College of Management of Technology

Derivatives, FIN-404

Oil futures and storage options

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Part 1. Documentation (☐ include links to sources)

Crude oil

Crude oil, formed from ancient organic matter under heat and pressure over millions of years, is a non-renewable resource refined into products like gasoline and diesel. Extracted via drilling, it significantly impacts the global economy through its pricing, driven by supply-demand dynamics and geopolitical factors. Historically crucial since the Industrial Revolution, major producers include the U.S., Saudi Arabia, and Russia. Notably, US oil imports have recently dropped to their lowest levels since 1985. ☐

Oil Future contracts

The official website of *IG Bank* ☐ defines futures in the following way :

> *A futures contract is a legal agreement between two parties to trade an asset at a predefined price, on a specific date in the future. Futures contracts are traded on exchanges and can be used to gain exposure to a variety of assets, such as commodities or indices. They are commonly referred to as 'futures'.*

In the case of oil, future contracts would then be a legal agreement to buy or sell a certain number of barrels of oil at a predetermined price, on a predetermined date. The agreement is signed and secured with a margin payment that covers a percentage of the total value of the contract. ☐
Forwards are also financial contracts closely related to futures, however, a futures contract is distinct from a forward contract in two important ways:

- A futures contract is a legally binding agreement to buy or sell a standardized asset on a specific date or during a specific month
- This kind of transactions are facilitated through a futures exchange ☐

Cash-flows and Payoff

When a trader takes a long position in a futures contract, an initial margin must be deposited with the broker, serving as collateral to cover potential losses. For example, if the initial margin requirement for a crude oil futures contract is \$5,000, this amount must be deposited to open the position. As the contract is marked-to-market daily, the trader's account will be adjusted based on changes in the settlement price. If the price increases by \$2 per barrel for a 1,000-barrel contract, the account is credited with \$2,000. Conversely, if the price drops by \$1 per barrel, the account is debited by \$1,000.
To avoid physical delivery and maintain market exposure, the trader may roll over the position as the contract nears maturity. This involves selling the current (near-month) contract and buying a new (next-month) contract. For instance, if the trader initially bought a July crude oil futures contract at \$70 per barrel and the price moves to \$71 per barrel before rolling over, the July contract is sold at \$71, realizing a gain of \$1,000. Then, the trader buys the August contract at the current market price, assume \$73 per barrel, and deposits the new margin requirement, potentially another \$5,000.
We can illustrate the process as the following :

- **Initial Margin:** -\$5,000
- **Day 1 Gain:** +\$2,000
- **Day 2 Loss:** -\$1,000
- **Final Settlement of July Contract:** +\$1,000

- **New Margin for August Contract:** -$5,000

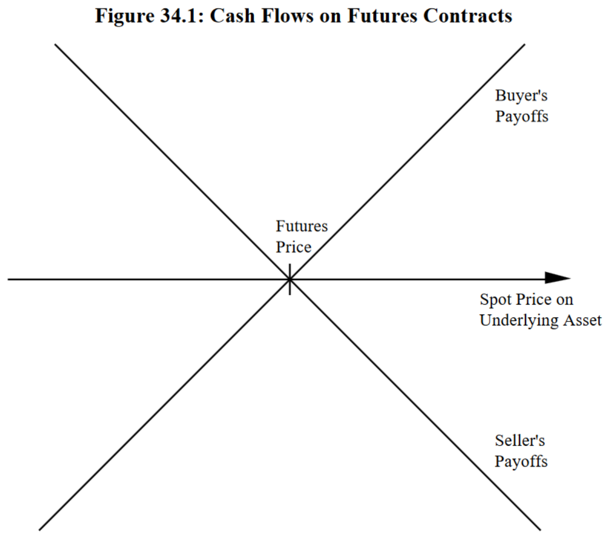


Figure 1: **Payoff related to the buyer and the seller of the future contract**

Existence and Importance to Investors

Oil futures exist to provide a standardized method for trading oil that facilitates market liquidity, price discovery, and risk management. Firstly, oil futures enable speculation on price movements, giving traders the opportunity to profit from fluctuations in oil prices without needing to own the physical commodity. Secondly, they serve as an effective tool for portfolio diversification, allowing investors to reduce overall risk by adding an asset with different performance characteristics compared to traditional stocks and bonds. Lastly, oil futures offer a means for hedging, protecting investors and companies against adverse price movements in related investments or operational costs, thereby stabilizing returns and managing financial risk in a volatile market.

Trading Volume and Open Interest

Volume represents the total number of contracts traded within a trading day, serving as a measure of market activity and liquidity. High volume indicates strong interest and can lead to better price discovery and narrower bid-ask spreads. Open interest (**OI**), on the other hand, represents the total number of outstanding futures contracts that have not been settled or closed by the end of the trading day. It measures the number of open positions in the market, reflecting ongoing interest. OI increases with the creation of new contracts and decreases when contracts are closed.

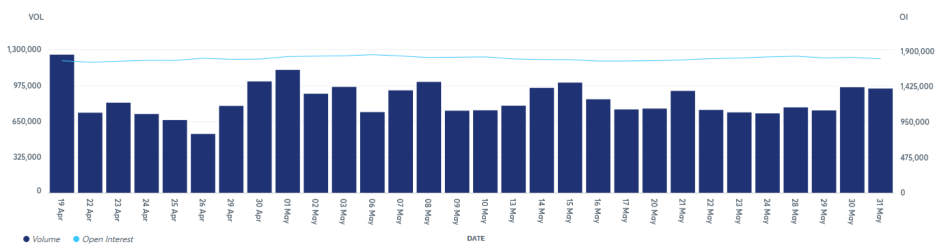


Figure 2: **OI and Volume**

Trading volumes in oil futures have generally increased over the years due to the growing importance of oil as a commodity and the increasing participation of financial players like hedge funds and institutional investors.

Open interest typically rises in conjunction with increased trading volumes, but can also provide insights into market sentiment. A rising open interest indicates new positions being added, while a falling open interest suggests positions are being closed.
Now, the evolution of trading volume and OI across different maturities will be addressed, highlighting the main sources of systematic seasonal variations in oil future contracts.

- **Short-term Contracts:** Higher trading volumes and open interest are often observed in near-month contracts due to their higher liquidity and lower risk compared to longer-dated contracts.
- **Long-term Contracts:** These tend to have lower trading volumes and open interest, but are crucial for hedging and for speculation based on long-term market outlooks.
- **Systematic Patterns:** Oil markets often exhibit seasonal variations due to predictable changes in supply and demand.
- **Winter Heating Season:** Higher demand for heating oil in the Northern Hemisphere during winter months can lead to increased trading volumes and open interest in heating oil futures.
- **Summer Driving Season:** Increased gasoline consumption during the summer travel season in the Northern Hemisphere often results in higher trading volumes and open interest in gasoline and crude oil futures.
- **Inventory Cycles:** Seasonal maintenance of refineries and changes in inventory levels can also contribute to seasonal trading patterns.
- **Holidays and Festivals:** Certain holidays and festivals (e.g. Chinese New Year, Diwali) can impact oil demand and futures trading activity.

Convenience Yield

The convenience yield is the benefit, or premium, associated with holding the physical commodity (such as oil) rather than holding a contract for future delivery. □
It is an adjustment to the cost of carrying the non-arbitrage pricing formula for forward prices in markets with trading constraints. It reflects the non-monetary advantages of owning the actual physical commodity.
Let $f_{t,T}$ be the future price of an asset with initial price S_t and maturity T. Suppose that r is the continuously compounded interest rate for one year. Then, the non-arbitrage pricing formula, including storage cost s, should be

$$f_{t,T} = S_t \cdot e^{(r+s)(T-t)}$$

However, this relationship does not hold in most commodity markets, partly because of the inability of investors and speculators to short the underlying asset S_t. Instead, there is a correction to the forward pricing formula given by the convenience yield c. Hence,

$$f_{t,T} = S_t \cdot e^{(r+s-c)(T-t)}$$

Therefore, the effect of a convenience yield is decreasing the futures prices. □

Backwardation and Contango

Contango and *Backwardation* are terms used to define the structure of the forward curve. □
When a market is in *backwardation*, the forward price of the futures contract is lower than the spot price. The resulting futures or forward curve would typically be downward sloping, i.e. "inverted", since contracts for further dates would typically trade at even lower prices. Hence,

$$f_{t,T} < S_t$$

It occurs for several reasons, such as

- **High Convenience Yield:** It occurs when holding the physical commodity provides significant benefits, such as immediate availability during times of high demand or supply disruptions.
- **Supply Shortages:** Anticipated short-term supply shortages can drive up the spot price as market participants are willing to pay a premium for immediate delivery.
- **Strong Current Demand:** It happens when current demand is strong relative to future expectations. It may signal a tight supply in the short term and can indicate that market participants expect prices to fall in the future due to potential supply increases or demand decreases.

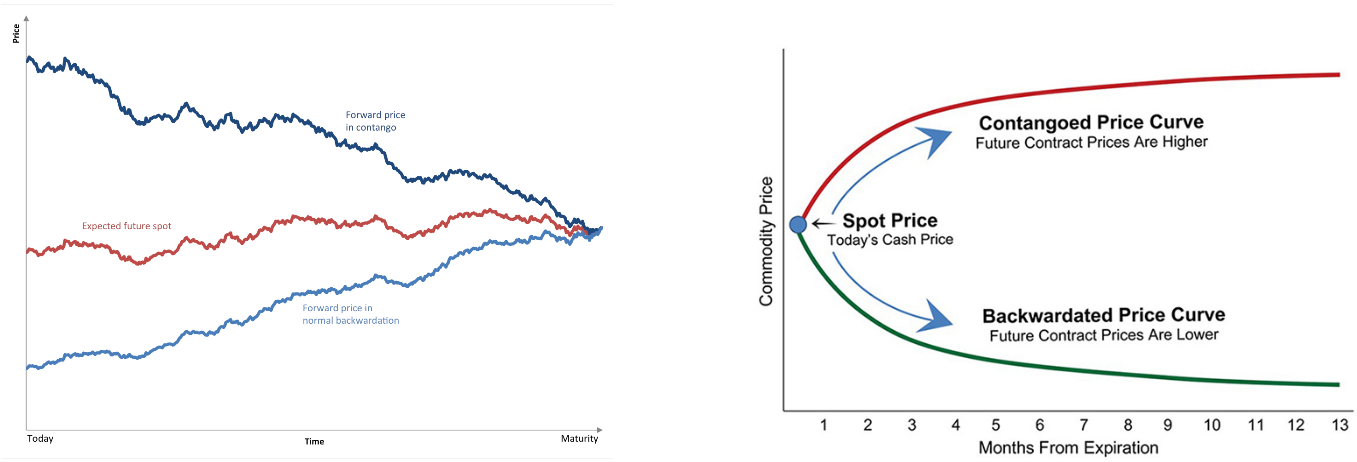
It may signal a tight supply in the short term and can indicate that market participants expect prices to fall in the future due to potential supply increases or demand decreases.
Conversely, when a market is in *contango*, the forward price of a futures contract is higher than the spot price. The futures, or forward curve, would typically be upward sloping, i.e., "normal", since contracts for further dates would typically trade at even higher prices. Hence,

$$f_{t,T} > S_t$$

It may occur due to the following reasons.

