习题2-1 (P24)

5. $M(t,u)=(t^2+1)\cos u$， $N(t,u)=2t\sin u$

$\frac{\partial M}{\partial t}=\frac{\partial N}{\partial u}=2t\cos u$，故原方程为恰当方程

$\int_{u_0}^{u}(t^2+1)\cos u\,du+\int_{t_0}^{t}2t\sin u_0\,dt=c$，取 $t_0=0$，$u_0=0$.

$\int_0^{u}(t^2+1)\cos u\,du+0=c \Rightarrow (t^2+1)\sin u=c$，$(t,x)\in D$，$\forall c\in R$

7. $M(x,y)=\frac{y}{x}+x^2$， $N(x,y)=\ln x-2y$

$\frac{\partial M}{\partial y}=\frac{1}{x}$， $\frac{\partial N}{\partial x}=\frac{1}{x} \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$，故原方程为恰当方程

$\int_{x_0}^{x}(\frac{y}{x}+x^2)dx+\int_{y_0}^{y}(\ln x_0-2y)dy=c$，取 $\cancel{t_0=0}$，$y_0=0$，$x_0=1$

$\int_{1}^{x}(\frac{y}{x}+x^2)dx+\int_0^{y}(-2y)dy=c \Rightarrow y\ln x+\frac{1}{3}x^3-y^2=c$，$(t,x)\in D$，$\forall c\in R$

9. $M(t,s)=\frac{2s-1}{t}$， $N(t,s)=\frac{s-s^2}{t^2}$

$\frac{\partial M}{\partial t}=\frac{1-2s}{t^2}$， $\frac{\partial N}{\partial s}=\frac{1-2s}{t^2} \Rightarrow \frac{\partial M}{\partial t}=\frac{\partial N}{\partial s}$，故原方程为恰当方程

$\int_{s_0}^{s}\frac{2s-1}{t}ds+\int_{t_0}^{t}\frac{s_0-s_0^2}{t^2}dt=c$，取 $s_0=0$，$t_0=1$

$\int_0^{s}\frac{2s-1}{t}ds=c \Rightarrow \frac{1}{t}(s^2-s)=c$，$(t,x)\in D$，$\forall c\in R$

10. $M(x,y)=xf(x^2+y^2)$，$N(x,y)=yf(x^2+y^2)$

$\frac{\partial M}{\partial y}=2xyf'(x^2+y^2)$，$\frac{\partial N}{\partial x}=2xyf'(x^2+y^2) \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$，故原方程为恰当方程

$\cancel{\int_{x_0}^{x}xf(x^2+y^2)dx+\int_{y_0}^{y}y_0f(x_0^2+y^2)dy=c}$

设 $F(x^2+y^2)$ 满足 $dF(x^2+y^2)=M(x,y)dx+N(x,y)dy$，

则解为 $F(x^2+y^2)=c$，$\forall(x,y)\in D$，$\forall c\in R$

习题2-2 (P30-P31)

1-(2) $ydy=\frac{x^2}{1+x^3}dx \Rightarrow 3ydy=\frac{1}{1+x^3}dx^3$

$\Rightarrow$ 解为：$3y^2-2\ln|1+x^3|=c$ $(y\neq0, x\neq-1)$

1-(5) <1> $\cos 2y\neq0$，$\frac{1}{(\cos 2y)^2}dy=(\cos x)^2dx \Rightarrow 2\tan 2y-2x-\sin 2x=c$ $(\forall x\in D, \forall c\in R$ $(\cos 2y\neq0))$

<2> $\cos 2y=0$，$y=\frac{\pi}{4}+\frac{k}{2}\pi$ $(k\in Z)$ 也是方程的解

1-(7) $(y+e^{y})dy=(x-e^{-x})dx$

$\Rightarrow \frac{y^2}{2}+e^{y}=\frac{x^2}{2}+e^{-x}+c \Rightarrow y^2-x^2+2(e^{y}-e^{-x})=c \quad (y+e^{y}\neq 0,\ \forall c\in\mathbb{R})$

