CS 135 Spring 2024

WRITTEN HOMEWORK ASSIGNMENT No. 2

Due Friday 2/9/2024 11:59 pm

Please adhere to the following rules:

- This assignment should be submitted via Canvas. Late assignments will not be accepted.
- You should attempt to solve the problems on your own. You are permitted to study with friends but you can have only high-level discussions about the problems. *You must write up your own solutions, in your own words.*
- Using large language models such as ChatGPT to find solutions, copying solutions from the Internet or asking students not enrolled in the class (or the class staff) is strictly prohibited.
- Please explain your answers. There will be penalties for unexplained solutions when the answers are not obvious.
- All submissions should be neat, preferably submitted either as Word or PDF files. We reserve the right to reject handwritten submissions if they are unacceptably messy or too difficult to read.
- I encourage you to learn how to typeset documents with LATEX . You can download Texmaker at http://www.xm1math.net/texmaker/. It is available for most platforms. Another option is to use online service https://www.overleaf.com/. There is a wealth of information online on how to format documents with LATEX and you can always post a question on Moodle's Technical Forum. Most scholarly articles in Mathematics and Computer Science, and even many books, are typeset with this tool. There is, however, an initial learning curve one needs to climb before getting comfortable with LATEX – hence this great tool is not required for this course.

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1. Find a Boolean statement B involving three logical variables p, q, r, whose truth table is:

   T =

p	q	r	**B**
1	1	1	0
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

2. (a) Is the below formula satisfiable? If yes, provide the satisfying assignment. If not, explain why.

$$(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee r) \wedge (p \vee \neg r)$$

(b) Does there exist a satisfying assignment in which $p = 0$. If yes, provide it, otherwise explain why.

3. A Boolean formula ϕ is in the conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals (a literal is either a Boolean variable or a negation of a Boolean variable). For example,

$$\phi(x_1, x_2, x_3) = (x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3)$$

is in CNF. Prove that every Boolean formula ϕ is equivalent to some CNF formula ϕ'. *Hint.* Consider a DNF formula for $\neg\phi$.

4. Consider the following quantified statement which describes a property of a certain (infinite) set of strings L:

$$\neg\Big(\underset{N\geq 1}{\exists} \quad \underset{\substack{w\in L\\ |w|\geq N}}{\forall} \quad \underset{\substack{x,y,z\\ w=xyz\\ |y|>0\\ |xy|\leq N}}{\exists} \quad \underset{i\geq 0}{\forall} \quad xy^iz \in L \Big) \qquad (\dagger)$$

Rewrite this statement in an equivalent form without the $\neg$ symbol. In other words, apply the formula for negation of quantified statements.
(Note that it is not necessary to understand the meaning of the statement to solve the problem. Nonetheless, here are some clarifications. w, x, y, z are some strings, for example, we could have $w =$ "*aabbccdd*", $x =$ "*aabb*", $y =$ "*cc*", $z =$ "*dd*". The $|x|$ denotes the length of a string; thus $|x| = 4, |y| = 2, |z| = 2, |w| = 8$. Writing symbols x, y, z one after another denotes concatenation of strings, e.g., $xyz =$ "*aabbccdd*" $= w$. Finally, y^i denotes the string y repeated (concatenated with itself) i times, e.g., $y^3 = yyy =$ "*cccccc*". You will encounter this statement in Theory of Computation under the name "the Pumping Lemma".)

5. Let S, Q be the following predicates defined on the set of positive integers $\mathbb{Z}^+ = \{n \mid n \text{ is an integer and } n > 0\}$

$$\begin{aligned} S(x) &= \text{" x is a square of a positive integer"} \\ Q(x) &= \text{" x is a cube of a positive integer"} \\ L_{25}(x) &= \text{" x is an integer greater than 25"} \end{aligned}$$

Determine whether the following quantified statements are true. Explain your answers.

(a)
$$\underset{x\in\mathbb{Z}^+}{\exists}\ \underset{y\in\mathbb{Z}^+}{\exists}\ \underset{z\in\mathbb{Z}^+}{\exists}\ L_{25}(x) \wedge L_{25}(y) \wedge L_{25}(z) \ \wedge\ S(x) \wedge S(y) \wedge S(z) \ \wedge\ x + y = z$$

(b)
$$\underset{x\in\mathbb{Z}^+}{\forall}\ \underset{y\in\mathbb{Z}^+}{\exists}\ Q(y) \wedge Q(xy)$$

6. Suppose that we have a binary predicate $P(x, y)$ defined for any pair of real numbers which satisfies

$$\underset{x\in\mathbb{R}}{\forall}\ \underset{y\in\mathbb{R}}{\exists} P(x, y) \qquad (\dagger)$$

John thought that such a predicate must also satisfy

$$\exists_{x \in \mathbb{R}} \exists_{y \in \mathbb{R}} \; y > x \wedge P(x, y) \qquad (\dagger\dagger)$$

Was John right? If yes, give an explanation why. If not, provide a predicate $P(x, y)$ defined for all pairs of real numbers which satisfies (†), but does not satisfy (††).

Hint. Consider predicates whose truth set forms a line. The truth set of a predicate $P(x, y)$ is the set of all pairs which make P true.