AP Physics C Mechanics notes: Kinematics

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Contents

1 Overview and Guide

This is a collection of notes on calculus-based physics at the **AP Physics C** level. They cover the entire 2023-2024 AP Physics C curriculum at a suitable depth for complete self-study, but since the underlying physics is the same, they can be used in accompaniment with classes or programs.

The first section of notes is on **mechanics,** split into seven units based on the official CollegeBoard course description. They assume fluency with single-variable calculus at the level of AP Calculus AB, and preferably some honors physics background. An approximate unit breakdown of the **AP Physics C Mechanics** exam is provided below for reference.

Mechanics unit	Approximate portion of exam grade (%)
Kinematics	**14-20**
Newton's Laws	16-20
Work, Energy and Power	14-17
Systems and Linear Momentum	14-17
Rotation	14-20
Oscillations	6-14
Newtonian Gravity	6-14

The second section of notes is on **electricity and magnetism,** split into five units based on the official CollegeBoard course description. They assume the same background as the first section notes, but more background with introductory physics is preferred, particularly with fields. An approximate unit breakdown of the **AP Physics C Electricity and Magnetism** exam is provided below for reference.

E & M unit	Approximate portion of exam grade (%)
Electrostatics	26-34
Electricity in matter	14-17
Circuits	17-23
Magnetism	17-23
Electromagnetism	14-20

We use the following common conventions.

- Arrowed quantities like $\vec{A}, \vec{p}, \vec{x}$ are vectors, whereas unbold quantities like v, r, t are scalars. If a vector quantity is introduced, its magnitude will often be noted by the unarrowed version of itself: $v = \|\vec{v}\|$.
- Using Newton's convention, time derivatives are denoted with dots: $\dot{v} = \frac{\mathrm{d}v}{\mathrm{d}t}$. Note that primes are *not* derivatives unless otherwise specified.
- Time-averaged quantities will be denoted with an overline; if r is precipitation, $\overline{r}$ is the time-averaged precipitation.
- Differential quantities will be treated like legitimate algebraic quantities; if $\frac{\mathrm{d}m}{\mathrm{d}t}$ is the rate of change of the mass of a snowball, then $\mathrm{d}m$ may be interpreted as the amount of mass accumulated in time $\mathrm{d}t$.

2 Mechanics

2.1 Kinematics

Kinematics is the study of motion of idealized objects without reference to their internal characteristics. We will first analyze motion in one dimension before generalizing to an arbitrary number of dimensions.

2.1.1 1D Kinematics

We characterize the motion of objects using their position, velocity, and acceleration.

- Positions of objects are measured with respect to a given coordinate system. In one dimension, the position of an object can be represented by a scalar x, where $x = 0$ denotes the origin. Changes in position Δx are called displacements.
- The velocity of an object is the time derivative of its position. If an object undergoes displacement Δx over a time Δt, its average velocity over that interval is $\overline{v} = \frac{\Delta x}{\Delta t}$.
- The acceleration of an object is the time derivative of its velocity. If an object changes in velocity by Δv over a time period Δt, its average acceleration over that period is $\overline{a} = \frac{\Delta v}{\Delta t}$.
- The higher-order derivatives of position include jerk ($\frac{\mathrm{d}^3 x}{\mathrm{d}t^3}$), snap ($\frac{\mathrm{d}^4 x}{\mathrm{d}t^4}$), crackle ($\frac{\mathrm{d}^5 x}{\mathrm{d}t^5}$), and pop[1] ($\frac{\mathrm{d}^6 x}{\mathrm{d}t^6}$). Questions about derivatives of position beyond third order are exceptionally rare on the exam.

