135 Discrete Structures Spring 2024

WRITTEN HOMEWORK ASSIGNMENT No. 6

Due Tuesday 03/26/2024 11:59 pm

Please adhere to the following rules:

- This assignment should be submitted via Canvas. Late assignments will not be accepted.
- You should attempt to solve the problems on your own. You are permitted to study with friends but you can have only high-level discussions about the problems. *You must write up your own solutions, in your own words.*
- Using large language models such as ChatGPT to find solutions, copying solutions from the Internet or asking students not enrolled in the class (or the class staff) is strictly prohibited.
- Please explain your answers. There will be penalties for unexplained solutions when the answers are not obvious.
- All submissions should be neat, preferably submitted either as Word or PDF files. We reserve the right to reject handwritten submissions if they are unacceptably messy or too difficult to read.
- I encourage you to learn how to typeset documents with LATEX . You can download Texmaker at http://www.xm1math.net/texmaker/. It is available for most platforms. Another option is to use online service https://www.overleaf.com/. There is a wealth of information online on how to format documents with LATEX and you can always post a question on Moodle's Technical Forum. Most scholarly articles in Mathematics and Computer Science, and even many books, are typeset with this tool. There is, however, an initial learning curve one needs to climb before getting comfortable with LATEX – hence this great tool is not required for this course.

1. Provide a one-to-one and onto mapping between the sets

$$A = \{x \in \mathbb{R} \mid 0 < x \leq 1\} \quad \text{and} \quad B = \{x \in \mathbb{R} \mid 0 < x < 1\}.$$

Explain why your mapping has the required properties. Note that such a mapping would establish that $|A| = |B|$. If you are unable to find the function asked for, use Schröder-Bernstein theorem to show that $|A| = |B|$ for partial credit.

2. Consider the following relation R on the set $\mathbb{Z}^+$

$$\forall_{m,n\in\mathbb{Z}^+} (m,n) \in R \iff \lceil \frac{m}{13} \rceil = \lceil \frac{n}{13} \rceil$$

Show that R is an equivalence relation and describe precisely what are its equivalence classes.

3. Consider the relation R on the set of all integer grid points $\mathbb{Z} \times \mathbb{Z}$ which is defined as follows

$$\forall_{(p,q),\,(r,s)\,\in\mathbb{Z}\times\mathbb{Z}} \big((p,q),\,(r,s)\big) \in R \iff \text{the largest of } |p|, |q| = \text{the largest of } |r|, |s|$$

For example, $(-2,-7)\,R\,(7,-5)$ because the largest of the two absolute values of the components of the grid point on the left is $7(= |-7| > |-2|)$ and is equal to the largest of the absolute values of the components of second grid point $(7 = |7| > |-5])$. Note that for grid points with coordinates of the same absolute value, say, (5, -5), any of the two components can be considered as having the largest absolute value.

   (a) Show that R is an equivalence relation
   (b) Describe equivalence classes of R
   (c) List all the grid points that belong to the the class $[\![(2,0)]\!]$ and draw them on the coordinate plane.

In questions 4-6 state and prove the basis and the inductive steps.

4. Prove by induction that for all integers $n \geq 5$

$$2^n > n^2 + n$$

5. Prove by induction that for all integers $n \geq 0$

$$13 \text{ divides } 2^{12n} - 1$$

6. Prove by induction that for all integers $n \geq 0$

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + (n+1)2^{n+1} = n2^{n+2} + 2$$

Note that the sum on the left can be written as

$$\sum_{k=1}^{n+1} k \cdot 2^k$$