解析函数

Domain coloring: [Complex Function Plotter (samuelj.li)]

1. Analytic function. 解析函数

<u>Defn</u>: $f: \Omega^{\text{open}} \subseteq \mathbb{C} \to \mathbb{C}$ is called analytic in Ω. if

$\forall\ z_0 \in \Omega$. $\displaystyle\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. which is denoted by $f'(z_0) \in \mathbb{C}$.

[$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall\ 0 < |z-z_0| < \delta$, we have

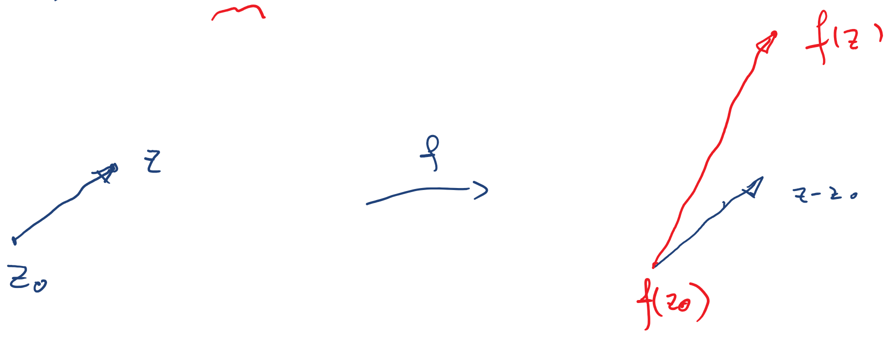
$$\left| \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \right| < \varepsilon$$]

If f is analytic in Ω. $\forall\ z_0 \in \Omega$. we have

$$f(z)-f(z_0) = f'(z_0)(z-z_0) + o(z-z_0) \quad \text{as } z \to z_0$$

[$o(z-z_0)$: $\displaystyle\lim_{z \to z_0} \frac{o(z-z_0)}{z-z_0} = 0$]

$$\Rightarrow \quad f(z)-f(z_0) \simeq f'(z_0) \cdot (z-z_0)$$



multiplying $f'(z_0)$ $\Leftrightarrow$ 拉伸, 旋转 (扭转)
Amplify, twist (伸扭)

f analytic $\Rightarrow$ $f'(z_0)$ is a 伸扭 on infinitesimal level.

2. Cauchy-Riemann equation

$f: \mathbb{C} \to \mathbb{C}$

$z \to (a+bi)\cdot z$.

$f(z) = (a+bi)\, z$.

$f'(z) = (a+bi)$

regarded as a map from $\mathbb{R}^2 \to \mathbb{R}^2$

$\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \longrightarrow (\underline{ax-by}, \underline{bx+ay})$

$$J\tilde{f} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}_{2\times 2}$$

$f: \Omega \to \mathbb{C}$

$f'(z_0)$

$\tilde{f}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\qquad f(z) = u(x,y) + i\,v(x,y)$

$(x,y) \to (u(x,y), v(x,y))$

$$\underline{J_{z_0}}\tilde{f} = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$$

If f is analytic, then near z_0

$f(z) - f(z_0) \in \mathbb{R}^2$

$f(z) - f(z_0) \simeq f'(z_0)(z - z_0)$

$$\longleftrightarrow \begin{pmatrix} \mathrm{Re}(f(z)-f(z_0)) \\ \mathrm{Im}(f(z)-f(z_0)) \end{pmatrix} \simeq J_{z_0}\tilde{f} \begin{pmatrix} \mathrm{Re}(z-z_0) \\ \mathrm{Im}(z-z_0) \end{pmatrix}$$

Prop 1. If f is analytic in Ω, then $f = u + iv$ satisfies

$$\boxed{\partial_x u = \partial_y v, \qquad \partial_y u = -\partial_x v}.$$

Cauchy-Riemann equation.

Prop 2. If $f = u + iv$ defined in Ω satisfies

$\partial_x u = \partial_y v$, $\partial_y u = -\partial_x v$, and u, v are continuous differentiable.

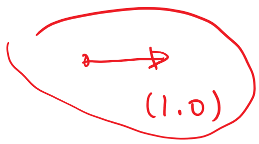
then f is analytic in Ω.

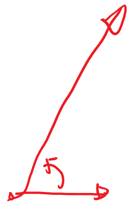
then f is analytic in Ω.

