共形与解析

1. Review.

Key Fact: $T: (\mathbb{R}^2 \simeq \mathbb{C}) \mapsto (\mathbb{R}^2 \simeq \mathbb{C})$ a linear transformation of the form $T\begin{pmatrix} x \\ y \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ can be written as a complex multiplication by z, $(x+yi) \cdot z = (u+vi)$. iff.

$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, moreover $z = a+bi$.

伸扭

$f: \Omega^{\text{open}} \subseteq \mathbb{C} \to \mathbb{C}$ is analytic, if $f'(z)$ exists!

$\Updownarrow$

$f: \Omega^{\text{open}} \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ $\quad Jf_{z_0}$ is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$

$\Downarrow$

f is analytic $\Rightarrow$ f satisfies C-R equation.

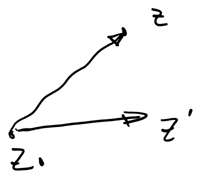
$$\boxed{\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v}$$

f is analytic. $\Leftarrow$ f satisfies C-R equation + u, v continuously differentiable

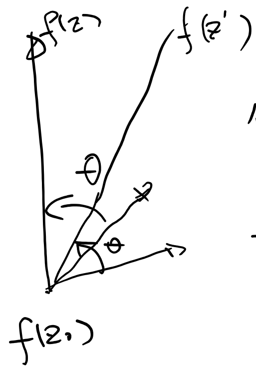
(2) 例子.

$$f(z) - f(z_0) \simeq f'(z_0) \cdot (z - z_0)$$

无穷小地来看.



Domain



Image

伸缩比: $\lim_{z \to z_0} \dfrac{|f(z) - f(z_0)|}{|z - z_0|}$

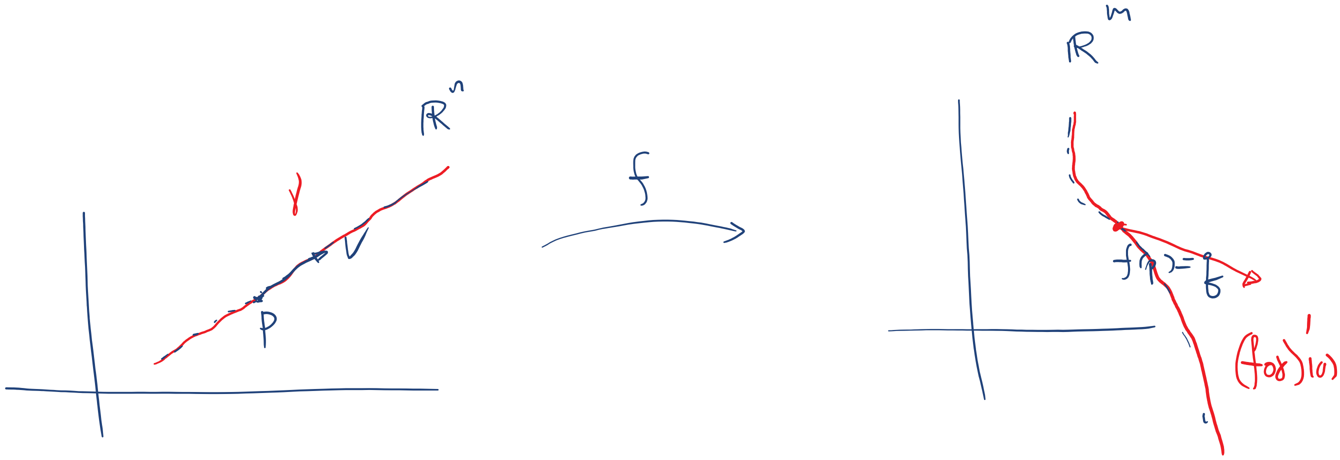
扭转角: θ

Remark. In fact, $f'(z_0)$ can be interpretted as the differential of the map f:

$f: \mathbb{R}^n \to \mathbb{R}^m$ smooth map.

$(df)_p : \underbrace{T_p\mathbb{R}^n}_{\cong \mathbb{R}^n} \to \underbrace{T_{f(p)}\mathbb{R}^m}_{\cong \mathbb{R}^m}$ is a linear transformation

given by $J_p f$.



$$v \xrightarrow{(df)_p} (f(\gamma))'(0)$$

$f(\gamma)$

choose γ s.t. $\gamma(0)=p$, $\gamma'(0)=v$

Note; independent of choice of γ

$(df)_p$ is matrix representation under canonical basis is $J_p f$.

