CHAPTER 10

Herbrand Proofs

10.1 Introduction

Logical entailment for Herbrand Logic is defined the same as for Propositional Logic and Relational Logic. A set of premises logically entails a conclusion if and only if every truth assignment that satisfies the premises also satisfies the conclusions. In the case of Propositional Logic and Relational Logic, we can check logical entailment by examining a truth table for the language. With finitely many proposition constants, the truth table is large but finite. For Herbrand Logic, things are not so easy. It is possible to have Herbrand bases of infinite size; and, in such cases, truth assignments are infinitely large and there are infinitely many of them, making it impossible to check logical entailment using truth tables.

All is not lost. As with Propositional Logic and Relational Logic, we can establish logical entailment in Herbrand Logic by writing proofs. In fact, it is possible to show that, with a few simple restrictions, a set of premises logically entails a conclusions if and only if there is a finite proof of the conclusion from the premises, even when the Herbrand base is infinite. Moreover, it is possible to find such proofs in a finite time. That said, things are not perfect. If a set of sentences does *not* logically entail a conclusion, then the process of searching for a proof might go on forever. Moreover, if we remove the restrictions mentioned above, we lose the guarantee of finite proofs. Still, the relationship between logical entailment and finite provability, given those restrictions, is a very powerful result and has enormous practical benefits.

In this chapter, we talk about the non-compactness of Herbrand Logic and the loss of completeness in our proof procedure. In the next chapter, we look at an extension to Fitch, called Induction, that allows us to prove more results in Herbrand Logic.

10.2 Non-Compactness and Incompleteness

In light of the negative results above, namely that Herbrand Logic is inherently incomplete, it is not surprising that Herbrand Logic is not compact. Recall that Compactness says that if an infinite set of sentences is unsatisfiable, there is some finite subset that is unsatisfiable. It guarantees finite proofs.

Non-Compactness Theorem: Herbrand Logic is not compact.

Proof. Consider the following infinite set of sentences.

$$p(a), p(f(a)), p(f(f(a))), ...$$

Assume the vocabulary is $\{p, a, f\}$. Hence, the ground terms are $a, f(a), f(f(a))$, ... This set of sentences entails $\forall x.p(x)$. Add in the sentence $\exists x.\neg p(x)$. Clearly, this infinite set is unsatisfiable. However, every finite subset is satisfiable. (Every finite subset is missing either $\exists x.\neg p(x)$ or one of the sentences above. If it is the former, the set is satisfiable; and, if it is the latter, the set can be satisfied by making the missing sentence false.) Thus, compactness does not hold.

Corollary (Infinite Proofs): In Herbrand Logic, some entailed sentences have only infinite proofs.

Proof. The above proof demonstrates a set of sentences that entail $\forall x.p(x)$. The set of premises in any finite proof will be missing one of the above sentences; thus, those premises do not entail $\forall x.p(x)$. Thus no finite proof can exist for $\forall x.p(x)$.

The statement in this Corollary was made earlier with the condition that checking whether a candidate proof actually proves a conjecture is decidable. There is no such condition on this theorem.