Time Series Analysis

Time Series Models of Heteroscedasticity

Hossein Moradi Rekabdarkolaee

South Dakota State University

email: *hossein.moradirekabdarkolaee@sdstate.edu*.

Office: CAME Building, *Room: 240*.

Introduction

- So far, we have learned
  - Different components of time series
    - Trend,
    - Seasonality,
    - Error.
  - Different type of time series
    - Stationary (weak and strong)
    - Non-stationary.
  - Different models for time series
    - Autoregressive,
    - Moving Average,
    - ARMA,
    - ARIMA.

Introduction

- Different way to find the order of time series
  - ACF,
  - PACF,
  - EACF,
  - Information Criteria-based method (AIC, BIC, AICc, etc.)
- Different approaches to estimate the parameters of the model we choose to fit the time series.
  - Method of Moment,
  - Least Squares,
  - Maximum Likelihood.

Introduction

- After fitting the model, we did
  - Model diagnostics or model criticism,
  - What does it mean
  - How to do it.
- We studied the prediction at a future time for different time series models.
- We also studied how to incorporate seasonality in the model.
- We studied how to incorporate external information into time series modeling.
- In this section, we study modeling the conditional variance of a time series.

Introduction

- The conditional variance of $\{Y_t\}$ given the past values, $Y_{t-1}, Y_{t-2}, \ldots$, measures the uncertainty in the deviation of Y_t from $E(Y_t|Y_{t-1}, Y_{t-2}, \ldots)$.
- If $\{Y_t\}$ follows some ARIMA model, the (one-step- ahead) conditional variance is always equal to the noise variance.
- The constancy of the conditional variance is true for predictions of any fixed number of steps ahead for an ARIMA process.
- In practice, the (one-step-ahead) conditional variance may vary with the current and past values of the process i.e. the conditional variance is also a random process, often referred to as the conditional variance process.

Modeling Volatility

- Volatility is the degree of variation of a trading price series over time, usually measured by the (conditional) standard deviation of (log) returns
- Why is volatility important?
  - Option pricing, e.g., Black-Scholes formula
  - Risk management, e.g., value at risk (VaR)
  - Asset allocation, e.g., minimum-variance portfolio
  - Interval forecasts
- A key challenge: Not directly observable

How to Model Volatility?

- We will take an econometric approach by modeling the conditional standard deviation (σ_t) of daily or monthly returns
- Basic structure:

$$r_t = \mu_t + a_t$$

- where

$$\mu_t = E(r_t|F_{t-1}) = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j a_{t-j}$$

- Volatility models are concerned with the time-evolution of

$$Var(r_t|F_{t-1}) = Var(a_t|F_{t-1}) = \sigma_t^2$$

- the conditional variance of a return

Literature on Univariate Volatility Modeling

- Autoregressive conditional heteroscedastic (ARCH) model [Engle, 1982]
- Generalized Autoregressive Conditional Heteroskedasticity (GARCH) Model [Bollerslev, 1986]
- Integrated Generalized Autoregressive Conditional heteroskedasticity (IGARCH) model
- Exponential general autoregressive conditional heteroskedastic (EGARCH) model [Nelson, 1991]
- Asymmetric parametric ARCH models [Ding, Granger, and Engle, 1994]
- Stochastic volatility (SV) models [Melino and Turnbull, 1990; Harvey, Ruiz, and Shephard, 1994; Jacqier, Polson. and Rossi, 1994]

Autoregressive Conditional Heteroscedastic (ARCH) Model

- An ARCH(m) model:

$$a_t = \sigma_t \epsilon_t, \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \ldots + \alpha_m a_{t-m}^2, \quad \alpha_i \geq 0 \text{ for } 1 \leq i \leq m$$

- where $\{\epsilon_t\}$ is a sequence of i.i.d. r.v. with

$$E(\epsilon_t) = 0$$

$$Var(\epsilon_t) = 1.$$

- Distribution: standard normal, standardized Student-t, generalized error distribution, or their skewed counterparts

Properties of ARCH Models

- Consider an ARCH(1) model

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2,$$

- where $\alpha_0 > 0$ and $\alpha_1 \geq 0$.
- We have the following properties:
  - $E(a_t) = E(E(a_t | F_{t-1})) = E(\sigma_t E(\epsilon_t)) = 0$
  - $Var(a_t) = \frac{\alpha_0}{1-\alpha_1}$ if $0 < \alpha_1 < 1$.
  - Under normality assumptions

$$E(a_t^2) = \frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}.$$

  - Therefore, $0 < \alpha_1^2 < \frac{1}{3}$ which implies heavy tails.