- **Lower Convenience Yield:** When the benefits of holding the physical commodity are minimal.
- **Storage Costs:** The cost of storing the commodity until the future delivery date.
- **Financing Costs:** The cost of financing the purchase of the commodity until delivery.

It may signal that there is sufficient supply or that future supplies are expected to be more abundant. This can indicate that market participants expect prices to rise in the future, possibly due to increasing demand or decreasing supply.



(a) Behaviour of Price of a Single Forward Contract in relation to the Expected Future Price

(b) Hypothetical Forward Curve

Figure 3: **Backwardation and Contango**

Today, oil prices are in *backwardation* as can be analyzed by checking lower futures prices compared to underlying crude oil spot prices for different tickers in CME. *Backwardation* has often occurred in the crude oil market, particularly over the past decade or so.
Some analysts point to the fracking and shale revolution as a cause of this. Smaller oil companies

often have limited balance sheets, and thus need to convince banks that they are creditworthy to obtain loans for new drilling operations.[1] In 2020, the market swung to sharp contango during the Covid-19 pandemic as the market became vastly oversupplied. Demand dried up virtually overnight while supply was harder to curtail. □

As the economy reopened, the oil market swung into backwardation once again. In late 2020, demand had started rising again as Covid restrictions eased, but production was returning slowly because of OPEC cuts and gradual growth from U.S. producers. It made sense for companies to produce oil and sell it as soon as possible. The value of oil investments surged in 2021 and 2022, with prices and stocks rising for most of that period. Oil companies posted record earnings in 2022. □

However, it's not hard to imagine a hypothetical new recession or other negative catalyst that would cause demand to drop sharply. Changes in the regulatory environment around fossil fuels or a big increase in the demand for electric vehicles could be other factors that could cause the curve to flip once again.

An unusual *contango* pattern was observed at the end of 2023, and was probably caused by the increasing popularity of exchange-traded funds that buy oil futures and an increase in oil speculators have changed how the commodity is traded. Companies like oil producers and refiners have been hedging less as prices have stayed high and refinery margins have been strong. So oil futures were used more for speculation. □

Thus, a futures curve controlled by speculators wasn't a real signal about supply and demand.

In the next section, we discuss the specifications of the 3 most liquid derivative contracts of every type (futures, options and micros) displayed on CME. □

Type of Oil Derivatives on CME

- **Futures:** Crude Oil Futures, WTI Houston (Argus) vs. WTI Trade Month Futures, and WTI Midland (Argus) vs. WTI Trade Month Futures are financial instruments traded on the CME platforms. All three contracts are based on 1,000 barrels and are quoted in U.S. dollars and cents per barrel.
  Each contract has a minimum price fluctuation of $0.01 per barrel, equivalent to $10.00. Crude Oil Futures contract is deliverable and includes Trading at Settlement (TAS) and Trading at Marker (TAM) rules. TAS trades off a "Base Price" of zero to create a differential versus the daily settlement price, while TAM uses a market price instead of the daily settlement price. The others two contracts are financially settled.

- **Options:** Crude oil options are essential tools for market participants and there are three key types of crude oil options traded on CME Group: Crude Oil Options, Crude Oil Financial Calendar Spread Option 1 Month, and WTI Average Price Option.
  Also these contracts follow the same price regulations listed for futures.
  Crude Oil Options is deliverable instead the other two contracts are financially settled.

- **Micros:** Micro-sized contracts offer traders the flexibility to manage crude oil price exposure with smaller margin requirements. These contracts are ideal for those looking for exposure to crude oil markets without the larger financial commitment required by standard contracts and they are Micro WTI Crude Oil Futures (1/10 size of WTI Crude Oil Futures) and E-mini Crude Oil Futures.

[1] Historically, oil producer hedging has had a major impact on the shape of the futures curve for that commodity.

Part 2. Analysis

Spot Price Modelling

1.

We have:

$$\frac{dO_t}{O_t} = (r_t - \delta_t)dt + \sigma_O^T dB_t^Q$$
$$dr_t = \lambda_r(r - r_t)dt + \sigma_r^T dB_t^Q$$
$$d\delta_t = \lambda_\delta(\delta - \delta_t)dt + \sigma_\delta^T dB_t^Q$$

Dividing by the norm of each volatility, we get:

$$\frac{dO_t}{O_t} = (r_t - \delta_t)dt + \frac{\sigma_O^T}{\|\sigma_O\|}\|\sigma_O\| dB_t^Q$$

$$dr_t = \frac{\sigma_r^T}{\|\sigma_r\|}\|\sigma_r\| dB_t^Q + \lambda_r(r - r_t)dt$$

$$d\delta_t = \frac{\sigma_\delta^T}{\|\sigma_\delta\|}\|\sigma_\delta\| dB_t^Q + \lambda_\delta(\delta - \delta_t)dt$$

Let's call:

$$dW_t^O = \frac{\sigma_O^T}{\|\sigma_O\|} dB_t^Q$$

$$dW_t^r = \frac{\sigma_r^T}{\|\sigma_r\|} dB_t^Q$$

$$dW_t^\delta = \frac{\sigma_\delta^T}{\|\sigma_\delta\|} dB_t^Q$$

Then we have:

$$\frac{dO_t}{O_t} = (r_t - \delta_t)dt + \|\sigma_O\| dW_t^O$$
$$dr_t = \|\sigma_r\| dW_t^r + \lambda_r(r - r_t)dt$$
$$d\delta_t = \|\sigma_\delta\| dW_t^\delta + \lambda_\delta(\delta - \delta_t)dt$$

Now we need to show that dW_t^O, dW_t^r, and dW_t^δ are indeed Brownian motions using Levy's characterisation theorem.
According to Levy's characterisation theorem, a continuous local martingale that starts at zero is a Brownian motion if its quadratic variation is t. We know that for $i, j \in \{O, r, \delta\}$:

$$d\langle W^i, W^j\rangle_t = \frac{\sigma_i^T \sigma_j}{\|\sigma_i\|\|\sigma_j\|} dt = \rho_{ij} dt$$

In particular, when $i = j$, we have:

$$d\langle W^i, W^i\rangle_t = \frac{\sigma_i^T \sigma_i}{\|\sigma_i\|\|\sigma_i\|} dt = dt$$

which shows that the quadratic variation is t.
Since $dW_t^i = \frac{\sigma_i^T}{\|\sigma_i\|} dB_t^Q$, the processes W_t^O, W_t^r, and W_t^δ are continuous local martingales that vanish at zero and have the correct quadratic variation. Therefore, by Levy's characterisation theorem, they are Brownian motions.

Next, we need to verify the correlation coefficients:

$$d\langle W^r, W^O\rangle_t = \rho_{rO} dt$$

$$d\langle W^r, W^\delta\rangle_t = \rho_{r\delta} dt$$

$$d\langle W^\delta, W^O\rangle_t = \rho_{\delta O} dt$$

By definition:

$$\rho_{ij} = \frac{\sigma_i^T \sigma_j}{\|\sigma_i\|\|\sigma_j\|}$$

By the Cauchy-Schwarz inequality, we know that:

$$|\rho_{ij}| \leq 1$$

Thus, these coefficients lie between -1 and 1, as required.

2.

The way to proceed in order to prove this expectation is first to find the dynamics of $\hat{O}_t$. We have by Itô's lemma:

$$d\left(\frac{O_t}{S_{0t}}\right) = \frac{dO_t}{S_{0t}} + O_t d\left(\frac{1}{S_{0t}}\right)$$

where $dS_{0t} = r_t S_{0t} dt$. Then:

$$d\left(\frac{O_t}{S_{0t}}\right) = \frac{dO_t}{S_{0t}} - \frac{O_t}{S_{0t}^2} dS_{0t}$$

Since:

$$dO_t = O_t\left((r_t - \delta_t)dt + \sigma_O^T dB_t^Q\right)$$

and

$$d\left(\frac{1}{S_{0t}}\right) = -\frac{1}{S_{0t}^2} dS_{0t} = -\frac{1}{S_{0t}^2} r_t S_{0t} dt = -\frac{r_t}{S_{0t}} dt$$

We get:

$$d\left(\frac{O_t}{S_{0t}}\right) = \frac{O_t}{S_{0t}}\left((r_t - \delta_t)dt + \sigma_O^T dB_t^Q\right) - \frac{O_t}{S_{0t}} r_t dt$$

Simplifying, we obtain:

$$d\left(\frac{O_t}{S_{0t}}\right) = \frac{O_t}{S_{0t}}\left(-\delta_t dt + \sigma_O^T dB_t^Q\right)$$

Thus:

$$d\left(\frac{O_t}{S_{0t}}\right) + \delta_t \frac{O_t}{S_{0t}} dt = \sigma_O^T \frac{O_t}{S_{0t}} dB_t^Q$$

This shows that the process $\left(\frac{O_t}{S_{0t}} + \int_0^t \delta_s \frac{O_s}{S_{0s}} ds\right)$ is a local martingale, but more specifically a martingale since:

$$\int_0^T \|\sigma_O\|^2 ds < \infty$$

Hence, for $t < T$:

$$\frac{O_t}{S_{0t}} + \int_0^t \delta_s \frac{O_s}{S_{0s}} ds = \mathbb{E}_t\left[\frac{O_T}{S_{0T}} + \int_0^T \delta_s \frac{O_s}{S_{0s}} ds\right]$$

Therefore:

$$\frac{O_t}{S_{0t}} = \mathbb{E}_t\left[\frac{O_T}{S_{0T}} + \int_t^T \delta_s \frac{O_s}{S_{0s}} ds\right]$$

Since:

$$S_{0s} = e^{\int_0^s r_u du}$$
$$S_{0t} = e^{\int_0^t r_u du}$$

Then:

$$\frac{S_{0t}}{S_{0s}} = e^{\int_s^t r_u du} = e^{-\int_t^s r_u du}$$

The final form is then:

$$O_t = \mathbb{E}_t^Q \left[\int_t^T e^{-\int_t^s r_u du} \delta_s O_s ds + e^{-\int_t^T r_u du} O_T \right]$$

The Gibson and Schwartz model is referred to as a stochastic convenience yield model because it incorporates a stochastic process for the convenience yield, δ_t. The convenience yield represents the non-monetary return from holding the physical commodity, such as ensuring availability, avoiding stockouts, and potentially benefiting from future price increases. This yield reflects the benefits of having immediate access to the commodity.
In the model, δ_t evolves randomly over time according to specified dynamics and the stochastic convenience yield directly influences the spot price of oil, as indicated by the expectation term in the equation, showing that $\mathcal{O}_t$ depends on the expected future values of the convenience yield, appropriately discounted.

