7 Many-valued Logics

7.1 Introduction

7.1.1 In this chapter, we leave possible-world semantics for a time, and turn to the subject of propositional many-valued logics. These are logics in which there are more than two truth values.

7.1.2 We have a look at the general structure of a many-valued logic, and some simple but important examples of many-valued logics. The treatment will be purely semantic: we do not look at tableaux for the logics, nor at any other form of proof procedure. Tableaux for some many-valued logics will emerge in the next chapter.

7.1.3 We also look at some of the philosophical issues that have motivated many-valued logics, how many-valuedness affects the issue of the conditional, and a few other noteworthy issues.

7.2 Many-valued Logic: The General Structure

7.2.1 Let us start with the general structure of a many-valued logic. To simplify things, we take, henceforth, $A \equiv B$ to be defined as $(A \supset B) \wedge (B \supset A)$.

7.2.2 Let $\mathcal{C}$ be the class of connectives of classical propositional logic $\{\wedge, \vee, \neg, \supset\}$. The classical propositional calculus can be thought of as defined by the structure $\langle \mathcal{V}, \mathcal{D}, \{f_c; c \in \mathcal{C}\} \rangle$. $\mathcal{V}$ is the set of truth values {1,0}. $\mathcal{D}$ is the set of *designated* values {1}; these are the values that are preserved in valid inferences. For every connective, c, f_c is the truth function it denotes. Thus, $f_\neg$ is a one-place function such that $f_\neg(0) = 1$ and $f_\neg(1) = 0$; $f_\wedge$ is a two-place function such that $f_\wedge(x, y) = 1$ if $x = y = 1$, and $f_\wedge(x, y) = 0$ otherwise; and so on. These functions can be (and often are) depicted in the following 'truth tables'.

$f_\neg$	
1	0
0	1

$f_\wedge$	1	0
1	1	0
0	0	0

7.2.3 An interpretation, ν, is a map from the propositional parameters to $\mathcal{V}$. An interpretation is extended to a map from all formulas into $\mathcal{V}$ by applying the appropriate truth functions recursively. Thus, for example, $\nu(\neg(p \wedge q)) = f_\neg(\nu(p \wedge q)) = f_\neg(f_\wedge(\nu(p), \nu(q)))$. (So if $\nu(p) = 1$ and $\nu(q) = 0$, $\nu(\neg(p \wedge q)) = f_\neg(f_\wedge(1, 0)) = f_\neg(0) = 1$.) Finally, an inference is semantically valid just if there is no interpretation that assigns all the premises a value in $\mathcal{D}$, but assigns the conclusion a value not in $\mathcal{D}$.

7.2.4 A many-valued logic is a natural generalisation of this structure. Given some propositional language with connectives $\mathcal{C}$ (maybe the same as those of the classical propositional calculus, maybe different), a logic is defined by a structure $\langle \mathcal{V}, \mathcal{D}, \{f_c; c \in \mathcal{C}\}\rangle$. $\mathcal{V}$ is the set of truth values: it may have any number of members (≥ 1). $\mathcal{D}$ is a subset of $\mathcal{V}$, and is the set of designated values. For every connective, c, f_c is the corresponding truth function. Thus, if c is an n-place connective, f_c is an n-place function with inputs and outputs in $\mathcal{V}$.

7.2.5 An interpretation for the language is a map, ν, from propositional parameters into $\mathcal{V}$. This is extended to a map from all formulas of the language to $\mathcal{V}$ by applying the appropriate truth functions recursively. Thus, if c is an n-place connective, $\nu(c(A_1, \ldots, A_n)) = f_c(\nu(A_1), \ldots, \nu(A_n))$. Finally, $\Sigma \models A$ iff there is no interpretation, ν, such that for all $B \in \Sigma$, $\nu(B) \in \mathcal{D}$, but $\nu(A) \notin \mathcal{D}$. A is a logical truth iff $\phi \models A$, i.e., iff for every interpretation $\nu(A) \in \mathcal{D}$.

7.2.6 If $\mathcal{V}$ is finite, the logic is said to be *finitely many-valued*. If $\mathcal{V}$ has n members, it is said to be an n-valued logic.

7.2.7 For any finitely many-valued logic, the validity of an inference with finitely many premises can be determined, as in the classical propositional calculus, simply by considering all the possible cases. We list all the possible combinations of truth values for the propositional parameters employed.

Then, for each combination, we compute the value of each premise and the conclusion. If, in any of these, the premises are all designated and the conclusion is not, the inference is invalid. Otherwise, it is valid. We will have an example of this procedure in the next section.

7.2.8 This method, though theoretically adequate, is often impractical because of exponential explosion. For if there are m propositional parameters employed in an inference, and n truth values, there are n^m possible cases to consider. This grows very rapidly. Thus, if the logic is 4-valued and we have an inference involving just four propositional parameters, there are already 256 cases to consider!

7.3 The 3-valued Logics of Kleene and Łukasiewicz

7.3.1 In what follows, we consider some simple examples of the above general structure. All the examples that we consider are 3-valued logics. The language, in every case, is that of the classical propositional calculus.

7.3.2 A simple example of a 3-valued logic is as follows. $\mathcal{V} = \{1, i, 0\}$. 1 and 0 are to be thought of as *true* and *false*, as usual. i is to be thought of as *neither true nor false*. $\mathcal{D}$ is just $\{1\}$. The truth functions for the connectives are depicted as follows:

$f_\neg$	
1	0
i	i
0	1

$f_\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$f_\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$f_\supset$	1	i	0
1	1	i	0
i	1	i	i
0	1	1	1

Thus, if $\nu(p) = 1$ and $\nu(q) = i$, $\nu(\neg p) = 0$ (top row of $f_\neg$), $\nu(\neg p \vee q) = i$ (bottom row, middle column of $f_\vee$), etc.

