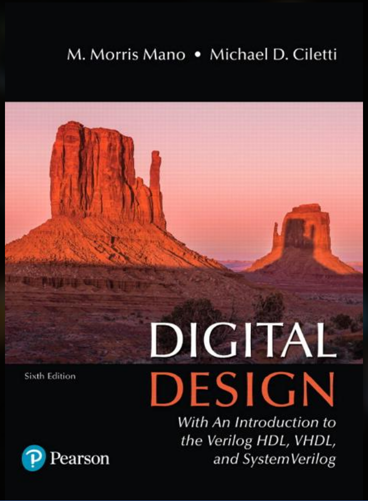



# Chapter 3
# Gate-Level Minimization

# MINIMIZATION

## II) Map (Karnaugh map, K-map)

aka. Graphical Manipulation

# MINIMIZATION

## II) Map (Karnaugh map, K-map)

aka. Graphical Manipulation

Algorithm; Straightforward, up to six variables

**Result is always minimal**

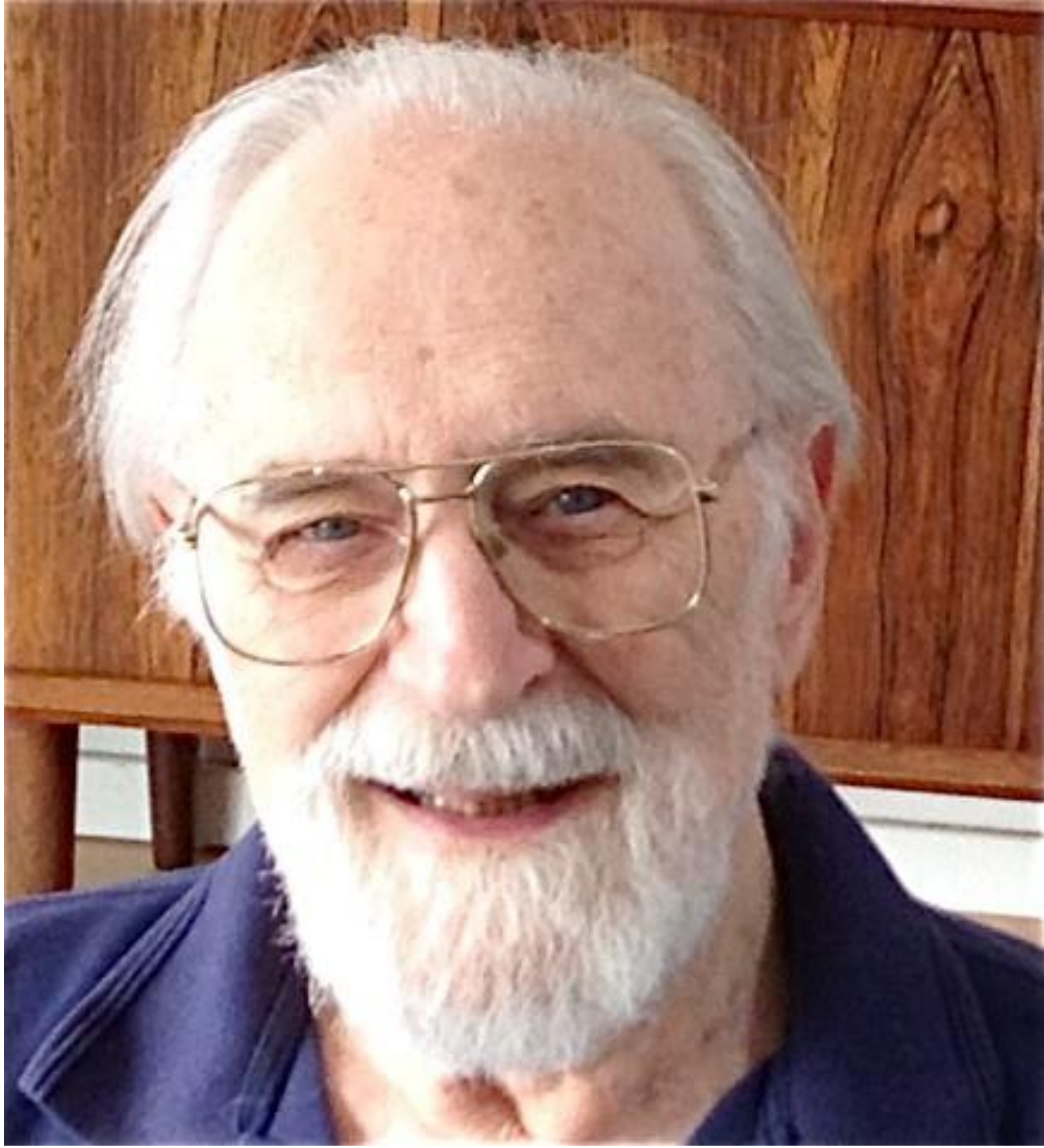
SOP

POS

# Maurice Karnaugh

Physicist
Mathematician
Inventor

Bell Labs (1954)
"The Map Method for Synthesis of Combinational Logic Circuits”



# KARNAUGH MAP

/ˈkɑːrnɔː/

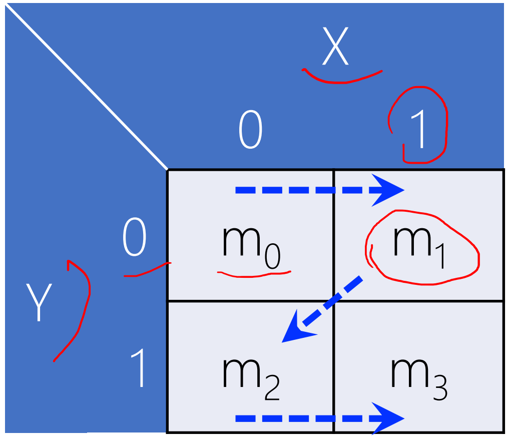
# 1-Variable KARNAUGH MAP

| X | F |
|---|---|
| 0 | $m_0$ |
| 1 | $m_1$ |

| X | |
|---|---|
| 0 | 1 |
| $m_0$ | $m_1$ |

# 2-Variable KARNAUGH MAP

| Y | X | F |
|---|---|---|
| 0 | 0 | $m_0$ |
| 0 | 1 | $m_1$ |
| 1 | 0 | $m_2$ |
| 1 | 1 | $m_3$ |



| Y | X | F |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

$$
\begin{aligned}
F(Y,X) &= m_0 + m_1 + m_2 \\
&= Y'X' + Y'X + YX' \\
&= Y'X' + Y'X' + Y'X + YX' \\
&= Y'(X' + X) + Y'X' + YX' \\
&= Y' + Y'X' + YX' \\
&= Y' + X'(Y' + Y) \\
&= Y' + X'
\end{aligned}
$$

| Y \ X | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 0 |

$$F(Y,X) = Y' + X'$$

| Y | X | F |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

$$F(Y,X) = m_1 + m_2$$
$$= Y'X + YX'$$

# XOR

| Y \ X | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

$$F(Y,X) = m_1 + m_2$$
$$= \underbrace{Y'X + YX'}$$
$$= Y \oplus X$$

# 3-Variable KARNAUGH MAP

| Z | Y | X | F |
|---|---|---|---|
| 0 | 0 | 0 | $m_0$ |
| 0 | 0 | 1 | $m_1$ |
| 0 | 1 | 0 | $m_2$ |
| 0 | 1 | 1 | $m_3$ |
| 1 | 0 | 0 | $m_4$ |
| 1 | 0 | 1 | $m_5$ |
| 1 | 1 | 0 | $m_6$ |
| 1 | 1 | 1 | $m_7$ |

| Y \ X | 0 | 1 |
|---|---|---|
| 0 | $m_0$ | $m_1$ |
| 1 | $m_2$ | $m_3$ |


| Y \ X | 0 | 1 |
|---|---|---|
| 0 | $m_0$ | $m_1$ |
| 1 | $m_2$ | $m_3$ |


ZX

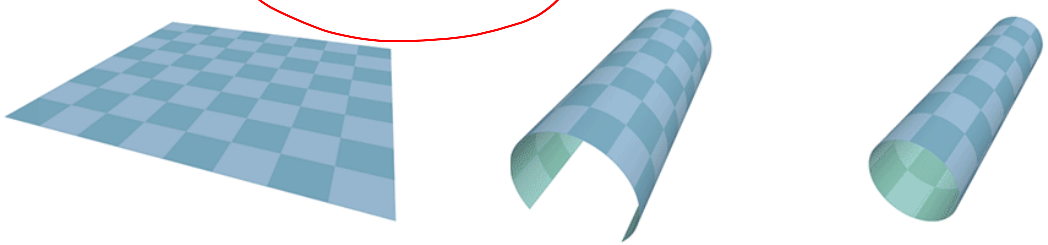
10 11

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $m_0$ | $m_1$ | $m_3$ | $m_2$ |
| 1 | $m_4$ | $m_5$ | $m_7$ | $m_6$ |

Y

| Z | Y | X | F |
|---|---|---|---|
| 0 | 0 | 0 | $m_0$ |
| 0 | 0 | 1 | $m_1$ |
| 0 | 1 | 0 | $m_2$ |
| 0 | 1 | 1 | $m_3$ |
| 1 | 0 | 0 | $m_4$ |
| 1 | 0 | 1 | $m_5$ |
| 1 | 1 | 0 | $m_6$ |
| 1 | 1 | 1 | $m_7$ |