3.

Equivalence of the Gabillon Model
We have:

$$\frac{dO_t}{O_t} = (r - \delta_t)dt + \sigma_O^T dB_t^Q$$

The goal would be to prove that.
We have:

$$\frac{dO_t}{O_t} = \phi \log\left(\frac{\ell_t}{O_t}\right) dt + \sigma_o^T dB_t^Q$$

subject to

$$\frac{d\ell_t}{\ell_t} = \mu_\ell dt + \sigma_\ell^T dB_t^Q$$

A natural way to proceed would be to consider $r - \delta_t = \bar{\Phi} \log\left(\frac{l_t}{O_t}\right)$ and for the volatility we have no other choice but to impose $\sigma_O^T = \sigma_o^T$ to prove the statement. Let us then start and let us find the dynamics of l_t. We have:

$$l_t = O_t \cdot e^{\frac{r-\delta_t}{\bar{\Phi}}}$$

for some constant $\bar{\Phi}$. Applying Itô's Lemma to l_t:

$$dl_t = (dO_t)\, e^{\frac{r-\delta_t}{\bar{\Phi}}} + O_t d\left(e^{\frac{r-\delta_t}{\bar{\Phi}}}\right) + d\left\langle O_t, e^{\frac{r-\delta_t}{\bar{\Phi}}}\right\rangle$$

$$= \left(O_t(r-\delta_t)dt + O_t\sigma_O^T dB_t^Q\right) e^{\frac{r-\delta_t}{\bar{\Phi}}} + O_t e^{\frac{r-\delta_t}{\bar{\Phi}}} \left(-\frac{1}{\bar{\Phi}}\lambda_\delta(\bar{\delta}-\delta_t)dt - \frac{1}{\bar{\Phi}}\sigma_\delta^T dB_t^Q + \frac{1}{2}\frac{1}{\bar{\Phi}^2}\|\sigma_\delta\|^2 dt\right) -$$
$$-\frac{1}{\bar{\Phi}} O_t e^{\frac{r-\delta_t}{\bar{\Phi}}} \sigma_O^T \sigma_\delta dt$$

Combining terms, we get:

$$dl_t = l_t \left((r-\delta_t) - \frac{1}{\bar{\Phi}}\lambda_\delta(\bar{\delta}-\delta_t) + \frac{1}{2}\frac{1}{\bar{\Phi}^2}\|\sigma_\delta\|^2 - \frac{1}{\bar{\Phi}}\sigma_O^T\sigma_\delta \right) dt + \left(\sigma_O - \frac{1}{\bar{\Phi}}\sigma_\delta\right) dB_t^Q$$

By setting $\bar{\Phi} = \lambda_\delta$, we simplify the equation:

$$dl_t = l_t \left(r - \bar{\delta} + \frac{1}{2} \frac{1}{\lambda_\delta^2} \|\sigma_\delta\|^2 - \frac{1}{\lambda_\delta} \sigma_O^T \sigma_\delta \right) dt + \left(\sigma_O - \frac{\sigma_\delta}{\lambda_\delta} \right) dB_t^Q$$

Thus, we identify the constants for the Gabillon model:

$$\mu_l = r - \bar{\delta} + \frac{1}{2} \frac{1}{\lambda_\delta^2} \|\sigma_\delta\|^2 - \frac{1}{\lambda_\delta} \sigma_O^T \sigma_\delta$$

$$\sigma_l = \sigma_O - \frac{\sigma_\delta}{\lambda_\delta}$$

Equivalence of the Schwartz and Smith Model

Given the initial model:

$$\frac{dO_t}{O_t} = (r - \delta_t)dt + \sigma_O^T dB_t^Q$$

and the dynamics for δ_t:

$$d\delta_t = \lambda_\delta(\bar{\delta} - \delta_t)dt + \sigma_\delta^T dB_t^Q$$

we aim to show the equivalence to the Schwartz and Smith model, which is given by:

$$\log O_t = x_t + \ell_t$$

with:

$$dx_t = \mu_x dt + \sigma_x^T dB_t^Q$$
$$d\ell_t = \lambda_\ell(\bar{\ell} - \ell_t)dt + \sigma_\ell^T dB_t^Q$$

We express O_t in terms of x_t and ℓ_t:

$$O_t = e^{x_t + \ell_t}$$

Applying Itô's Lemma to O_t:

$$dO_t = O_t(dx_t + d\ell_t + \frac{1}{2} d\langle x_t + \ell_t \rangle)$$

We find the dynamics for x_t and ℓ_t:

$$dx_t = \mu_x dt + \sigma_x^T dB_t^Q$$

$$d\ell_t = \lambda_\ell(\bar{\ell} - \ell_t)dt + \sigma_\ell^T dB_t^Q$$

Next, we compute the quadratic covariations:

$$d\langle x_t \rangle = \sigma_x^T \sigma_x dt$$

$$d\langle \ell_t \rangle = \sigma_\ell^T \sigma_\ell dt$$

$$d\langle x_t, \ell_t \rangle = \sigma_x^T \sigma_\ell dt$$

Combining the dynamics, we have:

$$dO_t = O_t \left((\mu_x dt + \sigma_x^T dB_t^Q) + (\lambda_\ell(\bar{\ell} - \ell_t)dt + \sigma_\ell^T dB_t^Q) + \frac{1}{2}(\sigma_x^T \sigma_x + \sigma_\ell^T \sigma_\ell + 2\sigma_x^T \sigma_\ell)dt \right)$$

Simplifying the equation, we get:

$$dO_t = O_t \left((\mu_x + \lambda_\ell(\bar{\ell} - \ell_t) + \frac{1}{2}(\sigma_x^T \sigma_x + \sigma_\ell^T \sigma_\ell + 2\sigma_x^T \sigma_\ell))dt + (\sigma_x^T + \sigma_\ell^T)dB_t^Q \right)$$

Since the parts that depend on t in the drift must be equal for the models to be equivalent, we set:

$$\lambda_\ell \ell_t = \delta_t$$

Hence, we can express ℓ_t as:

$$\ell_t = \frac{\delta_t}{\lambda_\ell}$$

Substituting $\ell_t = \frac{\delta_t}{\lambda_\ell}$ into the differential, we get:

$$d\ell_t = \frac{1}{\lambda_\ell} d\delta_t$$

Given the dynamics for δ_t:

$$d\delta_t = \lambda_\delta(\bar{\delta} - \delta_t)dt + \sigma_\delta^T dB_t^Q$$

We substitute this into the expression for $d\ell_t$:

$$d\ell_t = \frac{1}{\lambda_\ell}\left(\lambda_\delta(\bar{\delta} - \delta_t)dt + \sigma_\delta^T dB_t^Q\right)$$

Simplifies to:

$$d\ell_t = \frac{\lambda_\delta}{\lambda_\ell}(\bar{\delta} - \delta_t)dt + \frac{\sigma_\delta^T}{\lambda_\ell} dB_t^Q$$

To match the form, we impose:

$$\lambda_\ell = \lambda_\delta$$

Then, we have:

$$d\ell_t = (\bar{\delta} - \ell_t)dt + \frac{\sigma_\delta^T}{\lambda_\delta} dB_t^Q$$

Since $\ell = \frac{\bar{\delta}}{\lambda_\delta}$, we define:

$$\bar{\ell} = \frac{\bar{\delta}}{\lambda_\delta}$$

We also have:

$$\sigma_\ell^T = \frac{\sigma_\delta^T}{\lambda_\delta}$$

The dynamics of O_t then become:

$$dO_t = O_t\left(\mu_x + \lambda_\delta\left(\frac{\bar{\delta}}{\lambda_\delta} - \frac{\delta_t}{\lambda_\delta}\right) + \frac{1}{2}(\sigma_x^T\sigma_x + \sigma_\ell^T\sigma_\ell + 2\sigma_x^T\sigma_\ell)\right)dt + (\sigma_x^T + \sigma_\ell^T)dB_t^Q$$

Simplifying, we get:

$$dO_t = O_t\left(\mu_x + (\bar{\delta} - \delta_t) + \frac{1}{2}(\sigma_x^T\sigma_x + \sigma_\ell^T\sigma_\ell + 2\sigma_x^T\sigma_\ell)\right)dt + (\sigma_x^T + \sigma_\ell^T)dB_t^Q$$

Since $\sigma_\ell^T = \frac{\sigma_\delta^T}{\lambda_\delta}$:

$$\sigma_\ell^T = \frac{\sigma_\delta^T}{\lambda_\delta}$$

And combining $\sigma_x^T + \sigma_\ell^T$ must equal σ_O^T:

$$\sigma_x^T + \frac{\sigma_\delta^T}{\lambda_\delta} = \sigma_O^T$$

Therefore, the constants are identified as:

$$\mu_x = r - \bar{\delta} - \frac{1}{2}(\sigma_x^T \sigma_x + \sigma_\ell^T \sigma_\ell + 2\sigma_x^T \sigma_\ell)$$

$$\sigma_x^T + \frac{\sigma_\delta^T}{\lambda_\delta} = \sigma_O^T$$

Conclusion:

- $\mu_x = r - \bar{\delta} - \frac{1}{2}(\sigma_x^T \sigma_x + \sigma_\ell^T \sigma_\ell + 2\sigma_x^T \sigma_\ell)$
- $\lambda_\ell = \lambda_\delta$ to ensure mean-reverting terms match
- $\bar{\ell} = \frac{\bar{\delta}}{\lambda_\delta}$ to match long-term mean levels
- $\sigma_\ell^T = \frac{\sigma_\delta^T}{\lambda_\delta}$ to ensure the volatility terms match
- The total volatility $\sigma_O^T = \sigma_x^T + \sigma_\ell^T$ matches the given model

This completes the characterization of all constants for the models to be equivalent.

Bond Pricing

4.