7.3.3 Note that if the inputs of any of these functions are classical (1 or 0), the output is exactly the same as in the classical case. We compute the other entries as follows. Take $A \wedge B$ as an example. If A is false, then, whatever B is, this is (classically) sufficient to make $A \wedge B$ false. In particular, if B is neither true nor false, $A \wedge B$ is false. If A is true, on the other hand, and B is neither true nor false, there is insufficient information to compute the (classical) value of $A \wedge B$; hence, $A \wedge B$ is neither true nor false. Similar reasoning justifies all the other entries.

7.3.4 The logic specified above is usually called the (strong) Kleene 3-valued logic, often written K_3.[1]

7.3.5 The following table verifies that $p \supset q \models_{K_3} \neg q \supset \neg p$:

p	q	$p \supset q$	$\neg q$	$\supset$	$\neg p$
1	1	**1**	0	**1**	0
1	i	**i**	i	**i**	0
1	0	**0**	1	**0**	0
i	1	**1**	0	**1**	i
i	i	**i**	i	**i**	i
i	0	**i**	1	**i**	i
0	1	**1**	0	**1**	1
0	i	**1**	i	**1**	1
0	0	**1**	1	**1**	1

In the last three columns, the first number is the value of $\neg q$; the last number is that of $\neg p$, and the central number (printed in bold) is the value of the whole formula. As can be seen, there is no interpretation where the premise is designated, that is, has the value 1, and the conclusion is not.

7.3.6 In checking for validity, it may well be easier to work backwards. Consider the formula $p \supset (q \supset p)$. Suppose that this is undesignated. Then it has either the value 0 or the value i. If it has the value 0, then p has the value 1 and $q \supset p$ has the value 0. But if p has the value 1, so does $q \supset p$. This situation is therefore impossible. If it has the value i, there are three possibilities:

p	$q \supset p$
1	i
i	i
i	0

The first case is not possible, since if p has the value 1, so does $q \supset p$. Nor is the last case, since if p has the value i, $q \supset p$ has value either i or 1. But the

[1] Weak Kleene logic is the same as K_3, except that, for every truth function, if any input is i, so is the output.

second case is possible, namely when both p and q have the value i. Thus, $\nu(p) = \nu(q) = i$ is a counter-model to $p \supset (q \supset p)$, as a truth-table check confirms. So $\not\models_{K_3} p \supset (q \supset p)$.

7.3.7 A distinctive thing about K_3 is that the law of excluded middle is not valid: $\not\models_{K_3} p \vee \neg p$. (Counter-model: $\nu(p) = i$.) However, K_3 is distinct from intuitionist logic. As we shall see in 7.10.8, intuitionist logic is not the same as any finitely many-valued logic.

7.3.8 In fact, K_3 has no logical truths at all (7.14, problem 3)! In particular, the law of identity is not valid: $\not\models_{K_3} p \supset p$. (Simply give p the value i.) This may be changed by modifying the middle entry of the truth function for $\supset$, so that $f_\supset$ becomes:

$f_\supset$	1	i	0
1	1	i	0
i	1	1	i
0	1	1	1

(The meaning of $A \supset B$ in K_3 can still be expressed by $\neg A \vee B$, since this has the same truth table, as may be checked.) Now, $A \supset A$ always takes the value 1.

7.3.9 The logic resulting from this change is one originally given by Łukasiewicz, and is often called $Ł_3$.

7.4 *LP* and *RM*$_3$

7.4.1 Another 3-valued logic is the one often called *LP*. This is exactly the same as K_3, except that $\mathcal{D} = \{1, i\}$.

7.4.2 In the context of *LP*, the value i is thought of as *both true and false*. Consequently, 1 and 0 have to be thought of as *true and true only*, and *false and false only*, respectively. This change does not affect the truth tables, which still make perfectly good sense under the new interpretation. For example, if A takes the value 1 and B takes the value i, then A and B are both true; hence, $A \wedge B$ is true; but since B is false, $A \wedge B$ is false. Hence, the value of $A \wedge B$ is i. Similarly, if A takes the value 0, and B takes the value i, then A

and B are both false, so $A \wedge B$ is false; but only B is true, so $A \wedge B$ is not true. Hence, $A \wedge B$ takes the value 0.

7.4.3 However, the change of designated values makes a crucial difference. For example, $\models_{LP} p \vee \neg p$. (Whatever value p has, $p \vee \neg p$ takes either the value 1 or i. Thus it is always designated.) This fails in K_3, as we saw in 7.3.7.

7.4.4 On the other hand, $p \wedge \neg p \not\models_{LP} q$. Counter-model: $\nu(p) = i$ (making $\nu(p \wedge \neg p) = i$), $\nu(q) = 0$. But $p \wedge \neg p$ can never take the value 1 and so be designated in K_3. Thus, the inference is valid in K_3.

7.4.5 A notable feature of LP is that *modus ponens* is invalid: $p, p \supset q \not\models_{LP} q$. (Assign p the value i, and q the value 0.)

7.4.6 One way to rectify this is to change the truth function for $\supset$ to the following:

$f_\supset$	1	i	0
1	1	0	0
i	1	i	0
0	1	1	1

(As in 7.3.8, the meaning of $A \supset B$ in LP can still be expressed by $\neg A \vee B$.) Now, if A and $A \supset B$ have designated values (1 or i), so does B, as a moment checking the truth table verifies.

7.4.7 This change gives the logic often called RM_3.

7.5 Many-valued Logics and Conditionals

7.5.1 Further details of the properties of $\wedge$, $\vee$ and $\neg$ in the logics we have just met will emerge in the next chapter. For the present, let us concentrate on the conditional.

