A New Keynesian Perspective on the Great Recession

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1 The Model

The particular variant of the New Keynesian model used here takes its basic features from those of Ireland (2004, 2007). The economy consists of a representative household, a representative finished goods-producing firm, a continuum of intermediate goods-producing firms indexed by $i \in [0, 1]$, and a central bank. Habit formation introduced through the representative household's preferences implies that the model's version of the New Keynesian IS curve is partially backward and partially forward looking. Partial indexation of sticky nominal goods prices set by monopolistically competitive goods-producing firms implies that the model's version of the New Keynesian Phillips curve is partially backward and partially forward looking as well. The central bank conducts monetary policy according to Taylor (1993) rule for setting the nominal interest rate.

During each period $t = 0, 1, 2, ...$, each intermediate goods-producing firm produces a distinct, perishable intermediate good. Hence, intermediate goods may also be indexed by $i \in [0, 1]$, where firm i produces good i. The model features enough symmetry, however, to allow the analysis to focus on the behavior of a representative intermediate goods-producing firm, identified by the generic index i. The activities of each agent, and their implications for the evolution of equilibrium prices and quantities, will now be described in turn.

1.1 The Representative Household

The representative household enters each period $t = 0, 1, 2, ...$ with money M_{t-1} and bonds B_{t-1}. At the beginning of period t, the household receives a lump-sum nominal transfer T_t from the central bank. Next, the household's bonds mature, providing B_{t-1} additional units of money. The household uses some of its money to purchase B_t new bonds at the price of $1/r_t$ units of money per bond, where r_t denotes the gross nominal interest rate between t and $t + 1$.

During period t, the household supplies $h_t(i)$ units of labor to each intermediate goods-producing firm $i \in [0, 1]$, for a total of

$$h_t = \int_0^1 h_t(i)\, \mathrm{d}i.$$

The household gets paid at the nominal wage rate W_t, earning $W_t h_t$ in total labor income during the period. Also during the period, the household consumes C_t units of the finished good, purchased at the nominal price P_t from the representative finished goods-producing firm.

At the end of period t, the household receives nominal profits $D_t(i)$ from each intermediate goods-producing firm $i \in [0,1]$, for a total of

$$D_t = \int_0^1 D_t(i)\,\mathrm{d}i.$$

The household then carries M_t units of money into period $t+1$, where its budget constraint dictates that

$$\frac{M_{t-1} + T_t + B_{t-1} + W_t h_t + D_t}{P_t} \geq C_t + \frac{M_t + B_t/r_t}{P_t} \tag{1}$$

for all $t = 0, 1, 2, ...$

The household's preferences are described by the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t a_t [\ln(C_t - \gamma C_{t-1}) + \ln(M_t/P_t) - h_t],$$

where both the discount factor and the habit formation parameter lie between zero and one: $0 < \beta < 1$ and $0 \leq \gamma < 1$. The preference shock a_t follows the stationary autoregressive process

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at} \tag{2}$$

for all $t = 0, 1, 2, ...$, with $0 \leq \rho_a < 1$, where the serially uncorrelated innovation ε_{at} is normally distributed with mean zero and standard deviation σ_a. Utility is additively separable in consumption, real balances, and hours worked so as to imply a conventional specification for the model's IS curve that, in particular, does not include additional terms involving real balances or employment. Given this additive separability, the logarithmic specification over consumption is needed, as shown by King, Plosser, and Rebelo (1988), for the model to be consistent with balanced growth. And, as noted above, habit formation is introduced into preferences to allow for partially backward-looking behavior in consumption.

Thus, the household chooses C_t, h_t, B_t, and M_t for all $t = 0, 1, 2, ...$ to maximize its utility subject to the budget constraint (1) for all $t = 0, 1, 2, ...$ The first-order conditions for this problem can be written as

$$\Lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta\gamma E_t \left(\frac{a_{t+1}}{C_{t+1} - \gamma C_t} \right), \tag{3}$$

$$a_t = \Lambda_t (W_t/P_t), \tag{4}$$

$$\Lambda_t = \beta r_t E_t (\Lambda_{t+1}/\pi_{t+1}), \tag{5}$$

$$M_t/P_t = (a_t/\Lambda_t)[r_t/(r_t - 1)], \tag{6}$$

and (1) with equality for all $t = 0, 1, 2, ...$, where Λ_t denotes the nonnegative Lagrange multiplier on the budget constraint for period t and $\pi_t = P_t/P_{t-1}$ denotes the gross inflation rate between t and $t+1$.

1.2 The Representative Finished Goods-Producing Firm

During each period $t = 0, 1, 2, ...$, the representative finished goods-producing firm uses $Y_t(i)$ units of each intermediate good $i \in [0, 1]$, purchased at the nominal price $P_t(i)$, to manufacture Y_t units of the finished good according to the constant-returns-to-scale technology described by

$$\left[\int_0^1 Y_t(i)^{(\theta_t-1)/\theta_t}\,\mathrm{d}i\right]^{\theta_t/(\theta_t-1)} \geq Y_t,$$

where θ_t translates into a random shock to the intermediate goods-producing firms' desired markup of price over marginal cost and therefore acts like a cost-push shock of the kind introduced into the New Keynesian model by Clarida, Gali, and Gertler (1999). Here, this markup shock follows the stationary autoregressive process

$$\ln(\theta_t) = (1 - \rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{7}$$

for all $t = 0, 1, 2, ...$, with $0 \leq \rho_\theta < 1$ and $\theta > 1$, where the serially uncorrelated innovation $\varepsilon_{\theta t}$ is normally distributed with mean zero and standard deviation σ_θ.

