Complement the proof of soundness and completeness of propositional logic by proving the following cases:

For soundness proof, complete the proof of the following proof rule cases:

- $\wedge e1$

$$\frac{\Phi \wedge \psi}{\Phi} \wedge e_1:$$

It must be the case that some formula $x_1 \wedge x_2$ appears in the proof, with $x_k = x_1$ The formula $x_1 \wedge x_2$ have shorter proofs and therefore, using the hypothesis, it have the truth value T. Using the truth table for $\wedge$, we conclude that the truth value of x_k (x_1) is T

- $\wedge e2$

$$\frac{\Phi \wedge \psi}{\Phi} \wedge e_1:$$

It must be the case that some formula $x_1 \wedge x_2$ appears in the proof, with $x_k = x_2$ The formula $x_1 \wedge x_2$ have shorter proofs and therefore, using the hypothesis, it have the truth value T. Using the truth table for $\wedge$, we conclude that the truth value of x_k (x_2) is T

- $\perp e$

$\perp e: \dfrac{\perp}{\Phi}$

It must be the case that some formula $\perp$ appears in the proof. The proof of $\perp$ is shorter hence, according to the induction hypothesis, $\perp$ is true. According to the truth table of $\perp$, every formula is always true. So we conclude that the truth value of x_k is T.

There must be some pair $(d_i, \perp)$ where $i < k$ i.e. $\Gamma, d_i \vdash \perp$. According to the induction hypothesis, $\Gamma, d_i \vDash \perp$. Since $\perp$ is always F, there must be some contradiction in Γ, d_i. According to the definition of $\perp e$, $d_k \geq d_i$. Thus Γ, d_k is a contradiction, which means $\Gamma, d_k \vDash \Phi$. Thus x_k is T

- $\neg i$

$\neg i: \dfrac{\boxed{\begin{matrix}\Phi \\ \vdots \\ \perp\end{matrix}}}{\neg\Phi}$

It must be the case that some formula X_1 appears in the proof, and $X_k = \neg X_1$ and that we have one box with assumptions X_1 and conclusion $\perp$.

The proof of X_1 is shorter hence, according to the induction hypothesis, it has a truth value of T.

Then, the assumption X_1 of the box is true, using the induction hypothesis, its conclusion $\perp$ is true.

and according to the truth table of $\perp$, we conclude that the truth table of X_k is true

- $\neg\neg e$

$\neg\neg e$: $\dfrac{\neg\neg\Phi}{\Phi}$

It must be the case that some formula x_1 appears in the proof, and $x_K = \neg\neg x_1$. The proof of x_1 is shorter hence according to the induction hypothesis, it has a truth value of T

According to truth table of $\neg$, ~~x~~ $\neg\neg x_1$ is T.

Using the induction hypothesis, we conclude that the truth value of x_K is T

- $\vee\, i1$

$\vee i_1$: $\dfrac{\Phi}{\Phi \vee \psi}$

It must be the case that some formular X_1 appears in the proof, and $x_K = x_1 \vee x_2$.

The proof of x_1 is shorter hence according to the induction hypothesis, it has a truth value of T.

According to truth table of $\vee$, $x_1 \vee x_2$ is true,

Using the induction hypothesiss, we conclude that the truth value of x_k is T

- $\vee\, i2$

$$\vee i_1: \frac{\phi}{\phi \vee \psi}$$

It must be the case that some formular χ_2 appears in the proof, and $\chi_k = \chi_1 \vee \chi_2$.

The proof of χ_2 is shorter hence according to the indution hypothesis, it has a truth value of T.

According to truth table of $\vee$, $\chi_1 \vee \chi_2$ is true, using the indution hypothesiss, we conclude that the truth value of χ_k is T

- $\rightarrow i$

$$\rightarrow i: \frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \rightarrow \psi}$$

It must be the case that some formular χ_1 appears in the proof, and that we have one box with assumptions χ_1 and conclusion χ_2, and $\chi_k = (\chi_1 \rightarrow \chi_2)$

The proof of χ_1 is shorter hence according to the induction hypothesis, it has a truth value of T.

Then, the assumption χ_1 of the box is true, and using the indution hypothesis, its conclusion χ_2 has the truth value T.

According to the truth table of $\rightarrow$, we conclude that χ_k is T

- $\rightarrow e$

$\to e$: $\dfrac{\Phi \quad \Phi \to \psi}{\psi}$

It must be the case that some formular x_1, $x_1 \to x_2$ appears in the proof, and $x_K = x_2$.

The proof of x_1, $x_1 \to x_2$ is shorter hence x_1 is T, $x_1 \to x_2$ is T. According to the truth table of $\to$, we conclude that x_2 is T.

Hence x_K has the truth value of T

- ¬e

$\neg e$ $\dfrac{\Phi \quad \neg\Phi}{\perp}$

It must be the case that some formula x_1, $\neg x_1$ appears in the proof, and $x_K = \perp$.