We know that the risk free rate follows:

$$dr_t = \lambda_r \ (\bar{r} - r_t)d_t + \sigma_r^T \ dB_t^{\mathbb{Q}}$$

And the solution to this SDE is:

$$r_s = e^{-\lambda_r(s-t)} r_t + \bar{r}(1 - e^{-\lambda_r(s-t)}) + \int_t^s e^{-\lambda_r(s-u)} \sigma_r^T dB_u^{\mathbb{Q}}$$

At time t, r_t is known so the first two terms in the aforementioned equation are constant and the third term is a deterministic integral against the brownian motion so it follows a gaussian distribution. Hence, r_s follows a gaussian distribution and we can conclude that also $\int_t^T r_s d_s$ follows a gaussian distribution conditioned on t. We also see that since for all s, r_s only depends on r_t and the trajectory of the brownian motion (it is a markov process), $\int_t^T r_s d_s$ also only depends on r_t and $T - t$.
The price of a zero-coupon bond with maturity T is a derivative that pays $H = B_T(T) = 1$ at date T. So the price of this derivative is:

$$B_t(T) = \mathbb{E}_t^{\mathbb{Q}} \left[\frac{S_{0t}}{S_{0T}} B_T(T) \right] = \mathbb{E}_t^{\mathbb{Q}} \left[\frac{S_{0t}}{S_{0T}} \right] = \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r_s d_s} \right]$$

We showed that $\int_t^T r_s d_s$ only depends on r_t and $T - t$ so we can write $B_t(T) = \beta(T - t, r_t)$.
Next, we have that:

$$e^{-\int_0^t r_s d_s} \beta(T - t, r_t) = \mathbb{E}_t^{\mathbb{Q}} [e^{-\int_0^T r_s d_s}]$$

Which is a Levy martingale process under the EMM $\mathbb{Q}$. The boundary condition is $\beta(0, r) = 1$ because at the time to maturity, independent of the amount of the interest rate, the bond price is equal to one.
Since $e^{-\int_0^t r_s d_s} \beta(T - t, r_t)$ is a martingale, if we apply the Ito's lemma, the drift should be zero. This condition will give a PDE to solve (we replace $T - t$ with τ):

$$-\frac{\partial \beta}{\partial \tau} + \lambda_r(\bar{r} - r_t)\frac{\partial \beta}{\partial r} + \frac{1}{2}\|\sigma_r\|^2 \frac{\partial^2 \beta}{\partial^2 r} = r_t \beta \qquad s.t \qquad \beta(0, r) = 1$$

5.

We make a guess that $\beta(\tau, r_t) = e^{b_0(\tau)+b_r(\tau)r_t}$. After subsituting this into the PDE we get:

$$-b_0'(\tau) + \frac{1}{2}\|\sigma_r\|^2 b_r(\tau)^2 + \lambda_r \bar{r} b_r(\tau) = r_t(1 + \lambda_r b_r(\tau) + b_r'(\tau))$$

The equation above must hold for all $r_t \in \mathbb{R}$ (since r_t follows a normal distribution it can take all real values.) and $\tau \geq 0$ and Hence, using the separation of variables argument, both sides of the equation must be equal to zero which gives us a system of ODEs:

$$b_r'(\tau) = -1 - \lambda_r b_r(\tau), \qquad b_0'(\tau) = \lambda_r \bar{r} b_r(\tau) + \frac{1}{2}\|\sigma_r\|^2 b_r(\tau)^2$$

Now we need the boundary conditions. We know that:

$$\begin{aligned}
&\beta(0, r_t) = e^{b_0(0)+b_r(0)r_T} = 1 \quad \forall r_T \in \mathbb{R} \\
&\to - b_0(0) - b_r(0)r_T = 0 \quad \forall r_T \in \mathbb{R} \\
&\to \quad b_0(0) = 0, b_r(0) = 0
\end{aligned}$$

6.

The closed form solution for b_0 and b_r from are:

$$\begin{aligned}
b_r[\tau] &= \frac{e^{-\lambda_r \tau} - 1}{\lambda_r} \\
b_0[\tau] &= \frac{1}{4\lambda_r^3}(-e^{-2\lambda_r \tau}(e^{\lambda_r \tau} - 1)((3e^{\lambda_r \tau} - 1)\|\sigma_r\|^2 - 4e^{\lambda_r \tau}\bar{r}\lambda_r^2) + 2\lambda_r \tau(\|\sigma_r\|^2 - 2\bar{r}\lambda_r^2))
\end{aligned}$$

Futures Pricing

7.

We know that the futures price is a martingale under $\mathbb{Q}$. So we have:

$$f_t(T) = \mathbb{E}_t^{\mathbb{Q}}[f_T(T)] = \mathbb{E}_t^{\mathbb{Q}}[O_T]$$

Both δ and r solve an autonomous PDE, so both are Markov processes. We also have that the drift of O depends on (O, r, δ) so we can conclude that (O_t, r_r, δ_t) is a Markov process. So we have that $f_t = \mathbb{E}_t^{\mathbb{Q}}[O_T] = f(t, r_t, \delta_t, O_t)$. Since the futures price is a $\mathbb{Q}$-martingale, the drift of it would be equal to zero:

$$\begin{aligned}
0 = \frac{\partial f}{\partial t} + O(r - \delta)\frac{\partial f}{\partial O} + \lambda_r(\bar{r} - r)\frac{\partial f}{\partial r} + \lambda_\delta(\bar{\delta} - \delta)\frac{\partial f}{\partial \delta} + \frac{1}{2}O^2\|\sigma_O\|^2\frac{\partial^2 f}{\partial O^2} + \frac{1}{2}\|\sigma_\delta\|^2\frac{\partial^2 f}{\partial \delta^2} + \frac{1}{2}\|\sigma_r\|^2\frac{\partial^2 f}{\partial r^2} + \\
+ O\sigma_O^T\sigma_r\frac{\partial^2 f}{\partial O \partial r} + O\sigma_O^T\sigma_\delta\frac{\partial^2 f}{\partial O \partial \delta} + \sigma_r^T\sigma_\delta\frac{\partial^2 f}{\partial r \partial \delta}
\end{aligned}$$

such that $f(T, r, \delta, O) = O$.

8.

We make an initial guess that:

$$f_t = f(t, r_t, \delta_t, O_t) = e^{\phi_0(T-t)+\phi_r(T-t)r_t+\phi_\delta(T-t)\delta_t}O_t$$

We substitute the guess inside the PDE, divide the PDE by $f(t, r, \delta, o)$ (for more simplification), and replace t with $T - \tau$ to reach:

$$\frac{1}{2}(\|\sigma_r\|^2\phi_r(\tau)^2 + 2\phi_\delta(\tau)(\lambda_\delta(\overline{\delta} - \delta) + \sigma_O^T\sigma_\delta) + \|\sigma_\delta\|^2\phi_\delta(\tau)^2 + 2\phi_r(\tau)(\lambda_r(\overline{r} - r) + \sigma_O^T\sigma_r) + \\ + 2\phi_r(\tau)\phi_\delta(\tau)\sigma_r^T\sigma_\delta - 2(-r + \delta + \phi_0\prime(\tau) + r\phi_r\prime(\tau) + \delta\phi_\delta\prime(\tau)))$$

The equation above must hold for all $r_t, \delta_t \in \mathbb{R}$ and $\tau \geq 0$. Hence, using the separation of variables argument, the coefficient of r and δ in the equation above should be zero resulting in the following ODE's:

$$1 - \lambda_r\phi_r(\tau) - \phi_r\prime(\tau) = 0$$
$$-1 - \lambda_\delta\phi_\delta(\tau) - \phi_\delta\prime(\tau) = 0$$

Now we need the boundary conditions. We know that $f(T, r_T, \sigma_T, O_T) = O_T$ and so we have:

$$e^{\phi_0(0)+\phi_r(0)r_t+\phi_\delta(0)\delta_t} = 1 \quad \forall\delta_T, r_T \in \mathbb{R}$$
$$\rightarrow \phi_0(0) + \phi_r(0)r_t + \phi_\delta(0)\delta_t = 0\forall\delta_T, r_T \in \mathbb{R}$$
$$\rightarrow \phi_0(0) = \phi_r(0) = \phi_\delta(0) = 0$$

9.

The closed form solution for ϕ_r and ϕ_δ are:

$$\phi_r(\tau) = \frac{1 - e^{-\lambda_r\tau}}{\lambda_r}$$

$$\phi_\delta(\tau) = \frac{e^{-\lambda_\delta\tau} - 1}{\lambda_\delta}$$

$$\phi_0(\tau) = \int_0^\tau \frac{1}{2}(\|\sigma_r\|^2\phi_r(u)^2 + 2\phi_\delta(u)(\lambda_\delta\overline{\delta} + \sigma_O^T\sigma_\delta) + \|\sigma_\delta\|^2\phi_\delta(u)^2 + \\ + 2\phi_r(u)(\lambda_r\overline{r} + \sigma_O^T\sigma_r) + 2\phi_r(u)\phi_\delta(u)\sigma_r^T\sigma_\delta)du$$

For the futures price process under $\mathbb{Q}$, we know that the drift is 0 and hence the process is:

$$df = \frac{\partial f}{\partial r}\sigma_r + \frac{\partial f}{\partial \delta}\sigma_\delta + \frac{\partial f}{\partial O}\sigma_O O dB^{\mathbb{Q}} = \phi_r(\tau)f\sigma_r + \phi_\delta(\tau)f\sigma_\delta + f\sigma_O dB^{\mathbb{Q}}$$

10.

If we assume that the interest rate and convenience yield processes are drift-only processes (deterministic processes), meaning that $\sigma_\delta = \sigma_r = 0$, then from the previous part we would have:

$$df = f\sigma_O dB^{\mathbb{Q}} \rightarrow \frac{df}{f} = \sigma_O dB^{\mathbb{Q}}$$

meaning that the return process of the futures price only depends on σ_O. The first thing that this process tells us is that irrespective of the maturity date, all futures have the same return (because τ does not appear in the process unlike the general case where the interest rate and convenience yield are not deterministic). This means that the order of the futures prices with different maturities would not change in time. We know that in a contango, we usually have an upward-sloping futures curve while in backwardation we typically have a downward-sloping curve. Since the shape of the curve would not change, a transition between these two states would not happen. On the other hand we have:

$$\frac{dO}{O} = (r_t - \delta_t)dt + \sigma_O dB^{\mathbb{Q}}$$

Hence, we could write:

$$\frac{dO}{O} = (r_t - \delta_t)dt + \frac{df}{f}$$

This means that the return on the spot price at time t is the return on the futures price plus $r_t - \delta_t$. We know that both r_t and δ_t have deterministic values at each time. For example, we know that with $\sigma_r = 0$ we have $r_t = e^{-\lambda_r t} r_0 + \bar{r}(1 - e^{-\lambda_r t})$. Hence, we can conclude that the sign of $r_t - \delta_t$ can only change a small number of times. For example if we assume that $r_t - \delta_t > 0$, then the return for the spot price would be bigger than the return of the futures price. This will eventually lead to a spot price meaning that contango could not occur after a specific time.
Looking at the data, we see evidence of many occurrences of backwardation and contango. For example in 29th July 2020, we had contango and in 28th September 2021, we had backwardation. Many other backwardation and contango have happened. Having many transitions, is against the deterministic assumption of interest rates and convenience yield which would limit the number of transitions.
To sum up, relative changes in convenience yield and interest rates fundamentally drive the transitions between backwardation and contango. A stochastic model that accounts for the random, time-varying nature of these parameters provides a realistic framework to explain these transitions.

11.

