# Written Assignment 7

CS 135-B/LF

April 4, 2024

## 1

### 1.a

$$
\begin{aligned}
T(1) &= 2T(\lfloor \frac{1}{4} \rfloor) + \sqrt{1} \\
&= 2T(0) + 1 \\
&= 2(1) + 1 \\
T(1) &= 3
\end{aligned}
$$

$$
\begin{aligned}
T(2) &= 2T(\lfloor \frac{2}{4} \rfloor) + \sqrt{2} \\
&= 2T(0) + \sqrt{2} \\
T(2) &= 2 + \sqrt{2} \\
&\approx 3.41421
\end{aligned}
$$

### 1.b

We need to prove that $T(n) \leq 3\sqrt{n}\log_2 n$ for all $n \geq 2$.
Let's check this premise for basis $n = 2$:

$$
\begin{aligned}
T(2) &\leq 3\sqrt{2}\log_2(2) \\
2 + \sqrt{2} &\leq 3\sqrt{2} \\
2 &\leq 2\sqrt{2} \\
1 &\leq \sqrt{2} \\
&\top
\end{aligned}
$$

Inductive hypothesis: Let $k$ be an integer $\geq 2$. We'll assume that $T(i) \leq 3\sqrt{i}\log_2 i$ for all $2 \leq i \leq k, i \in \mathbb{Z}$.
Now we must prove that

$$T(k+1) \leq 3\sqrt{k+1}\log_2(k+1).$$

$$
\begin{aligned}
T(k+1) &= 2T(\left\lfloor \frac{k+1}{4} \right\rfloor) + \sqrt{k+1} && \text{By definition of } T \\
&\leq 2(3\sqrt{\left\lfloor \frac{k+1}{4} \right\rfloor} \log_2 \left\lfloor \frac{k+1}{4} \right\rfloor) + \sqrt{k+1} && \text{By the inductive hypothesis} \\
&\leq 2(3\sqrt{\frac{k+1}{4}} \log_2(\frac{k+1}{4})) + \sqrt{k+1} && \lfloor x \rfloor \leq x \\
&= 2(\frac{3}{2}\sqrt{k+1} \log_2(\frac{k+1}{4})) + \sqrt{k+1} && \text{By algebra} \\
&= 3\sqrt{k+1} \log_2(\frac{k+1}{4}) + \sqrt{k+1} \\
&= 3\sqrt{k+1}(\log_2(k+1) - \log_2(4)) + \sqrt{k+1} \\
&= 3\sqrt{k+1}(\log_2(k+1) - 2) + \sqrt{k+1} \\
&= 3\sqrt{k+1} \log_2(k+1) - 6\sqrt{k+1} + \sqrt{k+1} \\
&= 3\sqrt{k+1} \log_2(k+1) - 5\sqrt{k+1} \\
&< 3\sqrt{k+1} \log_2(k+1) && \blacksquare
\end{aligned}
$$

## 2

Let $a, b, c, d, m \in \mathbb{Z}$ with $m \geq 2$. We'll prove that
$a \equiv b (\text{mod } m) \wedge c \equiv d (\text{mod } m) \implies a - c \equiv b - d (\text{mod } m)$. Given

$$
\begin{cases} a \equiv b (\text{mod } m) \\ c \equiv d (\text{mod } m) \end{cases}
$$

we can break this down into

$$
\begin{cases} a \text{ mod } m = b \text{ mod } m \\ c \text{ mod } m = d \text{ mod } m \end{cases}
$$

by the definition of congruence. We'll scale the second equation by $-1$,

$$
\begin{cases} a \text{ mod } m & = b \text{ mod } m \\ (-c) \text{ mod } m & = (-d) \text{ mod } m \end{cases}
$$

add the equations together as one can do with a system,

$$
(a - c) \text{ mod } m = (b - d) \text{ mod } m
$$

and rewrite this as the congruence

$$
a - c \equiv b - d (\text{mod } m).
$$

$\blacksquare$

## 3

No. For $k = 6$, $2 * 3 * 5 * 7 * 11 * 13 + 1 = 30031$. 30031 has factors $\{59, 509\}$ which means 30031 is not prime. Therefore $p_1 p_2 \cdots p_k + 1$ is not always prime.

## 4

No. Since 30 is not divisible by 29, let $m = 30$:

$$\begin{aligned}
&2m^2 + 29 \\
&= 2(30)^2 + 29 \\
&= 1829
\end{aligned}$$

1829 has factors $\{31, 59\}$ which means 1829 is not prime. Therefore $2m^2 + 29$ does not always produce a prime number when m is not divisible by 29.

## 5

Let's prove a basis where $k = 1$: and since $1^4 - 1 = 0$ and 0 is divisible by 16, this basis holds.
Let's assume that $k$ is divisible by 16. With this we can say that there is some integer $s$ for which $k^4 - 1 = 16s$.
Let's prove that $k + 2$ is divisible by 16 ($k + 2$ because k and this equation's result must be odd). We can do this by proving that there is some integer which, when multiplied by 16, gives some expression of $k$. For some integer $t$:

$$\begin{aligned}
16t &= (k + 2)^4 - 1 \\
&= (k^2 + 4k + 4)^2 - 1 \\
&= k^4 + 8k^3 + 24k^2 + 32k + 16 - 1 \\
&= k^4 - 1 + 8k^3 + 24k^2 + 32k + 16
\end{aligned}$$

By the inductive hypothesis: there is some integer $s$ for which $k^4 - 1 = 16s$

$$\begin{aligned}
&= 16s + 8k^3 + 24k^2 + 32k + 16 \\
&= 16s + 8k^2(k + 3) + 16(2k + 1)
\end{aligned}$$

Since $k$ is an integer, we can reasonably expect $2k + 1$ to be an integer as well. $8k^2(k + 3)$ will be a multiple of 8 and some odd number $k^2(k + 3)$, which is guaranteed to be divisible by 16. We can think of this equation as

$$\text{(Some multiple of 16)} = \text{(The sum of multiples of 16)}$$

which proves that $k + 2$ produces a multiple of 16, which proves that for all odd integers $k \geq 1$, $k^4 - 1$ is divisible by 16.

## 6

$$\begin{aligned}
&10^{6,000,000,000,000,000,000,000,000,000,000,000,000,000,000,002} \bmod 13 \\
&= (10 \bmod 13)(10 \bmod 13)(10^{6,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000} \bmod 13) \bmod 13 \\
&= 10 \cdot 10 \cdot (10^{6 \times 10^{42}} \bmod 13) \bmod 13 \\
&= 100 \cdot (10^6 \bmod 13) \cdot (10^{10^{42}} \bmod 13) \bmod 13 \\
&= 100 \cdot 1 \cdot (10^{10^{21}} \bmod 13)^2 \bmod 13 \\
&= 100 \cdot ((10^{10^7} \bmod 13)^3 \bmod 13)^2 \bmod 13 \\
&= 100 \cdot (((10^{10} \bmod 13)^7 \bmod 13))^3 \bmod 13)^2 \bmod 13
\end{aligned}$$

Now we have a lot of easily-calculable modulus operations.

$$
\begin{aligned}
&= 100 \cdot ((3^7 \bmod 13)^3 \bmod 13)^2 \bmod 13 \\
&= 100 \cdot 1^2 \bmod 13 \\
&= 100 \cdot 1 \bmod 13 \\
&= 100 \bmod 13 \\
&= 9
\end{aligned}
$$