneously or within a very short period of time.

Examples of rare events : Arrivals of jobs, telephone calls, e-mail messages, traffic accidents, network blackouts, virus attacks, errors in software, floods, and earthquakes.

Probability mass function(pmf):

$$p(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & x = 0, 1, \ldots \\ 0, & otherwise \end{cases}$$

Where :
λ : is a rate, average number of rare events occuring within unit time.

Cumulative distribution function :

$$F(x) = \sum_{i=0}^{x} \frac{e^{-\lambda}\lambda^i}{i!}$$

$$\textbf{Mean} = \textbf{Variance} = \lambda$$
$$E(X) = V(X) = \lambda$$

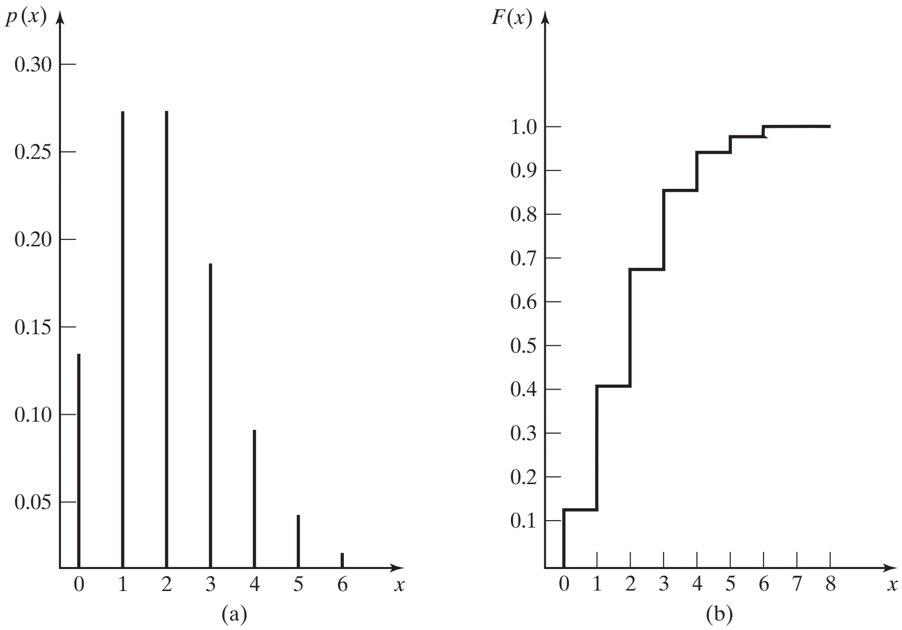


Figure 5.1: Poisson pmf and cdf
[courtesy of Banks et al., 2009]

5.2 Continuous Probability Distributions

5.2.1 Uniform Distribution

Usage : The uniform distribution plays a vital role in simulation. Random numbers, uniformly distributed between 0 and 1 (standard uniform), provide the means to generate random events. (See chapter 7 - Random Number Generation on page 11)

Probability distribution function(pdf):

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

Cumulative distribution function(cdf):

$$F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$

Where:
a : interval beginning.
b : interval end

Standard Uniform distribution : The Uniform distribution with a = 0 and b = 1.

$$\textbf{Mean} = E(X) = \frac{a+b}{2}$$
$$\textbf{Variance} = V(X) = \frac{(b-a)^2}{12}$$

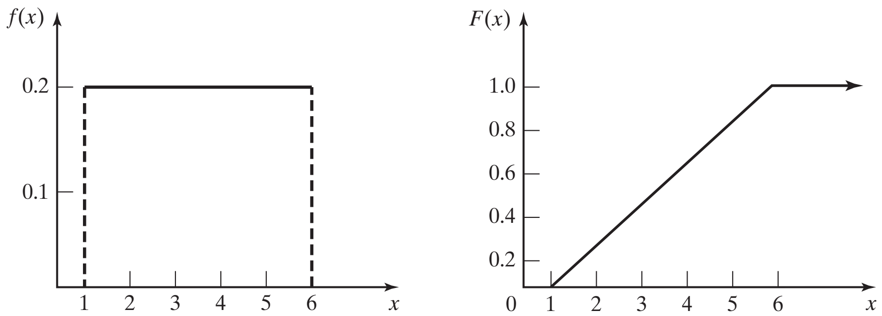


Figure 5.2: pmf and cdf for uniform distribution
[courtesy of Banks et al., 2009]