Assuming that the interest rate is constant (i.e. $\lambda_r = \|\sigma_r| = 0$), then solving the PDE again we would have:

$$\phi_r(\tau) = \tau$$

$$\phi_\delta(\tau) = \frac{-1 - e^{\lambda_\delta \tau}}{\lambda_\delta}$$

$$\phi_0(\tau) = \int_0^\tau \frac{1}{2}(2\phi_\delta(u)(\lambda_\delta \bar{\delta} + \sigma_O^T \sigma_\delta) + \|\sigma_\delta\|^2 \phi_\delta(u)^2)du$$

Hence, the futures price would be:

$$f_t(T) = e^{\phi_0(\tau) + \phi_\delta(\tau)\delta_t + r\tau} O_t$$

To find the relation between the implied and instantaneous yield, we set equal the price above and $f_t(T) = e^{(r - Y_t(T))\tau} O_t$. Equating the two prices we get:

$$\phi_0(\tau) + \phi_\delta(\tau)\delta_t = -\tau Y_t(T)$$
$$\rightarrow \frac{-1}{\tau}(\phi_0(\tau) + \phi_\delta(\tau)\delta_t) = Y_t(T)$$

In order to have $Y_t(T) = \delta_t$ we should have $\phi_0(\tau) = 0$ and $\phi_\delta(\tau) = -\tau$ which means $\phi_\delta\prime(\tau) = -1$. For $\phi_\delta(\tau)$ we have:

$$\phi_\delta\prime(\tau) = -1 - \lambda_\delta \phi_\delta \rightarrow -1 = -1 - \lambda_\delta \phi_\delta \rightarrow \lambda_\delta = 0$$

Now, by replacing $\lambda_\delta = 0$ into the equation for ϕ_0 we get:

$$\phi_0(\tau) = \frac{\tau^2}{6}(-3\sigma_O^T \sigma_\delta + \|\sigma_\delta\|^2 \tau)$$

To have $\phi_0(\tau) = 0$, we should have $\sigma_\delta = 0$. Hence, we showed that the drift and diffusion of the convenience yield is zero meaning that it is a constant process under $\mathbb{Q}$.

Storage Options

12.

Using the expectation formula derived for the spot oil price in **2.** and the definition of the storage price, we have that the storage option price is defined as:

$$P_{T_0} = \mathbb{E}^{\mathbb{Q}}_{T_0}\left[e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_1}\right] - \mathcal{O}_{T_0} - \mathbb{E}^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\xi_s ds\right]$$

$$= \mathbb{E}^{\mathbb{Q}}_{T_0}\left[e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_1}\right] - \mathbb{E}^{\mathbb{Q}}_{T_0}\left[e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_1}\right] - \mathbb{E}^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\delta_s\mathcal{O}_s ds\right] - \mathbb{E}^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\xi_s ds\right]$$

Thus it is equal to:

$$P_{T_0} = -\mathbb{E}^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\left(\delta_s\mathcal{O}_s + \xi_s\right) ds\right]$$

Then indeed

$$P^{+}_{T_0} = \max\left\{0, P_{T_0}\right\} = \max\left\{0, -\mathbb{E}^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\left(\xi_s + \delta_s\mathcal{O}_s\right) ds\right]\right\}$$

In this context, δ_t represents the convenience yield, which is the non-monetary return from holding the physical commodity, such as ensuring availability and avoiding stockouts.
The integral term, $\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u\, du}(\xi_s + \delta_s\mathcal{O}_s)\, ds$, combines the discounted future storage costs (ξ_s) and the convenience yield ($\delta_s\mathcal{O}_s$), with the discount factor $e^{-\int_{T_0}^{s} r_u\, du}$ bringing future costs to their present value.
The convenience yield $\delta_s\mathcal{O}_s$ reflects the additional value from holding the commodity, influencing the storage option's value. The max function ensures the option price is non-negative, reflecting the payoff structure of options. If the expected present value of these combined costs is negative (indicating the storage option is beneficial), the option is exercised.

13.

The oil price evolves as:

$$\frac{d\mathcal{O}_t}{\mathcal{O}_t} = (r_t - \delta_t)dt + \sigma_{\mathcal{O}}^{\top} dB_t^{\mathbb{Q}}$$

Then writing this SDE in an integral form we have:

$$\mathcal{O}_{T_1} = \mathcal{O}_{T_0} e^{\int_{T_0}^{T_1}(r_u-\delta_u)du - \frac{1}{2}\int_{T_0}^{T_1}\|\sigma_{\mathcal{O}}\|^2 du + \int_{T_0}^{T_1}\sigma_{\mathcal{O}}^{T} dB_u}$$

$$\mathcal{O}_{T_1} = \mathcal{O}_{T_0} e^{\int_{T_0}^{T_1}(r_u-\delta_u)du - \frac{1}{2}\int_{T_0}^{T_1}\|\sigma_{\mathcal{O}}\|^2 du + \int_{T_0}^{T_1}\sigma_{\mathcal{O}}^{T} dB_u}$$

So:

$$\mathcal{O}_{T_1} = \mathcal{O}_{T_0} e^{\int_{T_0}^{T_1}(r_u-\delta_u)du - \frac{1}{2}\int_{T_0}^{T_1}\|\sigma_{\mathcal{O}}\|^2 du + \int_{T_0}^{T_1}\sigma_{\mathcal{O}}^{T} dB_u} = \mathcal{O}_{T_0} e^{\int_{T_0}^{T_1}(r_u-\delta_u)du}\frac{Z_{T_1}}{Z_{T_0}}$$

where:

$$Z_t = e^{-\frac{1}{2}\int_0^t\|\sigma_{\mathcal{O}}\|^2 du + \sigma_{\mathcal{O}}^{T}(B_t - B_0)}$$

It remains to prove that it is a martingale: Applying Itô's Lemma we have:

$$dZ_t = Z_t\sigma_{\mathcal{O}}^{\top} dB_t$$

The process is a local martingale but using Novikov's condition it is a martingale. Then we have:

$$\mathbb{E}_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_1}\right] = \mathbb{E}_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_0}\frac{Z_{T_1}}{Z_{T_0}}\right]$$

Using Girsanov's Theorem, we indeed defined an equivalent probability measure to $\mathbb{Q}$ which is $\overline{\mathbb{Q}}$.

14.

Given the SDE under the new probability measure Q:

$$d\delta_t = \lambda_\delta(\bar{\delta} - \delta_t)dt + {\sigma_\delta}^T\sigma_O dt + \sigma_\delta^T dB_t^Q$$

We clearly see that δ_t is an autonomous SDE, hence δ_t is a Q-Markov process.
Then:

$$E_{T_0}^{Q}\left[e^{-\int_{T_0}^{T_1}\delta_u du}\right] = f(\Delta, \delta_{T_0})$$

Also:

$$e^{-\int_0^{T_0}\delta_u du} f(\Delta, \delta_{T_0}) = E_{T_0}^{Q}\left[e^{-\int_0^{T_1}\delta_u du}\right]$$

is a Lévy martingale, hence its drift must be zero. Applying Itô's Lemma to M_t:

$$dM_t = e^{-\int_0^{T_0}\delta_u du}\left(df + f\cdot d\left(e^{-\int_0^{T_0}\delta_u du}\right) + d\langle e^{-\int_0^{T_0}\delta_u du}, f\rangle\right)$$

Differentiating $e^{-\int_0^{T_0}\delta_u du}$:

$$d\left(e^{-\int_0^{T_0}\delta_u du}\right) = -\delta_t e^{-\int_0^{T_0}\delta_u du}dt$$

Applying Itô's Lemma to $f(\Delta, \delta_t)$:

$$df = \left(-\frac{\partial f}{\partial \Delta} + \lambda_\delta(\bar{\delta} - \delta_t)\frac{\partial f}{\partial \delta_t} + {\sigma_\delta}^T\sigma_O\frac{\partial f}{\partial \delta_t} + \frac{1}{2}\|\sigma_\delta\|^2\frac{\partial^2 f}{\partial \delta_t^2}\right)dt + \sigma_\delta\frac{\partial f}{\partial \delta_t}dB_t^Q$$

Combining these, we get:

$$dM_t = e^{-\int_0^{T_0}\delta_u du}\left(\left(-\frac{\partial f}{\partial \Delta} + \lambda_\delta(\bar{\delta} - \delta_t)\frac{\partial f}{\partial \delta_t} + {\sigma_\delta}^T\sigma_O\frac{\partial f}{\partial \delta_t} + \frac{1}{2}\|\sigma_\delta\|^2\frac{\partial^2 f}{\partial \delta_t^2}\right)dt - \delta_t f dt + \sigma_\delta\frac{\partial f}{\partial \delta_t}dB_t^Q\right)$$

For M_t to be a martingale, its drift must be zero:

$$-\frac{\partial f}{\partial \Delta} + \lambda_\delta(\bar{\delta} - \delta_t)\frac{\partial f}{\partial \delta_t} + {\sigma_\delta}^T\sigma_O\frac{\partial f}{\partial \delta_t} + \frac{1}{2}\|\sigma_\delta\|^2\frac{\partial^2 f}{\partial \delta_t^2} - \delta_t f = 0$$

Using the ansatz $f(\Delta, \delta_t) = e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_t}$ we replace it into the PDE and using the separation of variables argument we get the following system of ODE's:

$$\psi_\delta{\prime}(\Delta) = -1 - \lambda_\delta\psi_\delta(\Delta), \quad \psi_0(\Delta) = (\sigma_O^T\sigma_\delta + \bar{\delta}\lambda_\delta)\psi_\delta(\Delta) + \frac{1}{2}\|\sigma_\delta\|^2\psi_\delta(\Delta)^2$$

Following the same arguments as part 5, we get that the boundary conditions are:

$$\psi_\delta(0) = \psi_0(0) = 0$$

Solving these equations with Mathematica we get:

$$\psi_\delta(\Delta) = \frac{1}{\lambda_\delta}\left(e^{-\lambda_\delta\Delta} - 1\right)$$

$$\psi_0(\Delta) = \frac{1}{4\lambda_\delta^3}(-e^{-2\lambda_\delta\Delta}(e^{\lambda_\delta\Delta}-1)((3e^{\lambda_\delta\Delta}-1)\|\sigma_\delta\|^2-4e^{\lambda_\delta\Delta}\bar{\delta}\lambda_\delta^2-4e^{\lambda_\delta\Delta}\lambda_\delta\sigma_O^T\sigma_\delta)+2\lambda_\delta\Delta(\|\sigma_\delta\|^2-2\bar{\delta}\lambda_\delta^2-2\lambda_\delta\sigma_O^T\sigma_\delta))$$

15.