7.5.2 In past chapters, we have met a number of problematic inferences concerning conditionals. The following table summarises whether or not they hold in the various logics we have looked at. (A tick means *yes*; a cross means *no*.)

		K_3	$Ł_3$	LP	RM_3
(1)	$q \models p \supset q$	√	√	√	×
(2)	$\neg p \models p \supset q$	√	√	√	×
(3)	$(p \wedge q) \supset r \models (p \supset r) \vee (q \supset r)$	√	√	√	√
(4)	$(p \supset q) \wedge (r \supset s) \models (p \supset s) \vee (r \supset q)$	√	√	√	√
(5)	$\neg(p \supset q) \models p$	√	√	√	√
(6)	$p \supset r \models (p \wedge q) \supset r$	√	√	√	√
(7)	$p \supset q, q \supset r \models p \supset r$	√	√	×	√
(8)	$p \supset q \models \neg q \supset \neg p$	√	√	√	√
(9)	$\models p \supset (q \vee \neg q)$	×	×	√	×
(10)	$\models (p \wedge \neg p) \supset q$	×	×	√	×

(1) and (2) we met in 1.7, and (3)–(5) we met in 1.9, all in connection with the material conditional. (6)–(8) we met in 5.2, in connection with conditional logics. (9) and (10) we met in 4.6, in connection with the strict conditional. The checking of the details is left as a (quite lengthy) exercise. For K_3, a generally good strategy is to start by assuming that the premises take the value 1 (the only designated value), and recall that, in K_3, if a conditional takes the value 1, then either its antecedent takes the value 0 or the consequent takes the value 1. For $Ł_3$, it is similar, except that a conditional with value 1 may also have antecedent and consequent with value i. For LP, a generally good strategy is to start by assuming that the conclusion takes the value 0 (the only undesignated value), and recall that, in LP, if a conditional takes the value 0, then the antecedent takes the value 1 and the consequent takes the value 0. For RM_3, it is similar, except that if a conditional has value 0, the antecedent and consequent may also take the values 1 and i, or i and 0, respectively. And recall that classical inputs (1 or 0) always give the classical outputs.

7.5.3 As can be seen from the number of ticks, the conditionals do not fare very well. If one's concern is with the ordinary conditional, and not with conditionals with an enthymematic *ceteris paribus* clause, then one may ignore lines (6)–(8). But all the logics suffer from some of the same problems as the material conditional. K_3 and $Ł_3$ also suffer from some of the problems that the strict conditional does. In particular, even though (10) tells us that $(p \wedge \neg p) \supset q$ is not valid in these logics, contradictions still entail everything, since $p \wedge \neg p$ can never assume a designated value. By contrast, this is not true of *LP* (as we saw in 7.4.4), but this is so only because *modus ponens* is invalid, since $(p \wedge \neg p) \supset q$ is valid, as (10) shows. (*Modus ponens* is valid for the other logics, as may easily be checked.) About the best of the bunch is RM_3.

7.5.4 But there are quite general reasons as to why the conditional of any finitely many-valued logic is bound to be problematic. For a start, if disjunction is to behave in a natural way, the inference from A (or B) to $A \vee B$ must be valid. Hence, we must have:

(i) if A (or B) is designated, so is $A \vee B$

Also, $A \equiv A$ ought to be a logical truth. (Even if A is neither true nor false, for example, it would still seem to be the case that *if* A then A, and so, that A iff A.) Hence:

(ii) if A and B have the same value, $A \equiv B$ must be designated (since $A \equiv A$ is).

Note that both of these conditions hold for all the logics that we have looked at, with the exception of K_3, for which (ii) fails.

7.5.5 Now, take any n-valued logic that satisfies (i) and (ii), and consider $n+1$ propositional parameters, $p_1, p_2, \ldots, p_{n+1}$. Since there are only n truth values, in any interpretation, two of these must receive the same value. Hence, by (ii), for some j and k, $p_j \equiv p_k$ must be designated. But then the disjunction of all biconditionals of this form must also be designated, by (i). Hence, this disjunction is logically valid.

7.5.6 But this seems entirely wrong. Consider $n + 1$ propositions such as 'John has 1 hair on his head', 'John has 2 hairs on his head', ..., 'John has $n + 1$ hairs on his head'. Any biconditional relating a pair of these would appear to be false. Hence, the disjunction of all such pairs would also appear to be false – certainly not logically true.

7.6 Truth-value Gluts: Inconsistent Laws

7.6.1 Let us now turn to the issue of the philosophical motivations for many-valued logics and, in particular, the 3-valued logics we have met. Typically, the motivations for those logics that treat i as both true and false (a *truth-value glut*), like *LP* and RM_3, are different from those that treat i as neither true nor false (a *truth-value gap*), like K_3 and $Ł_3$. Let us start with the former. We will look at two reasons for supposing that there are truth-value gluts.[2]

7.6.2 The first concerns inconsistent laws, and the rights and obligations that agents have in virtue of these. We have already had an example of this in 4.8.3 concerning inconsistent traffic regulations.

7.6.3 Here is another example. Suppose that in a certain (entirely hypothetical) country the constitution contains the following clauses:

(1) No aborigine shall have the right to vote.
(2) All property-holders shall have the right to vote.

We may suppose that when the law was made, the possibility of an aboriginal property-holder was so inconceivable as not to be taken seriously. Despite this, as social circumstances change, aborigines do come to hold property. Let one such be John. John, it would appear, both does and does not have the right to vote.

7.6.4 Of course, if a situation of this kind comes to light, the law is likely to be changed to resolve the contradiction. The fact remains, though, that until the law is changed the contradiction is true.