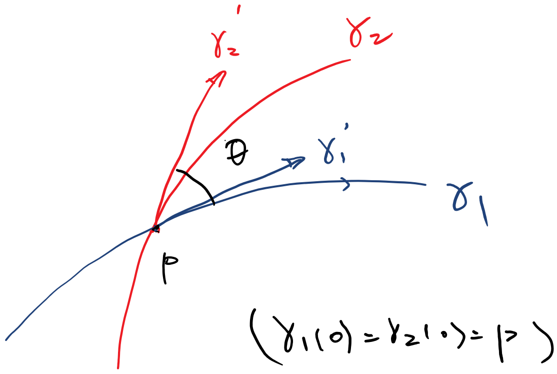
2. Analytic and conformal

Prop 1. If f is analytic in Ω, and $f'(z)\neq 0$ $\forall z\in\Omega$,

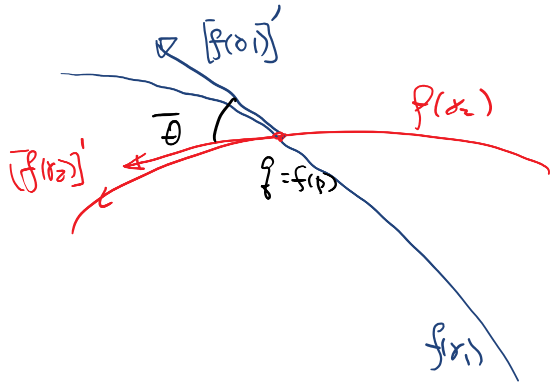
then f is a conformal map.

then f is a conformal map.

[Preserves oriented angles:]



$(\gamma_1(0)=\gamma_2(0)=p)$



Since f is analytic $\Rightarrow$

$$\left.\begin{array}{l} \gamma_1'(0)\cdot f'(p) = (f\circ\gamma_1)'(0) \\ \gamma_2'(0)\cdot f'(p) = (f\circ\gamma_2)'(0) \\ f'(p)\neq 0 \end{array}\right\} \Rightarrow \theta = \bar{\theta}.$$

Prop 2. If f is a conformal map: $\Omega^{open} \subseteq \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{R}^2$ and differentiable, then f is analytic.

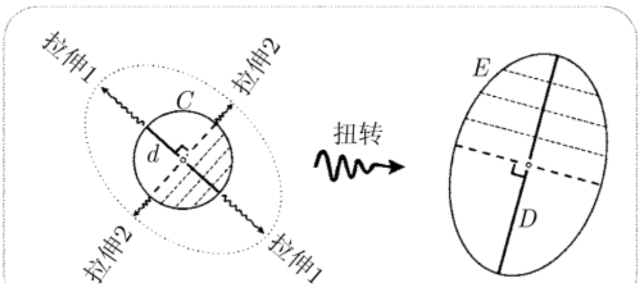
(Maybe conformal implies differentiable)

Proof: Goal: to show Jf is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ at every point of Ω.

保持角度 $\Rightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

Exercise: T is a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, and T preserves angles. i.e. $\measuredangle v,w = \measuredangle Tv, Tw \quad \forall v,w\in\mathbb{R}^2$.

then T must be of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.



J_f (f的微分)

局部线性变换就是在 d 方向做一个拉伸，在与它垂直的方向上做另一拉伸，最后再做一个扭转. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \left(A \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} A^{-1} \right)$

$A \cdot A^T = Id.$ $\lambda_1, \lambda_2 \geq 0$

Polar decomposition

From Wikipedia, the free encyclopedia

In mathematics, the **polar decomposition** of a square real or complex matrix A is a factorization of the form $A = UP$, where U is a unitary matrix and P is a positive semi-definite Hermitian matrix, both square and of the same size.[1]

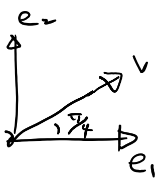
Intuitively, if a real $n \times n$ matrix A is interpreted as a linear transformation of n-dimensional space $\mathbb{R}^n$, the polar decomposition separates it into a rotation or reflection U of $\mathbb{R}^n$, and a scaling of the space along a set of n orthogonal axes.

