CS 135 Written Assignment 2

February 9, 2024

1

$$B \equiv (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$$

To get this formula, I combined all of the values of p, q, and r that return true into one statement in conjunctive normal form.

2

2.1

Yes. There is one set of values for p, q, and r that makes this equivalency true.
If $p = T$, $q = F$, and $r = T$:

$$\begin{aligned}
C(p,q,r) &\equiv (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee r) \wedge (p \vee \neg r) \\
C(T,F,T) &\equiv (T \vee F) \wedge (T \vee \neg F) \wedge (\neg T \vee \neg F) \wedge (\neg T \vee T) \wedge (T \vee \neg T) \\
&\equiv T \wedge (T \vee T) \wedge (F \vee T) \wedge (F \vee T) \wedge (T \vee F) \\
&\equiv T \wedge T \wedge T \wedge T \wedge T \\
&\equiv T
\end{aligned}$$

2.2

No. While $p = F$ the statement cannot be true.

$$\begin{aligned}
C(p,q,r) &\equiv (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee r) \wedge (p \vee \neg r) && \text{Original proposition} \\
C(F,q,r) &\equiv (F \vee q) \wedge (F \vee \neg q) \wedge (\neg F \vee \neg q) \wedge (\neg F \vee r) \wedge (F \vee \neg r) && p = F \\
&\equiv (F \vee q) \wedge (F \vee \neg q) \wedge (T \vee \neg q) \wedge (T \vee r) \wedge (F \vee \neg r) && \text{Complement Laws} \\
&\equiv (q) \wedge (\neg q) \wedge (T \vee \neg q) \wedge (T \vee r) \wedge (\neg r) && \text{Identity laws} \\
&\equiv (q) \wedge (\neg q) \wedge T \wedge T \wedge (\neg r) && \text{Domination laws} \\
&\equiv F \wedge T \wedge T \wedge (\neg r) && \text{Complement law} \\
&\equiv F && \text{Domination laws}
\end{aligned}$$

3

I propose the following steps to convert any Boolean formula ϕ into a CNF format.

1. Negate ϕ, finding $\neg\phi$.
2. Rewrite $\neg\phi$ in DNF.
3. Negate $\neg\phi$, finding $\neg\neg\phi$ which is logically equivalent to ϕ.

4. Convert $\neg\neg\phi$ into CNF using De Morgan's laws.

For example, let us take a look at this proposition:

$$\phi(a,b,c) \equiv (a \lor b) \lor (c \land \neg b) \lor (\neg b \implies c)$$
$$\neg\phi(a,b,c) \equiv \neg((a \land b) \lor (c \land \neg b) \lor (\neg b \implies c))$$

Now we can find $\neg\phi$ in DNF form using a truth table:

a	b	c	$\neg\phi(a,b,c)$
F	F	F	T
F	F	T	F
F	T	F	F
F	T	T	F
T	F	F	T
T	F	T	F
T	T	F	F
T	T	T	F

$$\neg\phi(a,b,c) \equiv (\neg a \land \neg b \land \neg c) \lor (a \land \neg b \land \neg c)$$

Negating $\neg\phi$ gives us $\neg\neg\phi$, which is logically equivalent to ϕ.

$$\begin{aligned}
\neg\phi(a,b,c) &\equiv (\neg a \land \neg b \land \neg c) \lor (a \land \neg b \land \neg c) && \text{Original proposition}\\
\neg\neg\phi(a,b,c) &\equiv \neg((\neg a \land \neg b \land \neg c) \lor (a \land \neg b \land \neg c)) && \text{Negation}\\
\phi(a,b,c) &\equiv \neg((\neg a \land \neg b \land \neg c) \lor (a \land \neg b \land \neg c)) && \text{Double negation law}\\
&\equiv \neg(\neg a \land \neg b \land \neg c) \land \neg(a \land \neg b \land \neg c) && \text{De Morgan's law}\\
&\equiv (a \lor b \lor c) \land (\neg a \lor b \lor c) && \text{De Morgan's laws}
\end{aligned}$$

The last equivalency listed is now in CNF.

