Time Series Analysis

Models for Stationary Time Series

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Review

- So far, we learned that a time series has 3 major components
  - Trend,
  - Seasonality,
  - Noise.
- Furthermore, we learned that there are various types of time series in terms of how their trends and covariance function's behavior:
  - Stationary (strong/weak)
  - Nonstationary.
- In this chapter, we are going to study different (weak/second-order) stationary processes.

Notation

- Throughout this chapter, I am going to use the following notation
  - $\{Y_t\}$ denotes the time series.
  - $\{\epsilon_t\}$ denotes the unobserved white noise series.
    - ϵ_ts are identically distributed.
    - This means $E(\epsilon_t) = 0$ and
    - ϵ_t and ϵ_s are uncorrelated for $t \neq s$.
  - We assume the $\{\epsilon_t\}$ is a white noise process.

General Linear Processes

- A *general linear process*, $\{Y_t\}$, is represented by

$$Y_t = \epsilon_t + \psi_1 \epsilon_{t-1} + \psi_2 \epsilon_{t-2} + \ldots = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}, \quad \epsilon_t \sim WN(0, \sigma^2).$$

- This form is the most general form of a famous process called Moving average process.
- In order to have convergence for this process, we have to impose some conditions on the ψ-weights.
- This condition is referred to as absolutely summable i.e.,

$$\sum_{i=0}^{\infty} |\psi_i| < \infty, \qquad \psi_0 = 1$$

- Note that a necessary condition for the ψs to be absolutely summable is that $\lim_{i \to \infty} \psi = 0$.

General Linear Processes

- A weaker condition that is enough for the MA(∞) process to be well defined is square summability, i.e.,

$$\sum_{i=0}^{\infty} \psi_i^2 < \infty, \qquad \psi_0 = 1$$

- Example: Consider the previous model when $\psi_j = \phi^j, \quad |\phi| < 1$, then we have

$$Y_t = \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + \dots, \qquad \epsilon_t \sim WN(0, \sigma^2)$$

- then, it can be shown that

$$E(Y_t) = 0, \quad \gamma_o = \frac{\sigma^2}{1-\phi^2}$$

$$\gamma_k = \frac{\phi^k\sigma^2}{1-\phi^2}$$

$$\rho_k = \phi^k$$

Time Series Modeling

- Wold's decomposition theorem (c.f. Fuller (1996) pg. 96) states that any covariance stationary time series $\{Y_t\}$ has a linear process or infinite order *moving average* representation of the form

$$Y_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}, \qquad \epsilon_t \sim WN(0, \sigma^2)$$

- So, why not model every stationary time series as a stationary 1-sided linear process?

Time Series Modeling

- Because the model is not tractable in practice:
  - Need to estimate the infinite number of parameters ψ_j.
  - But only have a finite number of data.
  - Can only estimate ACF ρ_k for $k = 0, \dots, n-1$.
- Instead, will try to use simpler (finite)
  - *autoregressive (AR)* and/or
  - *moving average (MA)*.
- models to describe the dependence structure of time series.

Moving Average Processes

- In the case where only a finite number of the ψ-weights are nonzero, we have what is called a *moving average process*. In this case, we change the notation

$$Y_t = \mu - \theta_1 \epsilon_{t-1} - \theta_2 \epsilon_{t-2} - \cdots - \theta_q \epsilon_{t-q} + \epsilon_t$$

- where $\theta_q \neq 0$. Often $\mu = 0$.
- We call such a series a **moving average of order** q and abbreviate the name to MA(q).

The First-Order Moving Average Process

- The first order Moving Average Process, denoted by MA(1), can be written as

$$Y_t = \epsilon_t - \theta\epsilon_{t-1}$$

- It can be shown that

$$\begin{aligned} E(Y_t) &= 0, & \gamma_0 &= \sigma^2(1+\theta^2) \\ \gamma_1 &= -\theta\sigma^2, & \rho_1 &= \frac{-\theta}{1+\theta^2} \\ \gamma_k &= \rho_k = 0, & k &\geq 2 \end{aligned}$$

- Note that form MA(1), the same value for ρ_1 will be obtained if θ be replaced by $1/\theta$.

- Therefore, if we knew that an MA(1) process had $\rho_1 = 0.4$, we still could not tell the precise value of θ. We will return to this troublesome point when we discuss *invertibility* later.

