HW 1

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September 25, 2022

Problem 1.1. Let $x, y, z \in \mathbb{R}$

1. $\forall x \exists y \ \ s.t. \ \ x + y = 1$
True
Negation: $\neg(\forall x \exists y \ \ s.t. \ \ x + y = 1) = \exists x \forall y \ \ s.t. \ \ x + y \neq 1$

2. $\exists x \forall y \ \ s.t. \ \ x + y = 1$
True
Negation: $\neg(\exists x \forall y \ \ s.t. \ \ x + y = 1) = \forall x \exists y \ \ s.t. \ \ x + y \neq 1$

3. $\exists x \exists y \forall z \ \ s.t. \ \ yz = x$
True
Negation: $\neg(\exists x \exists y \forall z \ \ s.t. \ \ yz = x) = \forall x \forall y \exists z \ \ s.t. \ \ yz \neq x$

Problem 1.2. Show that if a condition P is both necessary ($\overline{P} \Rightarrow \overline{Q}$) and sufficient (P $\Rightarrow$ Q), that this is logically equivalent to P = Q.

p	q	$p \Rightarrow q$	$\overline{p} \Rightarrow \overline{q}$	$(\overline{p} \Rightarrow \overline{q}) \wedge (p \Rightarrow q)$	$p = q$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	1	0	0	0
0	0	1	1	1	1

Since $(\overline{p} \Rightarrow \overline{q}) \wedge p \Rightarrow q$ shares the same truth table as $p = q$, they are logically equivalent.

Problem 1.3. Let c,d be two single digit numbers, $0 \leq$ c, d ≤ 9. We will create functions $f_1, f_2, f_3...$ that are as follows:

$$
\begin{aligned}
f_1(c,d) &= cd \\
f_2(c,d) &= cdcd \\
f_3(c,d) &= cdcdcd \\
f_4(c,d) &= cdcdcdcd \\
&\vdots
\end{aligned}
$$

For example, $f_3(4,7) = 474747$, as we are repeating the digits 4,7, 3 times. Prove that $\forall 0 \le c, d \le 9, 37 | f_9(c,d)$. In other words, prove that for any possible input into f_9, the output is divisible by 37.

Proof by exhaustion:

Let k be some number in $37k = 0$
$k = \frac{0}{37} = 0$, $k \in \mathbb{Z}$
$\therefore 37|0$

Let k be some number in $37k = 10101010101010101$
$k = \frac{10101010101010101}{37} = 273000273000273$, $k \in \mathbb{Z}$
$\therefore 37|10101010101010101$

Let k be some number in $37k = 20202020202020202$
$k = \frac{20202020202020202}{37} = 546000546000546$, $k \in \mathbb{Z}$
$\therefore 37|20202020202020202$

Let k be some number in $37k = 30303030303030303$
$k = \frac{30303030303030303}{37} = 819000819000819$, $k \in \mathbb{Z}$
$\therefore 37|30303030303030303$

Let k be some number in $37k = 40404040404040404$
$k = \frac{40404040404040404}{37} = 1092001092001092$, $k \in \mathbb{Z}$
$\therefore 37|40404040404040404$

Let k be some number in $37k = 50505050505050505$
$k = \frac{50505050505050505}{37} = 1365001365001365$, $k \in \mathbb{Z}$
$\therefore 37|50505050505050505$

Let k be some number in $37k = 60606060606060606$
$k = \frac{60606060606060606}{37} = 1638001638001638$, $k \in \mathbb{Z}$
$\therefore 37|60606060606060606$

Let k be some number in $37k = 70707070707070707$
$k = \frac{70707070707070707}{37} = 1911001911001911$, $k \in \mathbb{Z}$
$\therefore 37|70707070707070707$

Let k be some number in $37k = 80808080808080808$
$k = \frac{80808080808080808}{37} = 2184002184002184$, $k \in \mathbb{Z}$

$\therefore 37|80808080808080808$

Let k be some number in $37k = 90909090909090909$
$k = \frac{90909090909090909}{37} = 2457002457002457,\ k \in \mathbb{Z}$
$\therefore 37|90909090909090909$