Building an ARCH Model

1. Modeling the mean effect μ_t and testing for ARCH effects for a_t

$$\begin{cases} H_0 : & \text{no ARCH effects} \\ H_1 : & \text{ARCH effects} \end{cases}$$

2. Use Ljung-Box test to $\{a_t^2\}$ [McLeod and Li, 1983] or Lagrange multiplier test [Engle, 1982]
3. Order determination: use PACF of the squared residuals
4. Estimation: conditional MLE
5. Model checking: Q-statistics of standardized residuals and squared standardized residuals. Skewness and Kurtosis of standardized residuals.

The Advantages And Weaknesses of ARCH Models

- Advantages:
  1. Simplicity
  2. Generate volatility clustering
  3. Heavy tails
- Weaknesses:
  1. Symmetric between positive and negative returns
  2. Restrictive [e.g., for an ARCH(1) $\alpha_1^2 \in (0, 1/3)$].
  3. Not sufficiently adaptive in prediction

Generalized Autoregressive Conditional Heteroskedasticity (GARCH) Model

- For a log return series r_t, let $a_t = r_t - \mu_t$ be the innovation at time t.
- Then a_t follows a GARCH(m, s) model if

$$a_t = \sigma_t \epsilon_t, \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

- where $\{\epsilon_t\}$ is defined as before, $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$,and

$$\sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1.$$

Generalized Autoregressive Conditional Heteroskedasticity (GARCH) Model

- Re-parameterization:
- Let $\eta_t = a_t^2 - \sigma_t^2$.
- The GARCH model becomes

$$a_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) a_{t-i}^2 + \eta_t - \sum_{j=1}^{s} \beta_j \eta_{t-j}$$

- This is an ARMA form for the squared series a_t^2.

GARCH(1, 1) Model

- Model:

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

- Properties:

1. Weak stationarity if $0 \leq \alpha_1, \beta_1 \leq 1$, $(\alpha_1 + \beta_1) < 1$.
2. Volatility clusters.
3. Heavy tails if $1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0$, as

$$\frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3\left[1 - (\alpha_1 + \beta_1)^2\right]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2} > 3.$$

4. 1-step ahead forecast

$$\sigma_h^2(1) = \alpha_0 + \alpha_1 a_h^2 + \beta_1 \sigma_h^2.$$

Multi-Step Ahead Forecasts

- For multi-step ahead forecasts, use $a_t^2 = \sigma_t^2 \epsilon_t^2$ and rewrite the model as

$$\sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2 + \alpha_1 \sigma_t^2(\epsilon_t^2 - 1).$$

- We have 2-step ahead volatility forecast

$$\sigma_h^2 = \alpha_0 + (\alpha_1 + \beta_1)\sigma_h^2(1).$$

- In general, we have

$$\begin{aligned} \sigma_h^2(\ell) &= \alpha_0 + (\alpha_1 + \beta_1)\sigma_h^2(\ell - 1), \quad \ell > 1 \\ &= \frac{\alpha_0\left[1-(\alpha_1+\beta_1)^{\ell-1}\right]}{1-\alpha_1-\beta_1} + (\alpha_1 + \beta_1)^\ell \sigma_h^2(1). \end{aligned}$$

- Therefore

$$\sigma_h^2(\ell) \to \frac{\alpha_0}{1 - \alpha_1 - \beta_1}, \quad \text{as } \ell \to \infty$$

The Integrated GARCH Model

- If the AR polynomial of the GARCH representation has unit root then we have an IGARCH model.