Observation: If f is analytic, then $f'(z) = \partial_x u + i(\partial_x v)$.

$$f'(z) = \partial_y v - i(\partial_y u)$$

example $$(e^z)'\Big|_{z=z_0} = \lim_{z \to z_0} \frac{e^z - e^{z_0}}{z - z_0}$$





$$e^z = e^x e^{iy} = e^x \cos y + i e^x \sin y.$$

$u(x,y) = e^x \cos y$, $v(x,y) = e^x \sin y$ $\Rightarrow$ $\partial_x u \overset{?}{=} \partial_y v$ ✓

$\partial_y u \overset{?}{=} -\partial_x v$ ✓

Prop 2 $\Rightarrow$ e^z is analytic.

$$\Rightarrow (e^z)' = \partial_x u + i\,\partial_x v = e^x \cos y + i e^x \sin y = e^z.$$

$$e^{iz} = \cos(z) + i\sin(z)$$

$\Rightarrow$ $(e^{iz})' = i \cdot e^{iz} = i(\cos(z) + i\sin(z)) = -\sin(z) + i\cos(z)$ $\Big\}$ $\overset{?}{\Rightarrow}$

$(R.H.S)' = (\cos(z))' + i(\sin(z))'$

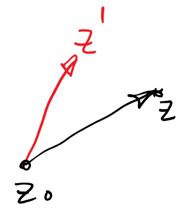
$$(\cos(z))' = -\sin(z).$$
$$(\sin(z))' = \cos(z).$$

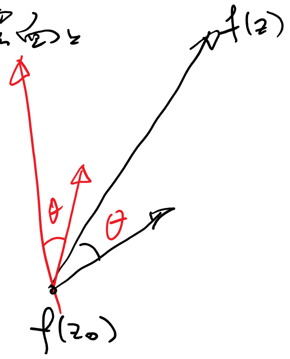
$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad (\cos(z))' = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin(z)$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \Rightarrow (\sin(z))' = \cos(z)$$

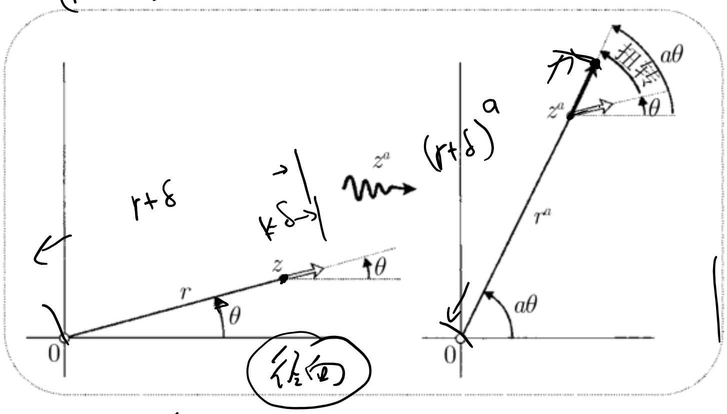
$$(z^n)'$$

f analytic. $f'(z_0)$ 伸扭. 在无穷小尺度上 $(z\to z_0)$





$$z^a = (re^{i\theta})^a = r^a e^{ia\theta}$$



扭转角度.

$a\theta - \theta = (a-1)\theta$.

$|\nearrow| = (r+\delta)^a - r^a$

伸缩比: $\lim_{\delta\to 0}\dfrac{(r+\delta)^a - r^a}{\delta}$

$$(x^a)'\big|_{x=r} = a r^{a-1}$$

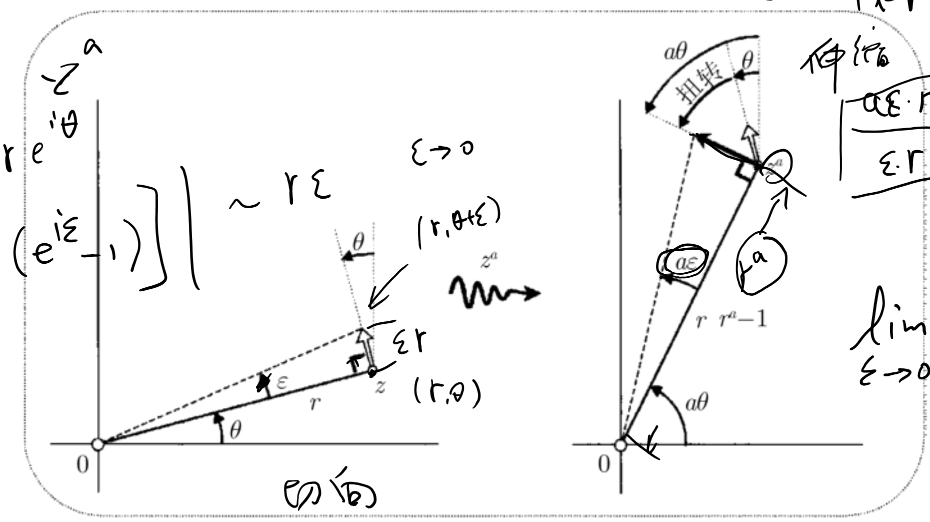
$$f'(z) = a r^{a-1}\cdot e^{i(a-1)\theta} = a z^{a-1}$$

z^a

$re^{i(\theta+\varepsilon)} - re^{i\theta}$

$= \left| r\left[e^{i\theta}(e^{i\varepsilon}-1)\right]\right| \sim r\varepsilon$