We will now derive the one-dimensional kinematic[2] equations. Consider a point object undergoing uniform acceleration $a(t) = a$, with initial position and velocity x_0 and v_0. Integrating $a(t)$ twice over time gives

$$\begin{aligned} v(t) &= C_1 + at \\ x(t) &= C_0 + C_1 t + \frac{1}{2}at^2, \end{aligned}$$

and the initial conditions $x(0) = x_0, v(0) = v_0$ imply $C_0 = x_0, C_1 = v_1$. Thus,

$$v(t) = v_0 + at \tag{1}$$

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2, \tag{2}$$

[1] Snap, Crackle, and Pop are the names of the three Rice Krispies mascots.

[2] Some people call these the SUVAT equations, where the five letters represent the displacement, initial velocity, final velocity, acceleration, and time.

which are the first[3] two kinematic equations. The displacement is $\Delta x = x - x_0 = v_0 t + \frac{1}{2}at^2$. Substituting the first kinematic equation,

$$\begin{aligned}
\Delta x &= v_0\left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v-v_0}{a}\right)^2 \\
&= \frac{v^2 - v_0^2}{2a} \\
\implies v^2 &= v_0^2 + 2a\Delta x, \qquad (3)
\end{aligned}$$

which is the third kinematic equation. Note the lack of time dependence - this one is useful when only distances and velocities are known. Since $v(t)$ is linear, its average value over a given interval is just the average of its values at the endpoints $\overline{v} = \frac{v+v_0}{2}$. Then the displacement covered over a time Δt is also

$$\begin{aligned}
\Delta x &= \overline{v}\Delta t \\
&= \left(\frac{v+v_0}{2}\right)\Delta t. \qquad (4)
\end{aligned}$$

These four equations encapsulate all of (one-dimensional, constant acceleration) kinematics.

Idea 1: Signed coordinates

Unlike in math, when defining a coordinate system you have freedom to choose the orientation of the axes and the direction of the positive axes. For example, consider a car starting from the origin at rest and accelerating southwards with magnitude of acceleration a. Suppose we want to find its position over time. If we orient the x axis parallel to the direction of the car's movement and let the positive direction face northwards, then we have

$$x = -\frac{1}{2}at^2$$

by the kinematic equations, where the negative sign is inserted because the acceleration oriented in the negative direction. Alternatively, if we let the positive direction face southwards, we have

$$x = \frac{1}{2}at^2,$$

since acceleration is now in the positive direction. Either of these approaches are correct as long as you clarify the sign of your coordinate system, but once you choose a sign for your coordinates, you **must** stick to it until the end for consistency.

[3] The ordering doesn't matter.

If instead of acceleration the nth time derivative of position is constant, then we instead have

$$x(t) = x_0 + x_0^{(1)}t + \frac{1}{2}x_0^{(2)}t^2 + \cdots + \frac{1}{(n-1)!}x_0^{(n-1)}t^{n-1} + \frac{1}{n!}x^{(n)}t^n,$$

where $x^{(n)}$ is the nth time derivative of position and $x_0^{(n)}$ is the initial value of the nth time derivative of position. The derivation is exactly the same as that of the constant-acceleration kinematic equations except that we now must integrate n times over time:

$$\underbrace{\int \cdots \int}_{n \text{ times}} x^{(n)} \underbrace{\mathrm{d}t \cdots \mathrm{d}t}_{n \text{ times}} = \frac{1}{n!}x^{(n)}t^n.$$

Idea 2: Freefall

Freefall problems involve objects falling close to the Earth's surface with no air resistance. It is experimentally verified that all objects accelerate downwards with magnitude $g \approx 9.8 \text{ m/s}^2$. Note that g is the *magnitude* of gravitational acceleration, and is always positive.

If you drop an object from rest, the height it falls in a time t is

$$h = \frac{1}{2}gt^2.$$

If you throw an object upwards with speed v_0, the maximum height it achieves is

$$h = \frac{v_0^2}{2g}.$$

These facts are just consequences of the kinematic equations.

Example 1: (Zhou)

A projectile is thrown upward and passes a point A and a point B a height h above. Let T_A and T_B be the time intervals between the two times the projectile passes A and B, respectively. Find the acceleration of gravity g in terms of T_A, T_B, and h.