The Second-Order Moving Average Process

- The Second Order Moving Average Process, MA(2), can be written as

$$Y_t = \epsilon_t - \theta_1 \epsilon_{t-1} - \theta_2 \epsilon_{t-2}$$

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2)\sigma^2$$
$$\gamma_1 = (-\theta_1 + \theta_1\theta_2)\sigma^2$$
$$\gamma_2 = -\theta_2\sigma^2$$

- Therefore, the autocorrelation function of an MAR(2) is given by

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}$$
$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$
$$\rho_k = 0, \quad \text{for } k = 3, 4, \ldots$$

The General MA(q) Process

- Following a similar patern, we have the MA(q) process which can be written as

$$Y_t = \epsilon_t - \theta_1\epsilon_{t-1} - \theta_2\epsilon_{t-2} - \ldots - \theta_q\epsilon_{t-q}$$

- It can be shown that

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2)\sigma^2$$

- and

$$\rho_k = \frac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \ldots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2}, \qquad \text{for } \; k = 1, 2, \ldots, q$$

$$\rho_k = 0, \qquad \text{for } \; k > q$$

- where the numerator of ρ_q is just $-\theta_q$.

Autoregressive Processes

- Autoregressive processes are as their name suggests - regressions on themselves. Specifically, a pth-order **autoregressive process AR(p)** $\{Y_t\}$ satisfies the equation

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \epsilon_t$$

  where $\phi_p \neq 0$. Often $\phi_0 = 0$.
- The order 1 Auto-Regressive process, denoted by **AR(1)**, is

$$Y_t = \phi_0 + \phi Y_{t-1} + \epsilon_t$$

- By taking variance on both sides of this model we have

$$\gamma_0 = \phi^2 \gamma_0 + \sigma^2 \qquad \Rightarrow \qquad \gamma_0 = \frac{\sigma^2}{1 - \phi^2}$$

Autoregressive Processes

- Now by multiply both sides of AR(1) by Y_{t-k}, and take expected values

$$\gamma_k = \phi\gamma_{k-1}$$

and thus

$$\gamma_k = \phi^k \frac{\sigma^2}{1-\phi^2}\ , \qquad \rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k$$

The General Linear Process Version of the AR(1) Model

- The recursive property of the AR(1) model is extremely useful for interpreting the model.
- For other purposes, it is convenient to express the AR(1) model as a general linear process.

$$\begin{aligned} Y_t &= \phi(\phi Y_{t-2} + \epsilon_{t-1}) + \epsilon_t \\ &= \epsilon_t + \phi\epsilon_{t-1} + \phi^2 Y_{t-2}. \end{aligned}$$

- Continuing this process for $k-1$ steps, we have

$$Y_t = \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + \ldots + \phi^k Y_{t-k}$$

and with condition $|\phi| < 1$ **(stationarity condition)**,

$$Y_t = \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + \phi^3\epsilon_{t-3} + \ldots$$

Stationarity Condition for AR(1)

- Recall AR(1) ($Y_t = \phi Y_{t-1} + \epsilon_t$) can be written as

$$Y_t = \sum_{j=0}^{n} \phi^j \epsilon_{t-j}$$

- When is this process stationary (stable)?
- Need coefficients to be square summable.

The Second-Order Autoregressive Process

- An $AR(2)$ process can be written as

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t$$

- For the stationarity of this process, we need to introduce the **AR characteristic polynomial**

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2$$

- and **AR characteristic equation**

$$1 - \phi_1 x - \phi_2 x^2 = 0$$

- Recall that a quadratic equation always has two roots (possibly complex).

Stationarity of the AR(2) Process

- A stationary solution of $AR(2)$ model exists if and only if the roots of the AR characteristic equation exceed 1 in absolute value (modulus).
- We sometimes say that the roots should lie outside the unit circle in the complex plane. This statement will generalize to the pth-order case without change.
- In the second-order case, the roots of the quadratic characteristic equation are

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

- For stationarity, we require that these roots exceed 1 in absolute value.
- It can be shown that this will be true if and only if three conditions are satisfied:

$$\phi_1 + \phi_2 < 1, \qquad \phi_2 - \phi_1 < 1, \qquad |\phi_2| < 1$$

The Autocorrelation Function for the AR(2) Process

- To derive the autocorrelation function for the AR(2) case, we multiply both sides of AR(2) by Y_{t-k}, and take expectations.
- Assuming stationarity, zero means, and that ϵ_t is independent of Y_{t-k}, we get

