Homework 8

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I pledge my honor that I have abided by the Stevens Honor System.

1

To find the least squares equation, we need to find the SSE. We can measure the residuals as $e_i = y_i - (\beta + \beta_1 x_i)$ with $i = 1, \ldots, n$. If we know $SSE = \sum_{i=1}^n e_i^2$ then $SSE = \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2$. If we set $\frac{\partial}{\partial \beta_j} SSE = 0$ for $j = 0, 1$, we can get the normal equations

$$\begin{cases} 1 \cdot \beta_0 + \frac{1}{n}\sum_{i=1}^n x_i + \beta_1 = \frac{1}{n}\sum_{i=1}^n y_i \\ \frac{1}{n}\sum_{i=1}^n x_i \cdot \beta_0 + \frac{1}{n}\sum_{i=1}^n x_i + \beta_1 = \frac{1}{n}\sum_{i=1}^n x_i y_i \end{cases}$$

2

We can use the equation from number 1 to get the Hassian matrix

$$\begin{pmatrix} \frac{\partial^2}{\partial \beta_0^2} SSE & \frac{\partial^2}{\partial \beta_0 \partial \beta_1} SSE \\ \frac{\partial^2}{\partial \beta_0 \partial \beta_1} SSE & \frac{\partial^2}{\partial \beta_1^2} SSE \end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{pmatrix}$$

The Hassian matrix is positive definitive whenever all x_is are non-zero.

3

$$Y = \tilde{X}\alpha + \varepsilon \implies \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_1 - \bar{x} \\ \vdots & \vdots \\ 1 & x_n - \bar{x} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} + \begin{pmatrix} \varepsilon_0 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

4

Using the equation from problem 3:

$$(\tilde{X}'\tilde{X})^{-1} = \frac{1}{|\tilde{X}'\tilde{X}|}(\tilde{X}'\tilde{X})^* = \frac{1}{n\sum_{i=1}^n (x_i^2 - \bar{x}^2) - (\sum_{i=1}^n x_i - \bar{x})^2} \begin{pmatrix} \sum_{i=1}^n x_i^2 - \bar{x}^2 & -\sum_{i=1}^n x_i - \bar{x} \\ -\sum_{i=1}^n x_i - \bar{x} & n \end{pmatrix}$$

and:

$$\tilde{X}'Y = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n (x_i - \bar{x}) y_i \end{pmatrix}$$

which lets us calculate:

$$\hat{\alpha} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}Y = \begin{cases} \bar{y} - \hat{\alpha}_1 \bar{x} \\ \frac{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x}) y_i - \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x}) \cdot \frac{1}{n}\sum_{i=1}^n y_i}{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2 - (\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x}))^2} \end{cases}$$

5

```
X <- c(1.41,1.50,1.53,1.55,1.57,1.60,1.62,1.65,1.68,1.71,1.73,1.76,1.78,1.80,1.82)
Y <- c(52.2,53.1,54.5,55.8,57.2,58.6,59.6,61.2,62.1,64.5,66.3,68.2,69.8,72.2,74.6)
n <- length(X)
Xbar <- X-mean(X)
a_1n <- ((1/n) * sum(Xbar * Y)) - ((1/n) * sum(Xbar)) * ((1/n) * sum(Y))
a_1d <- ((1/n) * sum(Xbar^2)) - ((1/n) * sum(Xbar))^2
a_1 <- a_1n / a_1d; a_1
```

```
## [1] 56.92512
```

```
a_0 <- mean(Y) - (a_1 * mean(Xbar)); a_0
```

```
## [1] 61.99333
```

```
sprintf('Our equation is: Y = %fX + %f', a_1, a_0)
```

```
## [1] "Our equation is: Y = 56.925123X + 61.993333"
```

6

i

$$Y = X\beta + \varepsilon \implies \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} \varepsilon_0 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

```
b_1n <- ((1/n) * sum(X * Y)) - ((1/n) * sum(X)) * ((1/n) * sum(Y))
b_1d <- ((1/n) * sum(X^2)) - ((1/n) * sum(X))^2
b_1 <- b_1n / b_1d
b_0 <- mean(Y) - (b_1 * mean(X))
sprintf('Our equation is: Y = %fX + %f', b_1, b_0)
```