Let k be some number in $37k = 101010101010101010$
$k = \frac{101010101010101010}{37} = 2730002730002730,\ k \in \mathbb{Z}$
$\therefore 37|101010101010101010$

Let k be some number in $37k = 111111111111111111$
$k = \frac{111111111111111111}{37} = 3003003003003003,\ k \in \mathbb{Z}$
$\therefore 37|111111111111111111$

Let k be some number in $37k = 121212121212121212$
$k = \frac{121212121212121212}{37} = 3276003276003276,\ k \in \mathbb{Z}$
$\therefore 37|121212121212121212$

Let k be some number in $37k = 131313131313131313$
$k = \frac{131313131313131313}{37} = 3549003549003549,\ k \in \mathbb{Z}$
$\therefore 37|131313131313131313$

Let k be some number in $37k = 141414141414141414$
$k = \frac{141414141414141414}{37} = 3822003822003822,\ k \in \mathbb{Z}$
$\therefore 37|141414141414141414$

Let k be some number in $37k = 151515151515151515$
$k = \frac{151515151515151515}{37} = 4095004095004095,\ k \in \mathbb{Z}$
$\therefore 37|151515151515151515$

Let k be some number in $37k = 161616161616161616$
$k = \frac{161616161616161616}{37} = 4368004368004368,\ k \in \mathbb{Z}$
$\therefore 37|161616161616161616$

Let k be some number in $37k = 171717171717171717$
$k = \frac{171717171717171717}{37} = 4641004641004641,\ k \in \mathbb{Z}$
$\therefore 37|171717171717171717$

Let k be some number in $37k = 181818181818181818$
$k = \frac{181818181818181818}{37} = 4914004914004914,\ k \in \mathbb{Z}$
$\therefore 37|181818181818181818$

Let k be some number in $37k = 191919191919191919$
$k = \frac{191919191919191919}{37} = 5187005187005187,\ k \in \mathbb{Z}$
$\therefore 37|191919191919191919$

Let k be some number in $37k = 202020202020202020$
$k = \frac{202020202020202020}{37} = 5460005460005460$, $k \in \mathbb{Z}$
$\therefore 37 | 202020202020202020$

Let k be some number in $37k = 212121212121212121$
$k = \frac{212121212121212121}{37} = 5733005733005733$, $k \in \mathbb{Z}$
$\therefore 37 | 212121212121212121$

Let k be some number in $37k = 222222222222222222$
$k = \frac{222222222222222222}{37} = 6006006006006006$, $k \in \mathbb{Z}$
$\therefore 37 | 222222222222222222$

Let k be some number in $37k = 232323232323232323$
$k = \frac{232323232323232323}{37} = 6279006279006279$, $k \in \mathbb{Z}$
$\therefore 37 | 232323232323232323$

Let k be some number in $37k = 242424242424242424$
$k = \frac{242424242424242424}{37} = 6552006552006552$, $k \in \mathbb{Z}$
$\therefore 37 | 242424242424242424$

Let k be some number in $37k = 252525252525252525$
$k = \frac{252525252525252525}{37} = 6825006825006825$, $k \in \mathbb{Z}$
$\therefore 37 | 252525252525252525$

Let k be some number in $37k = 262626262626262626$
$k = \frac{262626262626262626}{37} = 7098007098007098$, $k \in \mathbb{Z}$
$\therefore 37 | 262626262626262626$

Let k be some number in $37k = 272727272727272727$
$k = \frac{272727272727272727}{37} = 7371007371007371$, $k \in \mathbb{Z}$
$\therefore 37 | 272727272727272727$

Let k be some number in $37k = 282828282828282828$
$k = \frac{282828282828282828}{37} = 7644007644007644$, $k \in \mathbb{Z}$
$\therefore 37 | 282828282828282828$

Let k be some number in $37k = 292929292929292929$
$k = \frac{292929292929292929}{37} = 7917007917007917$, $k \in \mathbb{Z}$
$\therefore 37 | 292929292929292929$