Thus, during each period t, the finished goods-producing firm chooses $Y_t(i)$ for all $i \in [0, 1]$ to maximize its profits, which are given by

$$P_t\left[\int_0^1 Y_t(i)^{(\theta_t-1)/\theta_t}\,\mathrm{d}i\right]^{\theta_t/(\theta_t-1)} - \int_0^1 P_t(i)Y_t(i)\,\mathrm{d}i.$$

The first-order conditions for this problem are

$$Y_t(i) = [P_t(i)/P_t]^{-\theta_t} Y_t$$

for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$

Competition drives the finished goods-producing firm's profits to zero in equilibrium, determining P_t as

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta_t}\,\mathrm{d}i\right]^{1/(1-\theta_t)}$$

for all $t = 0, 1, 2, ...$

1.3 The Representative Intermediate Goods-Producing Firm

During each period $t = 0, 1, 2, ...$, the representative intermediate goods-producing firm hires $h_t(i)$ units of labor from the representative household to manufacture $Y_t(i)$ units of intermediate good i according to the constant-returns-to-scale technology described by

$$Z_t h_t(i) \geq Y_t(i). \tag{8}$$

The aggregate technology shock Z_t follows a random walk with drift:

$$\ln(Z_t) = \ln(z) + \ln(Z_{t-1}) + \varepsilon_{zt} \tag{9}$$

for all $t = 0, 1, 2, ...$, with $z > 1$ and where the serially uncorrelated innovation ε_{zt} is normally distributed with mean zero and standard deviation σ_z.

Since the intermediate goods substitute imperfectly for one another in producing the finished good, the representative intermediate goods-producing firm sells its output in a monopolistically competitive market: during period t, the firm sets its nominal price $P_t(i)$, subject to the requirement that it satisfy the representative finished goods-producing firm's demand at that price. And, following Rotemberg (1982), the intermediate goods-producing firm faces a quadratic cost of adjusting its nominal price between periods, measured in terms of the finished good and given by

$$\frac{\phi}{2}\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)}-1\right]^2 Y_t,$$

where $\phi \geq 0$ governs the magnitude of the price adjustment cost, α is a parameter that lies between zero and one, with $0 \leq \alpha \leq 1$, and π denotes the average, or steady-state, rate of inflation. According to this specification, the extent to which price setting is backward or forward-looking depends on the magnitude of the parameter α. When, in particular, $\alpha = 0$, then price setting is purely-forward looking in the sense that there is no indexation of prices to past inflation rates. When, on the other hand, $\alpha = 1$, then price setting is purely backward-looking, in the sense that there is full indexation of prices to the previous period's inflation rate.

In any case, the cost of price adjustment makes the intermediate goods-producing firm's problem dynamic: it chooses $P_t(i)$ for all $t = 0, 1, 2, ...$, to maximize its total real market value, proportional to

$$E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t [D_t(i)/P_t],$$

where $\beta^t \Lambda_t$ measures the marginal utility value to the representative household of an additional unit of real profits received in the form of dividends during period t and where

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta_t} Y_t - \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t}\left(\frac{W_t}{P_t}\right)\left(\frac{Y_t}{Z_t}\right) - \frac{\phi}{2}\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)}-1\right]^2 Y_t \tag{10}$$

measures the firm's real profits during the same period t. The first-order conditions for this problem are

$$\begin{aligned}
0 = (1-\theta_t)\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t} &+ \theta_t\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t-1}\left(\frac{W_t}{P_t}\right)\left(\frac{1}{Z_t}\right) \\
&- \phi\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)}-1\right]\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)}\right] \\
&+ \beta\phi E_t\left\{\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)\left[\frac{P_{t+1}(i)}{\pi_t^{\alpha}\pi^{1-\alpha}P_t(i)}-1\right]\left[\frac{P_{t+1}(i)}{\pi_t^{\alpha}\pi^{1-\alpha}P_t(i)}\right]\left[\frac{P_t}{P_t(i)}\right]\left(\frac{Y_{t+1}}{Y_t}\right)\right\}
\end{aligned} \tag{11}$$

and (8) with equality for all $t = 0, 1, 2, ...$

1.4 Symmetric Equilibrium

In a symmetric equilibrium, all intermediate goods-producing firms make identical decisions, so that $Y_t(i) = Y_t$, $h_t(i) = h_t$, $D_t(i) = D_t$, and $P_t(i) = P_t$ for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$ In addition, the market-clearing conditions $M_t = M_{t-1} + T_t$ and $B_t = B_{t-1} = 0$ for money and bonds must also hold for all $t = 0, 1, 2, ...$ After imposing these equilibrium conditions and using (4), (6), (8), and (10) to solve out for W_t/P_t, M_t/P_t, h_t, and D_t/P_t, the system consisting of (1)-(11) reduces to

$$Y_t = C_t + \frac{\phi}{2}\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}} - 1\right)^2 Y_t, \tag{1}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{2}$$

$$\Lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta\gamma E_t\left(\frac{a_{t+1}}{C_{t+1} - \gamma C_t}\right), \tag{3}$$

$$\Lambda_t = \beta r_t E_t(\Lambda_{t+1}/\pi_{t+1}), \tag{5}$$

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{7}$$

$$\ln(Z_t) = \ln(z) + \ln(Z_{t-1}) + \varepsilon_{zt}, \tag{9}$$

and

$$\begin{aligned}\theta_t - 1 = \theta_t\left(\frac{a_t}{\Lambda_t Z_t}\right) - \phi\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}} - 1\right)\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}}\right) \\ + \beta\phi E_t\left[\left(\frac{\Lambda_{t+1}Y_{t+1}}{\Lambda_t Y_t}\right)\left(\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}} - 1\right)\left(\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}}\right)\right]\end{aligned} \tag{11}$$