5.2.2 Exponential Distribution

Usage : The exponential distribution has been used to :

- Model interarrival times and service times. in these instances λ is a rate: arrival per unit time or service per unit time. (See chapter 6 - Queueing Theory on page 9)
- Model the lifetime of a component that fails catastrophically (instantaneously), such as a light bulb.

Probability distribution function(pdf):

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Cumulative distribution function(cdf):

$$F(x) = \begin{cases} 0, & x < 0 \\ \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

Mean $= E(X) = \dfrac{1}{\lambda}$

Variance $= V(X) = \dfrac{1}{\lambda^2}$

Memoryless property : One of the most important properties of the exponential distribution is that it is "memoryless", which means for all $s \geq 0$ and $t \geq 0$:

$$P(X > s + t | X > s) = P(X > t)$$

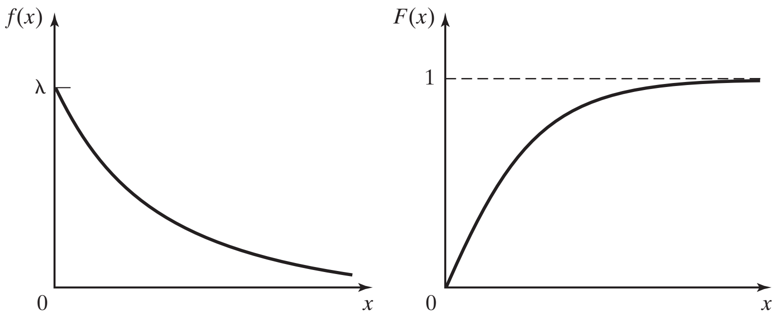


Figure 5.3: Exponential density function and cumulative density function
[courtesy of Banks et al., 2009]

5.2.3 Normal Distribution

Usage :

- sums and averages often have approximately Normal distribution. (See section 11.1 - Confidence-Interval Estimation on page 17)
- Normal distribution is often also found to be a good model for physical variables like weight, height, temperature, voltage, pollution level, and for instance, household incomes or student grades.

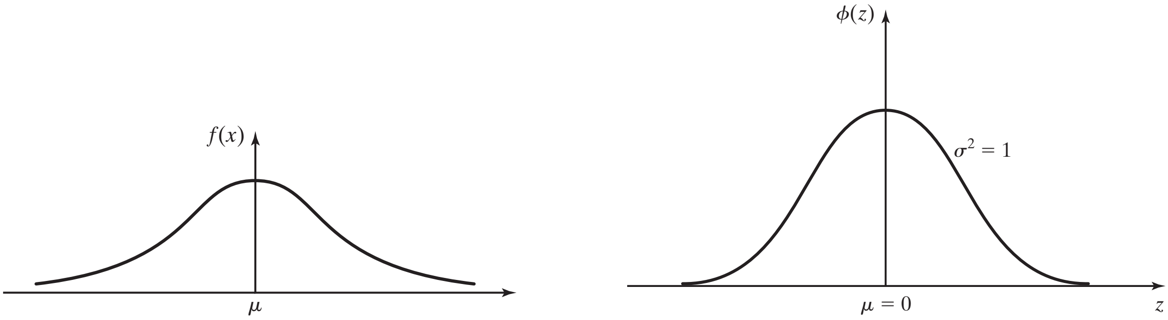
Definitions :

