Written Assignment 7

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1.a

$$\begin{aligned}
T(1) &= 2T(\lfloor\frac{1}{4}\rfloor) + \sqrt{1} \\
&= 2T(0) + 1 \\
&= 2(1) + 1 \\
T(1) &= 3
\end{aligned}$$

$$\begin{aligned}
T(2) &= 2T(\lfloor\frac{2}{4}\rfloor) + \sqrt{2} \\
&= 2T(0) + \sqrt{2} \\
T(2) &= 2 + \sqrt{2} \\
&\approx 3.41421
\end{aligned}$$

1.b

We need to prove that $T(n) \leq 3\sqrt{n}\log_2 n$ for all $n \geq 2$.
Let's check this premise for basis $n = 2$:

$$\begin{aligned}
T(2) &\leq 3\sqrt{2}\log_2(2) \\
2 + \sqrt{2} &\leq 3\sqrt{2} \\
2 &\leq 2\sqrt{2} \\
1 &\leq \sqrt{2} \\
&\top
\end{aligned}$$

Note the use of (a) in the second equation.
Inductive hypothesis: Let k be an integer ≥ 2. We'll assume that $T(i) \leq 3\sqrt{i}\log_2 i$ for all $2 \leq i \leq k, i \in \mathbb{Z}$.
Now we must prove that

$$T(k+1) \leq 3\sqrt{k+1}\log_2(k+1).$$

$$
\begin{aligned}
T(k+1) &= 2T(\left\lfloor \frac{k+1}{4} \right\rfloor) + \sqrt{k+1} && \text{By definition of } T \\
&\leq 2(3\sqrt{\left\lfloor \frac{k+1}{4} \right\rfloor} \log_2 \left\lfloor \frac{k+1}{4} \right\rfloor) + \sqrt{k+1} && \text{By the inductive hypothesis} \\
&\leq 2(3\sqrt{\frac{k+1}{4}} \log_2(\frac{k+1}{4})) + \sqrt{k+1} && \lfloor x \rfloor \leq x \\
&= 2(\frac{3}{2}\sqrt{k+1} \log_2(\frac{k+1}{4})) + \sqrt{k+1} && \text{By algebra} \\
&= 3\sqrt{k+1} \log_2(\frac{k+1}{4}) + \sqrt{k+1} \\
&= 3\sqrt{k+1}(\log_2(k+1) - \log_2(4)) + \sqrt{k+1} \\
&= 3\sqrt{k+1}(\log_2(k+1) - 2) + \sqrt{k+1} \\
&= 3\sqrt{k+1} \log_2(k+1) - 6\sqrt{k+1} + \sqrt{k+1} \\
&= 3\sqrt{k+1} \log_2(k+1) - 5\sqrt{k+1} \\
&< 3\sqrt{k+1} \log_2(k+1) && \blacksquare
\end{aligned}
$$