7.6.5 One way that one might object to this conclusion is as follows. The law contains a number of principles for resolving apparent contradictions, for example *lex posterior* (that a later law takes precedence over an earlier law), or that constitutional law takes precedence over statute law, which takes precedence over case law. One might insist that all contradictions are only apparent, and can be defused by applying one or other of these principles.

7.6.6 It is clear, however, that there could well be cases where none of these principles are applicable. Both laws are made at the same time; they are both laws of the same rank, and so on. Hence, though some legal contradictions may be only apparent, this need not always be the case.

[2] Other examples of truth-value gluts that have been suggested include the state of affairs realised at an instant of change; statements about some object in the border-area of a vague predicate; contradictory statements in the dialectical tradition of Hegel and Marx; statements with predicates whose criteria of application are over-dertermined; and certain statements about micro-objects in quantum mechanics.

7.7 Truth-value Gluts: Paradoxes of Self-reference

7.7.1 A second argument for the existence of truth-value gluts concerns the paradoxes of self-reference. There are many of these; some very old; some very modern. Here are a couple of well-known ones.

7.7.2 The Liar Paradox: Consider the sentence 'this sentence is false'. Suppose that it is true. Then what it says is the case. Hence it is false. Suppose, on the other hand, that it is false. That is just what it says, so it is true. In either case – one of which must obtain by the law of excluded middle – it is both true and false.

7.7.3 Russell's Paradox: Consider the set of all those sets which are not members of themselves, $\{x; x \notin x\}$. Call this r. If r is a member of itself, then it is one of the sets that is not a member of itself, so r is not a member of itself. On the other hand, if r is not a member of itself, then it is one of the sets in r, and hence it is a member of itself. In either case – one of which must obtain by the law of excluded middle – it is both true and false.

7.7.4 These (and many others like them) are both *prima facie* sound arguments, and have conclusions of the form $A \wedge \neg A$. If the arguments are sound, the conclusions are true, and hence there are truth-value gluts.

7.7.5 Many people have claimed that the arguments are not, despite appearances, sound. The reasons given are many and complex; let us consider, briefly, just a couple.

7.7.6 Some have argued that any sentence which is self-referential, like the liar sentence, is meaningless. (Hence, such sentences can play no role in logical arguments at all.) This, however, is clearly false. Consider: 'this sentence has five words', 'this sentence is written on page 129 of Part I of *An Introduction to Non-Classical Logic*', 'this sentence refers to itself'.

7.7.7 The most popular objection to the argument is that the liar sentence is neither true nor false. In this case, we can no longer appeal to the law of excluded middle, and so the arguments to contradiction are broken. (Thus, the paradoxes of self-reference are sometimes used as an argument for the existence of truth-value gaps, too.)

7.7.8 This suggestion does not avoid contradiction, however, because of 'extended paradoxes'.[3] Consider the sentence 'This sentence is either false or neither true nor false.' If it is true, it is either false or neither. In both cases it is not true. If, on the other hand, it is either false or neither (and so not true), then that is exactly what it claims, and so it is true. In either case, therefore, it is both true and not true.

7.8 Truth-value Gaps: Denotation Failure

7.8.1 Let us now turn to the question of why one might suppose there to be truth-value gaps. One reason for this, we saw in the last chapter. If one identifies truth with verification then, since there may well be sentences, A, such that neither A nor $\neg A$ can be verified, there may well be truth-value gaps. Intuitionism can be thought of as a particular case of this.[4] Since we discussed intuitionism in the last chapter, we will say no more about this argument here. Instead, we will look at two different arguments.[5]

7.8.2 The first concerns sentences that contain noun phrases that do not appear to refer to anything, like names such as 'Sherlock Holmes', and descriptions such as 'the largest integer' (there is no largest).

7.8.3 It was suggested by Frege that all sentences containing such terms are neither true nor false.[6] This seems unduly strong. Think, for example, of 'Sherlock Holmes does not really exist', or 'either 2 is even or the greatest prime number is'.

7.8.4 Still, there are some sentences containing non-denoting terms that can plausibly be taken as neither true nor false. One sort of example

[3] Moreover, and in any case, not all of the paradoxical arguments invoke the law of excluded middle. Berry's paradox, for example, does not.

[4] Though, note, in the Kripke semantics for intuitionist logic, every formula takes the value of either 1 or 0 at every world.

[5] Other examples of truth-value gaps that are sometimes given include category mistakes. Such as 'The number 3 is thinking about Sydney', and other 'nonsense' statements; statements in the border-area of some vague predicate; and cases of presupposition failure.

[6] Though he also thought that denotation failure ought not to arise in a properly constructed language. Non-denoting terms should be assigned an arbitrary reference.

concerns ‘truths of fiction’. It is natural to suppose that ‘Holmes lived in Baker Street’ is true, because Conan Doyle says so; ‘Holmes’ friend, Watson, was a lawyer’ is false, since Doyle tells us that Watson was a doctor; and ‘Holmes had three maiden aunts’ is neither true nor false, since Doyle tells us nothing about Holmes’ aunts or uncles.

7.8.5 This reason is not conclusive, though. An alternative view is that *all* such sentences are simply false. A fictional truth is really a shorthand for the truth of a sentence prefixed by ‘In the play/novel/film (etc.), it is the case that’. Thus, in Doyle’s stories (it is the case that) Holmes lived in Baker Street. Fictional falsities are similar. Thus, in Doyle’s stories it is not the case that Watson was a lawyer. And a fictional truth-value gap, A, is just something where neither A nor $\neg A$ holds in the fiction. Thus, it is not the case in Doyle’s stories that Holmes had three maiden aunts; and it is not the case that he did not.

