- Convert the following formula into CNF
  - $\neg\Big(p \to \big(\neg\big(q \wedge (\neg p \to (q \wedge r))\big)\big)\Big)$

Let $\Phi = \neg(p \to (\neg(q \wedge (\neg p \to (q \wedge r)))))$

1. First apply IMPL_FREE:

$$\text{IMPL\_FREE}(\Phi) = \neg(\text{IMPL\_FREE}(p \to (\neg(q \wedge (\neg p \to (q \wedge r))))))$$

$$= \neg(\neg\,\text{IMPL\_FREE}(p) \vee \text{IMPL\_FREE}(\neg(q \wedge (\neg p \to (q \wedge r)))))$$

$$= \neg(\neg p \vee \text{IMPL\_FREE}(\neg(q \wedge (\neg p \to (q \wedge r)))))$$

$$= \neg(\neg p \vee \neg\,\text{IMPL\_FREE}(q \wedge (\neg p \to (q \wedge r))))$$

$$= \neg(\neg p \vee \neg(\text{IMPL\_FREE}(q) \wedge \text{IMPL\_FREE}(\neg p \to (q \wedge r))))$$

$$= \neg(\neg p \vee \neg(q \wedge \text{IMPL\_FREE}(\neg p \to (q \wedge r))))$$

$$= \neg(\neg p \vee \neg(q \wedge (\neg\,\text{IMPL\_FREE}(\neg p) \vee \text{IMPL\_FREE}(q \wedge r))))$$

$$= \neg(\neg p \vee \neg(q \wedge (\neg\neg\,\text{IMPL\_FREE}(p) \vee (\text{IMPL\_FREE}(q) \wedge \text{IMPL\_FREE}(r)))))$$

$$= \neg(\neg p \vee \neg(q \wedge (\neg\neg p \vee (q \wedge r))))$$

2. The apply NNF:

$$\text{NNF}(\text{IMPL\_FREE}(\Phi)) = \text{NNF}(\neg(\neg p \vee \neg(q \wedge (\neg\neg p \vee (q \wedge r)))))$$

$$= \text{NNF}(\neg\neg p \wedge \neg\neg(q \wedge (\neg\neg p \vee (q \wedge r))))$$

$$= \text{NNF}(\neg\neg p) \wedge \text{NNF}(\neg\neg(q \wedge (\neg\neg p \vee (q \wedge r))))$$

$$= \text{NNF}(p) \wedge \text{NNF}(q \wedge (\neg\neg p \vee (q \wedge r)))$$

$$= p \wedge \text{NNF}(q \wedge (\neg\neg p \vee (p \wedge r))) = p \wedge (\text{NNF}(q) \wedge \text{NNF}(\neg\neg p \vee (q \wedge r)))$$

$= p \wedge (q \wedge NNF(\neg\neg p \vee (q \wedge r))) = p \wedge (q \wedge (NNF(\neg\neg p) \vee NNF(q \wedge r)))$

$= p \wedge (q \wedge (NNF(p) \vee (NNF(q) \wedge NNF(r))))$

$= p \wedge (q \wedge (p \vee (q \wedge r)))$

Then use CNF: $CNF(NNF(IMPL\_FREE(\Phi))) = CNF(p \wedge (q \wedge (p \vee (q \wedge r))))$

$= CNF(p) \wedge CNF(q \wedge (p \vee (q \wedge r)))$

$= p \wedge (CNF(q) \wedge CNF(p \vee (q \wedge r)))$

$= p \wedge (q \wedge CNF(p \vee (q \wedge r)))$

$= p \wedge (q \wedge DISTR(CNF(p), CNF(q \wedge r)))$

$= p \wedge (q \wedge DISTR(p, q \wedge r))$

$= p \wedge (q \wedge DISTR(p, q) \wedge DISTR(p, r))$

$= p \wedge (q \wedge (p \vee q) \wedge (p \vee r))$

$= p \wedge q \wedge (p \vee q) \wedge (p \vee r)$

Above is the result of CNF algorithm. Furthermore, this result can be simplified by:

$p \wedge q \wedge (p \vee q) \wedge (p \vee r) = p \wedge q \wedge (p \vee r)$.

- Determine the satisfiability of the following formula with Horn algorithm
  - $(p \wedge q \wedge s \rightarrow \bot) \wedge (q \wedge s \rightarrow p) \wedge (T \rightarrow s) \wedge (s \rightarrow q)$

$\Phi = (p \wedge q \wedge s \rightarrow \bot) \wedge (q \wedge s \rightarrow p) \wedge (T \rightarrow s) \wedge (s \rightarrow q)$

Φ is a Horn formula

Now begin Horn(Φ) function:

Marked initialy : $\top$

Since $\top \to s$, mark s (Now marked: $s, \top$)

Since $s \to q$, mark q (Now marked $s, q, \top$)

since $q \wedge s \to p$, mark p (Now marked $s, q, p, \top$)

Since $p \wedge q \wedge s \to \perp$, mark $\perp$ (Now marked $s, q, p, \top, \perp$)

Since $\perp$ is marked last, from Horn's algorithm, this formula is not satisfiable.