for all $t = 0, 1, 2, ...$

To help keep track of the model's observable variables, it is also helpful to define the growth rate of output as

$$g_t = Y_t/Y_{t-1} \tag{12}$$

for all $t = 0, 1, 2, ...$

1.5 The Efficient Level of Output and the Output Gap

A social planner for this economy who can overcome the frictions associated with monetary trade and sluggish price adjustment chooses Q_t and $n_t(i)$ for all $i \in [0, 1]$ to maximize the social welfare function

$$E_0 \sum_{t=0}^{\infty} \beta^t a_t \left[\ln(Q_t - \gamma Q_{t-1}) - \int_0^1 n_t(i)\,\mathrm{d}i\right]$$

subject to the aggregate feasibility constraint

$$Z_t \left[\int_0^1 n_t(i)^{(\theta_t-1)/\theta_t}\,\mathrm{d}i\right]^{\theta_t/(\theta_t-1)} \geq Q_t$$

for all $t = 0, 1, 2, ...$ The first-order conditions for this problem can be written as

$$\Xi_t = \frac{a_t}{Q_t - \gamma Q_{t-1}} - \beta\gamma E_t\left(\frac{a_{t+1}}{Q_{t+1} - \gamma Q_t}\right),$$

$$a_t = \Xi_t Z_t (Q_t/Z_t)^{1/\theta_t} n_t(i)^{-1/\theta_t}$$

for all $i \in [0, 1]$, and the aggregate feasibility constraint for all $t = 0, 1, 2, ...$, where Ξ_t denotes the nonnegative Lagrange multiplier on the aggregate feasibility constraint for period t.

The second optimality condition listed above implies that $n_t(i) = n_t$ for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$, where

$$n_t = (\Xi_t/a_t)^{\theta_t} Z_t^{\theta_t}(Q_t/Z_t).$$

Substituting this last relationship into the aggregate feasibility constraint yields

$$\Xi_t = a_t/Z_t.$$

Hence, the efficient level of output Q_t must satisfy

$$\frac{1}{Z_t} = \frac{1}{Q_t - \gamma Q_{t-1}} - \beta\gamma E_t\left[\left(\frac{a_{t+1}}{a_t}\right)\left(\frac{1}{Q_{t+1} - \gamma Q_t}\right)\right] \tag{13}$$

for all $t = 0, 1, 2, ...$ This definition of the efficient level of output implies a corresponding definition of the output gap as

$$x_t = Y_t/Q_t. \tag{14}$$

1.6 The Central Bank

The central bank conducts monetary policy according to a variant of the Taylor (1993) rule

$$\ln(r_t/r) = \rho_r \ln(r_{t-1}/r) + \rho_\pi \ln(\pi_t/\pi) + \rho_x \ln(x_t/x) + \rho_g \ln(g_t/g) + \varepsilon_{rt} \tag{15}$$

for all $t = 0, 1, 2, ...$, where r, π, x, and g denote the steady-state values of the interest rate r_t, the inflation rate π_t, the output gap x_t, and the output growth rate g_t and where the response coefficients $\rho_r \geq 0$, $\rho_\pi \geq 0$, $\rho_x \geq 0$, and $\rho_g \geq 0$ are chosen by the central bank. This policy rule allows the central bank to adjust the short-term interest rate in response to movements in inflation and two stationary measures of real economic activity, the output gap and the output growth rate. It also allows for a degree of interest rate smoothing, through the term involving the lagged interest rate. The serially uncorrelated monetary policy shock ε_{rt} is normally distributed with mean zero and standard deviation σ_r.

1.7 The Stationary System

Equations (1)-(3), (5), (7), (9), and (11)-(15) now form a system of 11 equations in the 11 variables Y_t, C_t, π_t, r_t, Q_t, x_t, g_t, Λ_t, a_t, θ_t, and Z_t. Some of the real variables on this list inherit unit roots from the random walk (9) in the technology shock. However, the variables

$y_t = Y_t/Z_t$, $c_t = C_t/Z_t$, $q_t = Q_t/Z_t$, $\lambda_t = Z_t\Lambda_t$, and $z_t = Z_t/Z_{t-1}$ remain stationary and, in terms of these stationary variables, the system can be rewritten as

$$y_t = c_t + \frac{\phi}{2}\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}} - 1\right)^2 y_t, \tag{1}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{2}$$

$$\lambda_t = \frac{a_t z_t}{z_t c_t - \gamma c_{t-1}} - \beta\gamma E_t\left(\frac{a_{t+1}}{z_{t+1}c_{t+1} - \gamma c_t}\right), \tag{3}$$

$$\lambda_t = \beta r_t E_t\left(\frac{\lambda_{t+1}}{z_{t+1}\pi_{t+1}}\right), \tag{5}$$

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{7}$$

$$\ln(z_t) = \ln(z) + \varepsilon_{zt}, \tag{9}$$

$$\begin{aligned} \theta_t - 1 = {} & \frac{\theta_t a_t}{\lambda_t} - \phi\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}} - 1\right)\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}}\right) \\ & + \beta\phi E_t\left[\left(\frac{\lambda_{t+1}y_{t+1}}{\lambda_t y_t}\right)\left(\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}} - 1\right)\left(\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}}\right)\right], \end{aligned} \tag{11}$$

$$g_t = (y_t/y_{t-1})z_t, \tag{12}$$

$$1 = \frac{z_t}{z_t q_t - \gamma q_{t-1}} - \beta\gamma E_t\left[\left(\frac{a_{t+1}}{a_t}\right)\left(\frac{1}{z_{t+1}q_{t+1} - \gamma q_t}\right)\right], \tag{13}$$

$$x_t = y_t/q_t, \tag{14}$$

and

$$\ln(r_t/r) = \rho_r \ln(r_{t-1}/r) + \rho_\pi \ln(\pi_t/\pi) + \rho_x \ln(x_t/x) + \rho_g \ln(g_t/g) + \varepsilon_{rt} \tag{15}$$

for all $t = 0, 1, 2, ...$

1.8 The Steady State

In the absence of shocks, the economy converges to a steady-state growth path, along which all of the stationary variables are constant, with $y_t = y$, $c_t = c$, $\pi_t = \pi$, $r_t = r$, $q_t = q$, $x_t = x$, $g_t = g$, $\lambda_t = \lambda$, $a_t = a$, $\theta_t = \theta$, and $z_t = z$ for all $t = 0, 1, 2, ...$