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $m_0$ | $m_1$ | $m_3$ | $m_2$ |
| 1 | $m_4$ | $m_5$ | $m_7$ | $m_6$ |



| Z | Y | X | F |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

$M_4$
$M_5$

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 |

$$F(Z,Y,X) = \sum m(0,1,2,3,6,7)$$
$$= Z' + Y$$

$$F(Z,Y,X) = \sum m(0,1,2,3,6,7)$$
$$= Z'Y'X' + Z'Y'X + Z'YX' + Z'YX + ZYX' + ZYX$$
$$= ?$$

# MAXTERMS

| Z | Y | X | F |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

$$F(Z,Y,X) = \prod M(4,5)$$
$$= (Z'+Y+X)\,(Z'+Y+X')$$
$$= ?$$

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | ~~1~~ 0 | ~~1~~ 0 | ~~1~~ 0 | ~~1~~ 0 |
| 1 | ~~0~~ 1 | ~~0~~ 1 | ~~1~~ 0 | ~~1~~ 0 |

$$F(Z,Y,X) = \prod M(4,5)$$
$$(F')' \qquad = ?$$

| Z | Y | X | F |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

$$F(Z,Y,X) = \prod M(4,5)$$
$$= (Z'+Y+X)\,(Z'+Y+X')$$
$$= ?$$

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 |

$$F'(Z,Y,X) = \sum m(4,5)$$
$$= ZY'$$

| Z | Y | X | F |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

$$F(Z,Y,X) = \prod M(4,5)$$
$$= (Z'+Y+X)(Z'+Y+X')$$
$$= ?$$

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 |

$$F(Z,Y,X) = \prod M(4,5)$$
$$= (F')'$$
$$= (ZY')'$$
$$= Z'+Y$$

| Z | Y | X | F |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

$$
\begin{aligned}
F(Z,Y,X) &= \sum m(0,1,2,3,6,7) \\
&= Z'Y'X' + Z'Y'X + Z'YX' + Z'YX + ZYX' + ZYX \\
&= ?
\end{aligned}
$$

$$
\begin{aligned}
F(Z,Y,X) &= \prod M(4,5) \\
&= (Z'+Y+X)\,(Z'+Y+X') \\
&= ?
\end{aligned}
$$

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 |

$$
\begin{aligned}
F(Z,Y,X) &= \sum m(0,1,2,3,6,7) \\
&= Z' + Y
\end{aligned}
$$

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 |

$$
\begin{aligned}
F(Z,Y,X) &= \prod M(4,5) \\
&= (F')' \\
&= (ZY')' \\
&= Z'+Y
\end{aligned}
$$

# WHAT IF

| Z | Y | X | F | |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | $m_0$ |
| 0 | 0 | 1 | 0 | |
| 0 | 1 | 0 | 1 | $m_2$ |
| 0 | 1 | 1 | 0 | |
| 1 | 0 | 0 | 1 | $m_4$ |
| 1 | 0 | 1 | 0 | |
| 1 | 1 | 0 | 1 | $m_6$ |
| 1 | 1 | 1 | 0 | |

$F(Z,Y,X) = \sum m(0,2,4,6)$

$= Z'Y'X' + Z'YX' + ZY'X' + ZYX'$

$= ?$

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 $m_0$ | 0 1 | 0 3 | 1 2 |
| 1 | 1 4 | 0 5 | 0 7 | 1 6 |

$F = Y'X' + YX' \Rightarrow X'$

| Z | Y | X | F |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 |

$$F(Z,Y,X) = \sum m(0,2,4,6)$$
$$= Z'Y'X' + Z'YX' + ZY'X' + ZYX'$$
$$= ?$$

| | | YX | | | |
|---|---|---|---|---|---|
| | | 00 | 01 | 11 | 10 |
| Z | 0 | 1 | 0 | 0 | 1 |
| | 1 | 1 | 0 | 0 | 1 |

$$F(Z,Y,X) = \sum m(0,2,4,6)$$
$$= X'$$

# MAXTERMS

| Z | Y | X | F |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 |

$$F(Z,Y,X) = \prod M(1,3,5,7)$$
$$= (Z+Y+X')(Z+Y'+X')(Z'+Y+X')(Z'+Y'+X')$$
$$= ?$$

| | | YX | | | |
|---|---|---|---|---|---|
| | | 00 | 01 | 11 | 10 |
| Z | 0 | 1 | 0 | 0 | 1 |
| | 1 | 1 | 0 | 0 | 1 |

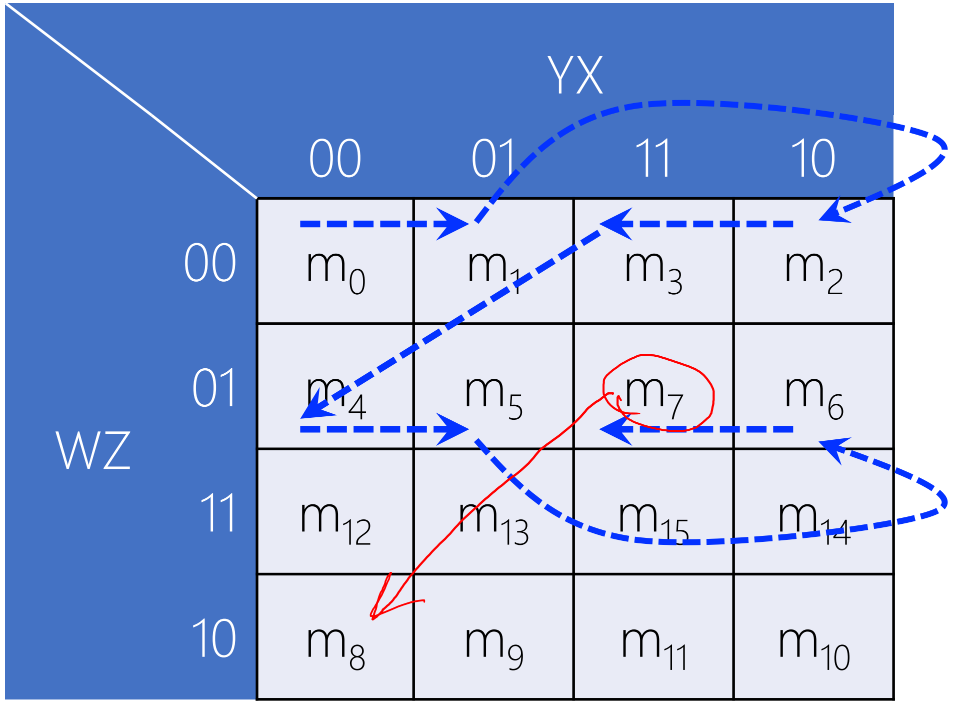
$$F(Z,Y,X) = \prod M(1,3,5,7)$$
$$= (X)'$$
$$= X'$$

# 4-Variable KARNAUGH MAP

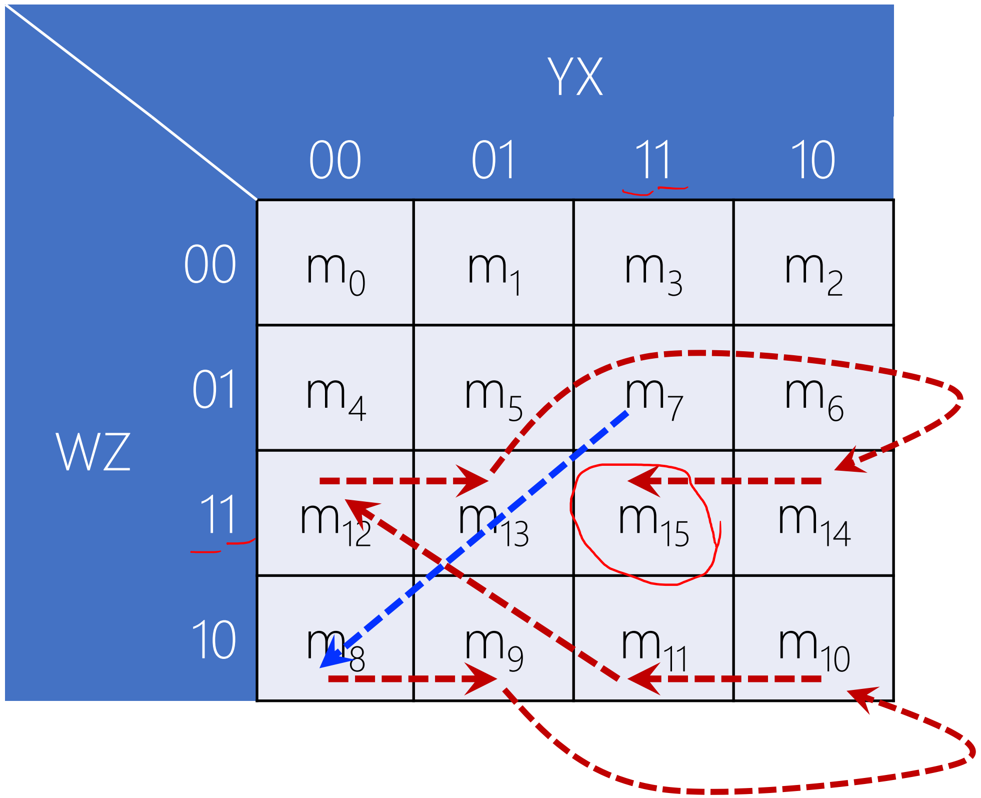
| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | $m_0$ | $m_1$ | $m_3$ | $m_2$ |
| 01 | $m_4$ | $m_5$ | $m_7$ | $m_6$ |
| 11 | $m_{12}$ | $m_{13}$ | $m_{15}$ | $m_{14}$ |
| 10 | $m_8$ | $m_9$ | $m_{11}$ | $m_{10}$ |

