逻辑用于描述现实世界

Logic in Computer Science

Lecture 01

Nature Deduction and Propositional Logic

自然演绎法

Li Qin

Associate Professor

Software Engineering Institute

Logic and Formal Systems — Syllabus

正确性和完备性

- ★ Propositional logic (natural deduction, semantics, soundness and completeness).

可满足性不可判定

- ★ Predicate logic (natural deduction, semantics, undecidability).
- ★ Logical Proof Tool (Rodin).
- ★ Model checking and Temporal logics (LTL, CTL)
- ★ Program verification (Floyd-Hoare logic).
- ★ Modal logic and agents. (optional)
- ★ Binary decision diagrams (optional)

推理模型 bpc
b belief
p phenomenon
c conclusion

Motivation for studying Logic: To acquire the ability to model real-life situations in a way that would allow us to reason about them formally.

即使没有学过逻辑也认可这个推导? 逻辑不是被创造的 而是被发现的 是经验总结

Example 1: If the train arrives late and there are no taxis at the station, then John is late for his meeting. John is not late for his meeting. The train did arrive late. ***Therefore***, there were taxis at the station.

b

c

p

Example 2: If it is raining and Jane does not have her umbrella with her, then she will get wet. Jane is not wet. It is raining. ***Therefore***, Jane has her umbrella with her.

公理的正确性还没办法证明

Can we verify the validity of these arguments formally?

- We need to turn the English sentences into formulas (***modeling***).
- Then, we can apply mathematical reasoning to formulas

Encoding:

	Example 1	Example 2
p	the train is late	it is raining
q	there are taxis at the station	Jane has her umbrella with her
r	John is late for his meeting	Jane gets wet

Pattern:

If p and not q, then r. Not r. p. Therefore q.

We shall study ***reasoning patterns***.

Declarative Sentences

Declarative sentences (we can consider whether they're true or not):

- The sum of the numbers 3 and 5 equals 8.
- Jane reacted violently to Jack's accusations.
- Every even natural number is the sum of two prime numbers.
- All Martians like peperoni on their pizza.

Non-declarative sentences (can't tell whether they're true or not):

- Could you please pass the salt.
- Ready, steady, go.
- May fortune come your way.

We want to turn declarative sentences into formulas and create a formalism to manipulate such formulas.

Turning English Phrases into Formulas

Atomic sentences:

如何证明推理正确

p: I won the lottery last week.

q: I purchased a lottery ticket.

r: I won last week's sweepstakes.

Connectives:

$\neg$: **negation** —$\neg p$: I did not win the lottery.

$\vee$: **disjunction** —$p \vee r$: I won the lottery last week or I won the last week's sweepstakes.

$\wedge$: **conjunction** —$p \wedge r$: I won the lottery and the sweepstakes last week.

$\rightarrow$: **implication** —$p \rightarrow q$: If I won the lottery last week, then I purchased a lottery ticket.

Composite formulas: $(p \wedge q) \rightarrow ((\neg r) \vee q)$; connective priority, $\neg$, $\wedge$, $\vee$, $\rightarrow$. By this convention, we can remove the brackets: $p \wedge q \rightarrow \neg r \vee q$.

证明规则是认知产生的，正确性没法证明，但已经被大家承认

- Collection of ***proof rules***, which allow to infer new formulas from existing formulas.

- Given the formulas $\Phi_1, \ldots, \Phi_n$, we intend to infer a conclusion Ψ. We denote this by

$$\Phi_1, \ldots, \Phi_n \vdash \Psi$$

This construct is called a ***sequent***.

逻辑与数学在各自的框架下都是自洽的

- Example:

$$p \wedge \neg q \to r, \neg r, p \vdash q$$

- There is no "perfect" set of proof rules. You can create your own (you can even invent your own logic). Such exercise resembles computer programming.