The steady-state values $a = 1$, θ, z, and π are determined exogenously by (2), (7), (9), and (15). Equations (1) and (12) imply that $c = y$ and $g = z$.

Equation (11) implies that

$$\lambda = \theta/(\theta - 1).$$

Equation (3) then implies that

$$y = \left(\frac{\theta - 1}{\theta}\right)\left(\frac{z - \beta\gamma}{z - \gamma}\right).$$

Equation (13) implies that

$$q = \frac{z - \beta\gamma}{z - \gamma},$$

to that (14) implies that

$$x = (\theta - 1)/\theta.$$

Finally, (5) implies that

$$r = (z/\beta)\pi.$$

1.9 The Linearized System

The system consisting of (1)-(3), (5), (7), (9), and (11)-(15) can be log-linearized around the steady state in order to describe how the economy responds to shocks. Let $\hat{y}_t = \ln(y_t/y)$, $\hat{c}_t = \ln(c_t/c)$, $\hat{\pi}_t = \ln(\pi_t/\pi)$, $\hat{r}_t = \ln(r_t/r)$, $\hat{q}_t = \ln(q_t/q)$, $\hat{x}_t = \ln(x_t/x)$, $\hat{g}_t = \ln(g_t/g)$, $\hat{\lambda}_t = \ln(\lambda_t/\lambda)$, $\hat{a}_t = \ln(a_t)$, $\hat{\theta}_t = \ln(\theta_t/\theta)$, and $\hat{z}_t = \ln(z_t/z)$ denote the percentage deviation of each variable from its steady-state level. A first-order Taylor approximation to (1) reveals that $\hat{c}_t = \hat{y}_t$, allowing $\hat{c}_t$ to be eliminated from the system. First-order approximations to the remaining ten equations then imply

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \tag{2}$$

$$(z - \beta\gamma)(z - \gamma)\hat{\lambda}_t = \gamma z \hat{y}_{t-1} - (z^2 + \beta\gamma^2)\hat{y}_t + \beta\gamma z E_t \hat{y}_{t+1} + (z - \beta\gamma\rho_a)(z - \gamma)\hat{a}_t - \gamma z \hat{z}_t, \tag{3}$$

$$\hat{\lambda}_t = \hat{r}_t + E_t \hat{\lambda}_{t+1} - E_t \hat{\pi}_{t+1}, \tag{5}$$

$$\hat{e}_t = \rho_e \hat{e}_{t-1} + \varepsilon_{et}, \tag{7}$$

$$\hat{z}_t = \varepsilon_{zt}, \tag{9}$$

$$(1 + \beta\alpha)\hat{\pi}_t = \alpha\hat{\pi}_{t-1} + \beta E_t \hat{\pi}_{t+1} - \psi\hat{\lambda}_t + \psi\hat{a}_t + \hat{e}_t, \tag{11}$$

$$\hat{g}_t = \hat{y}_t - \hat{y}_{t-1} + \hat{z}_t, \tag{12}$$

$$0 = \gamma z \hat{q}_{t-1} - (z^2 + \beta\gamma^2)\hat{q}_t + \beta\gamma z E_t \hat{q}_{t+1} + \beta\gamma(z - \gamma)(1 - \rho_a)\hat{a}_t - \gamma z \hat{z}_t, \tag{13}$$

$$\hat{x}_t = \hat{y}_t - \hat{q}_t, \tag{14}$$

and

$$\hat{r}_t = \rho_r \hat{r}_{t-1} + \rho_\pi \hat{\pi}_t + \rho_x \hat{x}_t + \rho_g \hat{g}_t + \varepsilon_{rt} \tag{15}$$

for all $t = 0, 1, 2, ...$ where, in (7) and (11), the cost-push shock $\hat{\theta}_t$ has been renormalized as $\hat{e}_t = -(1/\phi)\hat{\theta}_t$ and the new parameters ρ_e and ψ have been defined as $\rho_e = \rho_\theta$ and $\psi = (\theta - 1)/\phi$ so that ε_{et} is normally distributed with mean zero and standard deviation $\sigma_e = \sigma_\theta/\phi$.

2 Solving the Model

Let

$$s_t^0 = \begin{bmatrix} \hat{y}_{t-1} & \hat{\pi}_{t-1} & \hat{r}_{t-1} & \hat{q}_{t-1} & \hat{x}_t & \hat{g}_t & \hat{\lambda}_t & \hat{y}_t & \hat{\pi}_t & \hat{q}_t \end{bmatrix}'$$

and

$$\xi_t = \begin{bmatrix} \hat{a}_t & \hat{e}_t & \hat{z}_t & \varepsilon_{rt} \end{bmatrix}'.$$