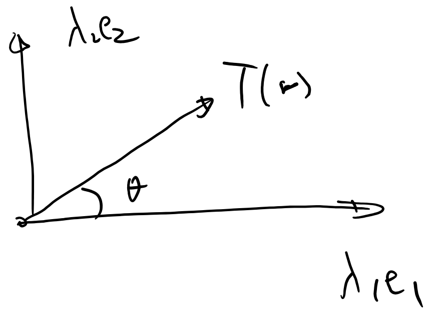
Since rotation preserves angles. we just need to show the following.

Claim: Suppose $\{e_1, e_2\}$ are orthonormal basis $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$T(e_1) = \lambda_1 e_1$ $T(e_2) = \lambda_2 e_2$. and T preserves angles. then

$\lambda_1 = \lambda_2$





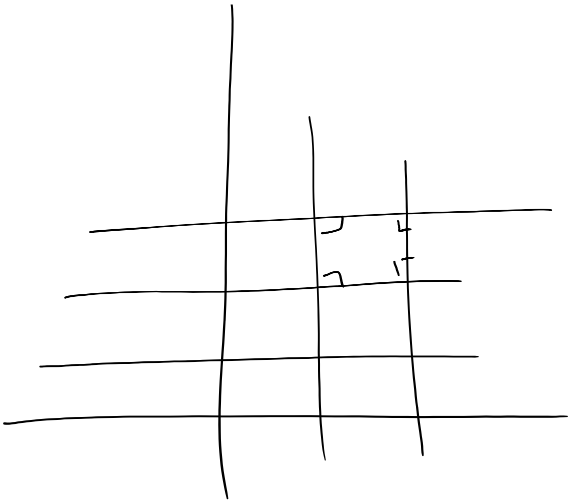
let $v = \cos\frac{\pi}{4} e_1 + \sin\frac{\pi}{4} e_2$

then $T(v) = \cos\frac{\pi}{4} \lambda_1 e_1 + \sin\frac{\pi}{4} \lambda_2 e_2$.

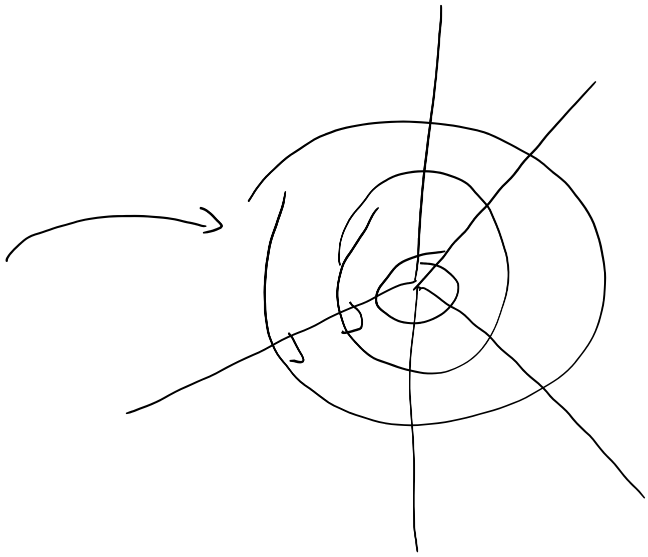
if $\lambda_1 \neq \lambda_2 \Rightarrow \theta \neq \frac{\pi}{4}$. contradiction.

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$= \begin{pmatrix} \lambda\cos\theta & -\lambda\sin\theta \\ \lambda\sin\theta & \lambda\cos\theta \end{pmatrix}$ is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$



Domain



解析延拓

1. Rigidity coming from conformal map.

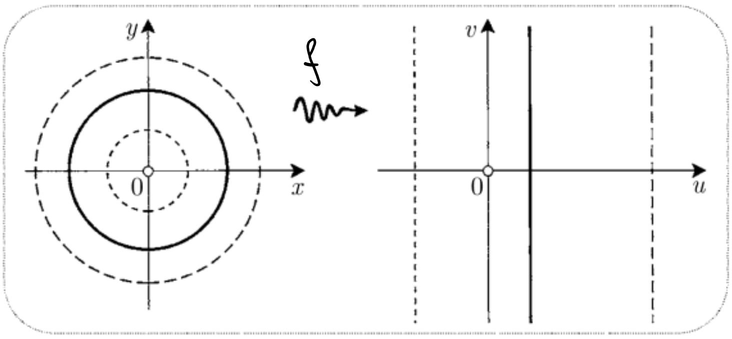
刚性.