$WZY'X = m_{13}$

1 1 0 1



YX
00
01
11
10
WZ
00
01
11
10
m0
m1
m3
m2
m4
m5
m7
m6
m12
m13
m15
m14
m8
m9
m11
m10


YX
00
01
11
10
WZ
00
01
11
10
$m_0$
$m_1$
$m_3$
$m_2$
$m_4$
$m_5$
$m_7$
$m_6$
$m_{12}$
$m_{13}$
$m_{15}$
$m_{14}$
$m_8$
$m_9$
$m_{11}$
$m_{10}$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | $m_0$ 1 | $m_1$ 1 | $m_3$ 0 | $m_2$ 1 |
| 01 | $m_4$ 1 | $m_5$ 1 | $m_7$ 0 | $m_6$ 1 |
| 11 | $m_{12}$ 1 | $m_{13}$ 1 | $m_{15}$ 0 | $m_{14}$ 1 |
| 10 | $m_8$ 1 | $m_9$ 1 | $m_{11}$ 0 | $m_{10}$ 0 |

$$F(W,Z,Y,X) = \sum m(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 0 $m_3$ | 1 $m_2$ |
| 01 | 1 $m_4$ | 1 $m_5$ | 0 $m_7$ | 1 $m_6$ |
| 11 | 1 $m_{12}$ | 1 $m_{13}$ | 0 $m_{15}$ | 1 $m_{14}$ |
| 10 | 1 $m_8$ | 1 $m_9$ | 0 $m_{11}$ | 0 $m_{10}$ |

$$F(W,Z,Y,X) = \sum m(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 0 $m_3$ | 1 $m_2$ |
| 01 | 1 $m_4$ | 1 $m_5$ | 0 $m_7$ | 1 $m_6$ |
| 11 | 1 $m_{12}$ | 1 $m_{13}$ | 0 $m_{15}$ | 1 $m_{14}$ |
| 10 | 1 $m_8$ | 1 $m_9$ | 0 $m_{11}$ | 0 $m_{10}$ |

$$F(W,Z,Y,X) = \sum m(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$$
$$= Y' +$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 0 $m_3$ | 1 $m_2$ |
| 01 | 1 $m_4$ | 1 $m_5$ | 0 $m_7$ | 1 $m_6$ |
| 11 | 1 $m_{12}$ | 1 $m_{13}$ | 0 $m_{15}$ | 1 $m_{14}$ |
| 10 | 1 $m_8$ | 1 $m_9$ | 0 $m_{11}$ | 0 $m_{10}$ |

Warning!

W={0,1}, the value of W matters in $m_{10}$

Z={0,1}

Y={1}

X={0}

$X = \{0, 1\}$ $2$

$YX = \{00, 01, 10, 11\}$ $2^2$

$2$

$2^3$

$ZYX = \{000, 001, \ldots, 111\}$ $8$

$$F(W,Z,Y,X) = \sum m(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$$

$$= Y' + ZYX'$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 0 $m_3$ | 1 $m_2$ |
| 01 | 1 $m_4$ | 1 $m_5$ | 0 $m_7$ | 1 $m_6$ |
| 11 | 1 $m_{12}$ | 1 $m_{13}$ | 0 $m_{15}$ | 1 $m_{14}$ |
| 10 | 1 $m_8$ | 1 $m_9$ | 0 $m_{11}$ | 0 $m_{10}$ |

$$F(W,Z,Y,X) = \sum m(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$$
$$= Y' + W'YX'$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 0 $m_3$ | 1 $m_2$ |
| 01 | 1 $m_4$ | 1 $m_5$ | 0 $m_7$ | 1 $m_6$ |
| 11 | 1 $m_{12}$ | 1 $m_{13}$ | 0 $m_{15}$ | 1 $m_{14}$ |
| 10 | 1 $m_8$ | 1 $m_9$ | 0 $m_{11}$ | 0 $m_{10}$ |

$$F(W,Z,Y,X) = \sum m(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$$
$$= Y' + W'YX' + WZYX'$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 0 $m_3$ | 1 $m_2$ |
| 01 | 1 $m_4$ | 1 $m_5$ | 0 $m_7$ | 1 $m_6$ |
| 11 | 1 $m_{12}$ | 1 $m_{13}$ | 0 $m_{15}$ | 1 $m_{14}$ |
| 10 | 1 $m_8$ | 1 $m_9$ | 0 $m_{11}$ | 0 $m_{10}$ |

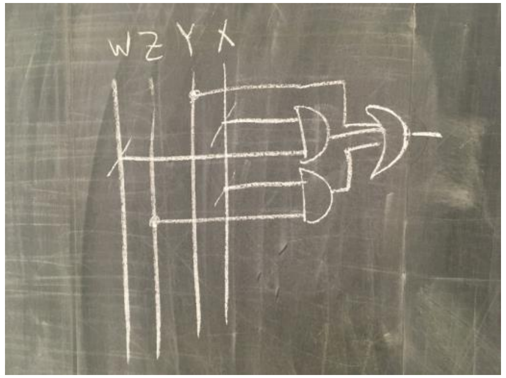
$$F(W,Z,Y,X) = \sum m(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$$
$$= Y' + W'YX' + ZYX'$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 0 $m_3$ | 1 $m_2$ |
| 01 | 1 $m_4$ | 1 $m_5$ | 0 $m_7$ | 1 $m_6$ |
| 11 | 1 $m_{12}$ | 1 $m_{13}$ | 0 $m_{15}$ | 1 $m_{14}$ |
| 10 | 1 $m_8$ | 1 $m_9$ | 0 $m_{11}$ | 0 $m_{10}$ |

$$F(W,Z,Y,X) = \sum m(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$$
$$= Y' + W'X' + WYX'$$

| WZ \ YX | 00 | 01 | 11 | 10 |
| --- | --- | --- | --- | --- |
| 00 | 1 $m_0$ | 1 $m_1$ | 0 $m_3$ | 1 $m_2$ |
| 01 | 1 $m_4$ | 1 $m_5$ | 0 $m_7$ | 1 $m_6$ |
| 11 | 1 $m_{12}$ | 1 $m_{13}$ | 0 $m_{15}$ | 1 $m_{14}$ |
| 10 | 1 $m_8$ | 1 $m_9$ | 0 $m_{11}$ | 0 $m_{10}$ |

$$F(W,Z,Y,X) = \sum m(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$$
$$= Y' + W'X' + ZX'$$



# MAXTERMS

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 0 $m_3$ | 1 $m_2$ |
| 01 | 1 $m_4$ | 1 $m_5$ | 0 $m_7$ | 1 $m_6$ |
| 11 | 1 $m_{12}$ | 1 $m_{13}$ | 0 $m_{15}$ | 1 $m_{14}$ |
| 10 | 1 $m_8$ | 1 $m_9$ | 0 $m_{11}$ | 0 $m_{10}$ |

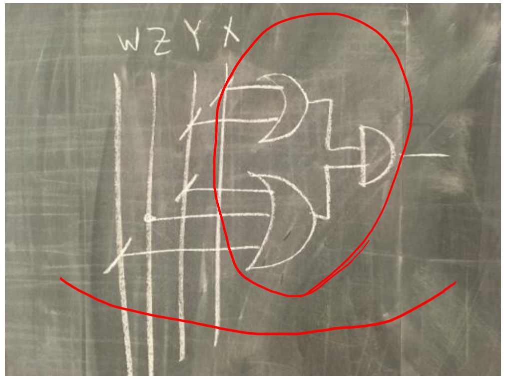
$$F(W,Z,Y,X) = \prod M(3, 7, 10, 11, 15)$$
$$= (YX)'(WZ'Y)'$$
$$= (Y'+X')(W'+Z+Y')$$

