HW 2

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Problem 2.1. Define the following sets $S_n \subset \mathbb{R}$

$$S_n = \{x | x \in \mathbb{R}, 0 < x < \tfrac{1}{n}\}.$$

Define the following set:

$$S = \bigcap_{n=1}^{\infty} S_n = S_1 \cap S_2 \cap S_3 \cap ....$$

Prove that $S = \emptyset$.

Proof by contradiction: Assume that there is a set with no intersecting values

$S_n = \{x | x \in \mathbb{R}, 0 < x < \frac{1}{n}\}$ where $\frac{1}{n}$ is the upper bound.

Since n goes on $\forall \mathbb{R}$ from 0 to ∞, $\frac{1}{n}$ will approach 0 but never = 0.

Consider a value j where $j = n + 1$ and $S_j = \{x | x \in \mathbb{R}, 0 < x < \frac{1}{j}\}$.
Since $\frac{1}{n} > \frac{1}{j}$, any value in S_j would be within S_n which would be replicated $\forall n$ since n approaches ∞.
$\therefore S \neq \emptyset$.

Problem 2.2. Let F_n be the n^{th} Fibonacci number. Prove that $\forall n \in \mathbb{N}$, $gcd(F_n, F_n + 1) = 1$.

In the Fibonacci sequence: $F_n + 1 = F_n + F_n - 1$.

$\therefore gcd(F_n + F_n - 1, F_n) = gcd(F_n + 1, F_n)$.

Base case: $gcd(F_1, F_2) = 1, \quad (F_1 = 1, F_2 = 1)$

Assume that: $(F_n - 1, F_n) = 1, \quad$ where $n \geq 2$.

Since: $F_n + 1 = F_n + F_n - 1$, and $gcd(F_n + F_n - 1, F_n) = gcd(F_n, F_n + 1)$

$\therefore gcd(F_n, F_n + 1) = 1 \quad \blacksquare$

Problem 2.3. $\forall n \in \mathbb{N}$, prove that

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k \right)^2.$$

Assume that: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Then: $\left(\displaystyle\sum_{k=1}^{n} k \right)^2 = \left(\frac{n(n+1)}{2} \right)^2 = \frac{n^2(n+1)^2}{4}$.

Now then, assume that: $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

Inductive step: Assume $\left(\frac{n(n+1)}{2} \right)^2$ to be true $\forall n$ up until $(n+1)$.

Use $P(n) \Rightarrow P(n+1)$.

$$P(n+1) = \left(\frac{n(n+1)}{2} + (n+1) \right)^2$$

$$\downarrow$$

$$\left(\frac{n^2+n}{2} + (n+1) \right)^2 = \left(\frac{n^2+3n+2}{2} \right)^2$$

$$= \frac{(n^2+3n+2)(n^2+3n+2)}{4} = \frac{n^4+3n^3+2n^2+3n^3+9n^2+6n+2n^2+6n+4}{4}$$

$$= \frac{n^4+6n^3+13n^2+12n+4}{4} = \frac{(n+1)(n+1)(n+2)(n+2)}{4} = \frac{(n+1)^2(n+2)^2}{4}$$

$P(n+1)$ for $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3$

$$= \frac{n^2(n^2+2n+1)}{4} + \frac{4(n+1)^3}{4} = \frac{n^4+2n^3+n^2}{4} + \frac{4(n^3+3n^2+3n+1)}{4}$$

$$= \frac{n^4+2n^3+n^2+4n^3+12n^2+12n+4}{4} = \frac{n^4+6n^3+13n^2+12n+4}{4}$$

$$= \frac{(n+1)(n+1)(n+2)(n+2)}{4} = \frac{(n+1)^2(n+2)^2}{4}$$

$\therefore$ Since $\left(\frac{n(n+1)}{2} + (n+1) \right)^2 = \frac{n^2(n+1)^2}{4} + (n+1)^3$

$\displaystyle\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k \right)^2 \ \blacksquare$

Problem 2.4. Prove that for a list of any size, the function below will return the largest item from the list.

```
if len(lst) == 0:
    return None
elif len(lst) == 1:
    return lst[0]
# lst[~0] represents the last item in the array
# equivilent to lst[-1]
maximum = maxItem(lst[:~0])
if lst[~0] < maximum:
    return maximum
else:
    return lst[~0]
```

Let the list foo = [4, 3, 2, 5, 1, 6, 2].

Following the code given: foo is broken up into pairs comparing each value with the next element, starting from index 0, and returning the largest value in each pair through every recursive iteration.

<u>Function calls</u>
1. $maximum = maxItem([4, 3, 2, 5, 1, 6])$
2. $maximum = maxItem([4, 3, 2, 5, 1])$
3. $maximum = maxItem([4, 3, 2, 5])$
4. $maximum = maxItem([4, 3, 2])$
5. $maximum = maxItem([4, 3])$
6. $maximum = maxItem([4])$ ← Base case reached, len(foo) = 1

<u>Returning</u>
1. $maximum = 4$
2. $maximum = 4$ $(3 < 4)$ ← pair
3. $maximum = 4$ $(2 < 4)$ ← pair
4. $maximum = 5$ $(5 \not< 4)$ ← pair
5. $maximum = 5$ $(1 < 5)$ ← pair
6. $maximum = 6$ $(6 \not< 5)$ ← pair

6 is the largest item in foo.

∴ Since $foo[\tilde{\ }0] < maximum$ will always check for the larger value in each pair, and return it if $foo[\tilde{\ }0]$ is not more than the current largest value in the list (or else, $foo[\tilde{\ }0]$ would be returned as the new current maximum value), the function will always return the largest item from the list of any size. ■