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} \qquad k = 1, 2, \ldots$$

then, dividing through by γ_0, we have

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \qquad k = 1, 2, \ldots$$

- These usually called the **Yule-Walker equations**

The Autocorrelation Function for the AR(2) Process

- **Yule-Walker equations**:

$$\begin{aligned}\gamma_k &= \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} \\ \rho_k &= \phi_1\rho_{k-1} + \phi_2\rho_{k-2} \qquad\qquad k = 1, 2, \ldots\end{aligned}$$

- Setting k = 1 and using $\rho_0 = 1$ and $\rho_{-1} = \rho_1$, we have

$$\begin{aligned}\rho_1 &= \frac{\phi_1}{1-\phi_2} \\ \rho_2 &= \phi_1\rho_1 + \phi_2\rho_0 = \frac{\phi_2(1-\phi_2)+\phi_1^2}{1-\phi_2} \\ &\ldots\end{aligned}$$

The Autocorrelation Function for the AR(2) Process

- **Characteristic Equation**:

$$1 - \phi_1 x - \phi_2 x^2 = 0$$

- Let the reciprocals of these roots by G_1 and G_2. It can be shown that if $G_1 \neq G_2$

$$G_1 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}, \qquad G_2 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

  then, it can be shown that

$$\rho_k = \frac{(1 - G_2^2)G_1^{k+1} - (1 - G_1^2)G_2^{k+1}}{(G_1 - G_2)(1 + G_1 G_2)}, \quad \text{for} \ \ k \geq 0$$

The Autocorrelation Function for the AR(2) Process

- If the roots are complex, i.e., $\phi_1^2 + 4\phi_2 < 0$, then

$$\rho_k = R^k \frac{\sin(\Theta k + \Phi)}{\sin(\Phi)}$$

where

$$R = \sqrt{-\phi_2}, \quad \cos(\Theta) = \frac{\phi_1}{2\sqrt{-\phi_2}}, \quad \tan(\Phi) = \frac{1-\phi_2}{1+\phi_2}$$

- This case displays a **damped** sine wave behavior with **damping factor** R, $0 \leq R < 1$,**frequency** Θ, and **phase** Φ.

- If the roost are equal, i.e., $\phi_1^2 + 4\phi_2 = 0$, then we have

$$\rho_k = \left(1 + \frac{1+\phi_2}{1-\phi_2} k\right)\left(\frac{\phi_1}{2}\right)^k, k = 0, 1, 2, \ldots$$

The Variance for the AR(2) Model

- **AR(2):**

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t$$

- By taking the variance of both sides of this we have

$$\gamma_0 = (\phi_1^2 + \phi_2^2)\gamma_0 + 2\phi_1\phi_2\gamma_1 + \sigma^2$$

- Also, we have

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}, \quad \text{set } k = 1, \gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1$$

- Hence,

$$\gamma_0 = \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma^2}{(1-\phi_2)^2 - \phi_1^2}$$

The General Linear Process Representation for an AR(2)

- Remember the general linear process of $\{Y_t\}$ has the following form

$$Y_t = \epsilon_t + \psi_1\epsilon_{t-1} + \psi_2\epsilon_{t-2} + \ldots = \sum_{i=0}^{\infty} \psi_i\epsilon_{t-i},$$

- we can substitute the general linear process representation into $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t$.
- If we then equate coefficients of ϵ_j, we get the recursive relationships

$$\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \phi_1 \\
\psi_2 &= \phi_1^2 + \phi_2 \\
&\vdots \\
\psi_j &= \phi_1\psi_{j-1} + \phi_2\psi_{j-2}, \qquad j = 3, 4, \ldots
\end{aligned}$$

The General Linear Process Representation for an AR(2)

- For **AR(2)** process is:

$$Y_t = \epsilon_t + \psi_1 \epsilon_{t-1} + \psi_2 \epsilon_{t-2} + \ldots = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i},$$