7.8.6 Another sort of example of a sentence that can plausibly be seen as neither true nor false is a subject/predicate sentence containing a non-denoting description, like ‘the greatest integer is even’. (Maybe not every predicate, though: ‘The greatest integer exists’ would seem to be false. But existence is a contentious notion anyway.)[7]

7.8.7 But again, this view is not mandatory. One may simply take such sentences to be false (so that their negations are true, etc.). This was, essentially, Russell’s view.

7.8.8 And Russell’s view would seem to work better than a truth-value gap view in many cases. Thus, let ‘Father Christmas’ be short for the description ‘the old man with a white beard who comes down the chimney at Christmas bringing presents’. Then the following would certainly appear to be false: ‘The Greeks worshipped Father Christmas’ and ‘Julius Caesar thought about Father Christmas.’

7.8.9 Note, though, that even Russell’s view appears to be in trouble with some similar examples. For example, it appears to be true that the Greeks

[7] A related suggestion concerns names that may denote objects, but not objects that exist in the world or situation at which truth is being evaluated. Thus, Aristotle exists in this world, but consider some world at which he does not exist. It may be suggested that ‘Aristotle is a philosopher’ is neither true nor false at that world.

worshipped the gods who lived on Mount Olympus, and that little Johnny does think about Father Christmas on 24 December.

7.8.10 Thus, though non-denotation does give some reason for supposing there to be truth-value gaps, the view has its problems, as do most views concerning non-denotation.[8]

7.9 Truth-value Gaps: Future Contingents

7.9.1 The second argument for the existence of truth-value gaps concerns certain statements about the future – future contingents. The suggestion is that statements such as 'The first pope in the twenty-second century will be Chinese' and 'It will rain in Brisbane some time on 6/6/2066' are now neither true nor false. The future does not yet exist; there are therefore, presently, no facts that makes such sentences true or false.

7.9.2 It might be replied that such sentences *are* either true or false; it's just that we do not know which yet. But there is a very famous argument, due to Aristotle, to the effect that this cannot be the case. It can be put in different ways; here is a standard version of it.

7.9.3 Let S be the sentence 'The first pope in the twenty-second century will be Chinese.' If S were true now, then it would necessarily be the case that the first pope in the twenty-second century will be Chinese. If S were false now, then it would necessarily be the case that the first pope in the twenty-second century will not be Chinese. Hence, if S were either true or false now, then whatever the state of affairs concerning the first pope in the twenty-second century, it will arise of necessity. But this is impossible, since what happens then is still a contingent matter. Hence, it is neither true nor false now.

7.9.4 One might say much about this argument, but a standard, and very plausible, response to it is that it hinges on a fallacy of ambiguity. Statements of the form 'if A then necessarily B' are ambiguous between 'if A, then, it necessarily follows that B' – $\Box(A \supset B)$ – and 'if A, then B is true of necessity' – $A \supset \Box B$. Moreover, neither of these entails the other (even in $K\upsilon$).

[8] We will meet the topic of denotation-failure again in chapter 21 (Part II).

7.9.5 Now, consider the sentence 'If S were true now, then it would necessarily be the case that the first pope in the twenty-second century will be Chinese', which is employed in the argument. If this is interpreted in the first way ($\Box(A \supset B)$), it is true, but the argument is invalid. (Since $A, \Box(A \supset B) \not\models \Box B$.) If we interpret it in the second way ($A \supset \Box B$), the argument is certainly valid, but now there is no reason to believe the conditional to be true (or, if there is, this argument does not provide it). Similar considerations apply to the second part of the argument. Aristotle's argument does not, therefore, appear to work.[9]

7.10 Supervaluations, Modality and Many-valued Logic

7.10.1 Let us finish with two other matters that arise in connection with Aristotle's argument of the previous section, though they have wider implications.

7.10.2 First, those who have taken future contingents to be neither true nor false, like Aristotle, have not normally taken all statements about the future to be truth-valueless – only statements about states of affairs that are as yet undetermined have that status. In particular, instances of the law of excluded middle, $S \vee \neg S$, are usually endorsed, even if S is a future contingent. Since this is not valid in K_3 or $Ł_3$, these logics do not appear to be the appropriate ones for future statements.

7.10.3 A logic better in this regard can be obtained by a technique called *supervaluation*. Let ν be any K_3 interpretation. Define $\nu \leq \nu'$ to mean that ν' is a classical interpretation that is the same as ν, except that wherever $\nu(p)$ is i, $\nu'(p)$ is either 0 or 1. (So ν' 'fills in all the gaps' in ν.) Call ν' a *resolution* of ν. Define the supervaluation of ν, ν^+, to be the map such that for every formula, A:

$\nu^+(A) = 1$ iff for all ν' such that $\nu \leq \nu'$, $\nu'(A) = 1$
$\nu^+(A) = 0$ iff for all ν' such that $\nu \leq \nu'$, $\nu'(A) = 0$
$\nu^+(A) = i$ otherwise

The thought here is that A is true on the supervaluation of ν; just in case however its gaps were to get resolved (and, in the case of future contingents,

[9] I will have more to say about the argument in 11a.7.

will get resolved), it would come out true. We can now define a notion of validity as something like 'truth preservation come what may', $\Sigma \models^S A$ (*supervalidity*), as follows:

$\Sigma \models^S A$ iff for every ν, if $\nu^+(B)$ is designated for all $B \in \Sigma$, $\nu^+(A)$ is designated

(where the designated values here are as for K_3).

