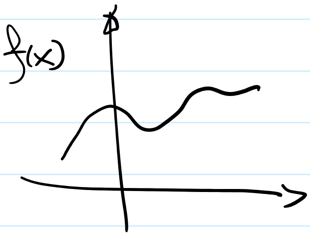
Complex function

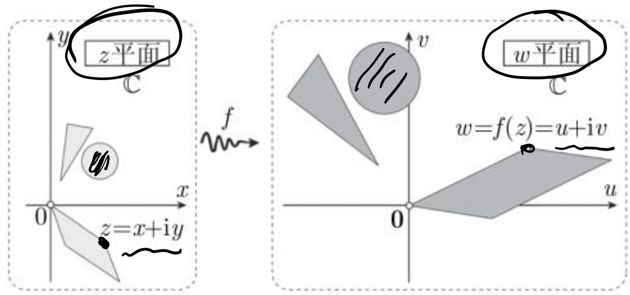
1. 复函数

$$z \in \mathbb{C} \xmapsto{\ f\ } w \in \mathbb{C}$$

$y=f(x)$



$$w=f(z)=u(z)+iv(z)$$



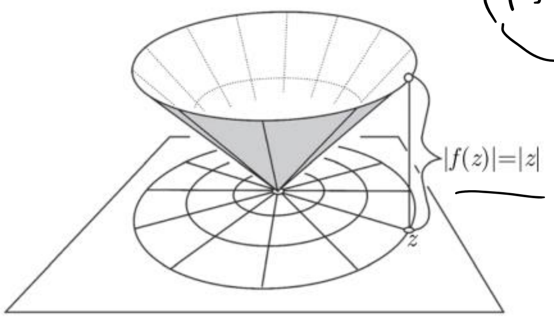
$\mathrm{Arg}(f(x))$

2. 模函数.

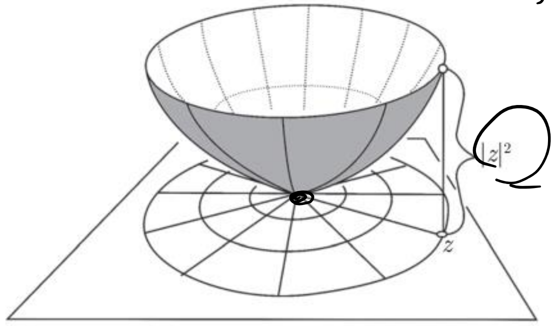
$f(z)=z$

$|f(z)|=|z|=\sqrt{x^2+y^2}$

$f(z)=z^2$



圆锥

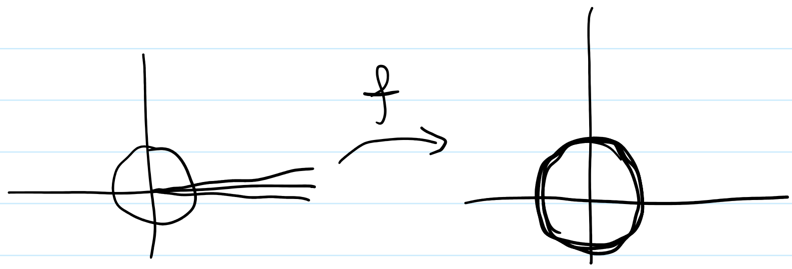


抛物面

3. $f(z)=z^n$. power function

$$z=re^{i\theta}$$

$$w=f(z)=r^n e^{in\theta}$$



4. 圆锥曲线

椭圆：到两点距离之和为定值.

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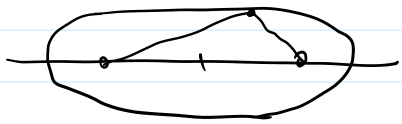
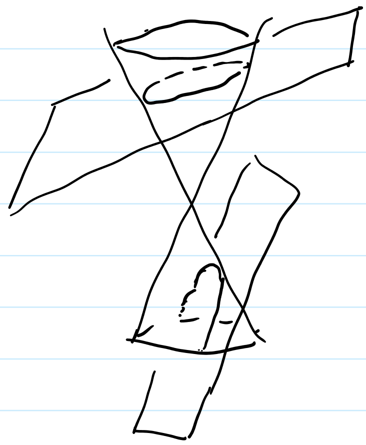
$$|z-a_1| + |z-a_2| = L.$$



圆锥曲线的一种.



$(L > |a_1-a_2|)$

卡西尼曲线：到两点距离之积为定值.

$$|z-a_1|\cdot|z-a_2| = k^2 \qquad (k>0)$$

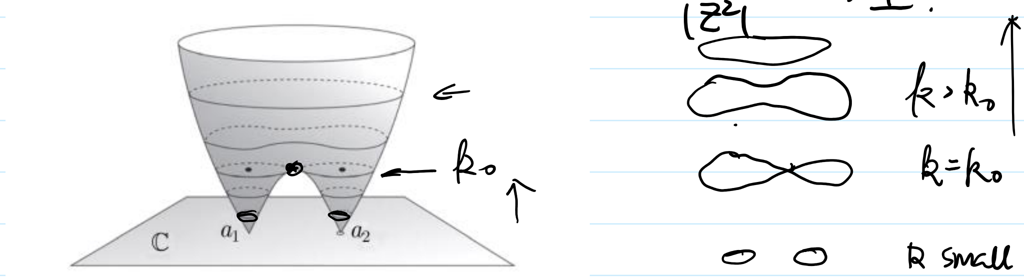
考虑 f 的模函数： $f(z)=(z-a_1)(z-a_2)$.