- It also can be shown that

$$\psi_j = \frac{G_1^{j+1} - G_2^{j+1}}{G_1 - G_2}, \qquad G_1 \neq G_2$$

$$\psi_j = R^j \left\{ \frac{\sin[(j+1)\Theta]}{\sin(\Theta)} \right\} \qquad \text{when roots are complex}$$

$$\psi_j = (1+j)(\phi_1)^j \qquad G_1 = G_2$$

The General Autoregressive Process

- A general AR process denoted by **AR(p)** can be represent as

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \epsilon_t.$$

- The AR(p) characteristic equation is

$$1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p = 0$$

- A stationary solution exists iff the roots of the AR characteristic equation each exceed 1 in absolute value (modulus).
- Two following inequalities are necessary for stationarity, but not sufficient, that both

$$\sum_{i=1}^{p} \phi_i < 1, \qquad |\phi_p| < 1$$

Yule-Walker equations-AR(p)

- Let multiply both sides of AR(p) with Y_{t-k}

$$Y_t Y_{t-k} = (\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \epsilon_t) Y_{t-k}$$

- Now, by taking the expectation on both sides, we have

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \ldots + \phi_p \gamma_{k-p}, \qquad \text{for }, k \geq 1$$

then divide both sides with γ_0, we get

$$\begin{aligned}
\rho_1 &= \phi_1 + \phi_2 \rho_1 + \phi_3 \rho_2 + \ldots + \phi_p \rho_{p-1} \\
\rho_2 &= \phi_1 \rho_1 + \phi_2 + \phi_3 \rho_1 + \ldots + \phi_p \rho_{p-2} \\
&\vdots \\
\rho_p &= \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + \phi_3 \rho_{p-3} + \ldots + \phi_p
\end{aligned}$$

- Given numerical values for $\phi_1, \phi_2, \ldots, \phi_p$, these linear equations can be solved to obtain numerical values for $\rho_1, \rho_2, \ldots, \rho_p$.

Yule-Walker equations- AR(p)

- Now if we multiply both sides of AR(p) with Y_t and take expectations, we have

$$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \ldots + \phi_p\gamma_p + \sigma^2$$

- then, using $\rho_k = \gamma_k/\gamma_0$, we have

$$\gamma_0 = \frac{\sigma^2}{1 - \phi_1\rho_1 - \phi_2\rho_2 - \ldots - \phi_p\rho_p}$$

The Mixed Autoregressive Moving Average Model

- A time series $\{Y_t\}$ is a mixed **autoregressive moving average** process of orders p and q, (**ARMA(p, q)**) if it can be written as

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \epsilon_t - \theta_1 \epsilon_{t-1} - \theta_2 \epsilon_{t-2} - \theta_q \epsilon_{t-q}$$

- **ARMA(1, 1)**:

$$Y_t = \phi Y_{t-1} + \epsilon_t - \theta \epsilon_{t-1}$$

- To derive Yule-Walker type equations, multiplying both sides with ϵ_t and ϵ_{t-1} then take expectation, we have

$$E[\epsilon_t Y_t] = E[\epsilon_t(\phi Y_{t-1} + \epsilon_t - \theta \epsilon_{t-1})] = \sigma^2$$

$$\begin{aligned} E[\epsilon_{t-1} Y_t] &= E[\epsilon_{t-1}(\phi Y_{t-1} + \epsilon_t - \theta \epsilon_{t-1})] \\ &= \phi\sigma^2 - \theta\sigma^2 = (\phi - \theta)\sigma^2 \end{aligned}$$

ARMA(1,1) Models

- The **ARMA$(1,1)$** process is:

$$Y_t = \phi Y_{t-1} + \epsilon_t - \theta \epsilon_{t-1}$$

- Now, if we multiply both sides by Y_{t-k} and take expectations, we have

$$\begin{aligned} \gamma_0 &= \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma^2 \\ \gamma_1 &= \phi\gamma_0 - \theta\sigma^2 \\ \gamma_k &= \phi\gamma_{k-1} \qquad \forall \;\; k \geq 2 \end{aligned}$$

then, solving these equations we have

$$\begin{aligned} \gamma_0 &= \frac{(1 - 2\theta\phi + \theta^2)}{1 - \phi^2}\sigma^2 \\ \rho_k &= \frac{(1 - \theta\phi)(\phi - \theta)}{1 - 2\theta\phi + \theta^2}\phi^{k-1} \quad \forall \; k \geq 1. \end{aligned}$$

The general linear process for ARMA(1,1)

- For ARMA(1, 1) it can be shown that

$$Y_t = \epsilon_t + (\phi - \theta) \sum_{i=1}^{\infty} \phi^{i-1} \epsilon_{t-i},$$

that is

$$\psi_i = (\phi - \theta)\phi^{i-1} \qquad \text{for } \; j \geq 1,$$

- The stationary condition is $|\phi| < 1$.
- This is equivalent to finding the root of the AR characteristics equation.

