Analytic function. $f: \Omega \overset{\text{open}}{\subseteq} \mathbb{C} \to \mathbb{C}$. $f'(z)$ exists

$$\begin{tikzcd}
 & & \text{conformal map } f: \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2 \arrow[dd, leftrightarrow] \\
\begin{matrix}\text{Analytic} \\ f'(z) \text{ exists}\end{matrix} \arrow[urr, "f'(z)\neq 0"] \arrow[drr, shift left] & & \\
 & & \begin{matrix} f(z) = u + iv \\ \text{C-R equation } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{matrix} \arrow[ull, shift left, "u, v \in C^1"] \arrow[uull, phantom]
\end{tikzcd}$$

1. 求导法则. (四则运算都成立)

- Product rule. $(f \cdot g)' = f' \cdot g + f \cdot g'$.
- Chain rule $(f(g(z)))' = f'(g(z)) \cdot g'(z)$

Building Block: z^n. e^z, $\sin(z)$. $\cos(z)$. z^a, $\mathrm{Log}(z)$

Special case of chain rule:

$$z = f^{-1}(f(z)) \Rightarrow (f^{-1})'(f(z)) \cdot f'(z) = 1.$$

$\Rightarrow$ $\mathrm{Log}(z)$ is inverse function of e^z.

$$\Rightarrow z = \mathrm{Log}(e^z) \Rightarrow (\mathrm{Log})'(e^z) \cdot e^z = 1$$

$$(\mathrm{Log})'(e^z) = \frac{1}{e^z} \Rightarrow (\mathrm{Log}(w))' = \frac{1}{w}.$$

2. Power series

<u>Prop</u>: $P(z) = \sum_{n=0}^{\infty} a_n z^n$, has radius of convergence R.

then $P(z)$ is analytic in $|z| < R$; moreover

Proof: Recall. $P(x)=\sum_{n=0} a_n x^n$ regarded as a real power series the above statement holds.

① $\sum_{n=0}^{\infty} a_n x^n \to P(x)$

② $\sum_{n=0}^{\infty} n a_n x^n$ has same convergent radius, R

$\sum_{n=0}^{\infty} n a_n x^n \rightrightarrows Q(x)$ for $x\in[-r,r]$, $\forall r<R$. (内闭一致收敛)

$\Longrightarrow P'(x)=Q(x)$, for $x\in[-r,r]$ $\forall r<R$

$\Rightarrow \forall x\in(-R,R)$.

Corollary. Power series is infinitely differentiable in $|z|<R$. where R is its radius of converge.

Remark: we will later that above property holds for any analytic funtions.

$f'(z)$ exists! $\Rightarrow f^{(n)}$ exist!

3. Analytic continuation. 解析延拓.

conformal map $\to$ $\boxed{\text{刚性 地 生长}}$

$\Updownarrow$

Analytic $\Leftrightarrow$ C-R equation

思想适应

解析刚性的本质特性尽纳于以下结果中：

如果哪怕是任意小一段曲线被一解析映射挤压成为一点，则其整个定义域也将坍缩于$\boxed{\text{该点}}$.

Prop. let $f:\Omega\to\mathbb{C}$ be an analytic funtion defined on an open and connected set Ω. suppose $\gamma\subset\Omega$ is a segment such that $f|_\gamma=0$ then, $f\equiv 0$.

Ω is connected if $\forall p,q\in\Omega\ \exists$ continuous path $\subset\Omega$ joins p & q.

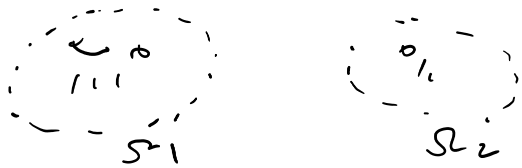
Ω is connected if $\forall$ $p, q \in \Omega$ $\exists$ path $\subset \Omega$ joins p & q.

an example of Ω which is not connected: $\Omega = \Omega_1 \cup \Omega_2$ (open, open)

$f: \Omega \to \mathbb{C}$.

$f|_{\Omega_1} = 0$ $f|_{\Omega_2} = 1$.

f is analytic



A connected open set is usually called a domain. ($\mathbb{C} + 1\,\text{dx}$)

In real analysis and complex analysis, branches of mathematics, the identity theorem for analytic functions states: given functions f and g analytic on a domain D (open and connected subset of $\mathbb{R}$ or $\mathbb{C}$), if $f = g$ on some $S \subseteq D$, where S has an accumulation point, then $f = g$ on D.

