

2. ④ ④ 往证: $P_0 = (x_0, y_1^0, \dots, y_n^0) \in G$, 则上述微分系统在 P_0 某个邻域 $G_0 \subset G$ 内有几个相互独立的首次积分.

• 考虑初值条件: $y_i(x_0) = c_i$ ($i=1, 2, \dots, n$)

且中 (x_0, c_1, \dots, c_n) 在 P_0 某个邻域 G^* 中

由初值问题连续可微性定理: $y_i = \varphi(x, c_1, \dots, c_n)$, $i=1, \dots, n$, 对所有变量连续可微而 $\frac{\partial(\varphi_1, \dots, \varphi_n)}{\partial(c_1, \dots, c_n)}|_{x=x_0} = 1$, 由隐函数定理,

可取解 $c_i = \phi_i(x, y_1, \dots, y_n)$ ($i=1, 2, \dots, n$)

且中 ϕ_i 在 P_0 某个邻域 G_0 连续可微, 且 $\frac{\partial(\phi_1, \dots, \phi_n)}{\partial(y_1, \dots, y_n)} \neq 0$.

故 $C_i = \phi_i(x, y_1, \dots, y_n)$ 为 n 个相互独立的首次积分

④ 往证: ϕ_1, \dots, ϕ_n 为 G_0 区域内 n 个相互独立的首次积分.

则该区域内任何一个首次积分都可表示为 $V = H(\phi_1, \dots, \phi_n)$

且中 $H: \mathbb{R}^n \rightarrow \mathbb{R}$ 连续可微

• 由于 $\frac{\partial(\phi_1, \dots, \phi_n)}{\partial(y_1, \dots, y_n)} \neq 0$, 由隐函数定理可得: 有解

$$y_i = y_i(x, \phi_1, \dots, \phi_n), \quad i=1, 2, \dots, n$$

设 V 是任意一个首次积分, 定义函数 $H: H(x, \phi_1, \dots, \phi_n) = V(x, y_1, \dots, y_n)$

下证 H 与 x 无关:

$$\begin{aligned} \frac{\partial H}{\partial x} &= \frac{\partial V}{\partial x} + \sum_{i=1}^n \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial x}, \text{ 由隐函数定理: } \frac{\partial y_i}{\partial x} = -\frac{1}{J} \frac{\partial(\phi_1, \phi_2, \dots, \phi_n)}{\partial(y_1, \dots, x, \dots, y_n)} \\ \Rightarrow \frac{\partial H}{\partial x} &= \frac{\partial V}{\partial x} - \frac{1}{J} \sum_{i=1}^n \frac{\partial V}{\partial y_i} \frac{\partial(\phi_1, \dots, \phi_i, \dots, \phi_n)}{\partial(y_1, \dots, x, \dots, y_n)} = \frac{1}{J} \frac{\partial(V, \phi_1, \dots, \phi_n)}{\partial(x, y_1, \dots, y_n)} \end{aligned}$$

由首次积分充要条件: $\frac{\partial V}{\partial x} + \sum_{i=1}^n \frac{\partial V}{\partial y_i} f_i = 0$

可得 $\frac{\partial(V, \phi_1, \dots, \phi_n)}{\partial(x, y_1, \dots, y_n)} = 0 = \frac{\partial H}{\partial x} \Rightarrow V = H(\phi_1, \dots, \phi_n) \quad \square$

综上: n 维微分系统在 G 内最多有 n 个函数独立的首次积分



3. (1) 必要性：已知其为首次积分，由首次积分为定义

该式任一解 $x_i = \psi_i(t)$ 代入 $F(t, x_1, \dots, x_n) = C$

~~② 充分性：~~ $0 \equiv \frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^n \frac{\partial F}{\partial x_i} f_i(t, \psi_1, \dots, \psi_n)$

$$= \frac{\partial F}{\partial t} + \sum_{i=1}^n \frac{\partial F}{\partial x_i} f_i(t, x_1, \dots, x_n)$$

(1) 充分性：由于 $\forall x_i \in G$, $\frac{\partial F}{\partial t} + \sum_{i=1}^n \frac{\partial F}{\partial x_i} f_i(t, x_1, \dots, x_n) = 0$

F 在任意方向偏导数恒为 0, 故 $\frac{\partial F}{\partial t} = 0$, $F(t, x_1, \dots, x_n) = C$

4. (1) $S(t) + I(t) + R(t) = C_1$ (三式做和)

(2) $I(t) = \frac{1}{P} \ln(S(t)) - S(t) + C_2$ (② 式以 ①)

5. $H(x, y) = C_1$ (① 式乘 $\frac{dy}{dt}$, ② 式乘 $\frac{dx}{dt}$)



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6. 1-(1) $y' = \frac{2u-x}{2x-y}$, 令 $y=ux$, 其中 u 为关于 x 的函数

$$x \frac{du}{dx} + u = \frac{2u-1}{2-u}, \text{ 即 } \frac{2-u}{u^2-1} du = \frac{dx}{x}, \text{ 积分得: } \ln|\frac{2u}{u^2-1}| - \frac{1}{2} \ln|u^2-1| = \ln|x| + C$$