Natural Deduction Rules — Conjunction

$$\frac{\Phi \quad \Psi}{\Phi \wedge \Psi} \wedge\mathrm{i}$$

***and*-introduction**

$$\frac{\Phi \wedge \Psi}{\Phi} \wedge\mathrm{e1} \qquad \frac{\Phi \wedge \Psi}{\Psi} \wedge\mathrm{e2}$$

***and*-elimination**

证明sequent的正确性

Example: Prove $p \wedge q, r \vdash q \wedge r$

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge$e2 1
4	$q \wedge r$	$\wedge$i 3,2

Alternate way to write the proof:

$$\frac{\dfrac{p \wedge q}{q}\ \wedge\mathrm{e2} \qquad r}{q \wedge r} \wedge\mathrm{i}$$

Natural Deduction Rules — Double Negation and Implication Elimination

$$\frac{\neg\neg\Phi}{\Phi}\ \neg\neg\mathrm{e}$$

double negation elimination

$$\frac{\Phi}{\neg\neg\Phi}\ \neg\neg\mathrm{i}$$

double negation introduction

$$\frac{\Phi \quad \Phi\rightarrow\Psi}{\Psi}\ \rightarrow\mathrm{e}$$

implication elimination

Example: $p, \neg\neg(q\wedge r) \vdash \neg\neg p \wedge r$

1	p	premise
2	$\neg\neg(q\wedge r)$	premise
3	$\neg\neg p$	$\neg\neg$i 1
4	$q\wedge r$	$\neg\neg$e 2
5	r	$\wedge$e2 4
6	$\neg\neg p\wedge r$	$\wedge$i 3,5

Justification:

$$\frac{p\text{: It rained} \qquad p\rightarrow q\text{: If it rained, the street is wet}}{q\text{: The street is wet}}$$

Example: $p, p\rightarrow q, p\rightarrow(q\rightarrow r) \vdash r$

1	$p\rightarrow(q\rightarrow r)$	premise
2	$p\rightarrow q$	premise
3	p	premise
4	$q\rightarrow r$	$\rightarrow$e 1,3
5	q	$\rightarrow$e 2,3
6	r	$\rightarrow$e 4,5

Natural Deduction Rules — Implication Introduction

$$\frac{\boxed{\begin{array}{c}\Phi\\ \vdots\\ \Psi\end{array}}}{\Phi \to \Psi} \to \mathrm{i}$$

In order to prove $\Phi \to \Psi$, we make the temporary assumption of Φ, and then prove Ψ. The scope of the assumption is indicated by the box.

Example: $p \wedge q \to r \vdash p \to (q \to r)$

1	$p \wedge q \to r$	premise
2	p	assumption
3	q	assumption
4	$p \wedge q$	$\wedge$ i 2,3
5	r	$\to$e 1,4
6	$q \to r$	$\to$i 3-5
7	$p \to (q \to r)$	$\to$i 2-6

Remark: We may transform any proof of

$$\Phi_1, \ldots, \Phi_n \vdash \Psi$$

into a proof of

$$\vdash \Phi_1 \to (\Phi_2 \to (\cdots (\Phi_n \to \Psi) \cdots))$$

Implication Introduction Examples

1 $p \to (q \to r)$ premise
2 $p \wedge q$ assumption
3 p $\wedge e1$ 2
4 $q \to r$ $\to e$ 1,3
r

1 $p \to q$ premise
2 $p \wedge r$ assumption
3 p $\wedge e1$ 2
4 r $\wedge e2$ 2
5 q $\to e$ 1,3
6 $q \wedge r$ $\wedge i$ 4,5

Example: $p \to (q \to r) \vdash \underline{p \wedge q \to r}$ $\to i$

1	$p \to (q \to r)$	premise
2	$p \wedge q$	assumption
3	p	$\wedge$e1 2
4	q	$\wedge$e2 2
5	$q \to r$	$\to$e 1,3
6	r	$\to$e 4,5
7	$p \wedge q \to r$	$\to$i 2–6

Example: $p \to q \vdash \underline{p \wedge r \to q \wedge r}$ 7 $\to i$ 2-6

1	$p \to q$	premise
2	$p \wedge r$	assumption
3	p	$\wedge$e1 2
4	r	$\wedge$e2 2
5	q	$\to$e 1,3
6	$q \wedge r$	$\wedge$i 4,5
7	$p \wedge r \to q \wedge r$	$\to$i 2–6

Natural Deduction Rules — Disjunction

$$\frac{\Phi}{\Phi \vee \Psi}\ \vee\text{i1} \qquad \frac{\Psi}{\Phi \vee \Psi}\ \vee\text{i2}$$

$$\frac{\Phi \vee \Psi \qquad \boxed{\begin{array}{c}\Phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\Psi\\ \vdots\\ \chi\end{array}}}{\chi}$$