Let k be some number in $37k = 303030303030303030$
$k = \frac{303030303030303030}{37} = 8190008190008190$, $k \in \mathbb{Z}$
$\therefore 37 | 303030303030303030$

Let k be some number in $37k = 313131313131313131$
$k = \frac{313131313131313131}{37} = 8463008463008463$, $k \in \mathbb{Z}$

$\therefore 37|3131313131313131313131$

Let k be some number in $37k = 323232323232323232$
$k = \frac{323232323232323232}{37} = 8736008736008736$, $k \in \mathbb{Z}$
$\therefore 37|323232323232323232$

Let k be some number in $37k = 333333333333333333$
$k = \frac{333333333333333333}{37} = 9009009009009008$, $k \in \mathbb{Z}$
$\therefore 37|333333333333333333$

Let k be some number in $37k = 343434343434343434$
$k = \frac{343434343434343434}{37} = 9282009282009282$, $k \in \mathbb{Z}$
$\therefore 37|343434343434343434$

Let k be some number in $37k = 353535353535353535$
$k = \frac{353535353535353535}{37} = 9555009555009556$, $k \in \mathbb{Z}$
$\therefore 37|353535353535353535$

Let k be some number in $37k = 363636363636363636$
$k = \frac{363636363636363636}{37} = 9828009828009828$, $k \in \mathbb{Z}$
$\therefore 37|363636363636363636$

Let k be some number in $37k = 373737373737373737$
$k = \frac{373737373737373737}{37} = 10101010101010100$, $k \in \mathbb{Z}$
$\therefore 37|373737373737373737$

Let k be some number in $37k = 383838383838383838$
$k = \frac{383838383838383838}{37} = 10374010374010374$, $k \in \mathbb{Z}$
$\therefore 37|383838383838383838$

Let k be some number in $37k = 393939393939393939$
$k = \frac{393939393939393939}{37} = 10647010647010648$, $k \in \mathbb{Z}$
$\therefore 37|393939393939393939$

Let k be some number in $37k = 404040404040404040$
$k = \frac{404040404040404040}{37} = 10920010920010920$, $k \in \mathbb{Z}$
$\therefore 37|404040404040404040$

Let k be some number in $37k = 414141414141414141$
$k = \frac{414141414141414141}{37} = 11193011193011192$, $k \in \mathbb{Z}$
$\therefore 37|414141414141414141$

Let k be some number in $37k = 424242424242424242$
$k = \frac{424242424242424242}{37} = 11466011466011466$, $k \in \mathbb{Z}$
$\therefore 37|424242424242424242$

Let k be some number in $37k = 434343434343434343$
$k = \frac{434343434343434343}{37} = 11739011739011740$, $k \in \mathbb{Z}$
$\therefore 37|434343434343434343$

Let k be some number in $37k = 444444444444444444$
$k = \frac{444444444444444444}{37} = 12012012012012012$, $k \in \mathbb{Z}$
$\therefore 37|444444444444444444$

Let k be some number in $37k = 454545454545454545$
$k = \frac{454545454545454545}{37} = 12285012285012284$, $k \in \mathbb{Z}$
$\therefore 37|454545454545454545$

Let k be some number in $37k = 464646464646464646$
$k = \frac{464646464646464646}{37} = 12558012558012558$, $k \in \mathbb{Z}$
$\therefore 37|464646464646464646$

Let k be some number in $37k = 474747474747474747$
$k = \frac{474747474747474747}{37} = 12831012831012832$, $k \in \mathbb{Z}$
$\therefore 37|474747474747474747$

Let k be some number in $37k = 484848484848484848$
$k = \frac{484848484848484848}{37} = 13104013104013104$, $k \in \mathbb{Z}$
$\therefore 37|484848484848484848$

Let k be some number in $37k = 494949494949494949$
$k = \frac{494949494949494949}{37} = 13377013377013376$, $k \in \mathbb{Z}$
$\therefore 37|494949494949494949$

Let k be some number in $37k = 505050505050505050$
$k = \frac{505050505050505050}{37} = 13650013650013650$, $k \in \mathbb{Z}$
$\therefore 37|505050505050505050$