- f, g analytic on D
- $f = g$ on $S \subset D$
- S has an accumulation point in D.

$\exists \{z_n\} \in S$ s.t. $\lim_{n \to \infty} z_n = z_\infty \in D$

$\Rightarrow$ $f = g$ on D.

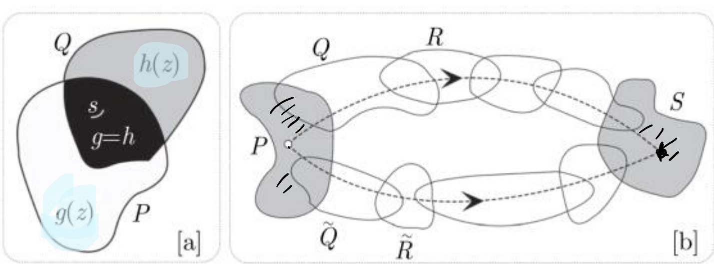
take S be a small piece of curve $\Rightarrow$ Prop

Defn. $f: \Omega \to \mathbb{C}$ is analytic

$F: \Omega' \to \mathbb{C}$ is also analytic. F is called an analytic continuation of f is $\Omega \subset \Omega'$. and $F = f$ on Ω.

FACT: By Identity Thm. if G is another analytic continuation of f defined on Ω' and Ω' is connected. then $G \equiv F$.

- F, G analytic in Ω', Ω' connected
- $F = G$ on $\Omega \subset \Omega'$ $\Rightarrow$ $F = G$ on Ω'.



analytic continuation along a path.

- $g(z)$ is analytic in P
- $h(z)$ is analytic in Q

- $P\cap Q$ is connected
- $g(z)=h(z)$ on $S\subset P\cap Q$

$\Rightarrow$ $g=h$ on $P\cap Q$.

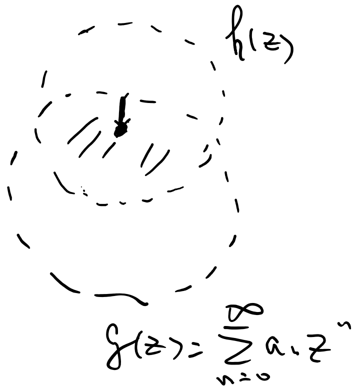
then we define

$$\tilde{g}(z)=\begin{cases} g(z) & z\in P \\ h(z) & z\in Q \end{cases}.$$

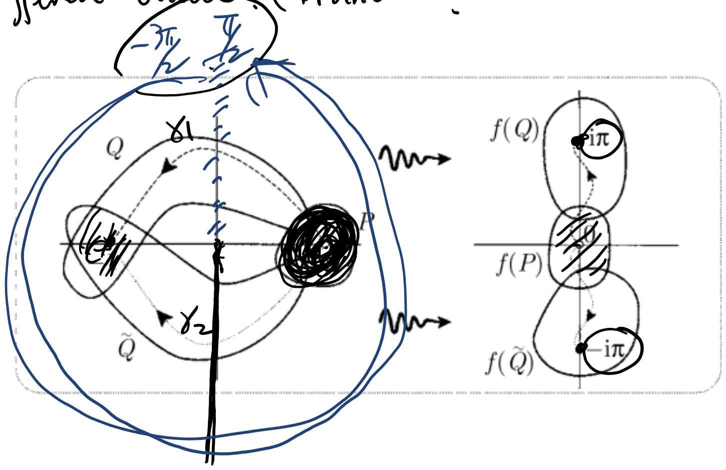
well-defined!

then it follow that

$\tilde{g}$ is an analytic extension of g on $P\cup Q$.



例: Analytic continuation along different paths may lead to different values (Branches)



$f(z)=\operatorname{Log}(z)$, on P. $\arg(z)\in(-\pi,\pi)$

$f(1)=0$.

$g(z)=\ln|z|+i\arg(z)$ on $P\cup Q$. $\arg z\in\left(-\frac{\pi}{2},\frac{3\pi}{2}\right)$

$g(z)=f(z)$ on P

$\Rightarrow g(z)$ is an analytic continuation of f on P:

$g(-1) = i \cdot \pi$.