还原变量并化简得: $(y-x) = c(x+y)^{\frac{1}{2}}$

$$1-(3) \text{ 设 } v = x+2y, \text{ 则 } \frac{dv}{dx} = 1+2 \frac{dy}{dx} = 1+2 \frac{v+1}{v-1}$$

$$\text{整理得: } \frac{dv}{dx} = \frac{4v+1}{v-1} \Rightarrow (2v-1)dv = (4v+1)dx \Rightarrow \frac{2v-1}{4v+1} dv = dx$$

$$\frac{1}{2}v^2 - \frac{3}{8} \ln|4v+1| = x + C \Rightarrow 8y - 4x - 3 \ln|4x+8y+1| = C$$

$$7. 2-(2) \text{ 左右同乘 } u: (3u^2v + uv^2)du + (u^3 + u^2v)dv = 0$$

$$\text{即: } (3u^2v du + u^3 dv) + (uv^2 du + u^2v dv) = 0$$

$$\Rightarrow u^3v + \frac{1}{2}u^2v^2 = C$$

$$2-(3) \text{ 原方程可化为: } \frac{2ydy}{2x+3x} = \frac{4y^2-2x^2}{x+y^2+3}, \text{ 令 } x^2 = v, y^2 = u$$

$$\text{则 } \frac{du}{dv} = \frac{4u-2v}{u+v+3}; \text{ 由 } \begin{cases} 4u-2v=0 \\ u+v+3=0 \end{cases} \Rightarrow \begin{cases} u=-1 \\ v=-2 \end{cases}$$

$$\text{令 } \begin{cases} m=u+1, \text{ 则 } \frac{dm}{dn} = \frac{4m-2n}{m+n} \\ n=v+2 \end{cases}, \text{ 令 } \frac{m}{n}=z, \text{ 则 } m=zn, \frac{dm}{dn} = \frac{dz}{dn}n+z = \frac{4z-2}{z+1}$$

故有: $\frac{dz}{dn}n = \frac{(1-z)(z-2)}{z+1}$, 此方程为分离变量方程

$$\frac{1}{n}dn = \frac{z+1}{(1-z)(z-2)} dz \Rightarrow \ln|\frac{(z-1)^2}{(z+2)^2}| = C + \ln n \Rightarrow (x^2 - y^2 + 1)^2 = C(-2x^2 + y^2 - 3)^2$$

$$8. 3-(1) y' = -y^2 - \frac{1}{4x^2}, \text{ 令 } z = xy, \frac{dz}{dx} = \frac{1}{x}(-z^2 + z - \frac{1}{4})$$

$$\text{① } z \neq \frac{1}{2} \text{ 时, 分离变量 } \frac{-1}{(z-\frac{1}{2})^2} dz = \frac{1}{x} dx \Rightarrow \frac{1}{z-\frac{1}{2}} = \ln|x| + C$$

还原变量并化简得: $y = \frac{1}{2x} + \frac{1}{x \ln|x| + cx}$

② $z = \frac{1}{2}$, $y = \frac{1}{2x}$ 为一特解

$$3-(2) xy' = x^2y^2 + xy + 1, \text{ 即为: } y' = y^2 + \frac{y}{x} + \frac{1}{x^2}$$

令 $z = xy$, 则 $\frac{dz}{dx} = \frac{1}{x}(z+1)^2$. 可分离变量.

$$-\frac{1}{z+1} = \ln|x| + C \Rightarrow y = -\frac{1}{x} - \frac{1}{x \ln|x| + cx}$$



$$9-4. y'' + p(x)y' + q(x)y = 0$$

$$\text{令 } y = e^{\int p(x)dx}, \quad y' = u e^{\int p(x)dx}, \quad y'' = u^2 e^{\int p(x)dx} + u' e^{\int p(x)dx}$$

$$\text{则 } y'' + p(x)y' + q(x)y = u^2 e^{\int p(x)dx} + p(x)u e^{\int p(x)dx} + q(x)e^{\int p(x)dx} = 0 \\ + u' e^{\int p(x)dx}$$

$$\text{即有: } u^2 + u' + p(x)u + q(x) = 0 \quad \square.$$

5. 设曲线 $y = y(x)$. 则:

$$\frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{dy}{dx} \frac{y}{x}} = 1 \Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y} \text{ 齐次方程.}$$

$$\text{解得 } \arctan \frac{y}{x} - \frac{1}{2} \ln(x^2 + y^2) = C$$

10.-4. 设 Q 的运动轨迹为 $y = y(x)$, 则 y 满足

$$\frac{dx}{dy} = \frac{y^{\frac{a}{b}} - y^{-\frac{a}{b}}}{2}, \quad x(1) = 0.$$

通过分离变量, 可得 $x = \frac{y^{\frac{a}{b}} - 1}{2(1 + \frac{a}{b})} - \frac{y^{(1-\frac{a}{b})} - 1}{2(1 - \frac{a}{b})}$

P,Q 在 T 时刻相遇时 $x(aT) = 0$,

$$\text{代入可得: } T = \frac{b}{2(b^2 - a^2)}$$

$$5. \frac{g m M}{R^2} = \frac{m v_0^2}{R} \Rightarrow v_0 = 7.94 \text{ km/s}$$



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