2-(1) $\sin 2x\,dx=-\cos 3y\,dy$

$\Rightarrow -\frac{1}{2}\cos 2x+\frac{1}{3}\sin 3y=c \Rightarrow 2\sin 3y-3\cos 2x=c$ ~~$(\forall c\in\mathbb{R})$~~

$y(\frac{\pi}{2})=\frac{\pi}{3}$，故 $c=3$，故解为：$2\sin 3y-3\cos 2x=3$

2-(3) ~~<1> r=0~~ $r\neq 0$. 否则与 $r(0)=2$ 矛盾

$\frac{1}{r}dr=d\theta \Rightarrow \ln r=\theta+c$，代入 $r(0)=2 \Rightarrow c=\ln 2$

$\ln r-\theta=\ln 2 \Rightarrow r=2e^{\theta}$

5. 设 $y(x_0)=a$，则常值函数 $y(x)=a$ 显然为方程的一个解

($\Leftarrow$) 已知瑕积分 $\left|\int_a^{a\pm\varepsilon}\frac{dy}{f(y)}\right|=\infty$. 仅证.

假设解局部不唯一，即有解 $y=g(x)\neq a$，$\exists x_1, h=g(x_1)-a\neq 0$

取 $\varepsilon=h$. $\left|\int_a^{a+h}\frac{dy}{f(y)}\right|=\left|\int_x^{x_1}dx\right|=|x_1-x|$ 为有限值，矛盾。故解局部唯一

($\Rightarrow$) 已知方程解 $y(x)=a$ 局部唯一. 仅证.

假设 $\left|\int_a^{a\pm\varepsilon}\frac{dy}{f(y)}\right|$ 收敛，则令 $g(x)=\int_a^{x}\frac{1}{f(y)}dy$，该变上限积分存在 $(y\neq a)$

$g'(y)=\frac{1}{f(y)}\ (y\neq a)$，构造隐函数 h 满足 $h(g)=y\neq a$

从而：$\frac{dh(g)}{dg}=\frac{dy}{dg(y)}=f(y)=f(h(g))$ 故 $y=h(g)$ 亦为解 $h(g)\neq a$. 与

与解的唯一性矛盾。故 $\left|\int_a^{a\pm\varepsilon}\frac{dy}{f(y)}\right|$ 发散

习题2-3.

1-(1) $p(x)=2$. $q(x)=xe^{-x}$

$y=e^{-2x}(\int xe^{-x}e^{2x}dx+c)=ce^{-2x}+xe^{-x}-e^{-x}$

1-(3) $\frac{dy}{dx}+\frac{2}{x}y=\frac{\sin x}{x}$，$p(x)=\frac{2}{x}$，$q(x)=\frac{\sin x}{x}$

$\int p(x)dx=\ln x^2+c$，故 $y=e^{-\ln x^2}(c+\int\frac{\sin x}{x}e^{\ln x^2}dx)$

$\Rightarrow y=\frac{1}{x^2}(c+\int x\sin x\,dx)=\frac{1}{x^2}(c-x\cos x+\sin x)$，由 $y(\pi)=\frac{1}{\pi}$ 得：$c=0$

故解为：$y=-\frac{1}{x}\cos x+\frac{1}{x^2}\sin x$

2-(2) $\frac{dx}{dy}=\frac{x+y^2}{y}=\frac{1}{y}x+y$

2-(4) $\frac{dy}{dx}=\frac{1}{\cos y}+x\frac{\sin y}{\cos y} \Rightarrow \frac{\cos y\,dy}{dx}=1+\sin y\cdot x$

令 $z=\sin y$，$\frac{dz}{dx}=1+z\cdot x$

3. $y'+a(x)y\le 0$. 两侧同乘 $e^{\int_0^x a(s)ds}$

$y'\cdot e^{\int_0^x a(s)ds}+y\left(e^{\int_0^x a(s)ds}\right)'\le 0$. $x=0$时等号成立.