$h(z) = \ln|z| + i\arg(z)$. on $P \cup \tilde{Q}$, $\arg z \in (-\frac{3}{2}\pi, \frac{\pi}{2})$

$h(z) = f(z)$ on P

$\Rightarrow h(z)$ is an analytic continuation of f on P

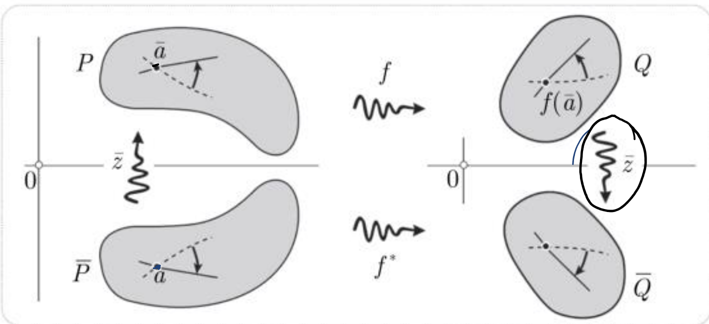
$h(-1) = -i\pi$.

Analytic continuation $\Rightarrow$ Multi-value function.

4. Schwartz reflection.

Prop: P domain $\subset \mathbb{R}^2_+ = \{\operatorname{Im}(z) > 0\}$

$f: P \to \mathbb{C}$ analytic function.



let $\bar{P} = P$ 关于x轴反射的像

Define

$$f^*(z) := \overline{f(\bar{z})}$$

Then f^* is analytic in $\bar{P}$.

$$f^*(z) = \overline{f(x-yi)} = \overline{u(x,-y) + i v(x,-y)}$$

$$= u(x,-y) - i v(x,-y)$$

$x-yi$

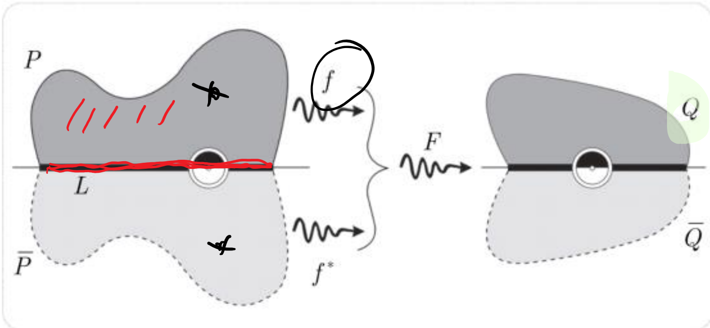
$\uparrow$

$z \quad x+yi$

check: C-R equation.

$f^*(z) = s(x,y) + i\,t(x,y)$

$$s(x,y) = u(x,-y)$$
$$t(x,y) = -v(x,-y)$$



Thm (Schwartz reflection Principle)

① $P \subset \mathbb{R}^2_+$ open set $\partial P \cap x\text{-axis} = L = (a,b) \subset \mathbb{R}$

② f is analytic in P and f is continuous on $P \cup L$

and $f|_L$ takes real value.

③ let $\bar{P}$ be P's reflection about x-axis.

set $f^*(z) = \overline{f(\bar{z})}$ for $z \in \bar{P}$.

④ Set
$$F(z) = \begin{cases} f(z) & z \in P \\ f(z) & z \in L \\ f^*(z) & z \in \bar{P} \end{cases}$$

So $F : P \sqcup L \sqcup \bar{P} \to \mathbb{C}$ is defined.

Then: F is analytic in $P \sqcup L \sqcup \bar{P}$.

Proof: Key. Note. F is analytic in $P \sqcup \bar{P}$ is clear!

The Difficulty lies on that $F'(z)$ exists for $z \in L$.

(we only know $F(z)$ is continuous at $z \in L$)

Note

Note

$\mathrm{Re}(f^*)$ is just an even extension of $\mathrm{Re}(f)$

$\mathrm{Im}(f^*)$ is just an odd extension of $\mathrm{Im}(f)$

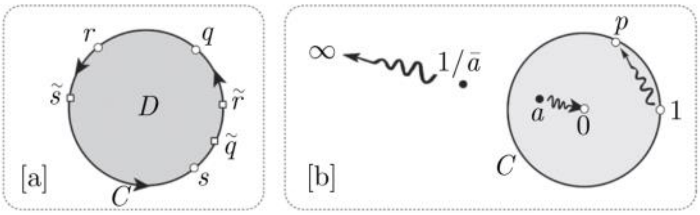
再谈莫比乌斯变换

1. Group structure.

2. linear action on Homogeneous coordinates.

3. Mobius Automorphism on Unit disk

D 的默比乌斯自同构的自由度为 3.



黎曼于 1851 年的博士论文包含了许多深刻的新结果, 其中最著名的就是以下的定理, 现在以**黎曼映射定理**著称于世:

任意单联通区域 R(除全平面外)都可以一一地共形映为任意另外一个这类区域 S. (3.54)