Notation	Definition
$N(\mu, \sigma^2)$	A normal distribution with mean $= \mu$ and variance $= \sigma^2$
Z	Standard Normal Distribution, with mean $\mu = 0$ and standard deviation $\sigma = 1$
$\phi(z)$	Probability distribution function of the standard normal distribution
$\Phi(z)$	Cumulative distribution function of the standard normal distribution, can be obtained from table A.3 [See Banks et al., 2009, p. 612]

$$Z = N(0,1)$$

If Random Variable X is a normal distribution with mean $= \mu$ and variance $= \sigma^2$, the value of Cumulative distribution function(cdf) for any value x can be obtained from standard normal distribution tables [See Table A.3 Banks et al., 2009, p. 612] by the following conversion :

$$\textbf{If } X \approx N(\mu, \sigma^2) \textbf{ ,Then } F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$



(a) pdf of the normal distribution (b) pdf of the **standard** normal distribution

Figure 5.4: probability density function of the normal distribution
[courtesy of Banks et al., 2009]

5.2.4 Triangular Distribution

Usage : used as a subjective description of a population for which there is only limited sample data(when the data is scarce). It is based on a knowledge of the minimum and maximum and an "inspired guess" of the modal value.
For these reasons, the triangular distribution has been called a "lack of knowledge" distribution.

Probability distribution function(pdf):

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)}, & a \le x \le b \\ \dfrac{2(c-x)}{(c-b)(c-a)}, & b < x \le c \\ 0, & \text{elsewhere} \end{cases}$$

Cumulative distribution function(cdf):

$$F(x) = \begin{cases} 0, & x \le a \\ \dfrac{(x-a)^2}{(b-a)(c-a)}, & a < x \le b \\ 1 - \dfrac{(c-x)^2}{(c-b)(c-a)}, & b < x \le c \\ 1, & x > c \end{cases}$$

$\textbf{Mean} = E(X) = \dfrac{a+b+c}{3}$

$\textbf{Variance} = V(X) = \dfrac{a^2+b^2+c^2-ab-ac-bc}{18}$

$\textbf{Mode} = b$

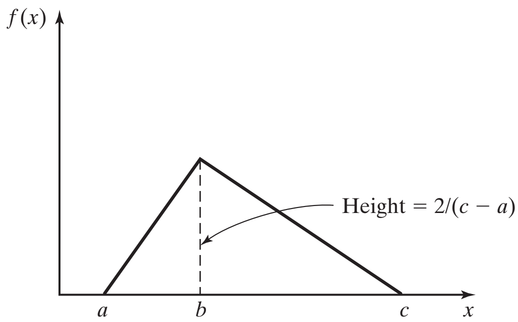


Figure 5.5: pdf of the triangular distribution
[courtesy of Banks et al., 2009]

Chapter 6

Queueing Theory

6.1 Queueing Notations

Table 6.1: Queueing Notation for Server Systems

Notation	**Definition**
$P_n(\pi_j)$	Steady-State **probability** of having n customers in the system.
λ	- Arrival **Rate**. - Throughput.
μ	Service **Rate**.
ρ	- Server Utilization. - Offered Load.
L	Long-run average **number** of customers in the ***system***.
L_Q	Long-run average **number** of customers in ***queue***.
w	- Long-run average **time** spent in the ***system*** per customer. - Response **time**.
w_Q	- Long-run average **time** spent in ***queue*** per customer. - Average waiting **time** in ***queue***.

[See Table 6.2 Banks et al., 2009, p. 253]