Then (3), (5), and (11)-(15) can be written as

$$AE_t s^0_{t+1} = Bs^0_t + C\xi_t, \tag{16}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta\gamma z & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta\gamma z \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -\gamma z & 0 & 0 & 0 & 0 & 0 & (z-\beta\gamma)(z-\gamma) & z^2+\beta\gamma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 & 0 & \psi & 0 & 1+\beta\alpha & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma z & 0 & 0 & 0 & 0 & 0 & z^2+\beta\gamma^2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_r & 0 & \rho_x & \rho_g & 0 & 0 & \rho_\pi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} -(z-\beta\gamma\rho_a)(z-\gamma) & 0 & \gamma z & 0 \\ 0 & 0 & 0 & 0 \\ -\psi & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\beta\gamma(z-\gamma)(1-\rho_a) & 0 & \gamma z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Meanwhile, (2), (7), and (9) can be written as

$$\xi_t = P\xi_{t-1} + \varepsilon_t, \tag{17}$$

where

$$P = \begin{bmatrix} \rho_a & 0 & 0 & 0 \\ 0 & \rho_e & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\varepsilon_t = \begin{bmatrix}\varepsilon_{at} & \varepsilon_{et} & \varepsilon_{zt} & \varepsilon_{rt}\end{bmatrix}'.$$

Equation (16) takes the form of a system of linear expectational difference equations, driven by the exogenous shocks in (17). This system can be solved by uncoupling the unstable and stable components and then solving the unstable component forward. There are a number of algorithms for working through this process; the approach taken here uses the methods outlined by Klein (2000).

Klein's method relies on the complex generalized Schur decomposition, which identifies unitary matrices Q and Z such that

$$QAZ = S$$

and

$$QBZ = T$$

are both upper triangular, where the generalized eigenvalues of B and A can be recovered as the ratios of the diagonal elements of T and S:

$$\lambda(B, A) = \{t_{ii}/s_{ii} | i = 1, 2, ..., 10\}.$$

The matrices Q, Z, S, and T can always be arranged so that the generalized eigenvalues appear in ascending order in absolute value. Note that there are four predetermined variables in the vector s_t^0. Thus, if four of the generalized eigenvalues in $\lambda(B, A)$ lie inside the unit circle and six of the generalized eigenvalues lie outside the unit circle, then the system has a unique solution. If more than six of the generalized eigenvalues in $\lambda(B, A)$ lie outside the unit circle, then the system has no solution. If less than six of the generalized eigenvalues in $\lambda(B, A)$ lie outside the unit circle, then the solution has multiple solutions. For details, see Blanchard and Kahn (1980) and Klein (2000).

Assume from now on that there are exactly six generalized eigenvalues that lie outside the unit circle, and partition the matrices Q, Z, S, and T conformably, so that

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where Q_1 is 4×10 and Q_2 is 6×10, and

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0_{(6\times 4)} & S_{22} \end{bmatrix},$$

and

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{(6\times 4)} & T_{22} \end{bmatrix},$$

where Z_{11}, S_{11}, and T_{11} are 4×4, Z_{12}, S_{12}, and T_{12} are 4×6, Z_{21} is 6×4, and Z_{22}, S_{22}, and T_{22} are 6×6.

Next, define the vector s_t^1 of auxilary variables as

$$s_t^1 = Z' s_t^0$$

so that, in particular,

$$s_t^1 = \begin{bmatrix} s_{1t}^1 \\ s_{2t}^1 \end{bmatrix},$$

where

$$s_{1t}^1 = Z_{11}' \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} + Z_{21}' \begin{bmatrix} \hat{x}_t \\ \hat{g}_t \\ \hat{\lambda}_t \\ \hat{y}_t \\ \hat{\pi}_t \\ \hat{q}_t \end{bmatrix} \tag{18}$$

is 4×1 and

$$s_{2t}^1 = Z_{12}' \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} + Z_{22}' \begin{bmatrix} \hat{x}_t \\ \hat{g}_t \\ \hat{\lambda}_t \\ \hat{y}_t \\ \hat{\pi}_t \\ \hat{q}_t \end{bmatrix} \tag{19}$$

is 6×1.

Since Z is unitary, $Z'Z = I$ or $Z' = Z^{-1}$ and hence $s_t^0 = Zs_t^1$. Use this fact to rewrite (16) as

$$AZE_t s_{t+1}^1 = BZs_t^1 + C\xi_t.$$

Premultiply this version of (16) by Q to obtain

$$SE_t s_{t+1}^1 = Ts_t^1 + QC\xi_t$$

or, in terms of the matrix partitions,

$$S_{11}E_t s_{1t+1}^1 + S_{12}E_t s_{2t+1}^1 = T_{11}s_t^1 + T_{12}s_{2t}^1 + Q_1C\xi_t \tag{20}$$

and

$$S_{22}E_t s_{2t+1}^1 = T_{22}s_{2t}^1 + Q_2C\xi_t. \tag{21}$$

Since the generalized eigenvalues corresponding to the diagonal elements of S_{22} and T_{22} all lie outside the unit circle, (21) can be solved forward to obtain

$$s_{2t}^1 = -T_{22}^{-1}R\xi_t,$$

where the 6×4 matrix R is given by

$$\begin{aligned}
\text{vec}(R) &= \text{vec}\sum_{j=0}^{\infty}(S_{22}T_{22}^{-1})^j Q_2 C P^j = \sum_{j=0}^{\infty}\text{vec}[(S_{22}T_{22}^{-1})^j Q_2 C P^j] \\
&= \sum_{j=0}^{\infty}[P^j \otimes (S_{22}T_{22}^{-1})^j]\text{vec}(Q_2C) = \sum_{j=0}^{\infty}[P \otimes (S_{22}T_{22}^{-1})]^j \text{vec}(Q_2C) \\
&= [I_{(24\times 24)} - P \otimes (S_{22}T_{22}^{-1})]^{-1}\text{vec}(Q_2C).
\end{aligned}$$

