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Query 3
I pledge my honor that I have abided by the Stevens Honor System.

Problem 1

Given this information, we can rewrite $Y = X\beta + \varepsilon$ as such:

$$\begin{pmatrix} y_1 \\ \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \times \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}.$$

Problem 2

We can gather this from slide deck 8 supplement:

$$X'X = \begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{pmatrix}$$

thus

$$(X'X)^{-1} = \frac{1}{|X'X|}(X'X)^* = \frac{1}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \begin{pmatrix} \sum_{i=1}^n x_i^2 & -\sum_{i=1}^n x_i \\ -\sum_{i=1}^n x_i & n \end{pmatrix}$$

where $(X'X)^*$ is the adjugate version of $X'X$.

On the same slide, we find that

$$X'Y = \begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{pmatrix}.$$

which lets us calculate

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = (X'X)^{-1}X'Y = \frac{1}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \begin{pmatrix} \sum_{i=1}^n x_i^2 & -\sum_{i=1}^n x_i \\ -\sum_{i=1}^n x_i & n \end{pmatrix} \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i - \sum_{i=1}^n x_i \sum_{i=1}^n x_i y_i}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \\ \frac{\frac{1}{n}\sum_{i=1}^n x_i y_i - \frac{1}{n}\sum_{i=1}^n x_i \frac{1}{n}\sum_{i=1}^n y_i}{\frac{1}{n}\sum_{i=1}^n x_i^2 - (\frac{1}{n}\sum_{i=1}^n x_i)^2} \end{pmatrix}$$

At this point, by plugging $\hat{\beta}_1$ into $\hat{\beta}_0$, we can simplify to $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$.

Problem 3

We can compare our final formula to equation 3 from slide deck 8:

$$\begin{cases} \hat{\beta}_1 = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \frac{1}{n}\sum_{i=1}^{n} y_i}{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\frac{1}{n}\sum_{i=1}^{n} x_i)^2} \\ \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \end{cases}$$

$$\begin{cases} \widehat{\beta}_1 & = & \dfrac{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i y_i - \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i \cdot \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} y_i}{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i^2 - \left(\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i\right)^2} \\ \widehat{\beta}_0 & = & \bar{y} - \widehat{\beta}_1 \bar{x}. \end{cases}$$

And find that they are identical.