The general linear process of ARMA(p, q)

- The **ARMA(p, q)** process is as follow:
- Let ϵ_t to be independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$, then the process can be presented as

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} - \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t,$$

- A stationary solution to this process exists if and only if all the roots of the AR characteristic equation $\phi(x) = 0$ exceed unity in modulus.

Finding ψ-weights of ARMA(p, q)

- If the stationarity conditions are satisfied, then the model can also be written as a general linear process with ψ-coefficients determined from

$$Y_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$$

- To obtain the general linear model, for the ARMA(1, 1) process, the ψ-coefficients has the following form

$$\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= -\theta_1 + \phi_1 \\
\psi_2 &= -\theta_2 + \phi_2 + \phi_1\psi_1 \\
&\vdots \\
\psi_j &= -\theta_j + \phi_p\psi_{j-p} + \phi_{p-1}\psi_{j-p+1} + \ldots + \phi_1\psi_{j-1},
\end{aligned}$$

where we take $\psi_j = 0$ for $j < 0$, and $\theta_j = 0$ for $j > q$.

The general linear process of ARMA(p, q)

- Again assuming stationarity, the autocorrelation function can easily be shown to satisfy

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \ldots + \phi_p \rho_{k-p}, \qquad \text{for } , k > q$$

- Similar equations can be developed for $k = 1, 2, 3, \ldots, q$ that involve $\theta_1, \theta_2, \ldots, \theta_q$ as follows:

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \ldots + \phi_p \gamma_p - \sigma^2 (\theta_0 + \theta_1 \psi_1 + \ldots + \theta_q \psi_q)$$
$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1 + \ldots + \phi_p \gamma_{p-1} - \sigma^2 (\theta_1 + \theta_2 \psi_1 + \ldots + \theta_q \psi_{q-1})$$
$$\vdots$$
$$\gamma_p = \phi_1 \gamma_{p-1} + \phi_2 \gamma_{p-2} + \ldots + \phi_p \gamma_0 - \sigma^2 (\theta_p + \theta_{p+1} \psi_1 + \ldots + \theta_q \psi_{q-p})$$

Invertibility

- We have seen that for the MA(1) process we get exactly the same autocorrelation function if θ is replaced by $1/\theta$.
- A similar problem of non-uniqueness exists for the MA(2) model.
- That is, we can have two different MA processes with the same autocovariance!
- This non-uniqueness is related to the following question:
  - An AR process can always be reexpressed as a general linear process so that an AR can be considered as an MA(∞).
  - Can a MA model be reexpressed as an AR model?

Invertibility

- To understand better the ideas, consider an MA(1) model:

$$Y_t = \epsilon_t - \theta\epsilon_{t-1} \quad \Rightarrow \quad \epsilon_t = Y_t + \theta\epsilon_{t-1}$$

  then

$$\begin{aligned}\epsilon_t &= Y_t + \theta(Y_{t-1} + \theta\epsilon_{t-2}) \\ &= Y_t + \theta Y_{t-1} + \theta^2\epsilon_{t-2}\end{aligned}$$

- If $|\theta| < 1$, we may continue this substitution “infinitely”, and have the following expression

$$\begin{aligned}\epsilon_t &= Y_t + \theta Y_{t-1} + \theta^2\epsilon_{t-2} + \dots \\ \Rightarrow Y_t &= (-\theta Y_{t-1} - \theta^2\epsilon_{t-2} - \theta^3\epsilon_{t-3} - \dots) + \epsilon_t.\end{aligned}$$

Invertibility

- We say that the MA(1) model is invertible if and only if $|\theta| < 1$.
- Every MA(q) process is stationary!
- This is because it is a finite linear process, therefore coefficients are always summable.
- However, as we've seen, we can have two different MA(q) processes with the same autocovariance.
- Therefore, MA models are not uniquely defined.
- To avoid this problem, we restrict attention to invertible MA processes.

Invertibility

- An MA(q) process is invertible if and only if the roots of its characteristic polynomial all lie outside the unit circle
- Note that, for a general MA(q) or ARMA(p, q) model, we define the **MA characteristic polynomial** as

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \ldots - \theta_q x^q$$

and the corresponding *MA characteristic equation*

$$1 - \theta_1 x - \theta_2 x^2 - \ldots - \theta_q x^q = 0$$