7.10.4 A fundamental fact is that $\Sigma \models^S A$ iff A is a classical consequence of Σ. (In particular, therefore, $\models^S A \vee \neg A$ even though A may be neither true nor false!) The argument for this is as follows. First, suppose that the inference is not classically valid; then there is a classical interpretation that makes all the members of Σ true and A false. But the only resolution of ν is ν itself. So every resolution of ν makes all the premises true and the conclusion false. That is, for all $B \in \Sigma$, $\nu^+(B) = 1$, and $\nu^+(A) = 0$. Hence, $\Sigma \not\models^S A$.[10] Conversely, suppose that $\Sigma \not\models^S A$. Then there is a ν such that for all $B \in \Sigma$, $\nu^+(B) = 1$ and $\nu^+(A) \neq 1$. Consequently, there is some resolution $\mu \geq \nu$ such that $\mu(A) = 0$, but for all $B \in \Sigma$, $\mu(B) = 1$. Since μ is a classical interpretation, the inference is not classically valid.

7.10.5 The alignment between classical validity and supervaluation validity is not, in fact, as clean as 7.10.4 makes it appear. For any logic, including classical logic, one can define a natural notion of multiple-conclusion validity. For this, the conclusions, like the premises, may be an arbitrary set of formulas (not just a single formula) and the inference is valid iff every interpretation (of the kind appropriate for the logic) that makes *every* premise true makes *some* conclusion true. Thus, in classical logic (and ignoring set braces for the conclusions as well as the premises), $A \vee B \models A, B$. This inference is not valid for $\models^S$. To see this, just consider an interpretation, ν, such that $\nu(p) = i$. Then $\nu^+(p \vee \neg p) = 1$, but $\nu^+(p) = \nu^+(\neg p) = i$.

7.10.5a A slightly different way of proceeding avoids this consequence. Define an inference to be valid iff, for every K_3 interpretation, ν, every resolution of ν that makes every premise true makes some (or, in the single conclusion case, the) conclusion true. Since the class of resolutions of all K_3

10 In certain contexts, there may be reason to suppose that not all resolutions of an evaluation are 'genuine possibilities'. In that case, one may wish to restrict the supervaluation of an evaluation to an appropriate subclass of its resolutions. If one does so, this half of the proof may break down, and the inferences that are supervaluation valid may actually *extend* the classically valid inferences.

interpretations is exactly the set of classical evaluations, this gives exactly classical logic (single or multiple conclusion, as appropriate).[11]

7.10.5b It is worth noting that there is a technique dual to supervaluation for the logic *LP*. Given any *LP* interpretation, define $\leq$ and validity exactly as in 7.10.3 (remembering that the designated values have now changed). In this context, it is usual to use the term *subvaluation* rather than *supervaluation*; correspondingly, we will use $\models_S$ instead of $\models^S$ (and call this *subvalidity*). This time, $A \models_S \Sigma$ iff the multiple conclusion inference from A to Σ is classically valid (and *a fortiori* for single conclusion inferences). The argument for this is as follows. First, suppose that the inference is not classically valid; then there is a classical interpretation that makes A true and every member of Σ false. But the only resolution of ν is ν itself. So every resolution of ν makes the premise true and all the conclusions false. That is, for all $B \in \Sigma$, $\nu^+(B) = 0$, and $\nu^+(A) = 1$. Hence, $A \not\models_S \Sigma$.[12] Conversely, suppose that $A \not\models_S \Sigma$. Then there is a ν such that $\nu^+(A) \neq 0$, and for all $B \in \Sigma$, $\nu^+(B) = 0$. Consequently, there is some resolution $\mu \geq \nu$ such that $\mu(A) = 1$, but for all $B \in \Sigma$, $\mu(B) = 0$. Since μ is a classical interpretation, the inference is not classically valid.

7.10.5c The result does not extend to multiple-premise inferences. Thus, in classical logic, $A, B \models A \wedge B$. This inference is not valid for $\models_S$. Just consider an interpretation, ν, such that $\nu(p) = i$. Then $\nu^+(p) = \nu^+(\neg p) = i$, but $\nu^+(p \wedge \neg p) = 0$. However, if validity is defined as in 7.10.5a, replacing K_3 with *LP*, then it coincides with classical validity, for the same reason.

7.10.5d Clearly, applying the super/subvaluation technique provides a number of different notions of validity. In deciding whether or not to apply the technique, and if so how, one has to decide what one wishes one's notion

11 Note that supervaluation techniques can be applied to the logic $Ł_3$, but are less appropriate. Supervaluation is essentially a gap-filling exercise. It should not destabilise things that already have a determinate truth. A resolution of a K_3 interpretation preserves classical truth values in the appropriate way. That is, if $\nu \leq \nu'$, and $\nu(A)$ is 0 or 1, $\nu'(A)$ has the same value. The same is not true of $Ł_3$. Similarly, subvaluations (about to be defined) do not destabilise classical values in *LP*, but they may do so in RM_3. See 7.14, problem 4.

12 Again, if one restricts the subvaluation to an appropriate class of its resolutions, this half of the proof may break down, and subvaluation validity may extend the classically valid inferences.

of validity to preserve: designated value under an interpretation, designated value under a super/subvaluation, or designated value under a resolution. In the case of future contingents, for example, are we interested in preserving actual truth value, truth value we can 'predict now', or 'eventual' truth value? Quite possibly, the answer may depend on why, exactly, gaps/gluts are supposed to arise in the application at hand. Conceivably, the answer may be different for different applications (e.g., future contingents and vagueness[13]).