Solution 1

Let d be the height the apple travels through to get to point A. We have the following system:

$$d = \frac{1}{2} g \left(\frac{T_A}{2}\right)^2$$
$$d - h = \frac{1}{2} g \left(\frac{T_B}{2}\right)^2.$$

Subtracting the two equations and solving for g gives

$$g = \frac{8h}{T_A^2 - T_B^2}.$$

This is actually a popular way to *measure* the value of g.

Example 2: $F = ma$ 2023

A projectile is thrown upward with speed v. By the time its speed has decreased to $v/2$, it has risen a height h. Neglecting air resistance, what is the maximum height reached by the projectile?

Solution 2

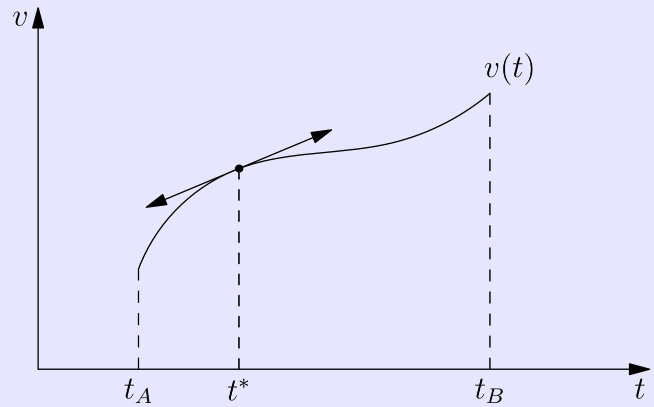
By the kinematic equations,

$$v^2 = (v/2)^2 - 2gh \implies \frac{3v^2}{8g} = h,$$

so $\frac{v^2}{2g} = \frac{4}{3}h$. But the maximum height is also $\frac{v^2}{2g}$, so the answer is $4h/3$.

Idea 3: Graphs of motion and review of calculus

Suppose you are given a graph of the one-dimensional velocity of an object with respect to time.



If the object travels with velocity $v(t)$, then between times t^* and $t^* + \mathrm{d}t$ the displacement it covers is approximately $\mathrm{d}x = v(t^*)\ \mathrm{d}t$. Summing all of the infinitesimal displacements and taking the limit $\mathrm{d}t \to 0$ gives $\Delta x = \int_{t_1}^{t_2} v(t)\ \mathrm{d}t$ between any times t_1, t_2. Graphically, this is the area under the curve of $v(t)$ bounded by the lines $t = t_1$ and $t = t_2$. Similarly, the acceleration $a(t^*)$ can be found by reading off the slope of the line tangent to $v(t)$ at t^*.

2.1.2 Multidimensional Kinematics

In two-or-higher dimensions we simply promote all scalar quantities to vectors:

$$\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}, \quad \vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}, \quad \vec{j} = \frac{\mathrm{d}\vec{a}}{\mathrm{d}t}, \quad \ldots$$

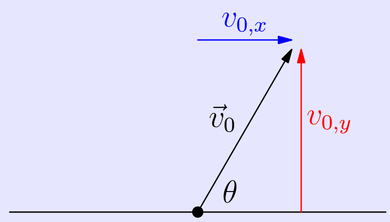
In n dimensions with constant acceleration $\vec{a} = (a_1, a_2, \ldots, a_n)$, we have n corresponding equations

$$\begin{aligned} r_1 &= r_{1,0} + v_1 t + \frac{1}{2} a_1 t^2 \\ r_2 &= r_{2,0} + v_2 t + \frac{1}{2} a_2 t^2 \\ &\vdots \\ r_n &= r_{n,0} + v_n t + \frac{1}{2} a_n t^2, \end{aligned}$$

where $\vec{r} = (r_1, r_2, \ldots, r_n)$ is the position. As usual the corresponding axes for $r_1, r_2, \ldots$ may be chosen freely for convenience.

