AP Physics C Mechanics notes: Work, Energy and Power

Roger Yang

September 2023

Contents

1 Overview and Guide

This is a collection of notes on calculus-based physics at the **AP Physics C** level. They cover the entire 2023-2024 AP Physics C curriculum at a suitable depth for complete self-study, but since the underlying physics is the same, they can be used in accompaniment with classes or programs.

The first section of notes is on **mechanics,** split into seven units based on the official CollegeBoard course description. They assume fluency with single-variable calculus at the level of AP Calculus AB, and preferably some honors physics background. An approximate unit breakdown of the **AP Physics C Mechanics** exam is provided below for reference.

Mechanics unit	Approximate portion of exam grade (%)
Kinematics	14-20
Newton's Laws	16-20
Work, Energy and Power	**14-17**
Systems and Linear Momentum	14-17
Rotation	14-20
Oscillations	6-14
Newtonian Gravity	6-14

The second section of notes is on **electricity and magnetism,** split into five units based on the official CollegeBoard course description. They assume the same background as the first section notes, but more background with introductory physics is preferred, particularly with fields. An approximate unit breakdown of the **AP Physics C Electricity and Magnetism** exam is provided below for reference.

E & M unit	Approximate portion of exam grade (%)
Electrostatics	26-34
Electricity in matter	14-17
Circuits	17-23
Magnetism	17-23
Electromagnetism	14-20

We use the following common conventions.

- Arrowed quantities like $\vec{A}, \vec{p}, \vec{x}$ are vectors, whereas unbold quantities like v, r, t are scalars. If a vector quantity is introduced, its magnitude will often be noted by the unarrowed version of itself: $v = \|\vec{v}\|$.
- Using Newton's convention, time derivatives are denoted with dots: $\dot{v} = \frac{\mathrm{d}v}{\mathrm{d}t}$. Note that primes are *not* derivatives unless otherwise specified.
- Time-averaged quantities will be denoted with an overline; if r is precipitation, $\overline{r}$ is the time-averaged precipitation.
- Differential quantities will be treated like legitimate algebraic quantities; if $\frac{\mathrm{d}m}{\mathrm{d}t}$ is the rate of change of the mass of a snowball, then $\mathrm{d}m$ may be interpreted as the amount of mass accumulated in time $\mathrm{d}t$.

2 Work and kinetic energy

We know that moving objects store energy because colliding objects dissipate energy as heat. Energy stored in the motion of objects is called *kinetic energy.* The kinetic energy of an object with mass m and speed v is $K = \frac{1}{2}mv^2$.

Remark 1

It's pretty intuitive that we should have $K \propto m$. There's a simple physical argument that shows we also must have $K \propto v^2$.

Let $K(m, v)$ be the kinetic energy as a function of mass and velocity, where we have $K(m, v) = mE(v)$ for some function E, since $K \propto m$. Consider two clay spheres which approach each other with opposite velocities and equal speed v, and suppose they stick together and remain at rest after colliding.

Let Q be the heat released by the collision. By conservation of energy,

$$K(m, v) + K(m, v) = 2K(m, v) = Q.$$

The key is that conservation of energy holds in *all* inertial frames, so we can also analyze the situation in the frame which follows one of the spheres initially. In this frame, one sphere is at rest, and the other moves with speed $2v$. They then collide and stick together, forming an object of mass $2m$ which travels at speed v. The heat dissipated in this frame is also Q, since all observers must agree on the heat released (it can be measured using a thermometer, which all observers must agree on). Conservation of energy then gives

$$K(m, 2v) = K(2m, v) + Q = 2K(m, v) + 2K(m, v) = 4K(m, v),$$

where we substituted the result from the lab frame and used the fact that $K \propto m$. This implies

$$E(2v) = 4E(v),$$

so kinetic energy is quadratic with speed.

Idea 1

If a force $\vec{F}$ is applied to an object and the *point of contact of the force* undergoes an infinitesimal displacement $\mathrm{d}\vec{x}$, the infinitesimal work done is

$$\mathrm{d}W = \vec{F} \cdot \mathrm{d}\vec{x} \implies W = \int \vec{F} \cdot \mathrm{d}\vec{x}.$$

Note that work is a scalar. If the work done on the object is only a function of the endpoints of the path $\vec{x}(t)$, the force is called *conservative*. Nonconservative forces generally depend on the object's velocity, like friction and air resistance.

Example 1: $F = ma$ 2019B

A uniform block of mass 10 kg is released at rest from the top of an incline of length 10 m and inclination 30°. The coefficients of static and kinetic friction between the incline and the block are $\mu_s = 0.15$ and $\mu_k = 0.1$. The end of the incline is connected to a frictionless horizontal surface. After a long time, what is the work done by friction?

Solution 1

It's easy to check that $\mu_s < \tan\theta$, so the block does slip, and friction does do work. Since the friction force is constant, the work done is

$$W = -\mu_k N\ell = -\mu_k mg\ell\cos\theta = -86.6 \text{ J}.$$

Note that the work done by friction is negative since the force of friction is directed opposite to the motion of the object.

Example 2

A wire is bent and suspended such that the left endpoint is h above the right endpoint. You attach a bead of mass m to the left endpoint and slide it to the right endpoint. How much work does gravity do during this process?