5 OR, 1 AND

$[(YX)+(WZ'Y)]'$

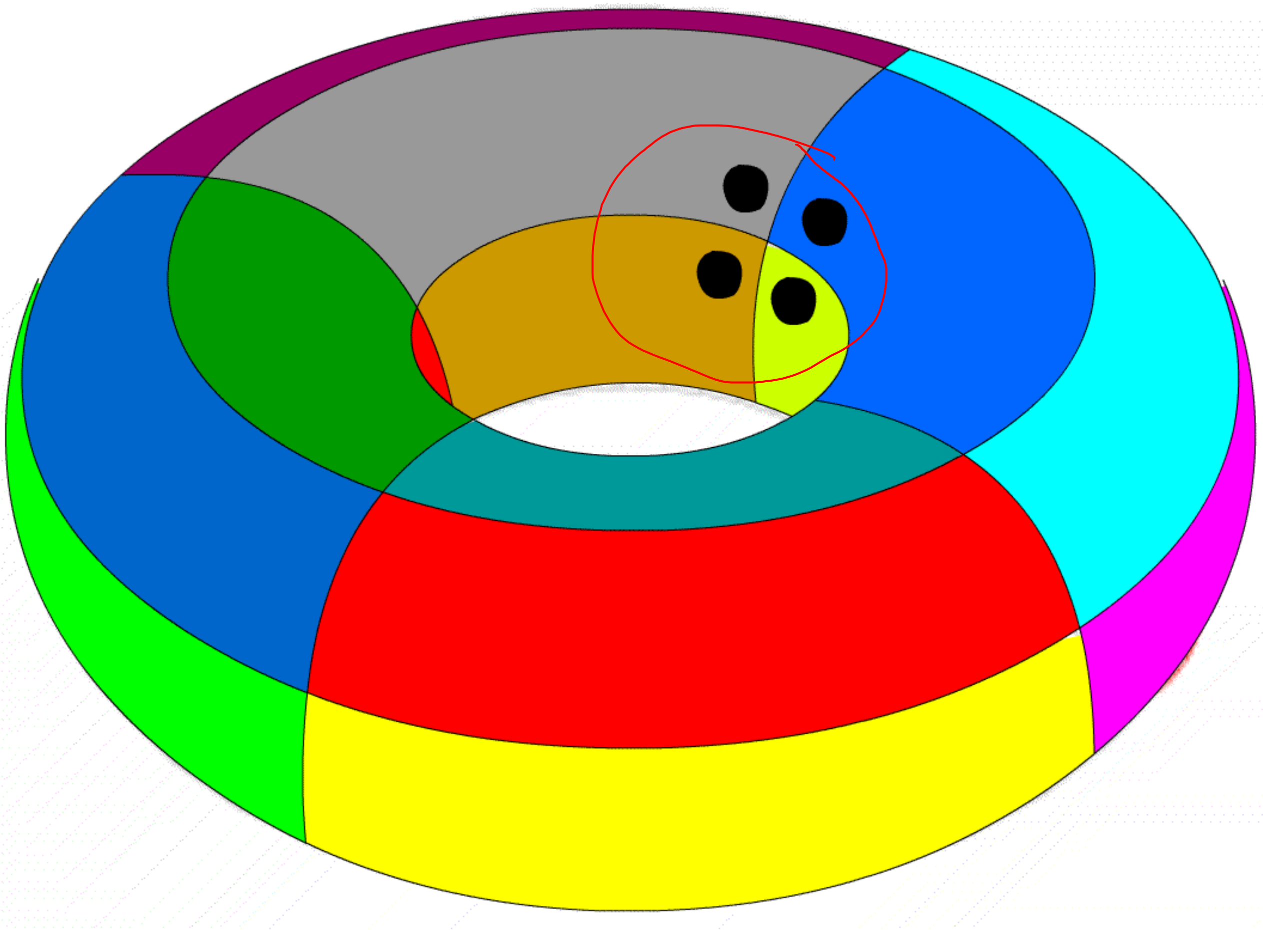
$= (YX)'(WZ'Y)'$

$= (Y'+X')(W'+Z+Y')$



# Click to Play!

https://en.wikipedia.org/wiki/Karnaugh_map#/media/File:Torus_from_rectangle.gif

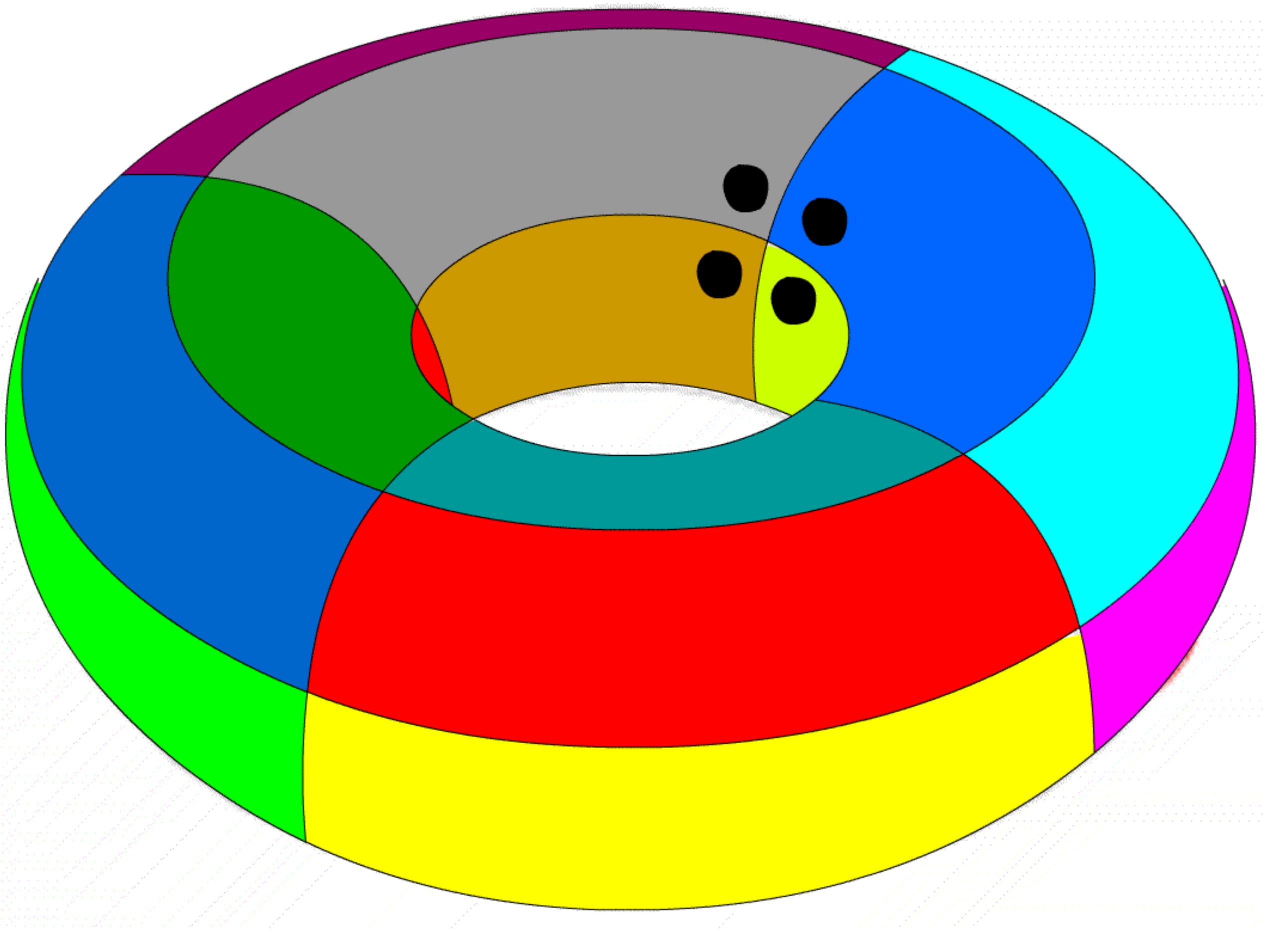


torus

| WZ \ YX | 00 | 01 | 11 | 10 |
| --- | --- | --- | --- | --- |
| 00 | 1 $m_0$ | 0 $m_1$ | 0 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 0 $m_5$ | 0 $m_7$ | 0 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 0 $m_{15}$ | 0 $m_{14}$ |
| 10 | 1 $m_8$ | 0 $m_9$ | 0 $m_{11}$ | 1 $m_{10}$ |

$$F(W,Z,Y,X) = \sum m(0, 2, 8, 10)$$
$$= Z'X'$$

# MAXTERMS



torus

$$(X+Z)' = X'Z'$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 ($m_0$) | 0 ($m_1$) | 0 ($m_3$) | 1 ($m_2$) |
| 01 | 0 ($m_4$) | 0 ($m_5$) | 0 ($m_7$) | 0 ($m_6$) |
| 11 | 0 ($m_{12}$) | 0 ($m_{13}$) | 0 ($m_{15}$) | 0 ($m_{14}$) |
| 10 | 1 ($m_8$) | 0 ($m_9$) | 0 ($m_{11}$) | 1 ($m_{10}$) |

$$
\begin{aligned}
F(W,Z,Y,X) &= \textstyle\sum m(0, 2, 8, 10) \\
&= Z'X' \\
&= \textstyle\prod M(1,3\text{-}7,9,11\text{-}15) \\
&= (X)'(Z)' \\
&= X'Z'
\end{aligned}
$$

Given two unsigned numbers x and y, design a logic circuit to see

$$x \overset{?}{\geq} y$$

| Y2 | Y1 | X2 | X1 | $F(Y2,Y1,X2,X1)=\Sigma m(0,1,2,3,5,6,7,10,11,15)$ | $F(Y2,Y1,X2,X1)=\Pi M(4,8,9,12,13,14)$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 |

| $Y_2Y_1$ \ $X_2X_1$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 1 $m_5$ | 1 $m_7$ | 1 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 1 $m_{15}$ | 0 $m_{14}$ |
| 10 | 0 $m_8$ | 0 $m_9$ | 1 $m_{11}$ | 1 $m_{10}$ |

$$F(Y_2,Y_1,X_2,X_1)=\boldsymbol{\Sigma}\, m(0,1,2,3,5,6,7,10,11,15)$$

$$F(Y_2,Y_1,X_2,X_1)=\boldsymbol{\Pi}\, M(4,8,9,12,13,14)$$

$X_2 \; X_1$

| $Y_2Y_1$ \ $X_2X_1$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 1 $m_5$ | 1 $m_7$ | 1 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 1 $m_{15}$ | 0 $m_{14}$ |
| 10 | 0 $m_8$ | 0 $m_9$ | 1 $m_{11}$ | 1 $m_{10}$ |

$$F(Y_2,Y_1,X_2,X_1)=\boldsymbol{\Sigma}\, m(0,1,2,3,5,6,7,10,11,15)$$

$$=Y'_2Y'_1 +$$

| $Y_2Y_1$ \ $X_2X_1$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 1 $m_5$ | 1 $m_7$ | 1 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 1 $m_{15}$ | 0 $m_{14}$ |
| 10 | 0 $m_8$ | 0 $m_9$ | 1 $m_{11}$ | 1 $m_{10}$ |