Let k be some number in $37k = 515151515151515151$
$k = \frac{515151515151515151}{37} = 13923013923013924$, $k \in \mathbb{Z}$
$\therefore 37|515151515151515151$

Let k be some number in $37k = 525252525252525252$
$k = \frac{525252525252525252}{37} = 14196014196014196$, $k \in \mathbb{Z}$
$\therefore 37|525252525252525252$

Let k be some number in $37k = 535353535353535353$
$k = \frac{535353535353535353}{37} = 14469014469014468$, $k \in \mathbb{Z}$
$\therefore 37|535353535353535353$

Let k be some number in $37k = 545454545454545454$
$k = \frac{545454545454545454}{37} = 14742014742014742$, $k \in \mathbb{Z}$

$\therefore 37|545454545454545454$

Let k be some number in $37k = 555555555555555555$
$k = \frac{555555555555555555}{37} = 15015015015015016,\ k \in \mathbb{Z}$
$\therefore 37|555555555555555555$

Let k be some number in $37k = 565656565656565656$
$k = \frac{565656565656565656}{37} = 15288015288015288,\ k \in \mathbb{Z}$
$\therefore 37|565656565656565656$

Let k be some number in $37k = 575757575757575757$
$k = \frac{575757575757575757}{37} = 15561015561015560,\ k \in \mathbb{Z}$
$\therefore 37|575757575757575757$

Let k be some number in $37k = 585858585858585858$
$k = \frac{585858585858585858}{37} = 15834015834015834,\ k \in \mathbb{Z}$
$\therefore 37|585858585858585858$

Let k be some number in $37k = 595959595959595959$
$k = \frac{595959595959595959}{37} = 16107016107016108,\ k \in \mathbb{Z}$
$\therefore 37|595959595959595959$

Let k be some number in $37k = 606060606060606060$
$k = \frac{606060606060606060}{37} = 16380016380016380,\ k \in \mathbb{Z}$
$\therefore 37|606060606060606060$

Let k be some number in $37k = 616161616161616161$
$k = \frac{616161616161616161}{37} = 16653016653016652,\ k \in \mathbb{Z}$
$\therefore 37|616161616161616161$

Let k be some number in $37k = 626262626262626262$
$k = \frac{626262626262626262}{37} = 16926016926016926,\ k \in \mathbb{Z}$
$\therefore 37|626262626262626262$

Let k be some number in $37k = 636363636363636363$
$k = \frac{636363636363636363}{37} = 17199017199017200,\ k \in \mathbb{Z}$
$\therefore 37|636363636363636363$

Let k be some number in $37k = 646464646464646464$
$k = \frac{646464646464646464}{37} = 17472017472017472,\ k \in \mathbb{Z}$
$\therefore 37|646464646464646464$

Let k be some number in $37k = 656565656565656565$
$k = \frac{656565656565656565}{37} = 17745017745017744,\ k \in \mathbb{Z}$
$\therefore 37|656565656565656565$

Let k be some number in $37k = 666666666666666666$
$k = \frac{666666666666666666}{37} = 18018018018018016$, $k \in \mathbb{Z}$
$\therefore 37 | 666666666666666666$

Let k be some number in $37k = 676767676767676767$
$k = \frac{676767676767676767}{37} = 18291018291018292$, $k \in \mathbb{Z}$
$\therefore 37 | 676767676767676767$

Let k be some number in $37k = 686868686868686868$
$k = \frac{686868686868686868}{37} = 18564018564018564$, $k \in \mathbb{Z}$
$\therefore 37 | 686868686868686868$

Let k be some number in $37k = 696969696969696969$
$k = \frac{696969696969696969}{37} = 18837018837018836$, $k \in \mathbb{Z}$
$\therefore 37 | 696969696969696969$

Let k be some number in $37k = 707070707070707070$
$k = \frac{707070707070707070}{37} = 19110019110019112$, $k \in \mathbb{Z}$
$\therefore 37 | 707070707070707070$