Solution 2

Let the path of the wire be $\vec{r}$. We have

$$\begin{aligned} W &= \int \vec{F} \cdot \mathrm{d}\vec{r} \\ &= \int mg \, \mathrm{d}y, \end{aligned}$$

since gravity is purely vertical. Then the work done is simply $-mgh$, independent of the path. Thus, gravity is conservative.

Idea 2: Work-energy theorem

Consider a point object of mass m which has acceleration $\vec{a}$. We have

$$\vec{a} \cdot \mathrm{d}\vec{x} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \cdot \mathrm{d}\vec{x} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \cdot \mathrm{d}\vec{v} = \vec{v} \cdot \mathrm{d}\vec{v},$$

where we used the chain rule in the middle equality. Multiplying through by m and integrating both sides gives

$$\int \vec{F}_{\text{net}} \cdot \mathrm{d}\vec{x} = m \int \vec{v} \cdot \mathrm{d}\vec{v} = \Delta\left(\frac{1}{2}mv^2\right),$$

so the net work done on the object (which is the sum of the work done by all forces) is the change in kinetic energy. Note that we only used the assumption $\vec{F}_{\text{net}} = m\vec{a}$, so the work-energy theorem is valid in all inertial frames. The theorem fails in general for extended objects, but a version of it can be rescued (see Remark 2).

Remark 2

Energy conservation works in all inertial reference frames of course, but changes in kinetic energy are frame-dependent!

Idea 3

The power delivered to an object is the rate of change of its energy with respect to time. Then,

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \vec{F} \cdot \vec{v}.$$

Biologically, power output roughly determines one's perceived rate of exertion.

Example 3

An astronaut is piloting a spacecraft in low-Earth orbit. He wants to increase his energy by firing a thruster that applies a fixed force over a short period of time. In what direction should he fire the thruster to maximize his final energy?

Solution 3

We want to maximize $\vec{F} \cdot \vec{v}$, which occurs when the force is parallel to the velocity. Thus, he should fire the thruster directly behind him.

Remark 3: CM Work (Zhou)

Consider a child accelerating on a bicycle from rest, and assume the bicycle wheels never slip on the ground. Suppose the child bikes a distance d, experiencing a constant static friction force F.

Suppose we wanted to find the work done on the child-bike system by static friction. Since the wheels never slip, the point of contact of static friction is always instantaneously at rest, so the work is zero. But doesn't that contradict the work-energy theorem, since the kinetic energy of the child-bike system can increase? No, because in general, the work-energy theorem states that work is the change in the *total* energy of the system, not just the kinetic energy. In other words,

$$W = \Delta(K + U_{\text{int}}),$$

where U_{int} is the internal energy of the system. This reduces to the normal work-energy theorem for point objects if we assume they can't store internal energy. For the biker, $W = 0$, so $\Delta K = -\Delta U_{\text{int}}$. Thus, the biker can increase their kinetic energy, but only by consuming internal energy from their bodily metabolism. So, the work-energy theorem actually doesn't tell us *anything* about the kinetic energy of the biker.

It turns out there is an alternative formulation of the work-energy theorem which is more helpful for these kinds of problems. Define the *center of mass work* by

$$W_{\text{cm}} = \int \vec{F} \cdot \mathrm{d}\vec{x}_{\text{cm}},$$

where $\mathrm{d}\vec{x}_{\text{cm}}$ is an infinitesimal displacement of the center of mass. Then we have the "center of mass work-energy theorem"

$$W_{\text{cm}} = \Delta\left(\frac{1}{2}Mv_{\text{cm}}^2\right).$$

In the case of the biker, the CM work is Fd, so $\frac{1}{2}Mv_f^2 = Fd$, allowing us to compute the change in speed of the biker.

The CollegeBoard doesn't make the distinction between normal work and CM work, which can lead to a lot of confusion for students. Always remember that work only cares about the displacement of the *point of contact,* and relates to a system's change in *total energy.*

3 Potential energy

Potential energy refers to the energy stored by objects which is not due to their internal energy or motion.

Idea 4

The difference in potential energy between points a and b due to a force $\vec{F}$ is

$$\Delta U = -\int_a^b \vec{F} \cdot \mathrm{d}\vec{r}.$$

For gravity close to the Earth, $F = -mg$, so the potential energy due to gravity (close to the Earth) is mgh, assuming zero potential at the surface. Similarly, for an ideal spring we have $F = -kx$, so $U_{\text{spring}} = \frac{1}{2}kx^2$, assuming zero potential at rest length.

Note that if the integral is path dependent, then it is impossible to define a potential. Only conservative forces allow potentials; there is no friction potential, for example.

We now have the tools to solve trickier problems with energy conservation.

Example 4: $F = ma$ 2022B

A ramp with height h is moving with fixed, uniform speed v to the right. A small block of mass m is placed at the top of the ramp, and is released at rest with respect to the ramp. The block slides smoothly to the bottom of the ramp and onto the floor. How much kinetic energy does it gain in this process? Neglect friction.

Solution 4

It's tempting to just say mgh, but that's not correct since the normal force of the ramp does work on the block. To get rid of the normal force, enter the frame of the ramp, in which the block starts from rest and ends with speed $\sqrt{2gh}$. Going back to the lab frame, the block has final speed $v + \sqrt{2gh}$, so the change in kinetic energy is

$$\frac{1}{2}m\left(v + \sqrt{2gh}\right)^2 - \frac{1}{2}mv^2 = mgh + mv\sqrt{2gh}.$$

We see that the normal force does $mv\sqrt{2gh}$ work on the block.