7.10.6 Let us now turn to the second matter. This concerns the connection between modality and many-valued logic. Notwithstanding the issue concerning the law of excluded middle that we have just discussed, Łukasiewicz was motivated to construct his logic $Ł_3$ by the problem about future contingents. According to him, statements about the past and present are now unalterable in truth value. If they are true, they are necessarily true; if they are false, they are necessarily false. But future contingents, those things taking the value i, are merely possible. Things that are true are also possible, of course. He therefore augmented the language with a modal possibility operator, $\Diamond$, and gave it the following truth table:

$f_\Diamond$	
1	1
i	1
0	0

Defining $\Box A$ in the standard way, as $\neg\Diamond\neg A$, gives it the truth table:

$f_\Box$	
1	1
i	0
0	0

7.10.7 These definitions give a modal logic that, in the light of modern modal logic, has some rather strange properties. For example, it is easy to check that $p \models_{Ł_3} \Box p$. (This is *not* the Rule of Necessitation.) Given the Aristotelian motivation, this may be acceptable. But there are other consequences that are certainly not. For example, it is easy to check that

[13] For vagueness, see 11.3.7.

$\Diamond A, \Diamond B \models_{Ł_3} \Diamond(A \wedge B)$. This is not acceptable – even to an Aristotelian. It is possible that the first pope in the twenty-second century will be Chinese and possible that she will not. But it is not possible that she both will and will not be.

7.10.8 In fact, none of the modal logics that we have looked at (nor conditional logics, nor intuitionist logic) is a finitely many-valued logic. The proof of this is essentially a version of the argument of 7.5.4, 7.5.5. The proof is given in 7.11.1–7.11.3.

7.10.9 There is a certain sense in which every logic can be thought of as an infinitely many-valued logic, however. A *uniform substitution* of a set of formulas is the result of replacing each propositional parameter uniformly with some formula or other (maybe itself). Thus, for example, a uniform substitution of the set $\{p, p \supset (p \vee q)\}$ is $\{r \wedge s, (r \wedge s) \supset ((r \wedge s) \vee q)\}$. A logic is *closed under uniform substitution* when any inference that is valid is also valid for every uniform substitution of the premises and conclusion. All standard logics are closed under uniform substitution.[14]

7.10.10 Now, it can be shown that every logical consequence relation, $\vdash$, closed under uniform substitution, is weakly complete with respect to a many-valued semantics. That is, $\vdash A$ iff A is logically valid in the semantics. This is proved in 7.11.5. The semantics is somewhat fraudulent, though, since it involves taking every formula as a truth value. Moreover, the result can be extended to strong completeness (that is, to inferences with arbitrary sets of premises – not just empty ones) only under certain conditions.[15]

7.11 *Proofs of Theorems

7.11.1 Definition: Let $A \mathrel{⥼\!⥽} B$ be $(A \mathrel{⥽} B) \wedge (B \mathrel{⥽} A)$, and let $A \sqsubset\!\sqsupset B$ be $(A \sqsupset B) \wedge (B \sqsupset A)$. Let D_{n+1} be the disjunction of all sentences of the form $p_j \mathrel{⥼\!⥽} p_k$ (if

[14] The general reason is as follows. Suppose that some substitution instance of an inference is invalid. Then there is some interpretation, $\mathcal{I}$ (appropriate for the logic in question), which makes the premises true and the conclusion untrue (at some world). Now consider the interpretation that is exactly the same as $\mathcal{I}$, except that it assigns to every parameter (at a world) the value of whatever formula was substituted for it (at that world) in $\mathcal{I}$. It is not difficult to check that the truth value of every formula (at every world) is the same in this interpretation as its substitution instance was in $\mathcal{I}$. Hence, the inference is invalid also.

[15] See Priest (2005b).

we are dealing with a modal logic), or $p_j \sqsubset\!\sqsupset p_k$ (if we are dealing with intuitionist logic), for $1 \leq j < k \leq n + 1$.

7.11.2 LEMMA: For no n is D_{n+1} a logical truth of any modal logic weaker than $K\upsilon$ or of intuitionist logic.

Proof:
The proof is by constructing counter-models in $K\upsilon$ and I, either directly or with the aid of tableaux. Details are left as an exercise. ■

7.11.3 THEOREM: No modal logic between L and $K\upsilon$ is a finitely many-valued logic.

Proof:
Suppose that it were, and that it had n truth values. Since $A \models_L A \vee B$:

(i) whenever $A \in \mathcal{D}$, $A \vee B \in \mathcal{D}$

Since $A \wedge B \models_L A$:

(ii) whenever $A \wedge B \in \mathcal{D}$, $A \in \mathcal{D}$

(and the same for B in both cases). Moreover, since $\models_L p \strictif p$:

(iii) for any $x \in \mathcal{V}$, $f_{\strictif}(x, x) \in \mathcal{D}$

Now, consider any interpretation, ν. Since there are only n truth values, for some $1 \leq j < k \leq n+1$, $\nu(p_j) = \nu(p_k)$. Hence, $\nu(p_j \strictif p_k) \in \mathcal{D}$ and $\nu(p_k \strictif p_j) \in \mathcal{D}$, by (iii), $\nu(p_j \strictiff p_k) \in \mathcal{D}$, by (ii), and $\nu(D_{n+1}) \in \mathcal{D}$, by (i). Thus, D_{n+1} is logically valid, which it is not, by the preceding lemma. ■

7.11.4 THEOREM: Intuitionist logic is not a finitely many-valued logic. Nor is any logic that extends intuitionist logic or any of the modal logics above with extra connectives. In particular, no conditional logic is a finitely many-valued logic.

Proof:
The proof for intuitionist logic is exactly the same, replacing $\strictiff$ with $\sqsubset\!\sqsupset$. The argument for any linguistic extension of the logics in question is also exactly the same. ■

7.11.5 THEOREM: Any logical consequence relation, $\vdash$, closed under uniform substitution, is weakly complete with respect to a many-valued semantics.

