135 Discrete Structures Spring 2024

WRITTEN HOMEWORK ASSIGNMENT No. 4

Due Monday 02/26/2022, 11:59 pm

Please adhere to the following rules:

- This assignment should be submitted via Canvas. Late assignments will not be accepted.
- You should attempt to solve the problems on your own. You are permitted to study with friends but you can have only high-level discussions about the problems. *You must write up your own solutions, in your own words.*
- Using large language models such as ChatGPT to find solutions, copying solutions from the Internet or asking students not enrolled in the class (or the class staff) is strictly prohibited.
- Please explain your answers. There will be penalties for unexplained solutions when the answers are not obvious.
- All submissions should be neat, preferably submitted either as Word or PDF files. We reserve the right to reject handwritten submissions if they are unacceptably messy or too difficult to read.
- I encourage you to learn how to typeset documents with LATEX . You can download Texmaker at http://www.xm1math.net/texmaker/. It is available for most platforms. Another option is to use online service https://www.overleaf.com/. There is a wealth of information online on how to format documents with LATEX and you can always post a question on Moodle's Technical Forum. Most scholarly articles in Mathematics and Computer Science, and even many books, are typeset with this tool. There is, however, an initial learning curve one needs to climb before getting comfortable with LATEX – hence this great tool is not required for this course.

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1. When working on a classified cryptographic project Prof. Drake considered the set $\mathcal{S}$ of all subsets of $\mathbb{Z}^+$ that contained only square-free numbers. Square free integers are those that are not divisible by a square of a prime number. For example, $\{6, 7, 10, 11, 13, 14, 15\}$ is an element of S, while $\{6, 9, 12\}$ is not. Prof. Drake considered the following predicate defined on S:

   $$P(X, Y) \text{ if and only if } X \supseteq Y$$

   Known for his attention to detail, the professor became angry at John, his inexperienced assistant, for making the following statements in their report:

(a)

$$P(\{26, 29, 32, 34, 35\}, \{32\})$$

(b)

$$P(\{17, 19, 22, 26, 30\}, 30)$$

For each of the statements explain why the professor was angry.

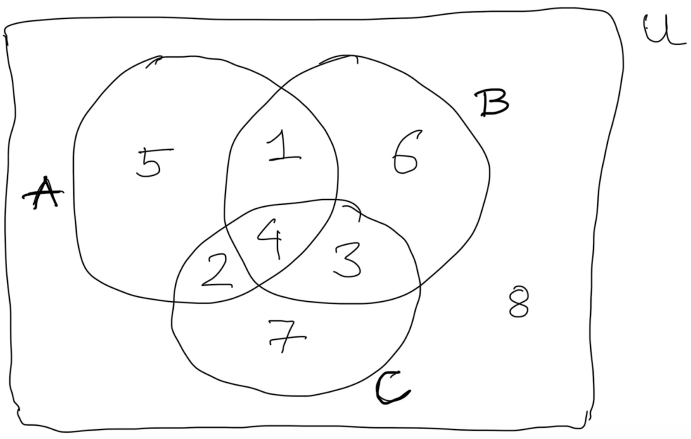
2. Let $U = \{x \in \mathbb{Z}^+ \mid \exists_{y \in \mathbb{Z}} \ y^2 = x\}$. Does there exist a subset $B \subseteq U$ which satisfies the following two conditions

   (a) $|B| > 7$

   (b) $\forall_{x \in B} \ 10 \leq x \leq 100$

   Explain. (Recall that for any finite set X, $|X|$ denotes the number of elements of X.)

3. a) Consider the following Venn diagram which represents the general relationships between three sets A, B, C that are subsets of some universe U. Express each of the numbered areas in terms of the sets A, B, C subject to the restriction that you can use only two operations $\cup$ (i.e., intersection) and $\overline{A}$ (i.e.,complementation). For example, area $1 = A \cap B \cap \overline{C}$.
   b) If you were asked to solve the same problem concerning a general arrangement of subsets A, B, C, D of some universe U, how many areas would there be? c) What would be the rule to generate each of these areas using only the operations of intersection and complementation?



4. For any set X, let's define the operation

$$S(X) = X \cup \{X\}$$

(a) What is $S(\{\sqrt{2}, \sqrt{3}, \sqrt{5}\})$

(b) Calculate $S(S(S(\emptyset)))$

(c) Let P be a finite set. How many elements are there in the set $S(P)$?

5. Let A, B, C be some sets. Is it true that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B \cap C|$$

Justify your answer.

6. Using the canonical set algebra identities show that for all sets A, B, C

$$(A \setminus B) \setminus C = A \setminus (B \cup C)$$

Cite a law you used for every step in your derivation.