1 $q \to r$
2 $p \vee q$ assumption
3 p assumption
4 $p \vee r$ $\vee$i1 3
5 q assumption
6 r $\to$e 1,5 $\vee$e
7 $p \vee r$ $\vee$i2 6
8 $p \vee r$ $\vee$e 2, 3-4, 5-7
$p \vee q \to p \vee r$ $\to$i 2-8

Example: $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee$i2 2
4	q	assumption
5	$q \vee p$	$\vee$i1 4
6	$q \vee p$	$\vee$e 1,2-3,4-5

Example: $q \to r \vdash p \vee q \to p \vee r$

1	$q \to r$ ✓	premise
2	$p \vee q$	assumption
3	p	assumption
4	$p \vee r$	$\vee$i1 3
5	q	assumption
6	r	$\to$e 1,5
7	$p \vee r$	$\vee$i2 6
8	$p \vee r$	$\vee$e 2,3-4,5-7
9	$p \vee q \to p \vee r$ ✓	$\to$i 2-8

Contradictions: formulas of the form $\Phi \wedge \neg\Phi$, $\neg\Phi \wedge \Phi$ —all such formulas shall be denoted by $\bot$ (bottom).

$$\frac{\bot}{\Phi}\ \bot\mathrm{e} \qquad\qquad \frac{\Phi \quad \neg\Phi}{\bot}\ \neg\mathrm{e}$$

Example: $p \to q, p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 1,3
5	$\neg q$	$\to$e 2,3
6	$\bot$	$\neg$e 4,5
7	$\neg p$	$\neg$i 3-6

1 $p \to q$ premise
2 $p \to \neg q$ premise
3 p assumption
4 q $\to$e 1,3
5 $\neg q$ $\to$e 2,3
6 $\bot$ $\neg$e 4,5
7 $\neg p$ $\neg$i 3-6

Example: $\neg p \vee q \vdash p \to q$

1 $\neg p \vee q$ premise
2 $\neg p$ assumption
3 p assumption
4 $\bot$ $\neg$e 2,3
5 q $\bot$e 4
6 $p \to q$ $\to$i 3-5

1	$\neg p \vee q$	premise
2	$\neg p$	assumption
3	p	assumption
4	$\bot$	$\neg$e 3,2
5	q	$\bot$e 4
6	$p \to q$	$\to$i 3-5
7	q	assumption
8	p	assumption
9	q	copy 7
10	$p \to q$	$\to$i 9-10
11	$p \to q$	$\vee$e 1,2-6,7-10

7 q assumption
8 p assumption
9 q copy 7
10 $p \to q$ $\to$i 8-9
11 $p \to q$ 1, 2-6, 7-10

Natural Deduction — Derived Rules 定理

$\to e$ MP

$$\frac{\Phi \to \Psi \qquad \neg\Psi}{\neg\Phi}\ \text{MT}$$

$p \to q, \neg q \vdash \neg p$

1	$p \to q$	premise
2	$\neg q$	premise
3	p	assumption
4	q	$\to e$ 1,3
5	$\bot$	$\neg e$ 2,4
6	$\neg p$	$\neg i$ 3-5

1	$\Phi \to \Psi$	premise
2	$\neg\Psi$	premise
3	Φ	assumption
4	Ψ	$\to e$ 1,3
5	$\bot$	$\neg e$ 4,2
6	$\neg\Phi$	$\neg i$ 3-5

Justification: If I am Chinese, then I am Asian. I am not Asian. Therefore, I'm not Chinese.

1	q	premise
2	$\neg q$	assumption
3	$\bot$	$\neg e$ 1,2
	$\neg\neg q$	$\neg i$ 2-3

$$\frac{\Phi}{\neg\neg\Phi}\ \neg\neg i$$

1	Φ	premise
2	$\neg\Phi$	assumption
3	$\bot$	$\neg e$ 1,2
4	$\neg\neg\Phi$	$\neg i$ 2-3

$$\frac{\boxed{\begin{array}{c}\neg\Phi \\ \vdots \\ \bot\end{array}}}{\Phi}\ \text{PBC}$$

$\neg i$ 3-5

PBC (Proof By Contradiction)

1	$\neg\Phi \to \bot$	given
2	$\neg\Phi$	assumption
3	$\bot$	$\to e$ 1,2
4	$\neg\neg\Phi$	$\neg i$ 2-3
5	Φ	$\neg\neg e$ 4