- An IGARCH(1, 1) model:

$$a_t = \sigma_t \epsilon_t, \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1) a_{t-1}^2.$$

- ℓ-step ahead forecasts

$$\sigma_h^2(\ell) = \sigma_h^2(1) + (\ell - 1)\alpha_0, \quad \ell \geq 1.$$

- Therefore, the effect of $\sigma_h^2(1)$ on future volatilities is persistent, and the volatility forecasts form a straight line with slope α_0.

The Exponential GARCH Model [Nelson, 1991]

- The EGARCH model is able to capture asymmetric effects between positive and negative asset returns by considering the weight innovation

$$g(\epsilon_t) = \theta\epsilon_t + \gamma\left[|\epsilon_t| - E(|\epsilon_t|)\right],$$

- with $E[g(\epsilon_t)] = 0$.

- We can see the asymmetry of $g(\epsilon_t)$ by rewriting it as

$$g(\epsilon_t) = \begin{cases} (\theta+\gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t \geq 0, \\ (\theta-\gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t < 0, \end{cases}$$

- An $EGARCH(m, s)$ model can be written as

$$a_t = \sigma_t\epsilon_t,, \quad \log(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 B + \ldots \beta_{s-1}B^{s-1}}{1 - \alpha_1 B - \ldots - \alpha_m B^m} g(\epsilon_{t-1})$$

EGARCH(1, 1) Model

- Model:

$$a_t = \sigma_t \epsilon_t, , \quad (1 - \alpha B) \log(\sigma_t^2) = (1 - \alpha)\alpha_0 + g(\epsilon_{t-1}),$$

- where ϵ_t are i.i.d. standard normal.

- In this case, $E(|\epsilon_t|) = \sqrt{\frac{2}{\pi}}$ and the model for $\log(\sigma_t^2)$ becomes

$$(1-\alpha B) \log(\sigma_t^2) = \begin{cases} (1 - \alpha)\alpha_0 - \sqrt{\frac{2}{\pi}} + (\gamma + \theta)\epsilon_{t-1} & \text{if } \epsilon_{t-1} \geq 0 \\ (1 - \alpha)\alpha_0 - \sqrt{\frac{2}{\pi}} + (\gamma - \theta)(-\epsilon_{t-1}) & \text{if } \epsilon_{t-1} < 0 \end{cases}$$

- Finally, we have

$$\sigma_t^2 = \sigma_{t-1}^{2\alpha} \exp\left((1 - \alpha)\alpha_0 - \sqrt{\frac{2}{\pi}}\gamma \right) = \begin{cases} \exp\left[(\gamma + \theta)\frac{a_{t-1}}{\sigma_{t-1}} \right] & \text{if } a_{t-1} \geq 0 \\ \exp\left[(\gamma - \theta)\frac{|a_{t-1}|}{\sigma_{t-1}} \right] & \text{if } a_{t-1} < 0 \end{cases}$$

Stochastic Volatility (SV) Model

- A (simple) SV model is

$$a_t = \sigma_t \epsilon_t, \quad (1 - \alpha_1 B - \ldots - \alpha_m B^m) \log(\sigma_t^2) = \alpha_0 + \nu_t$$

- where ϵ_t's are i.i.d. $N(0, 1)$, ν_t's are i.i.d. $N(0, \sigma_\nu^2)$, $\{\epsilon_t\}$, and $\{\nu_t\}$ are independent.

- Long-memory SV Model:

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \sigma \exp\left(\frac{u_t}{2}\right), \quad (1 - B)^d u_t = \eta_t$$

- where $\sigma > 0$, ϵ_t's are i.i.d. $N(0, 1)$, η_t's are i.i.d. $N(0, \sigma_\eta^2)$ and independent of ϵ_t, and $0 < d < 0.5$.

- In LMSV, we have

$$\begin{aligned}
\log(a_t^2) &= \log(\sigma_t^2) + u_t + \log(\epsilon_t^2) \\
&= \left[\log(\sigma_t^2) + E(\log(\epsilon_t^2))\right] + u_t + \left[\log(\epsilon_t^2) - E(\log(\epsilon_t^2))\right] \\
&= \mu + u_t + e_t.
\end{aligned}$$

- Thus, the $\log(a_t^2)$ series is a Gaussian long-memory signal plus a non-Gaussian white noise.