Use this result, along with (19), to solve for

$$\begin{bmatrix} \hat{x}_t \\ \hat{g}_t \\ \hat{\lambda}_t \\ \hat{y}_t \\ \hat{\pi}_t \\ \hat{q}_t \end{bmatrix} = -(Z_{22}')^{-1} Z_{12}' \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} - (Z_{22}')^{-1} T_{22}^{-1} R \xi_t.$$

Since Z is unitary, $Z'Z = I$ or

$$\begin{bmatrix} Z_{11}' & Z_{21}' \\ Z_{12}' & Z_{22}' \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} I_{(4\times 4)} & 0_{(6\times 4)} \\ 0_{(4\times 6)} & I_{(6\times 6)} \end{bmatrix}.$$

Hence, in particular,

$$Z_{12}' Z_{11} + Z_{22}' Z_{21} = 0$$

or

$$-(Z_{22}')^{-1} Z_{12}' = Z_{21} Z_{11}^{-1}$$

and

$$Z_{12}' Z_{12} + Z_{22}' Z_{22} = I$$

or

$$(Z_{22})^{-1} = Z_{22} + (Z_{22}')^{-1} Z_{12}' Z_{12} = Z_{22} - Z_{21} Z_{11}^{-1} Z_{12},$$

allowing this solution to be written more conveniently as

$$\begin{bmatrix} \hat{x}_t \\ \hat{g}_t \\ \hat{\lambda}_t \\ \hat{y}_t \\ \hat{\pi}_t \\ \hat{q}_t \end{bmatrix} = M_1 \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} M_2 \xi_t, \tag{22}$$

where

$$M_1 = Z_{21} Z_{11}^{-1}$$

and

$$M_2 = -[Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R.$$

Equation (18) now provides a solution for s_{1t}^1:

$$s_{1t}^1 = (Z_{11}' - Z_{21}' Z_{21} Z_{11}^{-1}) \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} - Z_{21}' [Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R \xi_t.$$

Using

$$Z_{11}' Z_{11} + Z_{21}' Z_{21} = I$$

or

$$Z_{11}' + Z_{21}' Z_{21} Z_{11}^{-1} = Z_{11}^{-1}$$

and

$$Z_{21}'[Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] = Z_{21}' Z_{22} - Z_{21}' Z_{21} Z_{11}^{-1} Z_{12} = -Z_{11}^{-1} Z_{12},$$

this last result can be written more conveniently as

$$s_{1t}^1 = Z_{11}^{-1} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} + Z_{11}^{-1} Z_{12} T_{22}^{-1} R \xi_t.$$

Substitute these results into (20) to obtain the solution

$$\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \hat{r}_t \\ \hat{q}_t \end{bmatrix} = M_3 \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} + M_4 \xi_t, \tag{23}$$

where

$$M_3 = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$$

and

$$M_4 = Z_{11} S_{11}^{-1} (T_{11} Z_{11}^{-1} Z_{12} T_{22}^{-1} R + Q_1 C + S_{12} T_{12}^{-1} R P - T_{12} T_{22}^{-1} R) - Z_{12} T_{22}^{-1} R P.$$

Equations (17), (22), and (23) now provide the model's solution:

$$s_{t+1} = \Pi s_t + W \varepsilon_{t+1} \tag{24}$$

and

$$f_t = U s_t, \tag{25}$$

where

$$s_t = \begin{bmatrix} \hat{y}_{t-1} & \hat{\pi}_{t-1} & \hat{r}_{t-1} & \hat{q}_{t-1} & \hat{a}_t & \hat{e}_t & \hat{z}_t & \varepsilon_{rt} \end{bmatrix}',$$

$$f_t = \begin{bmatrix} \hat{x}_t & \hat{g}_t & \hat{\lambda}_t & \hat{y}_t & \hat{\pi}_t & \hat{q}_t \end{bmatrix}',$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{et} & \varepsilon_{zt} & \varepsilon_{rt} \end{bmatrix}',$$

$$\Pi = \begin{bmatrix} M_3 & M_4 \\ 0_{(4\times4)} & P \end{bmatrix},$$

$$W = \begin{bmatrix} 0_{(4\times4)} \\ I_{(4\times4)} \end{bmatrix},$$

and

$$U = \begin{bmatrix} M_1 & M_2 \end{bmatrix}.$$

3 Estimating the Model

The model has implications for the behavior of three observable variables: the output growth rate, the inflation rate, and the short-term nominal interest rate. The empirical model has 16 parameters: z, β, π, γ, α, ψ, ρ_r, ρ_π, ρ_x, ρ_g, ρ_a, ρ_e, σ_a, σ_e, σ_z, and σ_r. Note that values can be assigned to z, β, and π to insure that the steady-state output growth rate, inflation rate, and short-term nominal interest rate in the model equal the corresponding average values in the data.

To estimate the remaining 13 parameters via maximum likelihood, let $\{d_t\}_{t=1}^T$ denote the series for the logarithmic deviations of the output growth rate, the inflation rate, and the short-term nominal interest rate from their average, or steady-state, values:

$$d_t = \begin{bmatrix} \hat{g}_t \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} \ln(Y_t) - \ln(Y_{t-1}) - \ln(z) \\ \ln(P_t) - \ln(P_{t-1}) - \ln(\pi) \\ \ln(R_t) - \ln(z) + \ln(\beta) - \ln(\pi) \end{bmatrix},$$

where Y_t is the level of real GDP per-capita, P_t is the level of the GDP deflator, and R_t is the gross nominal interest rate on three-month US Treasury bills.