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption
3	$\bot$	$\to e$ 1,2
4	$\neg\neg p$	$\neg e$ 2-3
5	p	$\neg\neg e$ 4

Natural Deduction Summary

Basic rules:

	Introd.	Elim.
$\wedge$	$\dfrac{\Phi \quad \Psi}{\Phi \wedge \Psi}\ \wedge\mathrm{i}$	$\dfrac{\Phi \wedge \Psi}{\Phi}\ \wedge\mathrm{e1}$ $\quad$ $\dfrac{\Phi \wedge \Psi}{\Psi}\ \wedge\mathrm{e2}$
$\vee$	$\dfrac{\Phi}{\Phi \vee \Psi}\ \vee\mathrm{i1}$ $\quad$ $\dfrac{\Psi}{\Phi \vee \Psi}\ \vee\mathrm{i2}$	$\dfrac{\Phi \vee \Psi \quad \boxed{\begin{array}{c}\Phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\Psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\Phi\\ \vdots\\ \Psi\end{array}}}{\Phi \rightarrow \Psi}\ \rightarrow\mathrm{i}$	$\dfrac{\Phi \quad \Phi \rightarrow \Psi}{\Psi}\ \rightarrow\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\Phi\\ \vdots\\ \bot\end{array}}}{\neg\Phi}\ \neg\mathrm{i}$	$\dfrac{\Phi \quad \neg\Phi}{\bot}\ \neg\mathrm{e}$

Basic rules (cont'd):

	Introd.	Elim.
$\bot$	no rule	$\dfrac{\bot}{\Phi}\ \bot\mathrm{e}$
$\neg\neg$	derived	$\dfrac{\neg\neg\Phi}{\Phi}\ \neg\neg\mathrm{e}$

Useful derived rules:

$$\dfrac{\Phi \rightarrow \Psi \quad \neg\Psi}{\neg\Phi}\ \mathrm{MT} \qquad \dfrac{\Phi}{\neg\neg\Phi}\ \neg\neg\mathrm{i}$$

$$\dfrac{\boxed{\begin{array}{c}\neg\Psi\\ \vdots\\ \bot\end{array}}}{\Phi}\ \mathrm{PBC} \qquad \dfrac{}{\Phi \vee \neg\Phi}\ \mathrm{LEM}$$

Execises

Prove the following Theorems with nature deduction:

$\equiv$ 等价

(1) $\neg(p \wedge q) \dashv\vdash \neg q \vee \neg p$

(2) $p \to q \dashv\vdash \neg q \to \neg p$

(3) $p \wedge q \to p \dashv\vdash r \vee \neg r$

语义等价

Definition: We say that two formulas Ψ and Φ are ***provably equivalent*** iff both $\Phi \vdash \Psi$ and $\Psi \vdash \Phi$. We denote this by $\Psi \dashv\vdash \Phi$.

Remark: We could define $\Psi \dashv\vdash \Phi$ to mean that $\vdash (\Phi \to \Psi) \wedge (\Psi \to \Phi)$ holds.

Interesting proof

Statement: There exist irrational numbers a and b such that a^b is rational.

Proof: Choose $b = \sqrt{2}$. We have two cases.

b^b可能是虚数? LEM

b^b *is rational.* Then choose $a = b$ and the statement is proven.

b^b *is irrational.* Then choose $a = b^b = (\sqrt{2})^{\sqrt{2}}$. We have

$$a^b = ((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^2 = 2 \quad \text{—rational.}$$

?

Proofs are in fact proof schemas.

$p \to q, p \vdash q$ $\qquad$ $r \lor \neg s \to s \to r, r \lor \neg s \vdash s \to r$

1	$p \to q$	premise	1	$r \lor \neg s \to s \to r$	premise
2	p	premise	2	$r \lor \neg s$	premise
3	q	$\to$e 1,2	3	$s \to r$	$\to$e 1,2

$p \rightsquigarrow r \lor \neg s$
$q \rightsquigarrow s \to r$

- We can build complicated formulas using our rules.
- What exactly are the formulas? We need to define a formal language.