Let us solve the first part.
Assuming $\xi_s = \alpha O_s$, and by applying the theorem of Fubini, we have:

$$\mathbb{E}_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u\,du}\xi_s\,ds\right] = \mathbb{E}_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u\,du}\alpha O_s\,ds\right] = \alpha\int_{T_0}^{T_1}\mathbb{E}_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{s} r_u\,du}O_s\right]ds$$

Then we apply the equality found in **13.** by replacing T_1 with s:

$$\alpha\int_{T_0}^{T_1} O_{T_0}\mathbb{E}_{T_0}^{\overline{\mathbb{Q}}}\left[e^{-\int_{T_0}^{s}\delta_u\,du}\right]ds = \alpha O_{T_0}\int_{T_0}^{T_1} e^{\psi_0(s-T_0)+\psi_\delta(s-T_0)\delta_{T_0}}\,d(s-T_0)$$

As we can notice, the integrand depends on the difference $s-T_0$. We substitute $s-T_0$ with τ, which is independent of the integral bounds T_0 and T_1. Thus, we can replace the bounds with 0 and Δ:

$$\mathbb{E}_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u\,du}\xi_s\,ds\right] = \alpha O_{T_0}\int_0^{\Delta} e^{\psi_0(\tau)+\psi_\delta(\tau)\delta_{T_0}}\,d\tau$$

Let us now move to the second part.
The value of the storage option is given by:

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_u du}P_{T_0}^+\right]$$

Let's characterize $P_{T_0}^+$ using the new measure. We have:

$$\mathbb{E}_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\xi_s ds\right] + \mathbb{E}_{T_0}^{\mathbb{Q}}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\delta_s O_s ds\right]$$

$$= \alpha O_{T_0}\int_0^{\Delta} e^{\psi_0(z)+\psi_\delta(z)\delta_{T_0}}dz + \left(O_{T_0} - \mathbb{E}_{T_0}^{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r_u du}O_{T_1}\right]\right)$$

$$= \alpha O_{T_0}\int_0^{\Delta} e^{\psi_0(z)+\psi_\delta(z)\delta_{T_0}}dz + O_{T_0} - O_{T_0}\left(e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}}\right)$$

Where in the first equality we used the previous part of the question as well as **2.** that states that :

$$O_t = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_u\,du}\delta_s O_s\,ds + e^{-\int_t^T r_u\,du}O_T\right], \quad 0 \le t \le T < \infty.$$

Continuing from the last step we get:

$$O_{T_0}\left(\alpha\int_0^{\Delta} e^{\psi_0(z)+\psi_\delta(z)\delta_{T_0}}dz + 1 - e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}}\right)$$

Hence:

$$P_{T_0}^+ = -O_{T_0}\left(\alpha\int_0^{\Delta} e^{\psi_0(z)+\psi_\delta(z)\delta_{T_0}}dz + 1 - e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}}\right)$$

$$= O_{T_0}\left(e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}} - 1 - \alpha\int_0^{\Delta} e^{\psi_0(z)+\psi_\delta(z)\delta_{T_0}}dz\right)$$

Thus let us consider H as the function defined as the following:

$$H(\delta_{T_0})^+ = \left(e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}} - 1 - \alpha \int_0^\Delta e^{\psi_0(z)+\psi_\delta(z)\delta_{T_0}} dz\right)^+$$

Hence:

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_u du} P_{T_0}^+\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_u du} O_{T_0} \cdot H(\delta_{T_0})^+\right]$$

We can use the result from **13.**, where we defined an equivalent measure $\mathbb{Q}$, and apply the formula:

$$\mathbb{E}_t^{\mathbb{Q}}[Y] = \mathbb{E}_t^{\mathbb{P}}\left[\frac{Z_s(\theta)}{Z_t(\theta)} Y\right] \quad \forall t \leq s \leq T \text{ and } Y \in \mathcal{L}_s^0$$

Since $H^+(\delta_{T_0})$ is measurable at the terminal time, which means it belongs to $\mathcal{L}'_{T_0}$, we can use this identity, which yields the desired result. We then conclude:

$$O_t \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} \delta_u \, du} H(\delta_{T_0})^+ \frac{Z_t}{Z_{T_0}}\right] = O_t \mathbb{E}_t^{\overline{\mathbb{Q}}}\left[e^{-\int_t^{T_0} \delta_u \, du} H(\delta_{T_0})^+\right]$$

The price of a storage option depends on the spot price of oil, the convenience yield, and the storage costs and we can indeed remark that it doesn't depend on the risk free rate. The intuitive explanation is that when the storage cost is a fixed percentage of the spot price, it aligns with the price movements of oil, making the option's valuation largely independent of the risk-free rate.
This is because both the storage cost and the spot price are discounted similarly, and the convenience yield's stochastic nature ensures that the valuation reflects real market conditions, reducing the sensitivity to the risk-free rate.

16.

Let us recall the function H defined previously :

$$H(\delta) = \left(e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}} - 1 - \alpha \int_0^\Delta e^{\psi_0(z)+\psi_\delta(z)\delta_{T_0}} dz\right)$$

We aim to demonstrate that $H'(\delta) < \psi_\delta(\Delta) H(\delta)$. Let's recall that:

$$\psi_\delta(\tau) = \frac{1}{\lambda_\delta}\left(e^{-\lambda_\delta \tau} - 1\right)$$

Given that $\psi_\delta(0) = 0$ and $\psi_\delta(\tau) < 0$ for $\forall \tau > 0$, and also $\psi_\delta(\Delta) < \psi_\delta(\tau)$, we can assert:

$$\int_0^\Delta \psi_\delta(\tau) e^{\psi_0(\tau)+\psi_\delta(\tau)\delta_{T0}} d\tau \geq \int_0^\Delta \psi_\delta(\Delta) e^{\psi_0(\tau)+\psi_\delta(\tau)\delta_{T0}} d\tau$$

Therefore, we derive that:

$$H'(\delta) \leq \psi_\delta(\Delta)\left(e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta} - \alpha \int_0^\Delta e^{\psi_0(\tau)+\psi_\delta(\tau)\delta_{T0}} d\tau - 1\right) + \psi_\delta(\Delta)$$

Since $\psi_\delta(\Delta) \leq 0$, it follows that $H'(\delta) < \psi_\delta(\Delta) H(\delta)$. This inequality $H'(\delta) < \psi_\delta(\Delta) H(\delta)$ indicates that if $H(\delta) > 0$, then $H'(\delta)$ is less than a negative quantity, making $H'(\delta)$ negative. Thus, $H(\delta)$ is a decreasing function wherever it is positive.
Given that $H'(\delta)$ is negative where $H(\delta) > 0$, $H(\delta)$ must decrease until it reaches zero. Assuming $H(\delta)$ starts positive at some initial point, it will continue to decrease until it hits zero at some $\delta = \delta^*$ and will not become positive again beyond this point.

Since $H(\delta)$ decreases and reaches zero at δ^*, and does not increase thereafter due to the negativity of $H'(\delta)$, we conclude:

$$\{\delta \in \mathbb{R} : H(\delta) \geq 0\} = (-\infty, \delta^*)$$

We aim to compute the conditional expectation:

$$\mathbb{E}_t \left[e^{-\int_t^{t+\Delta} \delta_s \, ds} H(\delta_{t+\Delta}) \mathbf{1}\{\delta_{t+\Delta} \leq \delta^*\} \right]$$

where $\Delta = T_0 - t$.

Using Lemma 1, the process $\left(\delta_{t+\Delta}, \int_t^{t+\Delta} \delta_s \, ds\right)$ is Gaussian conditional on δ_t with a certain mean and variance-covariance matrix. The process for δ_t is:

$$d\delta_t = \lambda_\delta(\bar{\delta} - \delta_t)dt + {\sigma_\delta}^T \sigma_O dt + \sigma_\delta^T dB_t^Q$$

Comparing this to the generic form in Lemma 1:

$$dx_t = a(b - x_t)dt + \sigma dW_t$$

Using the identifications:

$$a = \lambda_\delta, \quad b = \bar{\delta} + \frac{\sigma_\delta^T \sigma_O}{\lambda_\delta}, \quad \sigma = \sigma_\delta$$

This means the process can be rewritten as:

$$d\delta_t = \lambda_\delta \left(\left(\bar{\delta} + \frac{\sigma_\delta^T \sigma_O}{\lambda_\delta} \right) - \delta_t \right) dt + \sigma_\delta^T dB_t^Q$$

The mean vector $m(\delta_t)$ we are looking dfor, conditional on δ_t, is given by:

$$m(\delta_t) = \mathbb{E}\left[\left(\begin{matrix} \delta_{t+\Delta} \\ \int_t^{t+\Delta} \delta_s \, ds \end{matrix} \right) \middle| \delta_t \right] = \left(\begin{matrix} e^{-\lambda_\delta \Delta} \delta_t + (1 - e^{-\lambda_\delta \Delta}) \left(\bar{\delta} + \frac{\sigma_\delta^T \sigma_O}{\lambda_\delta} \right) \\ \left(\bar{\delta} + \frac{\sigma_\delta^T \sigma_O}{\lambda_\delta} \right) \Delta + \frac{1 - e^{-\lambda_\delta \Delta}}{\lambda_\delta} \left(\delta_t - \left(\bar{\delta} + \frac{\sigma_\delta^T \sigma_O}{\lambda_\delta} \right) \right) \end{matrix} \right)$$

The variance-covariance matrix V we are looking for is given by:

$$V = \mathrm{Var}\left(\left(\begin{matrix} \delta_{t+\Delta} \\ \int_t^{t+\Delta} \delta_s \, ds \end{matrix} \right) \middle| \delta_t \right) = \left(\begin{matrix} \frac{\sigma_\delta^2 (1 - e^{-2\lambda_\delta \Delta})}{2\lambda_\delta} & \frac{\sigma_\delta^2 (1 - e^{-\lambda_\delta \Delta})^2}{2\lambda_\delta^2} \\ \frac{\sigma_\delta^2 (1 - e^{-\lambda_\delta \Delta})^2}{2\lambda_\delta^2} & \frac{\sigma_\delta^2 e^{-2\lambda_\delta \Delta} \left(e^{2\lambda_\delta \Delta} (2\lambda_\delta \Delta - 3) + 4 e^{\lambda_\delta \Delta} - 1 \right)}{2\lambda_\delta^3} \end{matrix} \right)$$

To compute $\mathbb{E}_t \left[e^{-\int_t^{t+\Delta} \delta_s \, ds} H(\delta_{t+\Delta}) \mathbf{1}\{\delta_{t+\Delta} \leq \delta^*\} \right]$, we use the joint PDF $f_{X_1,X_2}(x_1, x_2)$ of the normal distribution. Let:

$$X_1 = \int_t^{t+\Delta} \delta_s \, ds, \quad X_2 = \delta_{t+\Delta}$$

We aim to find:

$$\mathbb{E}_t \left[e^{-X_1} H(X_2) \mathbf{1}\{X_2 \leq \delta^*\} \right]$$

This conditional expectation is given by :

$$\mathbb{E}_t \left[e^{-X_1} H(X_2) \mathbf{1}\{X_2 \leq \delta^*\} \right] = \iint_{-\infty}^{\delta^*} e^{-x_1} H(x_2) f_{X_1,X_2}(x_1, x_2) \, dx_1 \, dx_2$$

The joint PDF $f_{X_1,X_2}(x_1, x_2)$ is:

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi \sqrt{V_{11} V_{22} - V_{12}^2}} \exp \left(-\frac{1}{2} \left(\begin{matrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{matrix} \right)^T V^{-1} \left(\begin{matrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{matrix} \right) \right)$$