$$F(Y_2,Y_1,X_2,X_1)=\boldsymbol{\Sigma}\, m(0,1,2,3,5,6,7,10,11,15)$$
$$= Y'_2Y'_1 + X_2X_1$$

| $Y_2Y_1$ \ $X_2X_1$ | 00 | 01 | 11 | 10 |
| --- | --- | --- | --- | --- |
| 00 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 1 $m_5$ | 1 $m_7$ | 1 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 1 $m_{15}$ | 0 $m_{14}$ |
| 10 | 0 $m_8$ | 0 $m_9$ | 1 $m_{11}$ | 1 $m_{10}$ |

$$F(Y_2,Y_1,X_2,X_1) = \boldsymbol{\Sigma}\, m(0,1,2,3,5,6,7,10,11,15)$$

$$= Y'_2Y'_1 + X_2X_1 + Y'_2X_1$$

| $Y_2Y_1$ \ $X_2X_1$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 1 $m_5$ | 1 $m_7$ | 1 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 1 $m_{15}$ | 0 $m_{14}$ |
| 10 | 0 $m_8$ | 0 $m_9$ | 1 $m_{11}$ | 1 $m_{10}$ |

$$F(Y_2,Y_1,X_2,X_1)=\boldsymbol{\Sigma}\ m(0,1,2,3,5,6,7,10,11,15)$$
$$= Y'_2Y'_1 + X_2X_1 + Y'_2X_1 + Y'_2X_2$$

| $Y_2Y_1$ \ $X_2X_1$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 1 $m_5$ | 1 $m_7$ | 1 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 1 $m_{15}$ | 0 $m_{14}$ |
| 10 | 0 $m_8$ | 0 $m_9$ | 1 $m_{11}$ | 1 $m_{10}$ |

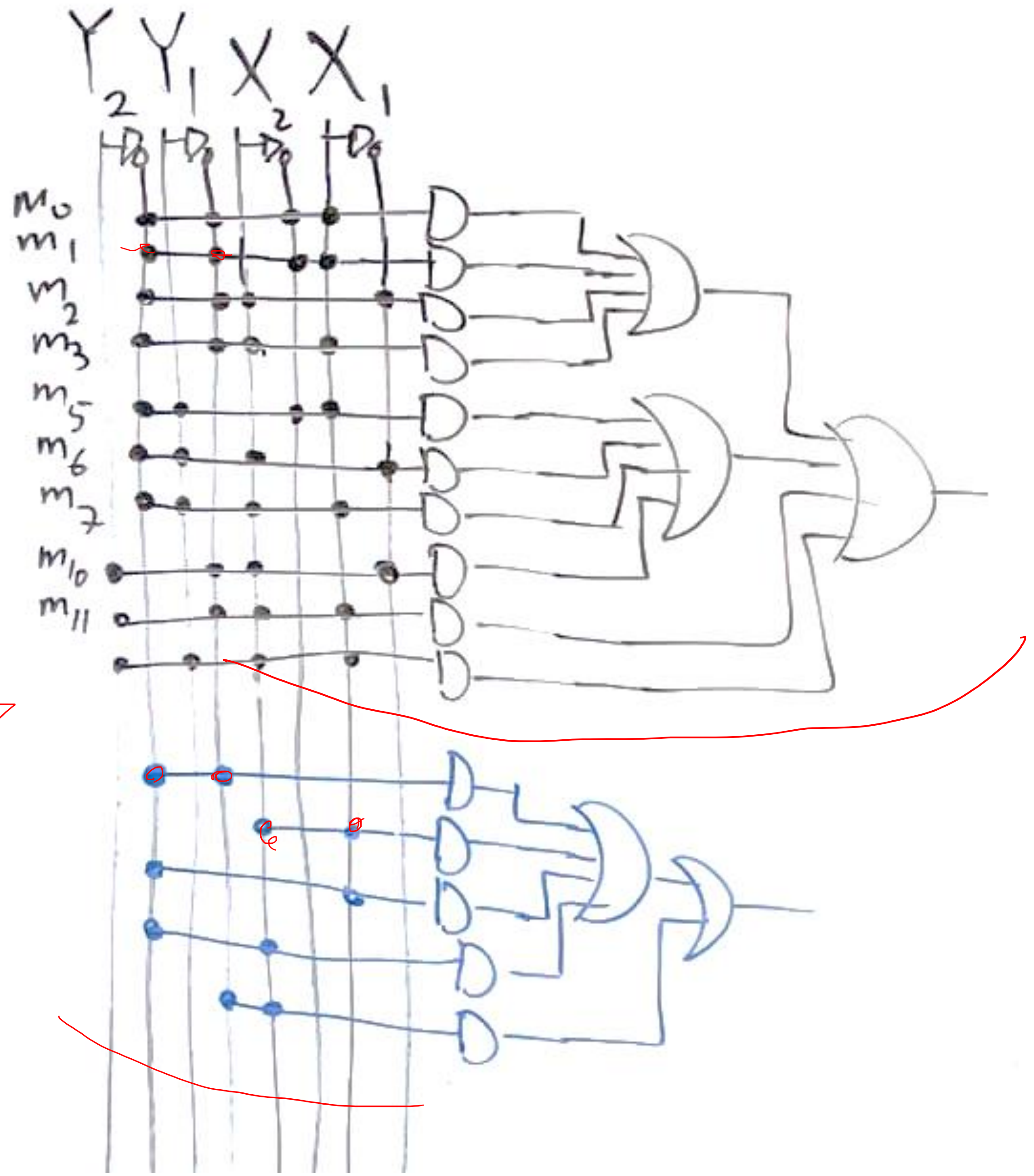
$$F(Y_2,Y_1,X_2,X_1)=\boldsymbol{\Sigma}\, m(0,1,2,3,5,6,7,10,11,15)$$

$$= Y'_2Y'_1 + X_2X_1 + Y'_2X_1 + Y'_2X_2 + Y'_1X_2$$

C SoP

Canonical SoP

Minimized SoP



# MAXTERMS

Change of Variable:

$X1 \rightarrow X$

$X2 \rightarrow Y$

$Y1 \rightarrow Z$

$Y2 \rightarrow W$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 1 $m_5$ | 1 $m_7$ | 1 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 1 $m_{15}$ | 0 $m_{14}$ |
| 10 | 0 $m_8$ | 0 $m_9$ | 1 $m_{11}$ | 1 $m_{10}$ |

$$F(Y_2,Y_1,X_2,X_1)=\boldsymbol{\Pi}\, M(4,8,9,12,13,14)$$

Change of Variable:

$X1 \rightarrow X$

$X2 \rightarrow Y$

$Y1 \rightarrow Z$

$Y2 \rightarrow W$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 1 $m_5$ | 1 $m_7$ | 1 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 1 $m_{15}$ | 0 $m_{14}$ |
| 10 | 0 $m_8$ | 0 $m_9$ | 1 $m_{11}$ | 1 $m_{10}$ |

$$F(Y_2,Y_1,X_2,X_1)=\mathbf{\Pi}\ M(4,8,9,12,13,14)$$

$$F(W,Z,Y,X)=?$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 1 $m_5$ | 1 $m_7$ | 1 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 1 $m_{15}$ | 0 $m_{14}$ |
| 10 | 0 $m_8$ | 0 $m_9$ | 1 $m_{11}$ | 1 $m_{10}$ |

$$F(Y_2,Y_1,X_2,X_1)=\boldsymbol{\Pi}\ M(4,8,9,12,13,14)$$

$$F(W,Z,Y,X)\ =(WY' +\ )'$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 1 $m_5$ | 1 $m_7$ | 1 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 1 $m_{15}$ | 0 $m_{14}$ |
| 10 | 0 $m_8$ | 0 $m_9$ | 1 $m_{11}$ | 1 $m_{10}$ |

$$F(Y_2,Y_1,X_2,X_1)=\mathbf{\Pi}\ M(4,8,9,12,13,14)$$

$$F(W,Z,Y,X)\ =(WY' + ZY'X' + )'$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 ($m_0$) | 1 ($m_1$) | 1 ($m_3$) | 1 ($m_2$) |
| 01 | 0 ($m_4$) | 1 ($m_5$) | 1 ($m_7$) | 1 ($m_6$) |
| 11 | 0 ($m_{12}$) | 0 ($m_{13}$) | 1 ($m_{15}$) | 0 ($m_{14}$) |
| 10 | 0 ($m_8$) | 0 ($m_9$) | 1 ($m_{11}$) | 1 ($m_{10}$) |

$$F(Y_2,Y_1,X_2,X_1)=\mathbf{\Pi}\ M(4,8,9,12,13,14)$$

$$F(W,Z,Y,X) = (WY' + ZY'X' + WZX')'$$

| WZ \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 01 | 0 $m_4$ | 1 $m_5$ | 1 $m_7$ | 1 $m_6$ |
| 11 | 0 $m_{12}$ | 0 $m_{13}$ | 1 $m_{15}$ | 0 $m_{14}$ |
| 10 | 0 $m_8$ | 0 $m_9$ | 1 $m_{11}$ | 1 $m_{10}$ |