Let k be some number in $37k = 717171717171717171$
$k = \frac{717171717171717171}{37} = 19383019383019384$, $k \in \mathbb{Z}$
$\therefore 37 | 717171717171717171$

Let k be some number in $37k = 727272727272727272$
$k = \frac{727272727272727272}{37} = 19656019656019656$, $k \in \mathbb{Z}$
$\therefore 37 | 727272727272727272$

Let k be some number in $37k = 737373737373737373$
$k = \frac{737373737373737373}{37} = 19929019929019928$, $k \in \mathbb{Z}$
$\therefore 37 | 737373737373737373$

Let k be some number in $37k = 747474747474747474$
$k = \frac{747474747474747474}{37} = 20202020202020200$, $k \in \mathbb{Z}$
$\therefore 37 | 747474747474747474$

Let k be some number in $37k = 757575757575757575$
$k = \frac{757575757575757575}{37} = 20475020475020476$, $k \in \mathbb{Z}$
$\therefore 37 | 757575757575757575$

Let k be some number in $37k = 767676767676767676$
$k = \frac{767676767676767676}{37} = 20748020748020748$, $k \in \mathbb{Z}$
$\therefore 37 | 767676767676767676$

Let k be some number in $37k = 777777777777777777$
$k = \frac{777777777777777777}{37} = 21021021021021020$, $k \in \mathbb{Z}$

$\therefore 37|777777777777777777$

Let k be some number in $37k = 787878787878787878$
$k = \frac{787878787878787878}{37} = 21294021294021296$, $k \in \mathbb{Z}$
$\therefore 37|787878787878787878$

Let k be some number in $37k = 797979797979797979$
$k = \frac{797979797979797979}{37} = 21567021567021568$, $k \in \mathbb{Z}$
$\therefore 37|797979797979797979$

Let k be some number in $37k = 808080808080808080$
$k = \frac{808080808080808080}{37} = 21840021840021840$, $k \in \mathbb{Z}$
$\therefore 37|808080808080808080$

Let k be some number in $37k = 818181818181818181$
$k = \frac{818181818181818181}{37} = 22113022113022112$, $k \in \mathbb{Z}$
$\therefore 37|818181818181818181$

Let k be some number in $37k = 828282828282828282$
$k = \frac{828282828282828282}{37} = 22386022386022384$, $k \in \mathbb{Z}$
$\therefore 37|828282828282828282$

Let k be some number in $37k = 838383838383838383$
$k = \frac{838383838383838383}{37} = 22659022659022660$, $k \in \mathbb{Z}$
$\therefore 37|838383838383838383$

Let k be some number in $37k = 848484848484848484$
$k = \frac{848484848484848484}{37} = 22932022932022932$, $k \in \mathbb{Z}$
$\therefore 37|848484848484848484$

Let k be some number in $37k = 858585858585858585$
$k = \frac{858585858585858585}{37} = 23205023205023204$, $k \in \mathbb{Z}$
$\therefore 37|858585858585858585$

Let k be some number in $37k = 868686868686868686$
$k = \frac{868686868686868686}{37} = 23478023478023480$, $k \in \mathbb{Z}$
$\therefore 37|868686868686868686$

Let k be some number in $37k = 878787878787878787$
$k = \frac{878787878787878787}{37} = 23751023751023752$, $k \in \mathbb{Z}$
$\therefore 37|878787878787878787$

Let k be some number in $37k = 888888888888888888$
$k = \frac{888888888888888888}{37} = 24024024024024024$, $k \in \mathbb{Z}$
$\therefore 37|888888888888888888$

Let k be some number in $37k = 898989898989898989$
$k = \frac{898989898989898989}{37} = 24297024297024296$, $k \in \mathbb{Z}$
$\therefore 37|898989898989898989$

Let k be some number in $37k = 909090909090909090$
$k = \frac{909090909090909090}{37} = 24570024570024568$, $k \in \mathbb{Z}$
$\therefore 37|909090909090909090$

Let k be some number in $37k = 919191919191919191$
$k = \frac{919191919191919191}{37} = 24843024843024844$, $k \in \mathbb{Z}$
$\therefore 37|919191919191919191$