$\mathbb{C} \to \mathbb{C}$

suppose f is a conformal map. which maps circles centered at 0 to vertical lines, as radius of circles increases the vertical lines move to right.

$\Downarrow$

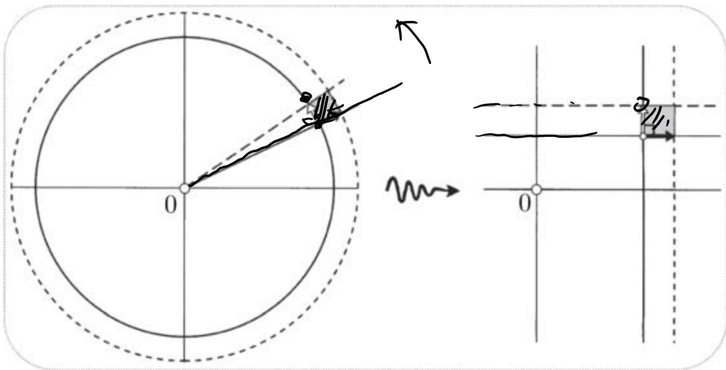
Can determine f !



Domain $\longrightarrow$ Image

$\Rightarrow$. radial lines mapped to horizontal lines.

. as radial lines rotates counterclockwise. horizontal lines move upward



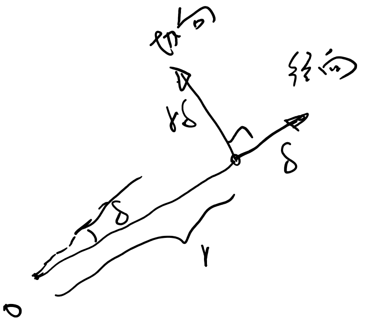
. f conformal $\Rightarrow$ f is analytic.

$\Rightarrow$ C-R equation. (Another version)

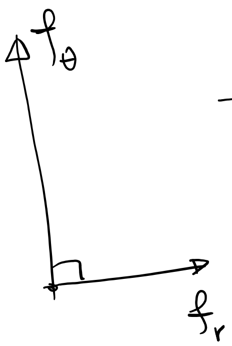
$f(r,\theta) = U(r,\theta) + i\,V(r,\theta)$.

$$\boxed{f_\theta = f_r \cdot (ir)}$$

$\Downarrow$



$\longrightarrow$



$$(U_\theta + iV_\theta) = (U_r + iV_r)(ir)$$

$\Rightarrow$ $\boxed{U_\theta = -V_r \cdot r}$

。Circles $\to$ vertical lines

$\Rightarrow$ $U_\theta = 0$ $\Rightarrow$ U depends on r alo...

$V_\theta = U_r \cdot r$?

$\forall \quad \underline{V_\theta = U_r \cdot r}$

L.H.S is a function of θ, R.H.S is a function of r. $\Rightarrow V_\theta = U_r \cdot r = C$.

$\Rightarrow U_\theta = 0 \Rightarrow U$ depends on r only

$\circ$ Radial lines $\to$ Horizontal lines

$\Rightarrow V_r = 0 \Rightarrow V$ depends on θ only

$$\Rightarrow \quad V = c\theta + a, \quad U = c\log r + b.$$

$$\Rightarrow \quad f(r,\theta) = c\log r + b + i(c\theta + a)$$

$$= c(\log r + i\theta) + (b + ai)$$

$$= c \cdot \mathrm{Log}(z) + c'$$

2. Analytic continuation.

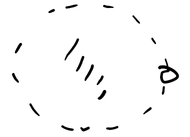
解析刚性的本质特性尽纳于以下结果中：

如果哪怕是任意小一段曲线被一解析映射挤压成为一点，则其整个定义域也将坍缩于该点.