$\varepsilon\to 0$



伸缩

$$\frac{a\varepsilon\cdot r^a}{\varepsilon\cdot r} = a\cdot r^{a-1}$$

$$\lim_{\varepsilon\to 0}\frac{\left|r^a e^{i(\theta+\varepsilon)a} - r^a e^{ia\theta}\right|}{\left|re^{i(\theta+\varepsilon)} - re^{i\theta}\right|} = a r^{a-1}.$$

e^z

起点 $x+i\theta$. 终点 $x+i(\theta+\delta)$

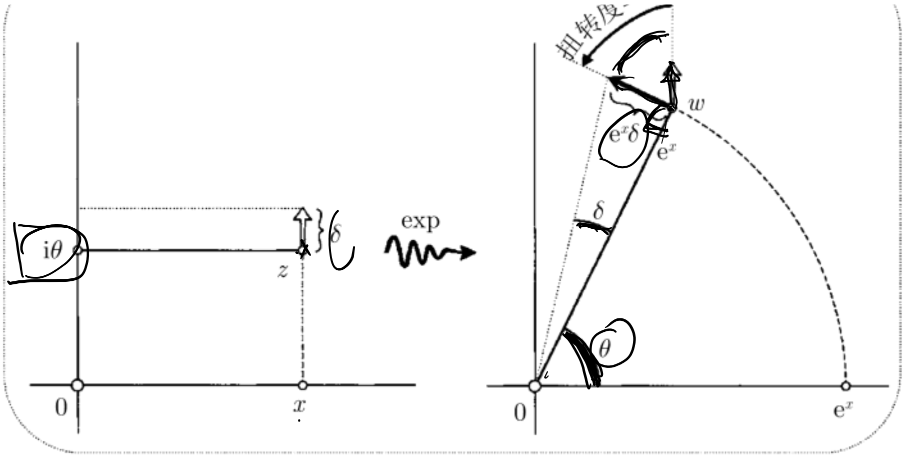
起点. $e^{x+i\theta}$ 终点 $e^{x+i(\theta+\delta)}$

$|\nwarrow| \sim \delta\cdot e^x$

as $\delta\to 0$

伸缩. δe^x

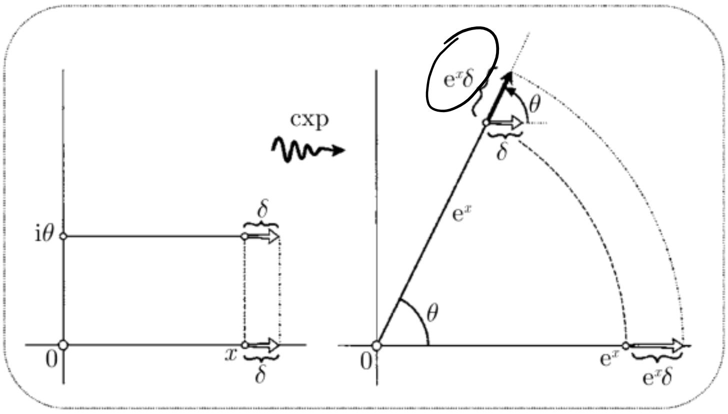
x



伸缩: $\frac{\delta e^x}{\delta} = e^x$

扭转: θ

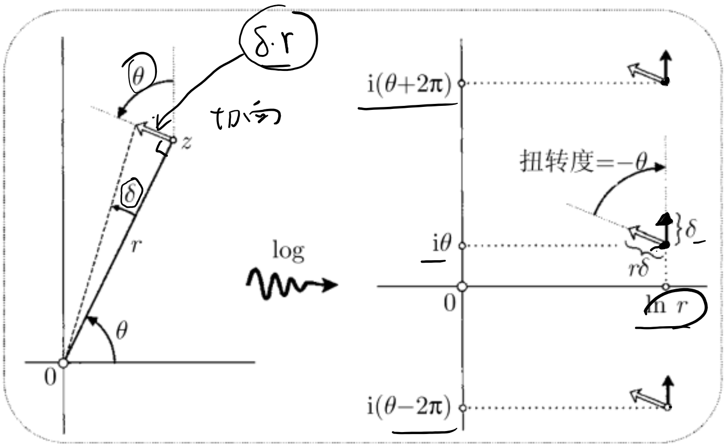
$f'(z) = e^x \cdot e^{i\theta}$
$= e^z$.



扭转: θ

$\underline{e^{x+\delta} - e^x} \underset{\delta\to 0}{\sim} e^x\cdot\delta$

伸缩 $\frac{e^x\cdot\delta}{\delta} = e^x$.



扭转度 $= -\theta$

伸缩度 $= \frac{\delta}{r\delta} = \frac{1}{r}$

$(\log(z))' = \frac{1}{r}\cdot e^{-i\theta}$
$= \frac{1}{z}$.

几何上看 $f'(z)$ = 伸扭 $(\delta\to 0)$

就是取过 z 处一段非常小的向量，观测其在 $f(z)$ 作用下产生的伸扭作用.

通常取 x 方向, y 方向, 径向 (r 方向)

切向 (的方向)

Observation: Analytic function $\Rightarrow$ conformal map.

(保持定向角度的映射)

$f(z)-f(z_0)\simeq \boxed{f'(z_0)}\cdot(z-z_0)$

$f(z')-f(z_0)\simeq \boxed{f'(z_0)}\cdot(z'-z_0)$

3. Conformal = Analyticity.