Definition:

atoms: propositional symbols p, q, p_1, p_2, ...

an atom is a ***well-formed formula (wff)***

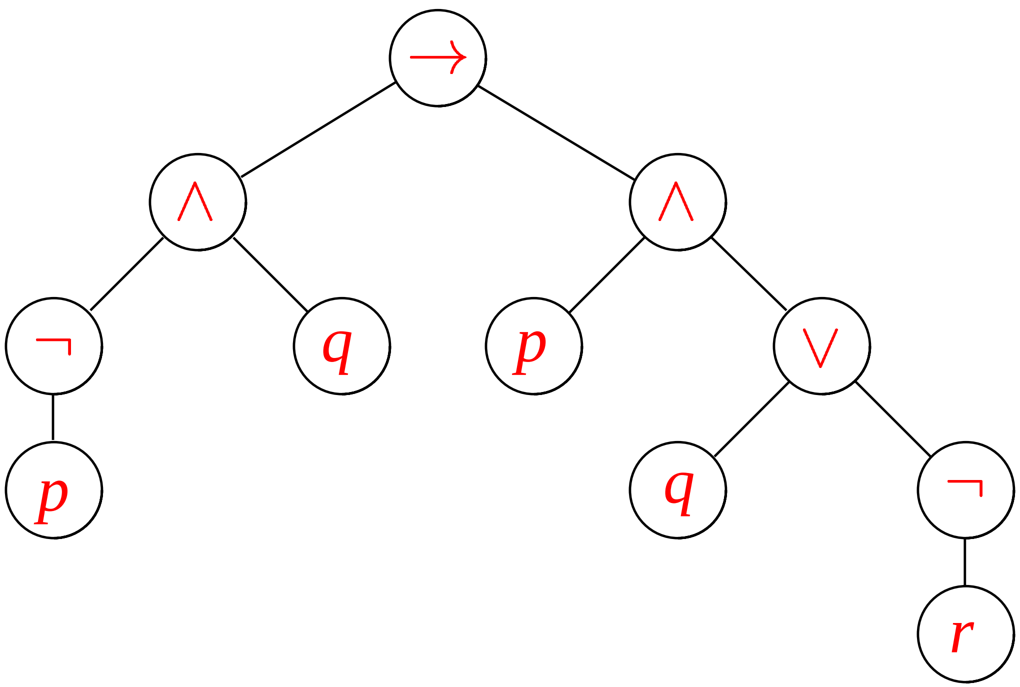
if Φ and Ψ are formulas, then so are $(\neg\Phi)$, $(\Phi \land \Psi)$, $(\Phi \lor \Psi)$, $(\Phi \to \Psi)$.

定义符号

BNF form: $\Phi ::= p \mid (\neg\Phi) \mid (\Phi \land \Phi) \mid (\Phi \lor \Phi) \mid (\Phi \to \Phi)$ 语法

Syntax Trees

Well-formed formula: $\underbrace{(((\neg p) \wedge q)}_{\substack{\text{subformula}\\ \text{corresponding to the}\\ \text{left subtree}}} \rightarrow (p \wedge (q \vee (\neg r))))$



All subformulas:

p
q
r
$(\neg p)$
$((\neg p) \wedge q)$
$(\neg r)$
$(q \vee (\neg r))$
$(p \wedge (q \vee (\neg r)))$
$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$

语义

The semantics of propositional logic is a mapping

wff

$$\text{Interpretation} : \text{WFF} \mapsto \{T, F\}$$

解释函数 $I(wff) = T/F.$

where T stands for ***true*** and F stands for ***false***. The semantics has to be consitent w.r.t. the connectives $\neg$, $\wedge$, $\vee$, and $\rightarrow$. This consitency is specified by the following ***truth table***.

如何用 truth table 表示出解释函数中

Φ	Ψ	$\neg\Psi$	$\Phi \wedge \Psi$	$\Phi \vee \Psi$	$\Phi \rightarrow \Psi$	$\top$	$\bot$
F	F	T	F	F	T	T	F
F	T	F	F	T	T		
T	F		F	T	F		
T	T		T	T	T		