6.2 Important Laws

6.2.1 Arrival Time and Service Time

$$\text{Arrival Time} = \frac{1}{\text{Arrival Rate}} = \frac{1}{\lambda}$$

$$\text{Service Time} = \frac{1}{\text{Service Rate}} = \frac{1}{\mu}$$

6.2.2 Little's Law

$$L = \lambda w$$

6.2.3 Server Utilization

$$\rho = \lambda E(S) = \frac{\lambda}{\mu}$$

For a single-server queue to be stable the arrival rate λ should be less than the service rate μ :

$$\lambda < \mu \implies \frac{\lambda}{\mu} < 1 \implies \rho < 1$$

6.3 Steady-State Parameters of the $M/M/1$ Queue

$$L = \lambda w = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1-\rho}$$

$$w = \frac{1}{\mu - \lambda} = \frac{1}{\mu(1-\rho)}$$

$$L_Q = L\rho = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{1-\rho}$$

$$w_Q = w\rho = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{\rho}{\mu(1-\rho)}$$

$$P_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n = (1-\rho)\rho^n$$

[See Table 6.4 Banks et al., 2009, p. 268]

Chapter 7

Random Number Generation

7.1 Linear Congruential Method

The linear congruential method generates a sequence of integers $X_1, X_2, \ldots$ Uniformly distributed between zero and $m-1$ by following a recursie relationship :

$$X_{i+1} = (aX_i + c) \mod m, \qquad i = 0, 1, 2, \ldots$$

Where :
X_0 : initial value(seed)
a : Multiplier
c : Increment
m : Modulus

Forms of linear congruential method:

- *Mixed congruential method* : when $c \neq 0$
- *Multiplicative congruential method* : when $c = 0$

Random Numbers R_i between zero and 1 can then be generated by setting :

$$R_i = \frac{X_i}{m} \qquad i = 1, 2, \ldots$$

7.2 Tests for Random Numbers

7.2.1 Frequency Tests

Kolmogorov-Smirnov Test

Step 1. Rank the data from samllest to largest. Let $R_{(i)}$ denote the ith smallest observation, so that :

$$R_{(1)} \leq R_{(2)} \leq \ldots \leq R_{(N)}$$

Place the sorted data in the following table and complete the calculations :

Table 7.1: Calculations for Kolmogorov-Smirnov Test

	i	1	2	3	$\ldots$	N
	$R_{(i)}$					$R_{(N)}$
	$\frac{i}{N}$					1
	$\frac{i-1}{N}$	0				
D^+	$\frac{i}{N} - R_{(i)}$					
D^-	$R_{(i)} - \frac{i-1}{N}$					

Where :
i : Rank of the observed number, from smallest to largest.
N : Sample size.

Step 2. Compute D^+ and D^-

- D^+ will be the maximum value in the row of $\frac{i}{N} - R_{(i)}$
- D^- will be the maximum value in the row of $R_{(i)} - \frac{i-1}{N}$

Step 3. Compute the largest absolute deviatin between The Uniform distribution cdf $F(x)$ and the empirical distribution of the data $S_n(x)$ over the range of the random variable. based on the statistic :

$$\begin{aligned} D &= \max |F(x) - S_n(x)| \\ &= \max\{D^+, D^-\} \end{aligned}$$

Step 4. Locate the critical value D_α in Table A.8 [See Banks et al., 2009, p. 619] for the specified significance level α and the given degree of freedom [degree of freedom = sample size (N)].

Step 5. If the sample statistic D is greater than the critical value D_α , the null hypothesis that the data are a sample from a uniform distribution is rejected.

Chi-Square Test

The chi-square test uses the sample statistic :

$$X_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

It is valid only for large samples, where $N \geq 50$.

Table 7.2: Computations for Chi-Square Test

Interval	O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
1					
2					
3					
...					
n					
	$\sum = N$	$\sum = N$			$\sum = X_0^2$

Where :
n : The number of classes
N : The sample size
O_i : The observed number in the ith class
E_i : The expected number in the ith class. For uniform distribution it equals $\frac{\text{sample size}}{\text{number of classes}} = \frac{N}{n}$

v : Degrees of freedom . for uniform distribution it equals $n - 1$

Step 1. Select the number of intervals n so that $E_i > 5$ (each group has to be of more than 5 samples)
Step 2. Perform the calculations according to the table above.
Step 3. Obtain the critical value $X_{0,v}^2$ from the table A.6 [See Banks et al., 2009, p. 617]
Step 4. If the calculated sample statistic X_0^2 is greater than the critical value obtained from table A.6 , the null hypothesis that the data are a sample from a uniform distribution is rejected.