Idea 4: Projectile motion on level surfaces

Consider throwing a projectile with initial velocity $\vec{v}_0 = (v_{0,x}, v_{0,y})$ over level ground with no air resistance.



Since acceleration due to gravity is $\vec{a} = (0, -g)$, the kinematic equations give

$$\begin{aligned} x &= v_{0,x}t \\ y &= v_{0,y}t - \frac{1}{2}gt^2, \end{aligned}$$

where I set $(x_0, y_0) = (0, 0)$ to be the origin. If the projectile is thrown at an angle θ above the horizontal, we have

$$\begin{aligned} v_{0,x} &= v_0 \cos\theta \\ v_{0,y} &= v_0 \sin\theta. \end{aligned}$$

To investigate the *shape* of the trajectory, we eliminate t from the x-kinematic equation and substitute it into the y-kinematic equation:

$$\begin{aligned} y(x) &= (v_0 \sin\theta)\left(\frac{x}{v_0\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2 \\ &= (\tan\theta)x - \left(\frac{1}{2}\frac{g}{v_0^2\cos^2\theta}\right)x^2, \end{aligned}$$

which is a parabola. Suppose we wanted to find the horizontal distance the projectile travels before hitting the ground again. To find the time it takes for the projectile to hit the ground again, we can set $y(t) = 0$. Alternatively, by symmetry, the final y velocity must be $-v_{0,y}$. Thus, the time it takes to hit the ground is

$$T = \frac{\Delta v}{a} = \frac{2v_{0,y}}{g} = \frac{2v_0\sin\theta}{g}.$$

Then, the horizontal distance traveled by the projectile at this time is

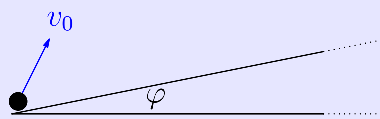
$$R = (v_0\cos\theta)\left(\frac{2v_0\sin\theta}{g}\right) = \frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin(2\theta)}{g},$$

where R is called the range. We see that for fixed v_0, the range is maximized at $\theta = \frac{\pi}{4}$. Similarly, the projectile reaches its apex at $t = \frac{v_0 \sin\theta}{g}$, so the maximum height is

$$H = (v_0 \sin\theta)\left(\frac{v_0 \sin\theta}{g}\right) - \left(\frac{1}{2}g\right)\left(\frac{v_0 \sin\theta}{g}\right)^2 = \frac{v_0^2 \sin^2\theta}{2g}.$$

Idea 5: Projectile motion on inclined surfaces

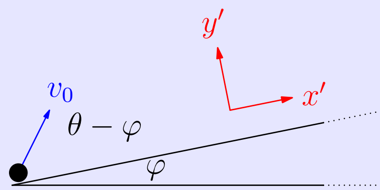
Sometimes you'll get questions about firing projectiles up inclined planes.



If θ is the angle at which the projectile is fired, we can analyze the motion by setting up the conventional kinematic equations

$$x(t) = v_0 t \cos\theta$$
$$y(t) = v_0 t \sin\theta - \frac{1}{2}gt^2,$$

and then noting that the ramp imposes the constraint $y \geq x \tan\varphi$. Alternatively, we can consider *rotated* coordinates parallel and perpendicular to the plane.



Geometry gives

$$v_{0,x'} = v_0 \cos(\theta - \varphi)$$
$$v_{0,y'} = v_0 \sin(\theta - \varphi).$$

Since gravity is tilted with respect to the primed coordinates, we have

$$x'(t) = v_0 \cos(\theta - \varphi)t - \frac{1}{2}gt^2 \sin\varphi$$
$$y'(t) = v_0 \sin(\theta - \varphi)t - \frac{1}{2}gt^2 \cos\varphi.$$

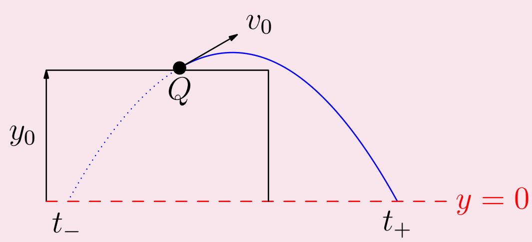
You can check the validity of the primed equations by plugging in $\varphi = 0$ and seeing that it reduces to the level-ground projectile equations. Now to find the time it takes for the projectile to land on the plane, we can simply set $y'(t) = 0$.