The empirical model then takes the form

$$s_{t+1} = As_t + B\varepsilon_{t+1} \tag{26}$$

and

$$d_t = Cs_t, \tag{27}$$

where $A = \Pi$, $B = W$, C is formed from the rows of U and Π as

$$C = \begin{bmatrix} U_2 \\ U_5 \\ \Pi_3 \end{bmatrix},$$

and the vector of zero-mean, serially uncorrelated innovations ε_{t+1} is normally distributed with diagonal covariance matrix

$$V = E\varepsilon_{t+1}\varepsilon_{t+1}' = \begin{bmatrix} \sigma_a^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & \sigma_r^2 \end{bmatrix}.$$

The model defined by (26) and (27) is in state-space form; hence the likelihood function for the sample $\{d_t\}_{t=1}^T$ can be constructed as outlined by Hamilton (1994, Ch.13). For $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t | d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|j-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t | d_{t-j}, d_{t-j-1}, ..., d_1).$$

Then, in particular, (26) implies that

$$\hat{s}_{1|0} = Es_1 = 0_{(8\times 1)} \tag{28}$$

and

$$\text{vec}(\Sigma_{1|0}) = \text{vec}(Es_1 s_1') = [I_{(64\times 64)} - A \otimes A]^{-1}\text{vec}(BVB'). \tag{29}$$

Now suppose that $\hat{s}_{t|t-1}$ and $\Sigma_{t|t-1}$ are in hand and consider the problem of calculating $\hat{s}_{t+1|t}$ and $\Sigma_{t+1|t}$. Note first from (27) that

$$\hat{d}_{t|t-1} = C\hat{s}_{t|t-1},$$

Hence,

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

is such that

$$Eu_t u_t' = C\Sigma_{t|t-1}C'.$$

Next, using Hamilton's (p.379, eq.13.2.13) formula for updating a linear projection.

$$\begin{aligned} \hat{s}_{t|t} &= s_{t|t-1} + [E(s_t - \hat{s}_{t|t-1})(d_t - \hat{d}_{t|t-1})'][E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1}u_t \\ &= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t. \end{aligned}$$

Hence, from (26),

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Using this last result, along with (26) again,

$$s_{t+1} - \hat{s}_{t+1|t} = A(s_t - \hat{s}_{t|t-1}) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Hence,

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t-1}A' - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1}A'.$$

These results can be summarized as follows. Let

$$\hat{s}_t = \hat{s}_{t|t-1} = E(s_t|d_{t-j}, d_{t-j-1}, ..., d_1)$$

and

$$\Sigma_t = \Sigma_{t|j-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})'.$$

Then

$$\hat{s}_{t+1} = A\hat{s}_t + K_t u_t$$

and

$$d_t = C\hat{s}_t + u_t,$$

where

$$u_t = d_t - E(d_t|d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$Eu_t u_t' = C\Sigma_t C' = \Omega_t,$$

the sequences for K_t and Σ_t can be generated recursively using

$$K_t = A\Sigma_t C'(C\Sigma_t C')^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_t A' - A\Sigma_t C'(C\Sigma_t C')^{-1}C\Sigma_t A',$$

and the initial conditions $\hat{s}_1$ and Σ_1 are provided by (28) and (29).

The innovations $\{u_t\}_{t=1}^T$ can then be used to form the log likelihood function for $\{d_t\}_{t=1}^T$ as

$$\ln(L) - \left(\frac{3T}{2}\right)\ln(2\pi) - \frac{1}{2}\sum_{t=0}^{T}\ln(|\Omega_t|) - \frac{1}{2}\sum_{t=0}^{T}u_t'\Omega_t^{-1}u_t.$$

4 Evaluating the Model

4.1 Variance Decompositions

Begin by considering (26), which can be rewritten as

$$s_t = As_{t-1} + B\varepsilon_t,$$

or

$$(I - AL)s_t = B\varepsilon_t,$$

or

$$s_t = \sum_{j=0}^{\infty} A^j B\varepsilon_{t-j}.$$

This last equation implies that

$$s_{t+k} = \sum_{j=0}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$E_t s_{t+k} = \sum_{j=k}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$s_{t+k} - E_t s_{t+k} = \sum_{j=0}^{k-1} A^j B\varepsilon_{t+k-j},$$

and hence

$$\begin{aligned}\Sigma_k^s &= E(s_{t+k} - E_t s_{t+k})(s_{t+k} - E_t s_{t+k})' \\ &= BVB' + ABVB'A' + A^2BVB'A^{2\prime} + ... + A^{k-1}BVB'A^{k-1\prime}.\end{aligned}$$

In addition, (26) implies that

$$\Sigma^s = \lim_{k\to\infty} \Sigma_k^s$$

is given by

$$\text{vec}(\Sigma^s) = [I_{(64\times 64)} - A \otimes A]^{-1}\text{vec}(BVB').$$

Next, consider (25), which implies that

$$\Sigma_k^f = E(f_{t+k} - E_t f_{t+k})(f_{t+k} - E_t f_{t+k})' = U\Sigma_k^s U'$$

and

$$\Sigma^f = \lim_{k\to\infty} \Sigma_k^f = U\Sigma^s U'.$$

Finally, note that

$$\ln(Y_t) - \ln(Y_{t-1}) - \ln(z) = U_2 s_t.$$

Hence,

$$\ln(Y_{t+k}) = \ln(Y_t) + k\ln(z) + U_2 \sum_{j=1}^{k} s_{t+j}.$$

Consequently,

$$\ln(Y_{t+k}) - E_t \ln(Y_{t+k}) = U_2 \sum_{j=1}^{k} (s_{t+j} - E_t s_{t+j}).$$