Chapter 8

Random-Variate Generation

8.1 Inverse-Transform Technique

8.1.1 Exponential Distribution

Random-variate generator for the exponential distribution :

$$X_i = -\frac{1}{\lambda}\ln(1 - R_i)$$

The fact that both R_i and $1 - R_i$ are uniformly distributed on $[0, 1]$ can be employed to simplify the previous equation to :

$$X_i = -\frac{1}{\lambda}\ln R_i$$

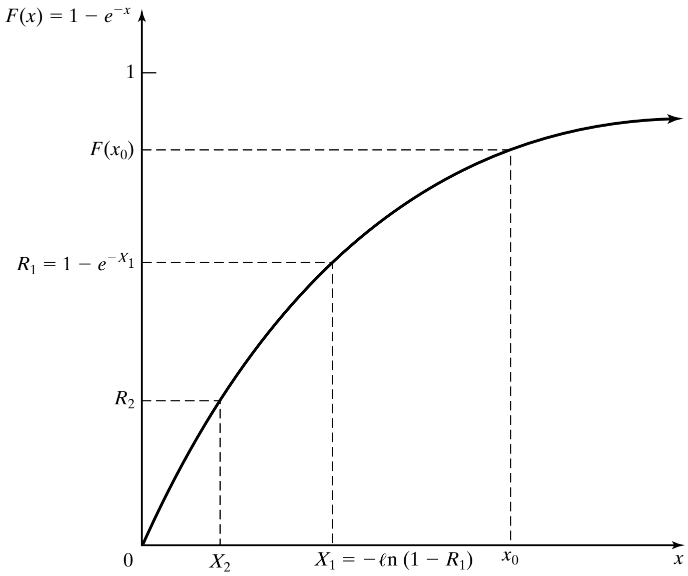


Figure 8.1: Graphical view of the inverse-transform technique
[courtesy of Banks et al., 2009]

The cdf shown is of an exponential distribution with rate $\lambda = 1$

8.1.2 Uniform Distribution

To generate a sequence of random numbers uniformly distributed on the interval $[a, b]$

$$X = a + (b - a)R$$

8.1.3 Geometric Distribution

$$X = \left\lceil \frac{\ln(1 - R)}{\ln(1 - p)} \right\rceil$$

Chapter 9

Input Modeling

9.1 Preliminary Statistics : Sample Mean and Sample Variance

- If the observations in a sample of discrete **ungrouped** data of size n are $X_1, X_2, \ldots, X_n$ then :

$$\textit{Sample Mean :} \qquad \widehat{\mu} = \bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \tag{9.1a}$$

$$= \frac{X_1 + X_2 + \ldots + X_n}{n} \tag{9.1b}$$

$$\textit{Sample Variance :} \qquad \widehat{\sigma} = S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1} \tag{9.2a}$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \tag{9.2b}$$

- If the data are discrete and have been **grouped in a frequency distribution** :

$$\textit{Sample Mean :} \qquad \widehat{\mu} = \bar{X} = \frac{\sum_{j=1}^{k} f_j X_j}{n} \tag{9.3}$$

$$\textit{Sample Variance :} \qquad \widehat{\sigma} = S^2 = \frac{\sum_{j=1}^{k} f_j X_j^2 - n\bar{X}^2}{n-1} \tag{9.4}$$

Where :

n : sample size.
k : the number of distinct groups of data.
X_j : the value of the jth group.
f_j : the observed frequency of the jth group.
S : the standard deviation.

9.2 Covariance and Correlation of Multivariate Input Models

Covariance between two random variables X_1 and X_2 :

$$\widehat{cov}(X_1, X_2) = \frac{1}{n-1}\sum_{j=1}^{n}(X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2) \tag{9.5a}$$

$$= \frac{1}{n-1}\left(\sum_{j=1}^{n} X_{1j}X_{2j} - n\bar{X}_1\bar{X}_2\right) \tag{9.5b}$$

Where $\bar{X}_1$ and $\bar{X}_2$ are the sample means.