Proof:
We define the components of a many-valued logic as follows. Let $\mathcal{V}$ be the set of formulas of the language. Let $\mathcal{D} = \{A : \vdash A\}$. For every n-place connective, c, let $f_c(A_1, \ldots, A_n) = c(A_1, \ldots, A_n)$. Now, suppose that $\vdash A$. Consider any interpretation, ν. Then it is easy to check that $\nu(A)$ is simply the formula A with every propositional parameter, p, replaced by $\nu(p)$. Call this A_ν. Since $\vdash$ is closed under uniform substitution, $\vdash A_\nu$. That is, $\nu(A) \in \mathcal{D}$. Conversely, suppose that $\nvdash A$. Consider the interpretation, ν, which maps every propositional parameter to itself. It is easy to check that $\nu(A) = A$. Hence, $\nu(A) \notin \mathcal{D}$. ■

7.12 History

The first many-valued logic was $Ł_3$. This, and its generalisation to n-valued logics, $Ł_n$, were invented by the Polish logician Łukasiewicz (pronounced Woo/ka/syey/vitz) around 1920. See Łukasiewicz (1967). (This paper also discusses future contingents and Łukasiewicz' modal logic.) At about the same time, the US mathematician Post (1921) was also constructing a many-valued logic. (Post's system has no simple philosophical motivation, though.) The logic K_3 was invented by Kleene (1952, sect. 64). He was brought to it by considering partial functions, that is, functions that may have no value for certain inputs (such as division when this is by 0). An expression such as 3/0 can be thought of as an instance of denotation failure. Some, such as Kripke (1975), have argued that i should be thought of as a lack of truth value, rather than as a third truth value; but this is a subtle distinction to which it is hard to give substance. *LP* (which stands for 'Logic of Paradox') was given by Priest (1979). RM_3 is one of a family of n-valued logics, RM_n, related to the logic *RM* (*R* Mingle), which we will meet in chapter 10. See Anderson and Belnap (1975, pp. 470f.).

The view that there are true contradictions, *dialetheism*, had a number of historical adherents; but, in its modern form, is relatively recent. For its history, see Priest (1998a). Kripke (1975) gave an influential account of the liar sentence as neither true nor false. Frege's views on non-denotation can be found in Frege (1970). A more nuanced defence of the same idea is in Strawson (1950). Russell's account of descriptions appeared in Russell (1905). Aristotle's argument for truth-value gaps is to be found in *De Interpretatione*, chapter 9.

Supervaluations were invented by van Fraassen (1969). For subvaluations see Varzi (2000). The proof that intuitionist logic is not many-valued was first given by Gödel (1933b). The idea was applied to modal logic by Dugunji (1940). The proof that every logic is weakly characterised by a many-valued logic is due to Lindenbaum (see Rescher 1969, p. 157).

7.13 Further Reading

For an excellent overview of many-valued logics, including their history, see Rescher (1969). Urquhart (1986) and Malinowski (2001) are shorter and also very good. The literature on the paradoxes of self-reference is enormous, but reasonable places to start are Haack (1979, ch. 8), Sainsbury (1995, ch. 5) and Priest (1987, chs. 1 and 2). Chapter 13 of the last of these also contains a discussion of inconsistent laws. The literature on non-denotation is also enormous. A suitable place to start is Haack (1979, ch. 5). A good discussion of Aristotle's argument for truth-value gaps, and its employment by Łukasiewicz, is Haack (1974, ch. 4). Many of the possible examples of truth-value gluts are discussed in Priest and Routley (1989a,b). Many of the possible examples of truth-value gaps are discussed in Blamey (1986, sect. 2). For multiple-conclusion logic, see Shoesmith and Smiley (1978).

7.14 Problems

1. Check all the details omitted in 7.5.2.
2. Call a many-valued logic in the language of the classical propositional calculus *normal* if, amongst its truth values are two, 1 and 0, such that 1 is designated, 0 is not, and for every truth function corresponding to a connective, the output for those inputs is the same as the classical output. (K_3, $Ł_3$, *LP* and RM_3 are all normal.) Show that every normal many-valued logic is a sub-logic of classical logic (i.e., that every inference valid in the logic is valid in classical logic).
3. Observe that in K_3 if an interpretation assigns the value i to every propositional parameter that occurs in a formula, then it assigns the value i to the formula itself. Infer that there are no logical truths in K_3. Are there any logical truths in $Ł_3$?

4. Let ν_1 and ν_2 be any interpretations of K_3 or LP. Write $\nu_1 \preceq \nu_2$ to mean that for every propositional parameter, p:

$$\text{if } \nu_1(p) = 1, \text{ then } \nu_2(p) = 1\text{ ; and if } \nu_1(p) = 0, \text{ then } \nu_2(p) = 0$$

Show by induction on the way that formulas are constructed, that if $\nu_1 \preceq \nu_2$, then the displayed condition is true for all formulas. Does the result hold for $Ł_3$ and RM_3?
5. By problem 2, if $\models_{LP} A$, then A is a classical logic truth. Use problem 4 to show the converse. (Hint: Suppose that ν is an LP interpretation such that $\nu(A) = 0$. Consider the interpretation, ν', which is the same as ν, except that if $\nu(p) = i$, $\nu'(p) = 0$.)
6. What is the truth value of 'this sentence is true'?
7. Tolkien tells us in *The Hobbit* that Bilbo Baggins is a hobbit, and all hobbits are short. Graham Priest is 6′4″. What is the truth value of 'Graham Priest is taller than Bilbo Baggins', and why?
8. Under what conditions is it appropriate to apply a super/subvaluation technique, and what determines the appropriate form to apply?
9. * Fill in the details omitted in 7.11.2.