$F(Y_2,Y_1,X_2,X_1) = \mathbf{\Pi}\ M(4,8,9,12,13,14)$

$F(W,Z,Y,X) = (WY' + ZY'X' + WZX')'$

$= (WY')'\ (ZY'X')'\ (WZX')'$

$= (W'+Y)\ (Z'+Y+X)' \cdot (W'+Z'+X)$

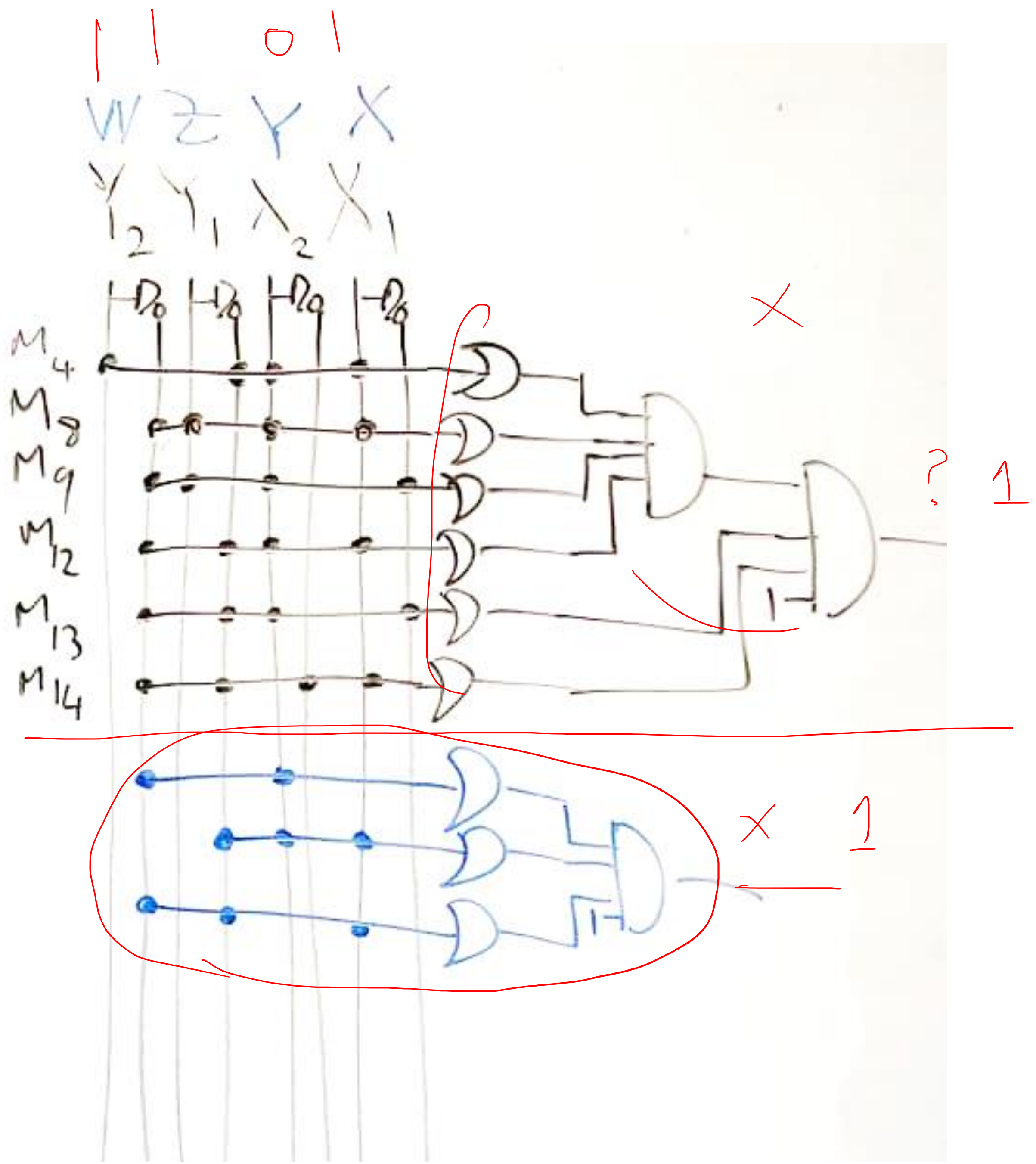
PoS

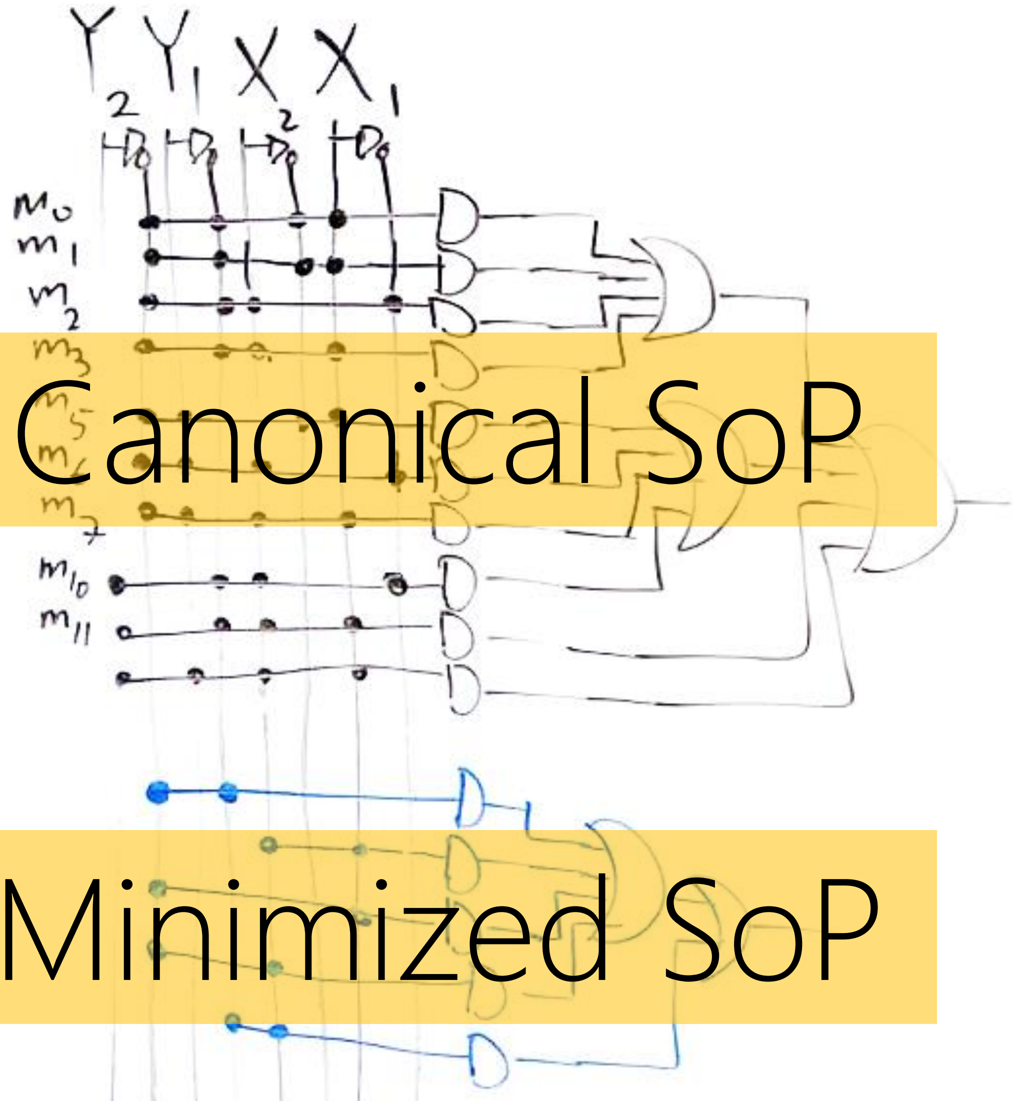
$3 \Rightarrow 1 \Rightarrow 1$

$(11)_2$ $(01)_2$