Let k be some number in $37k = 929292929292929292$
$k = \frac{929292929292929292}{37} = 25116025116025116$, $k \in \mathbb{Z}$
$\therefore 37|929292929292929292$

Let k be some number in $37k = 939393939393939393$
$k = \frac{939393939393939393}{37} = 25389025389025388$, $k \in \mathbb{Z}$
$\therefore 37|939393939393939393$

Let k be some number in $37k = 949494949494949494$
$k = \frac{949494949494949494}{37} = 25662025662025664$, $k \in \mathbb{Z}$
$\therefore 37|949494949494949494$

Let k be some number in $37k = 959595959595959595$
$k = \frac{959595959595959595}{37} = 25935025935025936$, $k \in \mathbb{Z}$
$\therefore 37|959595959595959595$

Let k be some number in $37k = 969696969696969696$
$k = \frac{969696969696969696}{37} = 26208026208026208$, $k \in \mathbb{Z}$
$\therefore 37|969696969696969696$

Let k be some number in $37k = 979797979797979797$
$k = \frac{979797979797979797}{37} = 26481026481026480$, $k \in \mathbb{Z}$
$\therefore 37|979797979797979797$

Let k be some number in $37k = 989898989898989898$
$k = \frac{989898989898989898}{37} = 26754026754026752$, $k \in \mathbb{Z}$
$\therefore 37|989898989898989898$

Let k be some number in $37k = 999999999999999999$
$k = \frac{999999999999999999}{37} = 27027027027027028$, $k \in \mathbb{Z}$
$\therefore 37|999999999999999999$

$\therefore \forall 0 \leq c, d \leq 9$, $37|f_9(c, d)$. $\blacksquare$

Problem 1.4. To ride the PATH train from Newark to New York, you must swipe a metro card that costs \$2.75 per ride. Suppose my friend is moving out of New York and they give me their old metro card that has some amount of money on it, given as M (can be any dollar/cent amount).

Part 1: Suppose every time I refill this card, I only refill with \$20.00 exactly. Assuming that I only refill when I do not have enough left in the card to pay for my next ride, provide and prove all values of M for which the metro card amount will eventually reach 0.

Theorem: The equation $ax + by = c$ has a solution iff $gcd(a, b)|c$.
Let $a = 2.75, \quad b = 20$
Let $M = 0.25k, \forall k \in \mathbb{Z} \quad gcd(2.75, 20) = 0.25 = d$
Let $c = M$
Proof: Suppose $gcd(a, b) = d \Rightarrow a = k_1 \cdot d, \;\; b = k_2 \cdot d, \;\; k \in \mathbb{Z}$

$$k_1dx - k_2dy = c$$
$$k_1x - k_2y = \frac{c}{d} \qquad (\forall x \text{ and } \forall y \in \mathbb{Z})$$

$\therefore d|c \;\; \blacksquare$

Part 2: Suppose instead of paying \$20.00, every time I refill using some amount of money R (represented in **cents**). Prove that so long as R is not divisible by 5 or 11, that the metro card will eventually reach 0 for any value of M.

Theorem: The equation $ax + by = c$ has a solution iff $gcd(a, b)|c$.
$k \in \mathbb{Z}$
Let $a = 5k$, $b = 11k$, $c = M$
Let $d_1 = 5$, $d_2 = 11$
Since $gcd(2.75, a) = d_1$ and $gcd(2.75, b) = d_2$ where both $(d_1 \text{ and } d_2) > 1)$.
Let $R = \forall r$ that is $(\overline{a} \wedge \overline{b})$.
Proof: Suppose $gcd(2.75, R) = d, \quad d = 1$

$$R = k_1d, \quad 2.75 = k_2d, \quad k_1, k_2 \in \mathbb{Z}$$
$$k_1dx - k_2dy = c \qquad \forall x, \forall y \in \mathbb{Z}$$
$$k_1x - k_2y = \frac{c}{d}$$

$\therefore d|c \; \blacksquare$

References:
https://math.stackexchange.com
Negation
Logical Equivalence