例1.

$$f(z) = 1 + z + z^2 + z^3 + \cdots z^n + \cdots \qquad R = 1$$

$$= \frac{1}{1-z}.$$



Domain of L.H.S. $|z|<1$. Domain of R.H.S $z \neq 1$.

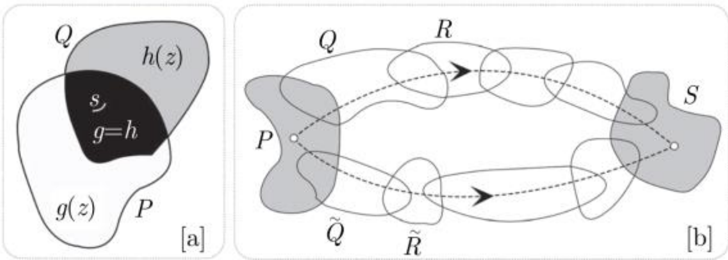
$\frac{1}{1-z}$ is an analytic extension of $\underset{f(z)}{\underbrace{1 + z + \cdots + z^n + \cdots}}$

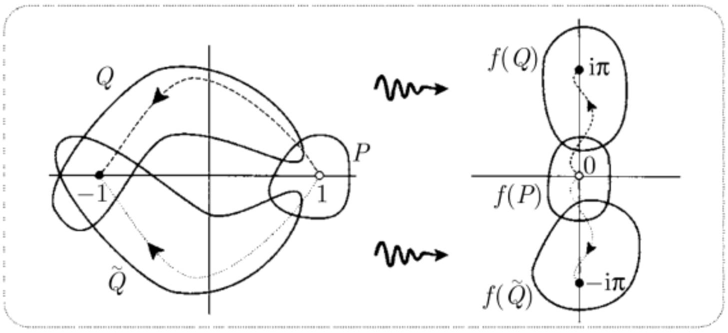
$\underline{\frac{1}{1-z}} - \underline{f(z)} = 0$ on $|z|<1$.

let $\tilde{f}$ be another $\overset{\text{analytic}}{\text{extension}}$ of f

$$\frac{1}{1-z} - \tilde{f} = 0 \quad \text{on } |z|<1 \Rightarrow \underline{\frac{1}{1-z}} - \tilde{f} \equiv 0$$

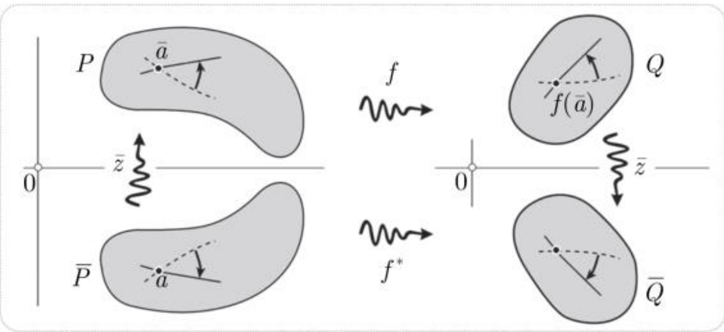
thus $\frac{1}{1-z}$ is called $\underline{\text{the}}$ analytic continuation of f.



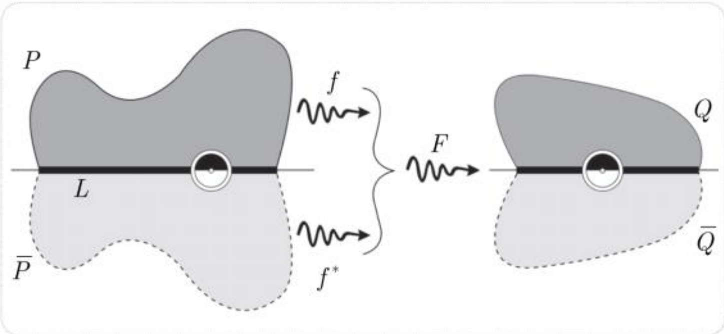


屏幕剪辑的捕获时间: 2022/10/10 9:49

3. Schwartz reflection.



P
$\bar{a}$
f
Q
$f(\bar{a})$
$\bar{z}$
0
$\bar{z}$
0
$\overline{P}$
a
f^*
$\overline{Q}$



P
f
Q
F
L
$\overline{Q}$
$\overline{P}$
f^*