# Canonical PoS

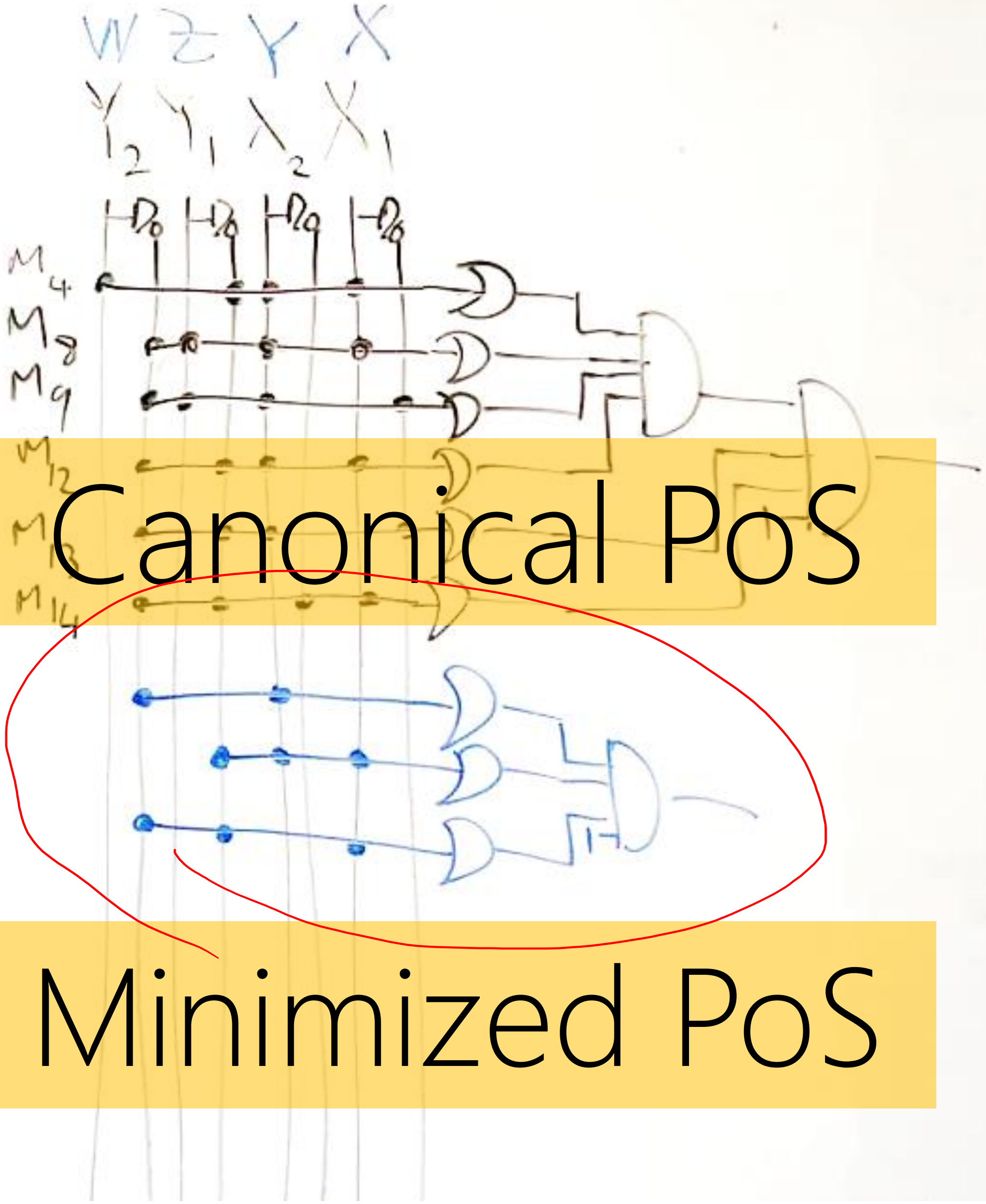
# Minimized PoS






Canonical SoP

Minimized SoP




Canonical PoS

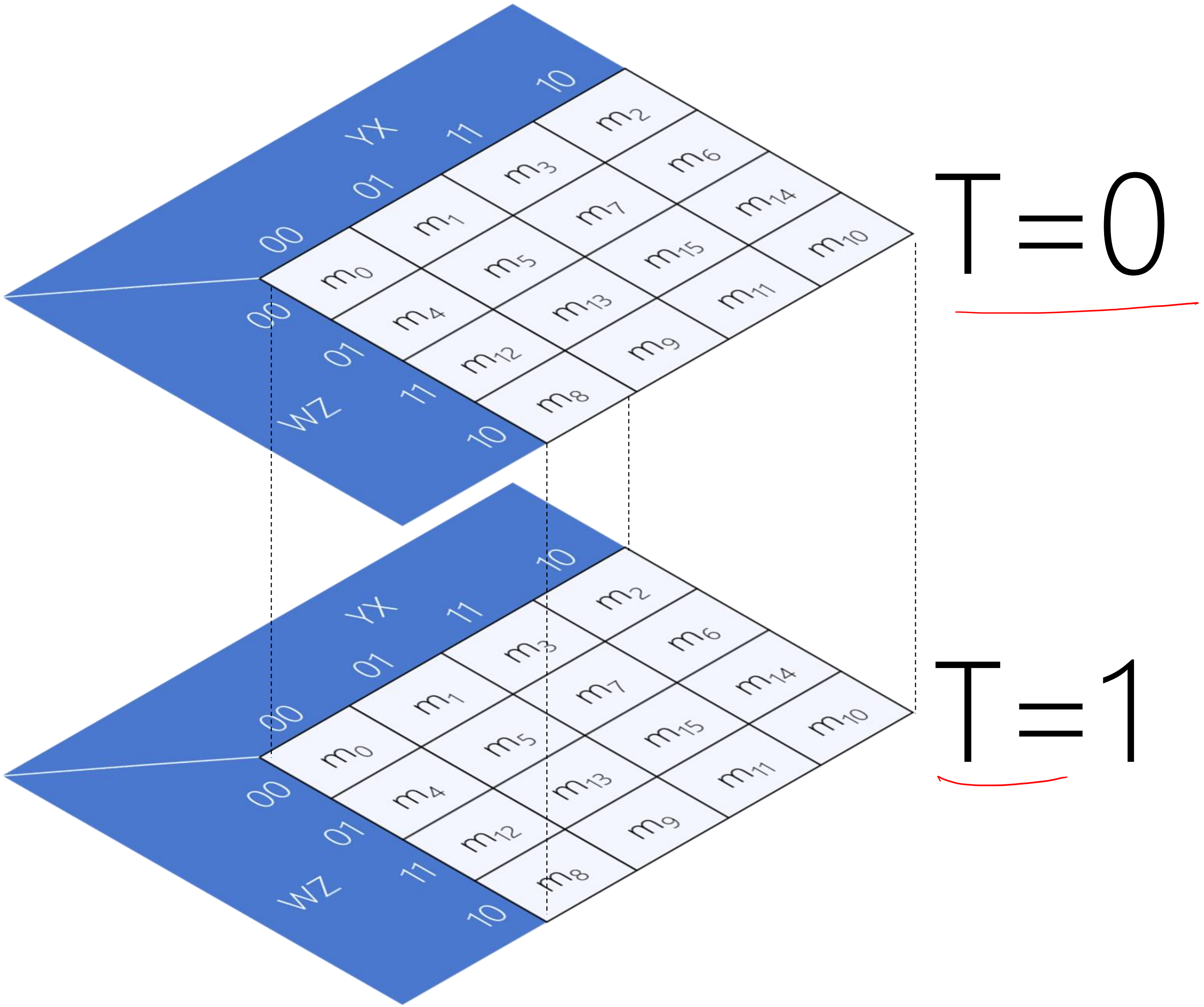
Minimized PoS

# Question

If |Canonical SoP| > |Canonical PoS| then |Minimized SoP| > |Minimized PoS|?
If yes, prove.
If no, bring a counter-example.