By substituting μ and V with the derived expressions, we get:

$$\mathbb{E}_t\left[e^{-X_1}H(X_2)\mathbf{1}\{X_2 \leq \delta^*\}\right] = \iint_{-\infty}^{\delta^*} e^{-x_1}H(x_2)\frac{1}{2\pi\sqrt{V_{11}V_{22}-V_{12}^2}}$$

$$\times \exp\left(-\frac{1}{2}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}^T V^{-1}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}\right) dx_1dx_2$$

Thus, the final form of the conditional expectation integral is:

$$\mathbb{E}_t\left[e^{-\int_t^{t+\Delta}\delta_s\, ds}H(\delta_{t+\Delta})\mathbf{1}\{\delta_{t+\Delta} \leq \delta^*\}\right] =$$

$$\iint_{-\infty}^{\delta^*} e^{-x_1}H(x_2)\frac{1}{2\pi\sqrt{V_{11}V_{22}-V_{12}^2}} \times \exp\left(-\frac{1}{2}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}^T V^{-1}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}\right) dx_1dx_2$$

where μ, and V are defined as above.
And as a conclusion, we have :

$$\mathbb{E}_t^Q\left[e^{-\int_t^{T_0} r_u du}\mathcal{P}_{T_0}^+\right] = \mathcal{O}_t\mathbb{E}_t^{\tilde{Q}}\left[e^{-\int_t^{T_0}\delta_u du}H(\delta_{T_0})^+\right] =$$

$$\mathcal{O}_t\iint_{-\infty}^{\delta^*} e^{-x_1}H(x_2)\frac{1}{2\pi\sqrt{V_{11}V_{22}-V_{12}^2}} \times \exp\left(-\frac{1}{2}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}^T V^{-1}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}\right) dx_1dx_2$$

Which indeed only solely depends on δ_t, and O_t as stated in the question.
Another approach to calculate the storage option price could involve changing the probability measure to a risk-neutral measure specifically tailored for the convenience yield. By defining a new measure $\tilde{Q}$ such that the drift of δ_t under $\tilde{Q}$ becomes zero, we can simplify the stochastic differential equation. This transformation allows us to focus on the diffusion by transforming into a martingale to derive a more tractable form of the expected payoff which can lead to closed-form solutions or easier numerical integration.

17.

For this question let's assume that r is constant. Meaning that:

$$P_{T_0}^+ = \max\left\{0, -\mathbb{E}_{T_0}^Q\left[\int_{T_0}^{T_1} e^{r(T_0-s)}(\alpha O_{T_0} + \delta O_s)\, ds\right]\right\}$$

We repeat the same reasoning but keeping the interest rate constant, we have:

$$\mathbb{E}_{T_0}^Q\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^s r_u\, du}\xi_s\, ds\right] + \mathbb{E}_{T_0}^Q\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^s r_u\, du}\delta O_s\, ds\right]$$

$$= \alpha O_{T_0}\mathbb{E}_{T_0}^Q\left[\int_{T_0}^{T_1} e^{-r(T_0-s)}\, ds\right] + O_{T_0} - \mathbb{E}_{T_0}^Q\left[e^{-\int_{T_0}^{T_1} r_u\, du}O_{T_1}\right]$$

Then:

$$= \alpha O_{T_0}\mathbb{E}_{T_0}^Q\left[\int_{T_0}^{T_1} e^{-r(T_0-s)}\, ds\right] + O_{T_0} - \mathbb{E}_{T_0}^Q\left[e^{-\int_{T_0}^{T_1} r_u\, du}\right]$$

$$= \alpha O_{T_0}\left[-e^{-r(T_0-s)}\right]_{T_0}^{T_1} + O_{T_0} - \mathbb{E}_{T_0}^Q\left[e^{-r(T_0-s)}\right]$$

Thus:

$$= \alpha O_{T_0}\left[-e^{-r(T_0-T_1)} + 1\right] + O_{T_0} - O_{T_0}e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}}$$

Then:

$$P_{T_0}^{+} = O_{T_0}\left[e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}} - 1 - \alpha e^{r(T_0-T_1)}\right]$$

As previously, let us again define : $\left[e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}} - 1 - \alpha e^{r(T_0-T_1)}\right]$ as our function $H(\delta_{T_0})$. Since $\psi_\delta(\Delta)$ is a negative function, then $H(\delta_{T_0})$ defined as the factor function after O_{T_0} is indeed decreasing. Using Lemma 1 we have:

$$\left(\delta_{T_0}, \int_t^{t+\Delta} \delta_s \, ds\right) \text{ is conditionally Gaussian on } \delta_t$$

Exactly as the previous question, when moving to the other equivalent measure $\overline{Q}$ we have then:

$$\mathbb{E}_t^{\overline{Q}}\left[e^{-\int_t^{T_0}\delta_u \, du}\overline{H}(\delta_{T_0})^{+}\right] = \mathbb{E}_t^{\overline{Q}}\left[e^{-\int_t^{T_0}\delta_u \, du}\overline{H}(\delta_{T_0})\mathbf{1}\{\delta_{T_0} < \delta^*\}\right]$$

However, this time we can easily compute our threshold, let us determine δ^*:

$$e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}} > 1 + \alpha - \alpha e^{r(T_0-T_1)}$$

$$\iff \delta_{T_0} < \frac{\ln\left(1+\alpha-\alpha e^{r(T_0-T_1)}\right) - \psi_0(\Delta)}{\psi_\delta(\Delta)}$$

because $T_1 > T_0$ to have the ln is well defined. Keeping again the same reasoning of **16.**. We aim to compute the expectation:

$$\mathbb{E}_t\left[e^{-X_1}\overline{H}(X_2)\mathbf{1}\{X_2 \le \delta^*\}\right] = \iint_{-\infty}^{\delta^*} e^{-x_1}\overline{H}(x_2) f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

where the vector: (X_1, X_2) is a gaussian process under our conditioning at time t and the joint PDF $f_{X_1,X_2}(x_1,x_2)$ using the same notations of **16.** is:

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sqrt{V_{11}V_{22}-V_{12}^2}} \exp\left(-\frac{1}{2}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix}^T V^{-1}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix}\right)$$

The fact that we can compute the expectation that way using an integral is justified by the fact that the transformative function of the process is a measurable and continuous function with respect to our sigma algebra condition on t.
By substituting μ and V that we used in **16.**, we get:

$$\mathbb{E}_t\left[e^{-X_1}\overline{H}(X_2)\mathbf{1}\{X_2 \le \delta^*\}\right] = \iint_{-\infty}^{\delta^*} e^{-x_1}\overline{H}(x_2)\frac{1}{2\pi\sqrt{V_{11}V_{22}-V_{12}^2}}$$

$$\times \exp\left(-\frac{1}{2}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix}^T V^{-1}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix}\right) dx_1\, dx_2$$

Thus, the final form of the conditional expectation integral is:

$$\mathbb{E}_t\left[e^{-\int_t^{t+\Delta}\delta_s \, ds}\overline{H}(\delta_{t+\Delta})\mathbf{1}\{\delta_{t+\Delta} \le \delta^*\}\right] =$$

$$\iint_{-\infty}^{\frac{\ln(1+\alpha-\alpha e^{r(T_0-T_1)})-\psi_0(\Delta)}{\psi_\delta(\Delta)}} e^{-x_1}\overline{H}(x_2)\frac{1}{2\pi\sqrt{V_{11}V_{22}-V_{12}^2}}\times\exp\left(-\frac{1}{2}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix}^T V^{-1}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix}\right) dx_1 dx_2$$

We see that indeed the integral only depends on where μ_1, μ_2, and V are defined as above **16.**
Hence as a conclusion :

$$\mathbb{E}_t^Q\left[e^{-\int_t^{T_0} r_u\,du}\mathcal{P}_{T_0}^+\right] = \mathcal{O}_t\mathbb{E}_t^Q\left[e^{-\int_t^{T_0}\delta_u\,du}\overline{H}(\delta_{T_0})^+\right] =$$

$$= \mathcal{O}_t\iint_{-\infty}^{\frac{\ln(1+\alpha-\alpha e^{r(T_0-T_1)})-\psi_0(\Delta)}{\psi_\delta(\Delta)}} e^{-x_1}\overline{H}(x_2)\frac{1}{2\pi\sqrt{V_{11}V_{22}-V_{12}^2}}\times\exp\left(-\frac{1}{2}\begin{pmatrix}x_1-\mu_1\\x_2-\mu_2\end{pmatrix}^T V^{-1}\begin{pmatrix}x_1-\mu_1\\x_2-\mu_2\end{pmatrix}\right)dx_1dx_2$$

This integral this time has an explicit $\delta*$ and also the new function $\overline{H}$ do not depend on an integral anymore and is a fully explicit function, hence the whole integral is defined and depends on δ_t, and O_t. It would be harder to solve with a stochastic risk-free rate and so without a fixed one because some steps would not have been possible, a major one would be the following:

$$\begin{aligned}
&\alpha O_{T_0}\mathbb{E}_{T_0}^Q\left[\int_{T_0}^{T_1} e^{-r(T_0-s)}\,ds\right] + O_{T_0} - \mathbb{E}_{T_0}^Q\left[e^{-\int_{T_0}^{T_1} r_u\,du}O_{T_1}\right]\\
&= \alpha O_{T_0}\mathbb{E}_{T_0}^Q\left[\int_{T_0}^{T_1} e^{-r(T_0-s)}\,ds\right] + O_{T_0} - \mathbb{E}_{T_0}^Q\left[e^{-\int_{T_0}^{T_1} r_u\,du}\right]\\
&= \alpha O_{T_0}\left[-e^{-r(T_0-s)}\right]_{T_0}^{T_1} + O_{T_0} - \mathbb{E}_{T_0}^Q\left[e^{-r(T_0-s)}\right]
\end{aligned}$$

And then we would not characterize the function $\overline{H}$ explicitly, which also leads to the fact that we would not be able to fully characterize $\delta*$

Part 3. Calibration and Implementation

Calibration

18.

Given the equations that we have in section , we have:

$$Y_0(\tau) = \frac{-1}{\tau}(b_0(\tau) + b_r(\tau)r_0)$$

The left-hand side of the equation is the yield for zero coupon bonds that we have in our dataset. First, we do a grid search on different values of r_0, λ_r, and $\overline{r}$. For each set of parameters, we calculate the right-hand side of the equation for all values of τ. We choose the set of parameters that leads to the smallest mean absolute percentage error (MAPE):

$$min_{\lambda_r,\overline{r},r_0} 100\times\frac{1}{n}\sum_{i=0}^{n}\left|\frac{Y_0(\tau_i)-\hat{Y}_0(\tau_i)}{Y_0(\tau_i)}\right|$$

where n is the number of different zero-coupon bonds.
When tuning parameters through grid search, it is essential to follow a systematic and iterative approach to refine the search space effectively. We started with a broad range for each parameter to get a general sense of where the optimal values might lie. Once the initial search was complete, we analyzed the results to identify the regions where the MAPE, indicated potential optimal values. Then we narrowed down the ranges step by step and performed the next rounds of grid search with finer intervals. For example, if in a step we find the optimal values on the boundary of our grid search, we have to expand the search in that direction.
Second, we used an optimizer and set the initial guess to the parameters found in the grid search to find the optimal set of parameters. Choosing bad initial guesses could result in bad results. Hence, the first step (grid search) was essential in our approach.