And hence

$$\Sigma_k^Y = E[\ln(Y_{t+k}) - E_t \ln(Y_{t+k})][\ln(Y_{t+k}) - E_t \ln(Y_{t+k})]' = U_2 \Sigma_k^S U_2',$$

where

$$\begin{aligned}
\Sigma_k^S &= \left[\sum_{j=1}^{k}(s_{t+j} - E_t s_{t+j})\right]\left[\sum_{j=1}^{k}(s_{t+j} - E_t s_{t+j})\right]' \\
&= \left[\sum_{j=1}^{k}\sum_{l=0}^{j-1} A^l B\varepsilon_{t+j-l}\right]\left[\sum_{j=1}^{k}\sum_{l=0}^{j-1} A^l B\varepsilon_{t+j-l}\right]' \\
&= [B\varepsilon_{t+k} + (I+A)B\varepsilon_{t+k-1} + (I+A+A^2)B\varepsilon_{t+k-2} + ... + (I+A+...+A^{k-1})B\varepsilon_{t+1}] \\
&\quad \times[B\varepsilon_{t+k} + (I+A)B\varepsilon_{t+k-1} + (I+A+A^2)B\varepsilon_{t+k-2} + ... + (I+A+...+A^{k-1})B\varepsilon_{t+1}]' \\
&= BVB' + (I+A)BVB'(I+A)' + (I+A+A^2)BVB'(I+A+A^2)' \\
&\quad +... + (I+A+...+A^{k-1})BVB'(I+A+...+A^{k+1})'.
\end{aligned}$$

4.2 Producing Smoothed Estimates of the Shocks

Hamilton (Ch.13, Sec.6, pp.394-397) shows how to generate a sequence of smoothed estimates $\{\hat{s}_{t|T}\}_{t=1}^{T}$ of the unobservable state vector, where

$$\hat{s}_{t|T} = E(s_t|d_T, d_{T-1}, ..., d_1).$$

As before for $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t|d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|j-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t | d_{t-j}, d_{t-j-1}, ..., d_1).$$

Also as before, let

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

so that again,

$$Eu_t u_t' = C\Sigma_{t|t-1}C'.$$

Then

$$\hat{s}_{1|0} = Es_1 = 0_{(8\times 1)}$$

and

$$\text{vec}(\Sigma_{1|0}) = \text{vec}(Es_1 s_1') = [I_{(64\times 64)} - A \otimes A]^{-1}\text{vec}(BVB').$$

From these starting values, the sequences $\{\hat{s}_{t|t}\}_{t=1}^T$, $\{\hat{s}_{t|t-1}\}_{t=1}^T$, $\{\Sigma_{t|t}\}_{t=1}^T$, and $\{\Sigma_{t|t-1}\}_{t=1}^T$ can be generated recursively using

$$u_t = d_t - C\hat{s}_{t|t-1},$$

$$\hat{s}_{t|t} = \hat{s}_{t|t-1} = \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t,$$

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t},$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1},$$

and

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t}A'$$

for $t = 1, 2, ..., T$.

Now, to begin, construct a sequence $\{J_t\}_{t=1}^{T-1}$ using Hamilton's equation (13.6.11):

$$J_t = \Sigma_{t|t}A'\Sigma_{t+1|t}^{-1}.$$

Then note that $\hat{s}_{T|T}$ is just the last element of $\{\hat{s}_{t|t}\}_{t=1}^T$. From this terminal condition, the rest of the sequence can be generated recursively using Hamilton's equation (13.6.16):

$$\hat{s}_{T-j|T} = \hat{s}_{T-j|T-j} + J_{T-j}(\hat{s}_{T-j+1|T} - \hat{s}_{T-j+1|T-j})$$

for $j = 1, 2, ..., T-1$. Kohn and Ansley (1983) show that in cases where $\Sigma_{t+1|t}$ turns out to be singular, its inverse can be replaced by its Moore-Penrose pseudoinverse in the expression of J_t.

Likewise, Hamilton (1994, Ch.13, eq.13.6.20) and Kohn and Ansley (1983) show that a sequence for $\{\Sigma_{t|T}\}_{t=1}^T$ can be generated recursively using

$$\Sigma_{T-j|T} = \Sigma_{T-j|T-j} + J_{T-j}(\Sigma_{T-j+1|T} - \Sigma_{T-j+1|T-j})J_{T-j}'$$

for $j = 1, 2, ..., T-1$.

4.3 Producing Counterfactual Series for the Observables

Once smoothed estimates of the shocks have been obtained, some of all of these shocks can be fed through the equations (24) and (25) describing the model's solution to trace out the behavior of the observable and unobservable variables in the presence or absence of specific shocks. In addition, the same shocks can be passed through (24) and (25) after some of the model's structural parameters have been been changed in order to characterize various counterfactual simulations.

More complicated counterfactual exercises can assume, for instance, a different sample path for the nominal interest rate when the preference, cost push, and technology shocks are drawn from the actual, historical experience. Tracing out these counterfactuals can be done using the same Kalman smoothing algorithm that, for instance, Clarida and Coyle (1984) recommend for generating conditional forecasts from more conventional vector autoregressive time series models.

To begin, assemble the counterfactual path for the interest rate and the smoothed estimates of the preference, cost-push, and technology shock into a new "data vector" as

$$\tilde{d}_t = \begin{bmatrix} \tilde{r}_t \\ \hat{a}_t \\ \hat{e}_t \\ \hat{z}_t \end{bmatrix}.$$

Next, form a new "empirical model" as

$$s_{t+1} = As_t + B\varepsilon_{t+1}$$

and

$$\tilde{d}_t = Cs_t,$$

where as before $A = \Pi$ and $B = W$ but where now

$$C = \begin{bmatrix} \Pi_3 \\ C_1 \end{bmatrix}$$

and

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then, by passing these data through the same Kalman smoothing algorithm outlined above, one can obtain counterfactual paths for the monetary policy shock and the model's observable and unobservable endogenous variables.