Remark 1: Unphysical solutions

Most quadratics you solve will have two solutions for t. For example, if you were solving $y_0 + v_0 t \sin\theta - \frac{1}{2}gt^2 = 0$, the solutions would be

$$t_\pm = \frac{v_0 \sin\theta \pm \sqrt{v_0^2 \sin^2\theta + 2gy_0}}{g}.$$

Note that if y_0 is positive, the solution t_- is actually negative, which of course doesn't make physical sense since time can't be negative. The reason why this occurs is because the mathematics doesn't care about physical boundary conditions - given the bottom diagram, for example:



where the projectile is launched at point Q, the negative solution t_- corresponds to the other root of the parabola which passes *through* the platform and extends down to the ground. Of course, this can't happen in real life because we assume the ball can't pass through the platform, and that the ball only passes forward through time. However, these extra assumptions are not built into the kinematic equations, so we must enforce them manually by rejecting unphysical solutions.

That's really it for multidimensional kinematics! On the AP test, you'll likely get a lot of projectile motion problems that require applying multiple equations in tandem.

Example 3: $F = ma$ 2023

A projectile is thrown from a horizontal surface, and reaches a maximum height h and also lands a distance h from the launch point. Neglecting air resistance, what is the maximum height for a projectile thrown directly upward with the same initial speed?

Solution 3

We have
$$h = \frac{2v_0^2 \sin\theta\cos\theta}{g} = \frac{v_0^2 \sin^2\theta}{2g},$$
which implies
$$\tan\theta = 4.$$
Drawing a right triangle gives
$$\sin\theta = \frac{4}{\sqrt{17}}$$
$$\cos\theta = \frac{1}{\sqrt{17}}$$
That means
$$\frac{v_0^2}{2g} = \frac{h}{\sin^2\theta} = \frac{17}{16}h,$$
so the maximum height achieved is $17h/16$.

Here's another, trickier example. It's doubtful anything on the AP test will be *this* hard, but it's useful to look at harder problems to understand how all the pieces fit together.

Example 4

Place a cannon at the origin that fires projectiles upwards at a variable angle with initial speed v_0. A point (x, y) with $x, y > 0$ is called *reachable* if a projectile fired from the cannon can hit it. Describe the locus of points reachable by the cannon. As usual, gravitational acceleration is $-g$ everywhere.

Solution 4

We only care about the shape of the trajectories so we need $y(x)$. We have
$$\begin{aligned} y(x) &= (\tan\theta)x - \left(\frac{1}{2}\frac{g}{v_0^2\cos^2\theta}\right)x^2 \\ &= (x)\tan\theta - \left(\frac{gx^2}{2v_0^2}\right)\sec^2\theta \\ 0 &= -\frac{gx^2}{2v_0^2} - y + x\tan\theta - \left(\frac{gx^2}{2v_0^2}\right)\tan^2\theta, \end{aligned}$$
which is a *quadratic* in $\tan\theta$. If (x, y) is reachable, then there is a solution for $\tan\theta$.

The boundary of reachable points can be found by setting the discriminant to zero:

$$x^2 - \frac{g^2x^4}{v_0^4} - \frac{2gx^2}{v_0^2}y = 0$$
$$y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}.$$

If you've taken conics, you may recognize this as a parabola with focus at the cannon and vertex height $\frac{v_0^2}{2g}$. Then the reachable points are located everywhere within this parabola.