	λ_r	r_0	$\bar{r}$	**MAPE**	**MSE**	**MAE**
Value	0.2593	0.0145	0.0098	0.2547	2.0654×10^{-9}	3.3533×10^{-5}

Table 1: **Final Parameters and Accuracies achieved**

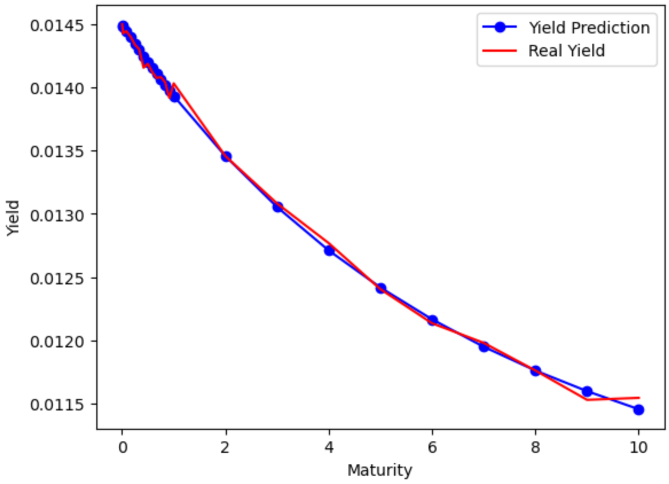


Figure 4: **Real Yield vs Predicted Yield**

19.

First, we will see what the quadratic variation matrix tells us. We know that for 2 processes A_t and B_t with constant diffusions (as in our case) σ_A and σ_B, we have:

$$< A, B >_t = \int_0^t \sigma_A^T \sigma_B dt = \sigma_A^T \sigma_B t$$

We also know that the diffusion for the $log(O_t)$ process is $\frac{1}{O_t} \times \sigma_O O_t = \sigma_O$. Given these two facts, the quadratic variation matrix for X_t is:

$$t \times \begin{bmatrix} \|\sigma_r\|^2 & \sigma_r^T \sigma_\delta & \sigma_r^T \sigma_O \\ \sigma_r^T \sigma_\delta & \|\sigma_\delta\|^2 & \sigma_\delta^T \sigma_O \\ \sigma_r^T \sigma_O & \sigma_\delta^T \sigma_O & \|\sigma_O\|^2 \end{bmatrix}$$

Hence, we have the dot product of each pair of diffusion coefficients and the norm-2 of each of them. Knowing these values and using the parameters we found for the Vasicek model in the previous part, we do a grid search on δ_0, λ_δ, and $\bar{\delta}$. For each set of parameters, we calculate the futures price from part for all values of τ:

$$f_t = f(t, r_t, \delta_t, O_t) = e^{\phi_0(T-t) + \phi_r(T-t) r_t + \phi_\delta(T-t)\delta_t} O_t$$

We then compare these calculated values with the real future prices $p_0(\tau)$ which we have in our dataset and choose the set of parameters that gives us the lowest amount of MAPE:

$$min_{\lambda_\delta, \bar{\delta}, \delta_0} 100 \times \frac{1}{n} \sum_{i=0}^{n} \left| \frac{p_0(\tau_i) - f_0(\tau_i)}{p_0(\tau_i)} \right|$$

We use the same grid search and optimizer approach as the previous part.

	λ_δ	δ_0	$\overline{\delta}$	**MAPE**	**MSE**	**MAE**
Value	0.5251	0.0197	-0.0102	0.0133	0.0002	0.0110

Table 2: **Final Parameters and Accuracies achieved**

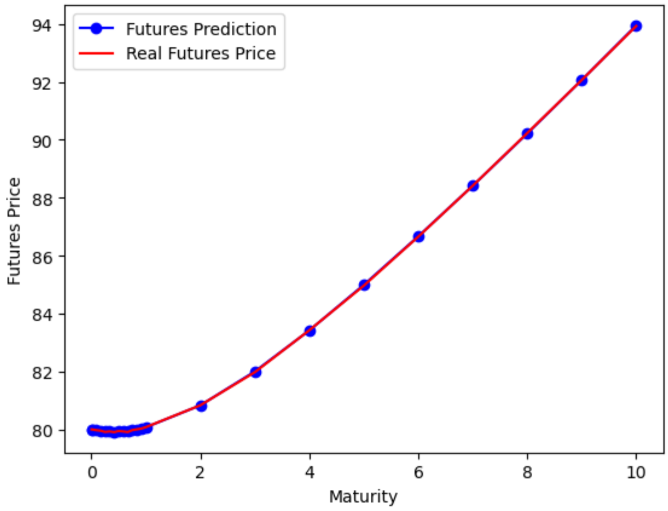


Figure 5: **Real Future Prices vs Predicted Future Prices**

Implementation

20.

The details of our implementation are in our code.

21.

In this part, for different values of α, Δ, δ_0, $\overline{\delta}$, $\|\sigma_\delta\|$, and $\rho_{O\delta}$. We set as a benchmark the calibrated values in the previous parts and the parameters given in the description of the question. Then at each time, we fix all the variables and change only one of them. We plot the payoff, price, and exercise threshold (critical value) for each.
For the storage period (Δ), as we see in **Figure 6**, the price and payoff remain zero except for huge values of Δ. The interpretation would be that for high values of Δ, the value of the option becomes more because the option seller is taking the risk of changes in the storage cost for a longer period. With increasing values of Δ, the critical value also increases. This is because when the storage period increases, the buyer of the option would exercise even with higher values of convenience yield.

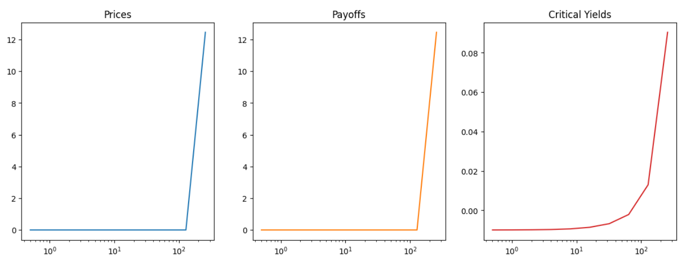


Figure 6: **Plot of Storage Period**

For all the values of α, we get zero price as shown in **Figure 7**. The reason is that even if we have very tiny values, still the benchmark storage period which is 0.25 is too small to give value to the option. When we increase the value of α, the buyer of the option is paying more for the cost of storage. Hence, it requires a lower convenience yield. That is why the critical values drop for bigger values of α.

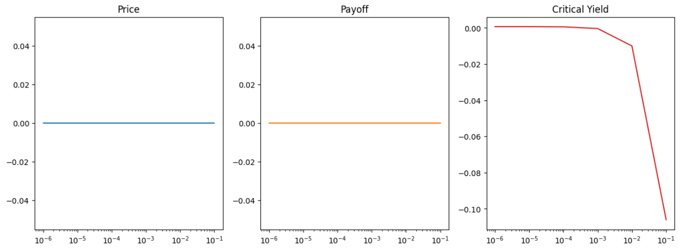


Figure 7: **Plot of** α

For δ_0, we observe that we get positive prices for very negative convenience yield starting points (see **Figure 8**). This is because the yield would not have enough time (in $\Delta = 0.25$) to reach the mean value $\overline{\delta}$, and so it would have negative values in most of the simulated paths, which makes the price positive. The reason is that with lower convenience yield the effective cost of holding the commodity increases. We can also see this because we discount with the convenience yield in pricing the option and so negative values means higher price. For the exercise threshold, we get a flat plot, which means that the threshold of δ_{T0} that we exercise the option does not depend on the δ_0 values that we experimented.

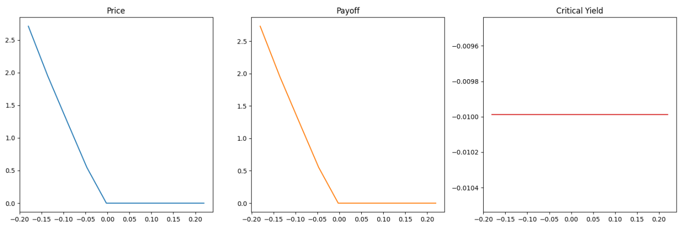


Figure 8: **Plot of** δ_0

For $\overline{\delta}$, we have the same interpretation as for δ_0, meaning that for a very negative convenience yield, the price becomes positive and can be seen in **Figure 9**. But this time we observe that the convenience yield will reduce with increasing the mean convenience yield value. This is because the lower convenience yield means, the investor would accept to exercise the option for higher critical values.

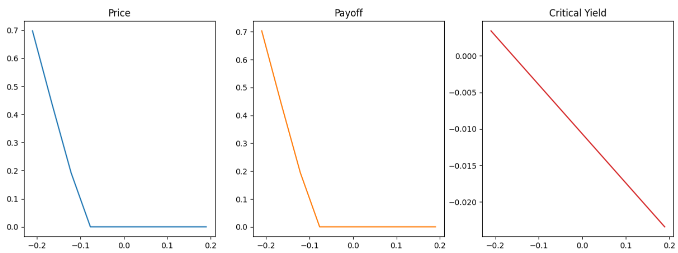


Figure 9: **Plot of** $\overline{\delta}$

For both the $\rho_{O\delta}$ and $\|\sigma_\delta\|$, we see a similar pattern (see **Figure 10 and 11**). The price is zero for all values that we tested and the critical value is decreasing. The reason the price is zero is that the magnitude of our changes to these variables is very small to offset the negative effect of the short period which makes the price zero.

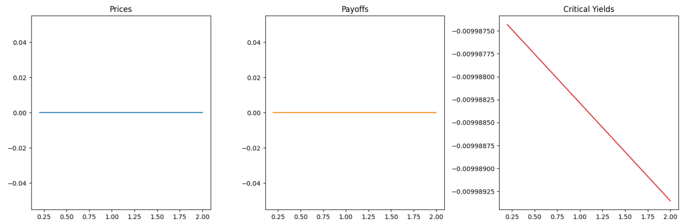


Figure 10: **Plot of** $\rho_{O\delta}$

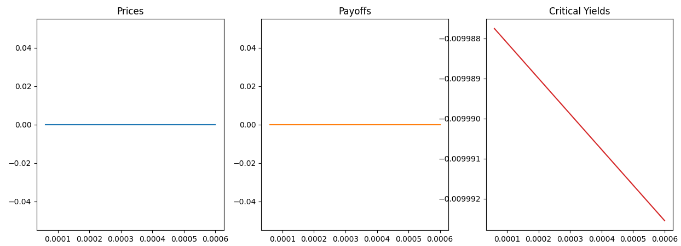


Figure 11: **Plot of** $\|\sigma_\delta\|$