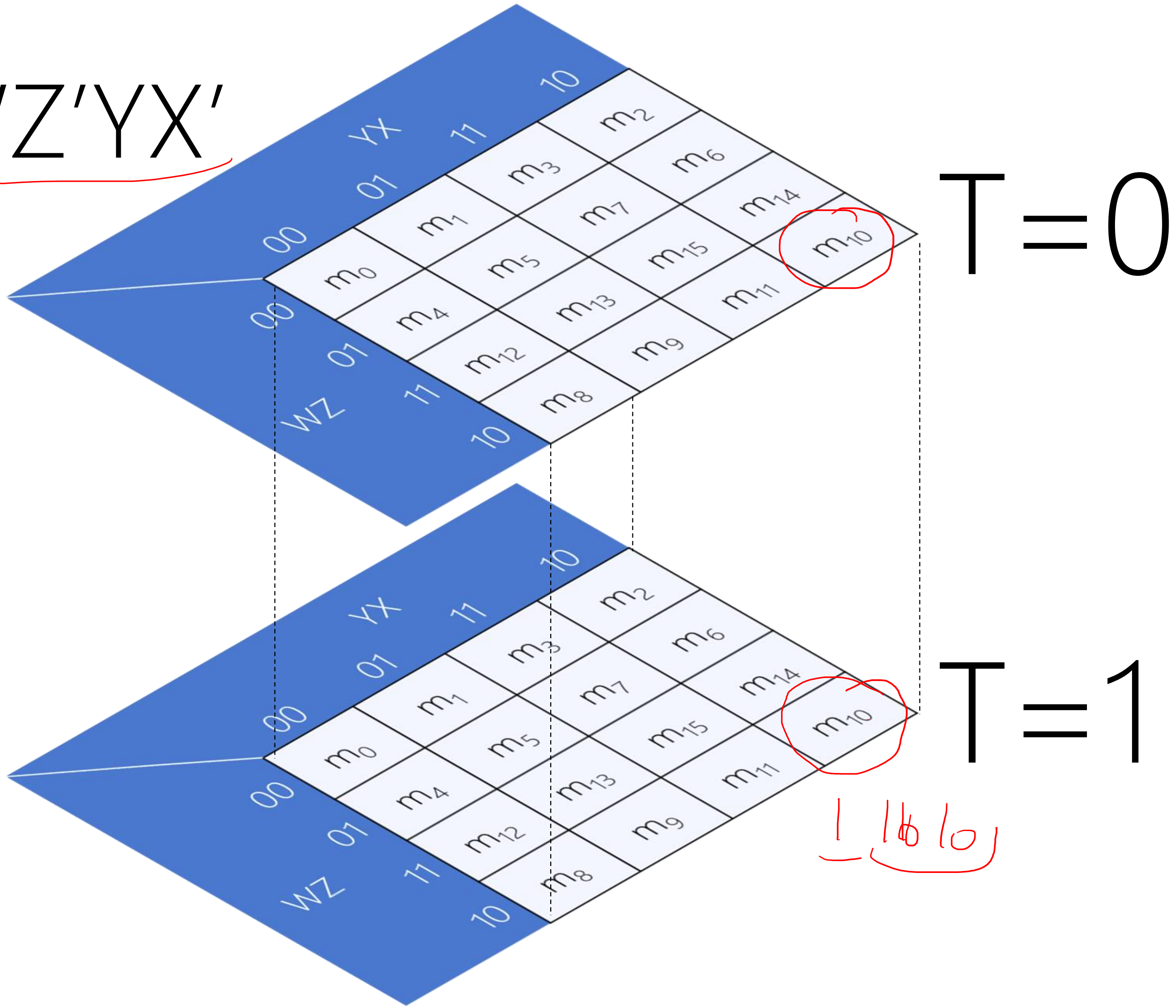
# 5-Variable KARNAUGH MAP

$F(T,W,Z,Y,X)$

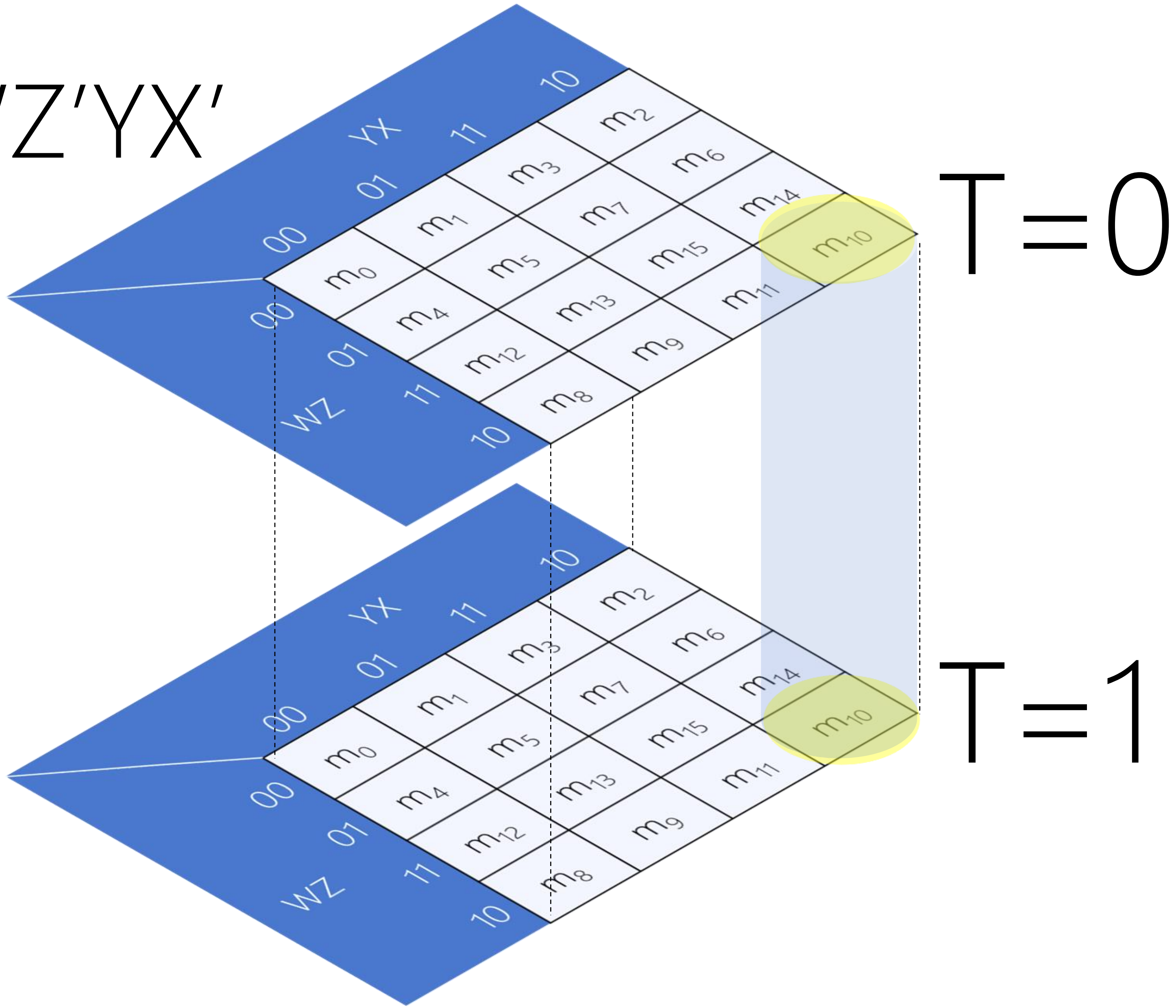


YX
00
01
11
10
WZ
00
01
11
10
m0 m1 m3 m2
m4 m5 m7 m6
m12 m13 m15 m14
m8 m9 m11 m10
T=0
YX
00
01
11
10
WZ
00
01
11
10
m0 m1 m3 m2
m4 m5 m7 m6
m12 m13 m15 m14
m8 m9 m11 m10
T=1

$$F(T,W,Z,Y,X)$$
$$= T'WZ'YX' + TWZ'YX'$$
$$= m_{01010} + m_{11010}$$
$$= m_{10} + m_{26}$$



$$F(T,W,Z,Y,X)$$
$$= T'WZ'YX' + TWZ'YX'$$
$$= m_{01010} + m_{11010}$$
$$= m_{10} + m_{26}$$
$$= WZ'YX'$$



# n-Variable ~~KARNAUGH MAP~~

**Willard Van Orman Quine** (/kwaɪn/; known to intimates as "Van"; 1908 – 2000) was an American philosopher and logician in the analytic tradition, recognized as "*one of the most influential philosophers of the twentieth century.*"

**Edward Joseph McCluskey** (1929 – 2016) was a Professor at Stanford University. He was a pioneer in the field of Electrical Engineering. McCluskey worked on electronic switching systems at the Bell Telephone Laboratories

# Quine–McCluskey Algorithm

https://en.wikipedia.org/wiki/Quine%E2%80%93McCluskey_algorithm

1878 ← 1937 ← 1952 ← 1956

# Demo
# Quine–McCluskey Algorithm

https://www.mathematatik.uni-marburg.de/~thormae/lectures/ti1/code/qmc/

# Don’t Care Conditions

In practice, in some applications the function is not specified for certain combinations of the variables.

| Z | Y | X | F=if input is positive(2's comp.) then 1 else 0 |
|---|---|---|---|
| 0 | 0 | 0 | ? |
| 0 | 0 | 1 | ? |
| 0 | 1 | 0 | ? |
| 0 | 1 | 1 | ? |
| 1 | 0 | 0 | ? |
| 1 | 0 | 1 | ? |
| 1 | 1 | 0 | ? |
| 1 | 1 | 1 | ? |

| Z | Y | X | F=if input is positive(2's comp.) then 1 else 0 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

In math, 0 is not positive neither negative!

$-4$   11 / 011 + 1 / 100

$-3$   011

$-2$   010

$-1$   001

| Z | Y | X | $F=\sum m(1,2,3)=\prod M(0,4,5,6,7)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 0 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 1 | 0 $m_4$ | 0 $m_5$ | 0 $m_7$ | 0 $m_6$ |

$$F(Z,Y,X) = \sum m(1, 2, 3)$$
$$= Z'X + Z'Y$$

Boolean algebra → $Z'(X+Y)$

# MAXTERMS

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 0 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 1 | 0 $m_4$ | 0 $m_5$ | 0 $m_7$ | 0 $m_6$ |

$$F(Z,Y,X) = \Pi M(0,4,5,6,7)$$
$$= (Z + Y'X')'$$
$$= Z'(Y+X)$$

| Z | Y | X | F=if positive(2's comp.) then 1 if negative 0 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

In math, 0 is not positive neither negative!

| Z | Y | X | F=if positive(2's comp.) then 1 if negative 0 |
|---|---|---|---|
| 0 | 0 | 0 | |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 1 | 0 $m_4$ | 0 $m_5$ | 0 $m_7$ | 0 $m_6$ |

$$F(Z,Y,X) = \sum m(1, 2, 3) + \sum d(0)$$
$$= Z'X + Z'Y$$

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 1 | 0 $m_4$ | 0 $m_5$ | 0 $m_7$ | 0 $m_6$ |

$$F(Z,Y,X) = \sum m(1, 2, 3) + \sum m(0)$$
$$= Z'$$

In this case, the don't care condition help to more simplification

# MAXTERMS

| Z \ YX | 00 | 01 | 11 | 10 |
| --- | --- | --- | --- | --- |
| 0 | $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 1 | 0 $m_4$ | 0 $m_5$ | 0 $m_7$ | 0 $m_6$ |

$$F(Z,Y,X) = \Pi\, M(4,5,6,7) + \sum D(0)$$

$$= (Z)'$$

$$= Z'$$

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 0 $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 1 | 0 $m_4$ | 0 $m_5$ | 0 $m_7$ | 0 $m_6$ |

$$F(Z,Y,X) = \prod M(0,4,5,6,7) + \sum M(0)$$
$$= (Z + Y'X')'$$
$$= Z'\,(Y+X)$$

In this case, the don't care condition does NOT help to more simplification

# Don't Care Conditions

Functions that have unspecified outputs for some input combinations are called *incompletely specified functions*.

Don't-care conditions can be used on a map to provide further simplification of the Boolean expression.

# Don’t Care Conditions

To distinguish the don’t-care condition from 1’s and 0’s, an $x$ is used.

Chi

| Z \ YX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $x$ $m_0$ | 1 $m_1$ | 1 $m_3$ | 1 $m_2$ |
| 1 | 0 $m_4$ | 0 $m_5$ | 0 $m_7$ | 0 $m_6$ |

$$F(Z,Y,X) = \sum m(1, 2, 3) + \sum d(0) \rightarrow 1$$

$$F(Z,Y,X) = \prod M(4,5,6,7) + \sum D(0) \rightarrow 0$$