无限個点

从简单到复杂
递归定义

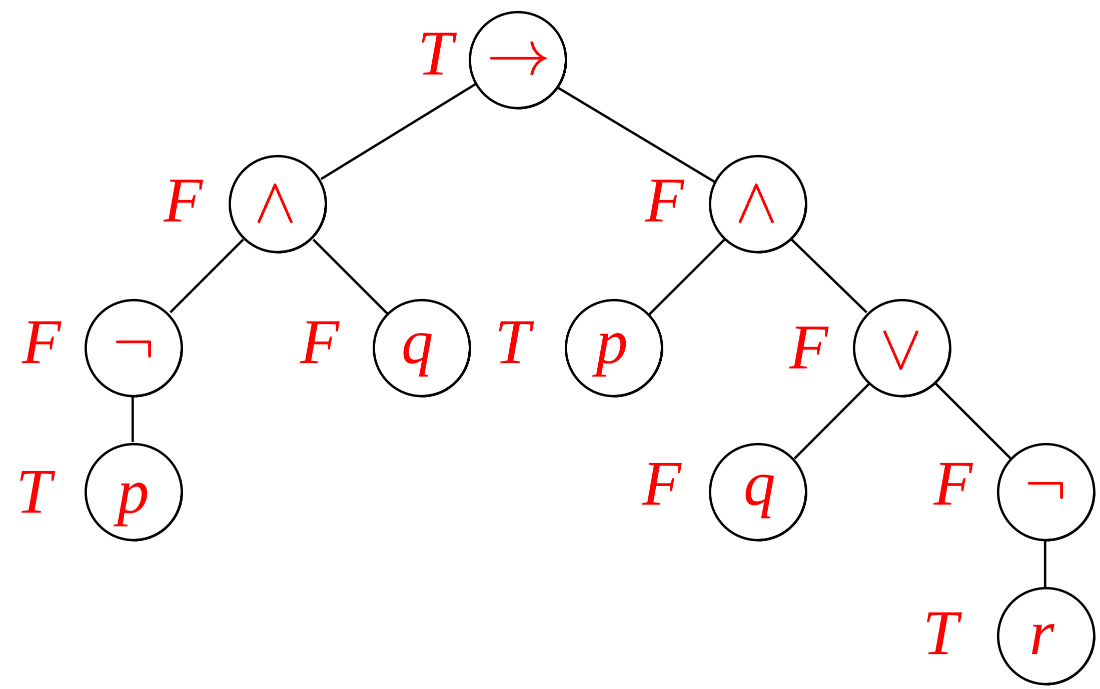
Truth tables are means of exploring all possible interpretations for a given formula.

$I(\neg p) = \neg I(p)$

$I(p \wedge q) = I(p) \wedge I(q)$

Truth Table Example

p	q	r	$p \wedge q \rightarrow p \wedge (q \vee \neg r)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T



Given a sequent $\Phi_1, \Phi_2, \ldots, \Phi_n \vdash \Psi$ (which we don't know whether it is valid), we denote by

$$\Phi_1, \Phi_2, \ldots, \Phi_n \models \Psi$$

a new kind of sequent, which is valid if for every semantics S such that $S(\Phi_i) = T$, $i = 1, \ldots, n$, we also have that $S(\Psi) = T$. The $\models$ relation is called ***semantic entailment***.

等价

$\forall i \in [1..n],\ \varphi_1(A_i)=T, ..\ \varphi_m(A_i)=T \longrightarrow \psi(A_i)=T$

A_i表示原子命题的取值组合.

φ_1 φ_2 ψ

Example: $p, q \models p \wedge (q \vee \neg r)$

$n = 2\times 2$

	p	q
A1	T	T
A2	T	F
A3	F	T
A4	F	F

$\phi_1(A_1)=T \quad \phi_2(A_1)=T \quad \psi(A_1)=T$

$\phi_1(A_2)=T \quad \phi_2(A_2)=F$

$\phi_1(A_3)=F \quad \phi_2(A_3)=T$

$\phi_1(A_4)=F \quad \phi_2(A_4)=F$

Soundness and Completeness of Propositional Logic

如何证明正确

$\phi_1 \ldots \phi_n \vdash \psi$
$\phi_1 \ldots \phi_n \models \psi$
等价的、证明等价可以增强正确自信

When we define a logic (or any type of calculus), we want to show that it is useful.

任何一个逻辑系统都具有的属性

eg. 深度学习的结果反映"real truth" = 测试集

- ***Soundness:*** Formulas that we derive using the calculus reflect a "real" truth. truth table 人为规定的真假
- ***Completeness:*** Every formula corresponding to a "real" truth can be inferred using the rules of the calculus.