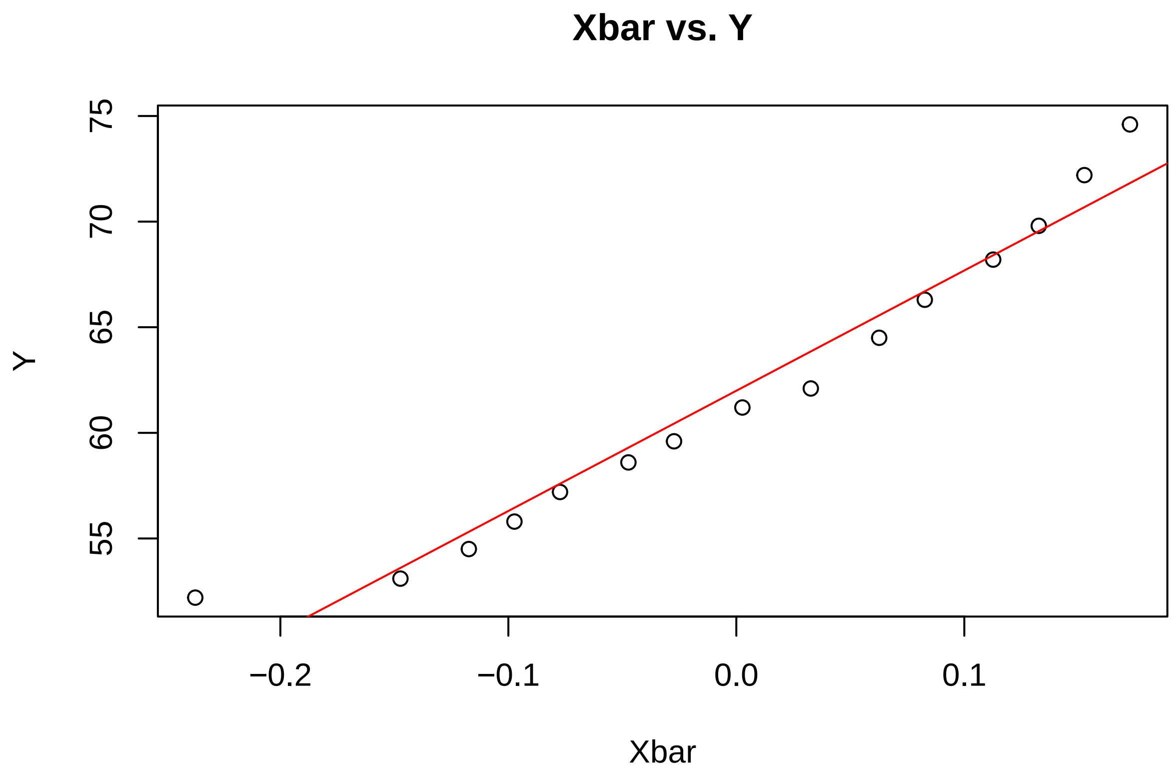
```
## [1] "Our equation is: Y = 56.925123X + -31.781320"
```

ii

If we compare the equation from 6.i ($Y = 56.925123X + -31.781320$) to the equation from 5 ($Y = 56.925123X + 61.993333$) we find that the slope $\hat{\alpha}_1 = \hat{\beta}_1$, but the intercepts $\hat{\alpha}_0 \neq \hat{\alpha}_1$. This is because centralizing X as we did in 5 preserves the relationship between X and Y, except it moves all of the points down by the mean of X.

7

```
plot(Xbar, Y, main="Xbar vs. Y")
abline(a = a_0, b = a_1, col="red")
```



8

According to slide deck 8, page 28, the CI of slope $\hat{\alpha}_1$ is

$$CI = \hat{\alpha}_1 \pm t_{1-\alpha/2}(n-2) * SE_{\hat{\beta}_1}$$

$$\text{where } SE_{\hat{\alpha}_0} = \sqrt{\frac{1}{\sum_{i=1}^n (x_i - \bar{x})^2} S^2}$$

$$\text{where } S^2 = \frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

```
SS <- sum((Y - (b_1*X + b_0))^2)/(n-2)
SE <- sqrt(SS / (sum((X-mean(X))^2)))
cl <- 0.925
sprintf('Our confidence interval is: [%f, %f]',
    b_1 - qt(1 - (cl/2), n-2) * SE,
    b_1 + qt(1 - (cl/2), n-2) * SE)
```

```
## [1] "Our confidence interval is: [56.593586, 57.256661]"
```

9

We have $H_0 : \beta_1 = 0$ vs. $H_a : \beta_1 \neq 0$. According to slide deck 8, page 31,

$$P_{H_0}(|T| > |t|) 2[1 - \texttt{pt}(|t|, n-2)] < \alpha$$

with $T = \frac{\hat{\beta}_1}{SE_{\hat{\beta}_1}} \sim T_{n-2}$ observed at t from data.

```
t = b_1/SE; t
```

```
## [1] 16.47985
```

```
p = 2 * (1 - pt(abs(t), n-2)); p
```

```
## [1] 4.30179e-10
```

Our p-value is smaller than all common p-values so we may reasonably reject H_0.

10

According to slide deck 8, page 38,

$$SE_{\hat{Y}^*} = \sqrt{1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_1 - \bar{x})^2} S^2}$$

where S^2 is the same as in 8. The prediction interval, according to page 39, is estimated as

$$\hat{\beta}_0 + \hat{\beta}_1 x^* \pm t_{1-\alpha/2}(n-2) \cdot SE_{\hat{Y}^*}.$$

```
xstar <- 1.85
cl <- 0.9
SEY <- sqrt(1 + (1 / n) + ((xstar - mean(X))^2)/(sum((X - mean(X))^2)))
sprintf('Our prediction interval is: [%f, %f]',
    b_0 + (b_1 * xstar) - pt(1 - (cl / 2), n-2) * SEY,
    b_0 + (b_1 * xstar) + pt(1 - (cl / 2), n-2) * SEY)
```

```
## [1] "Our prediction interval is: [72.737670, 74.322647]"
```