2.2 Reference frames and relative motion

Reference frames are an incredibly important part of Newtonian mechanics that the AP course syllabus strangely omits. The basic idea is that physics works no matter which frame[4] you take, so you have freedom to choose the frame you carry out the analysis in. Switching reference frames is useful because it converts a problem about motion into one about *relative* motion. If two objects undergoing one-dimensional motion in the same dimensions have positions $x_1(t)$ and $x_2(t)$, then:

- The relative position of 1 with respect to 2 is $x_1(t) - x_2(t)$.
- The relative velocity of 1 with respect to 2 is $\frac{\mathrm{d}}{\mathrm{d}t}(x_1(t) - x_2(t)) = v_1(t) - v_2(t)$.
- The relative acceleration of 1 with respect to 2 is $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x_1(t) - x_2(t)) = a_1(t) - a_2(t)$.

If higher-order derivatives of position are relevant (they won't be) then the pattern proceeds similarly. The reason why this works is because the derivative is linear, i.e. for *any* functions f and g, we have $\frac{\mathrm{d}}{\mathrm{d}x}(f(x) + g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{\mathrm{d}}{\mathrm{d}x}g(x)$.
If both objects have the same acceleration, you can get rid of acceleration by switching to a co-accelerating frame, in which the transformed objects move with constant velocity. The next two examples combine this idea with traditional projectile kinematics.

Example 5: (KoMaL 2019)

A cannon A is at the edge of a cliff with a 800 m drop. Cannon B is on the ground below the cliff and 600 m horizontally away from it. Cannon A shoots a cannonball directly towards cannon B at 60 m/s. Cannon B shoots a cannonball directly towards

[4] More precisely, physics works in all non-accelerating frames of reference, which are called *inertial* frames. However, we will see that even in accelerating frames, physical laws can be "rescued" by imposing *fictitious* quantities.

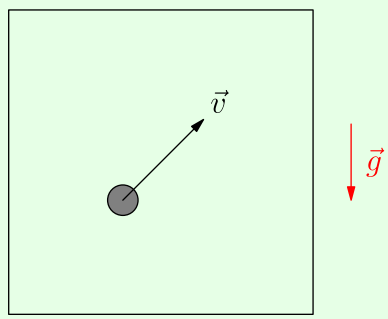
cannon A at 40 m/s. Will the two cannonballs hit each other in midair?

Solution 5: (Zhou)

Work in the frame freely falling with the cannonballs. In this case, the balls have a relative velocity of 100 m/s and initial separation of 1000 m, so it takes 10 s to collide. If there were no gravity, this collision would occur at a point $(2/5)(800 \text{ m}) = 320$ m above the ground. However, because of gravity both balls have fallen by an extra $gt^2/2 = 500$ m by this time. Hence the balls hit the ground before they can hit each other in midair.

Example 6: $F = ma$ 2016

Consider a particle in a box where the force of gravity is down as shown in the figure.



The particle has an initial velocity as shown, and the box has a constant acceleration to the right. Sketch the subsequent trajectory of the particle.

Solution 6

The box accelerates rightwards with respect to the particle, so the particle accelerates leftwards with respect to the box. Thus, in the box's frame, the particle undergoes constant downwards and leftwards acceleration, so its trajectory is a parabola tilted to the right.

Example 7: (Wang)

Two particles are released in gravitational acceleration g with leftward and rightward speeds v_1 and v_2. Find the distance between them when their velocities are perpendicular.

Solution 7: (Zhou)

After a time t, the velocities are $(-v_1, -gt)$ and $(v_2, -gt)$. These are perpendicular when their dot product is zero, so $v_1 v_2 = (gt)^2$. Thus,

$$t = \frac{\sqrt{v_1 v_2}}{g}.$$

To compute the distance, we can just work in the frame falling with the masses. Then it's clear that the acceleration g doesn't matter, and the distance is just

$$d = (v_1 + v_2)t = \frac{(v_1 + v_2)\sqrt{v_1 v_2}}{g}.$$

We will revisit reference frames in dynamics, which is the study of motion of systems taking into account their internal characteristics.

3 Appendix and test taking strategies