In the case of propositional logic, given the wffs Φ_1, Φ_2, . . . , Φ_n, and Ψ, we have

既然把 ⊨ 作为"real" truth，为什么还发明了 ⊢ ?
∵ ⊨ 指数级增长，尤其是在谓词逻辑中

- ***Soundness:*** if $\Phi_1, \ldots, \Phi_n \vdash \Psi$ holds, then $\Phi_1, \ldots, \Phi_n \models \Psi$ holds.
- ***Completeness:*** if $\Phi_1, \ldots, \Phi_n \models \Psi$ holds, then $\Phi_1, \ldots, \Phi_n \vdash \Psi$ holds.

适用于自然数上的问题

结构归纳法——递归定义

Base $x \in p, q, r$

Induction $\neg P(\phi) \to P(\neg \phi)$

$P(\phi_1) \wedge P(\phi_2) \to P(\phi_1 \wedge \phi_2)$

$P(\phi_1) \vee P(\phi_2) \to P(\phi_1 \vee \phi_2)$

无限的结论可以通过递归转化成有限的证明

How do we prove that $1+2+\cdots+n = \frac{n\cdot(n+1)}{2}$? **Answer:** Mathematical induction.

(Base case) We prove the statement for $n = 1$. Indeed, $1 = \frac{1\cdot 2}{2}$.

(Induction case) We assume that the statement is true for some general value of n, and we show that it implies the statement for $n+1$. In other words, we prove that

$$1+2+\cdots+n = \frac{n\cdot(n+1)}{2} \to 1+2+\cdots+n+(n+1) = \frac{(n+1)\cdot(n+2)}{2}$$

Indeed

表示出所有自然数

$$1+2+\cdots+n+(n+1) = \frac{n\cdot(n+1)}{2}+(n+1) = \frac{(n+1)\cdot(n+2)}{2}$$

皮亚诺的整数递归定义

$N = (0, S)$

$0 = 0$

$1 = S(0)$

$2 = S(S(0))$

...

$k+1 = S(k)$.

Base $n=0 \to P(0)$

Induction $(n=k \wedge P(k)) \to (n=S(k) \wedge P(k+1))$

General Mathematical Induction Principle

无限的结论可以通过递归转化成有限的证明

Given a statement $\eta(n)$ that depends on a natural number n, and whose validity we want to prove for all possible values of n, we proceed in the following two steps:

- ***Base case:*** prove that $\eta(1)$ holds.
- ***Induction case:*** prove that $\eta(n) \rightarrow \eta(n+1)$, for all natural numbers n. When proving such a statement, we call $\eta(n)$ the ***induction hypothesis***.
- These two conditions prove $\eta(n)$ for all n.

Course of Values Induction

归纳：特殊→一般

推理：一般→特殊

Given a statement $\eta(n)$ that depends on a natural number n, and whose validity we want to prove for all possible values of n, we proceed in the following two steps:

- ***Base case:*** prove that $\eta(1)$ holds.
- ***Induction case:*** prove that $\eta(1) \wedge \eta(2) \wedge \cdots \wedge \eta(n) \to \eta(n+1)$, for all natural numbers n. When proving such a statement, we call $\eta(1) \wedge \eta(2) \wedge \cdots \wedge \eta(n)$ the ***induction hypothesis***.
- These two conditions prove $\eta(n)$ for all n.

Course of Values Induction Example

Definition: Given a well-formed formula Φ, we define its height to be 1 plus the length of its largest path of its parse tree.

Theorem: For every well-formed propositional logic formula, the number of left brackets is equal to the number of right brackets.

Proof: Denote by $\eta(n)$ the statement "all formulas Φ of height n have the same number of left and right brackets."

> ***Base case:*** $n = 1$. $\eta(1)$ applies to all propositional formulas p, q, $\ldots$ and obvioulsy holds.
>
> ***Induction case:*** $n > 1$. Then the root of the parse tree of Φ is one of the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$. We assume that it is $\rightarrow$ (the other cases are proved in a similar manner.) Then $\Phi = \Phi_1 \rightarrow \Phi_2$ for some wffs Φ_1 and Φ_2, whose heights are strictly smaller than n. Using the induction hypothesis, the number of left and right brackets is equal for both Φ_1 and Φ_2. Φ adds only two brackets, one $'('$ and one $')'$. Therefore, the statement is correct.