两侧积分：$y\cdot e^{\int_0^x a(s)ds}\le C$，代入$x=0$得：$C=\varphi(0)$

故 $\varphi(x)\le\varphi(0)e^{-\int_0^x a(s)ds}$

~~5. (1) $q(x)=0$，则方程解形如 $y=ce^{-\int_{x_0}^x p(x)dx}$，解以$w$为周期~~

~~$\Leftrightarrow e^{-\int_{x_0}^x p(x)dx}=e^{-\int_{x_0}^{x+w}p(x+w)dx}=e^{-\int_{x_0}^x p(x)dx}e^{-\int_x^{x+w}p(x)dx}$~~

~~$\Leftrightarrow e^{-\int_x^{x+w}p(x)dx}=1 \Leftrightarrow \int_x^{x+w}p(x)dx=0$~~

~~5. (1) $q(x)\equiv0$，则方程解形如 $y=ce^{-\int_{x_0}^x p(t)dt}$，解以$w$为周期~~

~~$\Leftrightarrow e^{-\int_{x_0}^x p(t)dt}=e^{-\int_{x_0}^{x+w}p(x+w)dx}=e^{-\int_{x_0}^x p(x)dx}e^{-\int_x^{x+w}p(t)dt}$~~

~~$\Leftrightarrow e^{-\int_x^{x+w}p(t)dt}=1 \Leftrightarrow$~~

5. (1) $q(x)\equiv0$. 则方程解形如 $y=ce^{-\int_{x_0}^x p(t)dt}$，解以$w$为周期

$\Leftrightarrow e^{-\int_{x_0}^x p(t)dt}=e^{-\int_{x_0}^{x+w}p(t)dt}=e^{-\int_{x_0}^x p(t)dt}\cdot e^{-\int_x^{x+w}p(t)dt}$

$\Leftrightarrow e^{-\int_x^{x+w}p(t)dt}=1 \Leftrightarrow -\int_x^{x+w}p(t)dt=0 \Leftrightarrow \int_0^w p(t)dt=0$

$\Leftrightarrow \bar{p}=\frac{1}{w}\int_0^w p(t)dt=0$

(2) 通解形如：$y(x)=e^{-\int_{x_0}^x p(t)dt}\left[c+\int_{x_0}^x q(s)e^{\int_{x_0}^s p(t)dt}ds\right]$ $(x_0\in I)$

$y(x+w)=e^{-\int_{x_0}^x p(t)dt}e^{-\int_x^{x+w}p(t)dt}\left[c+\int_{x_0}^{x+w}q(s)e^{\int_{x_0}^s p(t)dt}ds\right]$

由于$q(x)$不恒为0，故 $\int_x^{x+w}q(s)ds$ 不恒为0，而 $\int_x^{x+w}e^{\int_{x_0}^s p(t)dt}ds>0$

故 $\int_x^{x+w}q(s)e^{\int_{x_0}^s p(t)dt}ds$ 不恒为0，故 $\int_{x_0}^x q(s)e^{\int_{x_0}^s p(t)dt}ds$ 不恒等于

$\int_{x_0}^{x+w}q(s)e^{\int_{x_0}^s p(t)dt}ds$

故 $y(x)=y(x+w)\Leftrightarrow e^{-\int_x^{x+w}p(t)dt}\ne1 \Leftrightarrow \bar{p}(t)\ne0$

取 $x_0=0$ 代入 $y(x)$ 与 $y(x+w)$ 表达式，可得：

$$c+\int_0^x q(s)e^{\int_0^s p(t)dt}ds = e^{-\int_x^{x+w}p(t)dt}\left[c+\int_0^{x+w}q(s)e^{\int_0^s p(t)dt}ds\right]$$

$$\Rightarrow c=\frac{\left(\int_0^{w+x}-e^{\bar{P}w}\int_0^x\right)q(s)e^{\int_0^s p(t)dt}ds}{e^{\bar{P}w}-1}$$

通解：$y(x)=e^{-\int_0^x p(t)dt}\left[c+\int_0^x q(s)e^{\int_0^s p(t)dt}ds\right]$