The formulas x_1, $\neg x_1$ have shorter proofs, and using induction hypothesis, x_1 and $\neg x_1$ have the truth value T.

According to the truth table of $\neg$, we conclude x_1 is F which contradicts with x_1 have the truth value T.

According to the truth value of $\perp$, we conclude $\perp$ is T.
Hence x_K has the truth value of T

For completeness proof, complete the proof of the following cases:

- 1.

$\Phi_1 \wedge \Phi_2 \vdash \Phi_1 \vee \Phi_2$ natural deduction

1	$\Phi_1 \wedge \Phi_2$	premise
2	Φ_1	$\wedge e_1$
3	$\Phi_1 \vee \Phi_2$	$\vee i_1$

- 2.

$\Phi_1 \wedge \neg\Phi_2 \vdash \Phi_1 \vee \Phi_2$

1	$\Phi_1 \wedge \neg\Phi_2$	premise
2	Φ_1	$\wedge e_1$
3	$\Phi_1 \vee \Phi_2$	$\vee i_1$

- 3.

$\neg\Phi_1 \wedge \Phi_2 \vdash \Phi_1 \vee \Phi_2$

1	$\neg\Phi_1 \wedge \Phi_2$	premise
2	Φ_2	$\wedge e_2$
3	$\Phi_1 \vee \Phi_2$	$\vee i_2$

- 4.

$\neg\Phi_1 \wedge \neg\Phi_2 \vdash \neg(\Phi_1 \vee \Phi_2)$

1	$\neg\Phi_1 \wedge \neg\Phi_2$	premise
2	$\neg\Phi_1$	$\wedge e_1$
3	$\neg\Phi_2$	$\wedge e_2$
4	$\Phi_1 \vee \Phi_2$	assumption
5	Φ_1	assumption
6	$\perp$	$\neg e$ 2, 5
7	$\neg(\Phi_1 \vee \Phi_2)$	$\neg i$ 4-6
8	$\Phi_1 \vee \Phi_2$	assumption, 4
9	Φ_2	assumption
10	$\perp$	$\neg e$
11	$\neg(\Phi_1 \vee \Phi_2)$	$\neg i$
12	$\neg(\Phi_1 \vee \Phi_2)$	$\vee e$

- 5.

$\neg\Phi_1 \wedge \neg\Phi_2 \vdash \neg(\Phi_1 \vee \Phi_2)$

1	$\neg\Phi_1 \wedge \neg\Phi_2$	premise
2	$\neg\Phi_1$	$\wedge e_1$
3	$\neg\Phi_2$	$\wedge e_2$
4	$\Phi_1 \vee \Phi_2$	assumption
5	Φ_1	assumption
6	$\perp$	$\neg e$ 2, 5
7	$\neg(\Phi_1 \vee \Phi_2)$	$\neg i$ 4-6
8	$\Phi_1 \vee \Phi_2$	assumption, 4
9	Φ_2	assumption
10	$\perp$	$\neg e$ 3, 9
11	$\neg(\Phi_1 \vee \Phi_2)$	$\neg i$ 8-10
12	$\neg(\Phi_1 \vee \Phi_2)$	$\vee e$ 4, 5-7, 9-11

- 6.

$\Phi_1 \land \Phi_2 \vdash \Phi_1 \to \Phi_2$

1	$\Phi_1 \land \Phi_2$	premise
2	Φ_1	assumption
3	Φ_2	$\land e_2$ 1
4	$\Phi_1 \to \Phi_2$	$\to i$ 2-3

- 7.

$\Phi_1 \land \neg\Phi_2 \vdash \neg(\Phi_1 \to \Phi_2)$

1	$\Phi_1 \land \neg\Phi_2$	premise
2	$\Phi_1 \to \Phi_2$	assumption
3	Φ_1	$\land e_1$ 1
4	Φ_2	$\to e$ 2,3
5	$\neg\Phi_2$	$\land e_2$ 1
6	$\bot$	$\neg e$ 4,5
7	$\neg(\Phi_1 \to \Phi_2)$	$\neg i$ 2-6

- 8.

$\neg\Phi_1 \wedge \Phi_2 \vdash \Phi_1 \to \Phi_2$

1	$\neg\Phi_1 \wedge \Phi_2$	premise
2	Φ_1	assumption
3	Φ_2	$\wedge e_2$ 1
4	$\Phi_1 \to \Phi_2$	$\to i$ 2-3

- 9.

$\neg\Phi_1 \wedge \neg\Phi_2 \vdash \Phi_1 \to \Phi_2$

1	$\neg\Phi_1 \wedge \neg\Phi_2$	premise
2	$\neg\Phi_1$	$\wedge e_1$
3	Φ_1	assumption
4	$\bot$	$\neg e$ 2,3
5	Φ_2	$\bot e$ 4
6	$\Phi_1 \to \Phi_2$	$\to i$ 3-5