4

We can use De Morgan's law on quantifiers as such:

$$\neg(\underset{N\geq 1}{\exists}\ \underset{\substack{w\in L\\|w|\geq N}}{\forall}\ \underset{\substack{x,y,z\\w=xyz\\|y|>0\\|xy|\leq N}}{\exists}\ \underset{i\geq 0}{\forall}\ xy^iz\in L) \qquad \text{Original proposition}$$

$$\equiv \underset{N\geq 1}{\forall}\ \neg(\underset{\substack{w\in L\\|w|\geq N}}{\forall}\ \underset{\substack{x,y,z\\w=xyz\\|y|>0\\|xy|\leq N}}{\exists}\ \underset{i\geq 0}{\forall}\ xy^iz\in L) \qquad \text{De Morgan's Law}$$

$$\equiv \underset{N\geq 1}{\forall}\ \underset{\substack{w\in L\\|w|\geq N}}{\exists}\ \neg(\underset{\substack{x,y,z\\w=xyz\\|y|>0\\|xy|\leq N}}{\exists}\ \underset{i\geq 0}{\forall}\ xy^iz\in L) \qquad \text{De Morgan's Law}$$

$$\equiv \underset{N\geq 1}{\forall}\ \underset{\substack{w\in L\\|w|\geq N}}{\exists}\ \underset{\substack{x,y,z\\w=xyz\\|y|>0\\|xy|\leq N}}{\forall}\ \neg(\underset{i\geq 0}{\forall}\ xy^iz\in L) \qquad \text{De Morgan's Law}$$

$$\equiv \underset{N\geq 1}{\forall}\ \underset{\substack{w\in L\\|w|\geq N}}{\exists}\ \underset{\substack{x,y,z\\w=xyz\\|y|>0\\|xy|\leq N}}{\forall}\ \underset{i\geq 0}{\exists}\ \neg(xy^iz\in L) \qquad \text{De Morgan's Law}$$

$$\equiv \underset{N\geq 1}{\forall}\ \underset{\substack{w\in L\\|w|\geq N}}{\exists}\ \underset{\substack{x,y,z\\w=xyz\\|y|>0\\|xy|\leq N}}{\forall}\ \underset{i\geq 0}{\exists}\ xy^iz\notin L \qquad \text{De Morgan's Law}$$

5

5.1

To start interpreting this statement I will convert it into plain English:
`There exist some positive integers x, y, and z for which all of the following are true:`
`x, y, and z are integers greater than 25.`
`x, y, and z are squares of a positive integer.`
`The sum of x and y is equal to z.`
Because only existential quantifiers are used, we only need to find one set of (x, y, z) where all of these conditions are met to prove this statement true.
Take for example $(36, 64, 100)$:
`36, 64, and 100 are positive integers.`
`36, 64, and 100 are integers greater than 25.`
`36, 64, and 100 are squares of a positive integer.`
`The sum of 36 and 64 is equal to 100.`
With this example we can prove this statement true.

5.2

Again I will write out the conditions in plain English:
`For all positive integers x, there exists a positive integer y for which all of the following are true:`
`y is a cube of a positive integer.`
`The product of x and y is a cube of a positive integer.`
Because a universal quantifier is present, we can disprove this statement by finding a value x that makes the statement false.

Let us check $(3, 8)$:
`3 is a positive integer, 8 is a positive integer.`
`8 is a cube of a positive integer.`
`The product of 3 and 8 is NOT a cube of a positive integer.`
$3 \times 8 = 24$, and the cube root of 24 is $2.88449\ldots$ which is not a positive integer. As such, the statement is false.

6

John is incorrect. Here is an example: if $P(x, y) \equiv x = y$, (†) is satisfied but (††) is not.
Let us prove (†) true:

$$\underset{x\in\mathbb{R}}{\forall}\ \underset{y\in\mathbb{R}}{\exists}\ x = y$$

Because x and y are in the same set, $x = y$ will always be true. Now to prove (††) false:

$$\underset{x\in\mathbb{R}}{\exists}\ \underset{y\in\mathbb{R}}{\exists}\ y > x \land x = y$$

This conjunction is always false because no y can be both exclusively greater than and equal to x.