The correlation is estimated by :

$$\widehat{\rho} = \frac{\widehat{cov}(X_1, X_2)}{\widehat{\sigma}_1\widehat{\sigma}_2} \tag{9.6}$$

Where $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are the sample variances.

[Refer to Banks et al., 2009, p. 388]

9.3 Suggested Maximum Likelihood Estimators(MLE)

Table 9.1: Suggested Estimators for Distributions Often Used in Simulation

Distribution	*Parameter(s)*	*Suggested Estimator(s)*
Poisson	λ	$\widehat{\lambda} = \bar{X}$
Exponential	λ	$\widehat{\lambda} = \dfrac{1}{\bar{X}}$
Normal	μ, σ^2	$\widehat{\mu} = \bar{X}$
		$\widehat{\sigma}^2 = S^2(unbiased)$

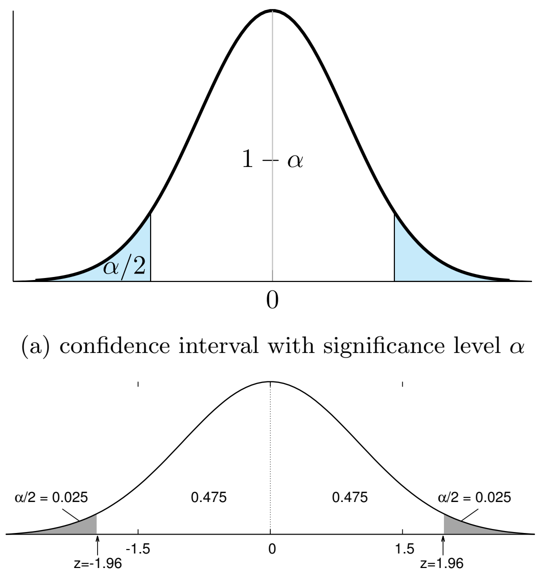
Chapter 11

Estimation of Absolute Performance(Output Analysis)

11.1 Confidence-Interval Estimation

Table 11.1: Confidence Intervals for some values of α

Tail Area($\alpha/2$)	α	*Confidence Level*	$Z_{\alpha/2}$
0.05	0.1	90%	1.645
0.025	0.05	95%	1.96
0.01	0.02	98%	2.33
0.005	0.01	99%	2.58



(b) confidence interval for $\alpha = 0.05$

Figure 11.1: confidence intervals

Table 11.2: Point Estimator, Standard Error, Margin of Error and Confidence Interval for Quantitative Population and Binomial Population

	Quantitative Population	Binomial Population
Point Estimator ($\widehat{\theta}$)	Point Estimator of Population Mean ($\widehat{\mu}$) : $\bar{X}$	Point Estimator of population proportion(p) : $\widehat{p} = \frac{x}{n}$ **Where :** x : number of successes. n : total number of trials.
Standard Error(SE)	$SE = \frac{S}{\sqrt{n}}$	$SE = \sqrt{\frac{\widehat{pq}}{n}}$
Margin of Error ($n \geq 30$) $= \pm 1.96 \times SE$	$\pm 1.96 \frac{S}{\sqrt{n}}$	$\pm 1.96 \sqrt{\frac{\widehat{pq}}{n}}$
$(1-\alpha)100\%$ confidence interval $= \widehat{\theta} \pm z_{\alpha/2} \times SE$	$\bar{X} \pm z_{\alpha/2} \frac{S}{\sqrt{n}}$	$\widehat{p} \pm z_{\alpha/2} \sqrt{\frac{\widehat{pq}}{n}}$

Bibliography

Jerry Banks, John S. Carson, Barry L. Nelson, and David M. Nicol. *Discrete-Event System Simulation:International Edition*. Pearson, 5th edition, August 2009.