$$|f(z)|=|z-a_1||z-a_2|$$

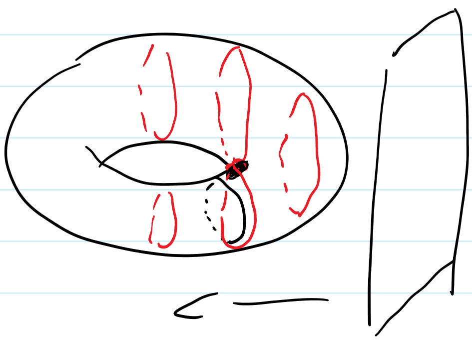
l: a_1 —)| ~ - - , ,

$z\to\infty$ 指 $|z|\to\infty$

1. $z\to a_1$ $\quad \dfrac{|f(z)|}{|z-a_1|}\to|a_1-a_2|$.
2. $z\to a_2$ $\quad \dfrac{|f(z)|}{|z-a_2|}\to|a_1-a_2|$
3. $z\to\infty$ $\quad \dfrac{|f(z)|}{|z^2|}\to 1$.



卡西尼曲线：环面截线.



4. 幂级数 (Power series)

Defn. (无穷多项式)

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots \qquad = \sum_{n=0}^{+\infty} a_n z^n . \qquad z \in \mathbb{C}$$

$a_n \in \mathbb{C}$.

let $\quad P_n(z) = \sum_{k=0}^{n} a_k z^k$

Defn.

$P(z)$ is said to be convergent at z

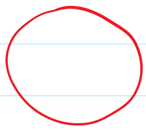
if $\forall \varepsilon > 0$. $\exists N$. s.t. $\forall n, m \geq N$. $\quad |P_n(z) - P_m(z)| < \varepsilon$.

Thm given $P(z)$. $\exists R$ (radius of convergence) $(0 \leq R \leq \infty)$

s.t. $P(z)$ converges absolutely for $|z| < R$.

$P(z)$ diverges for $|z| > R$

$|z| = R$ 一切皆有可能



compact set

有界闭集.

Moreover, for any compact set $K \subset |z| < R$.

$P(z)$ converges uniformly on K. [内闭一致收敛].

$\exists$ uniform $\delta > 0$ s.t.

$|z| \leq R - \delta \quad \forall z \in K$.

Defn. $P(z)$ is said to converge absolutely at z if

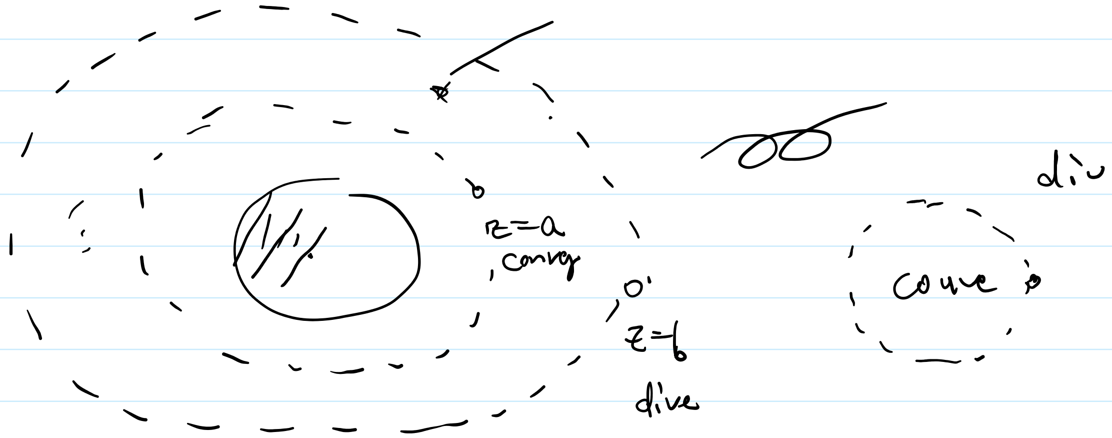
$$\sum_{n=0}^{\infty} |a_n| |z|^n < +\infty$$

On existence of radius of convergence.

1. if $P(z)$ converges at $z = a$, then $P(z)$ converges for $|z| < |a|$.

2. If $P(z)$ diverges at $z = a$. then $P(z)$ diverges for $|z| > |a|$.

2. If $P(z)$ diverges at $z=a$, then $P(z)$ diverges for $|z|>|a|$.



Proof of 1. $(m>n)$

$$|P_n(z)-P_m(z)| = |a_{n+1}z^{n+1}+\cdots+a_m z^m|$$

$$= \Big|\underbrace{a_{n+1}a^{n+1}}\left(\frac{z}{a}\right)^{n+1}+\cdots\underbrace{a_m a^m}\cdot\left(\frac{z}{a}\right)^m\Big|$$

Since $P(a)$ converges. $\leq M\left(\left|\frac{z}{a}\right|^{n+1}+\cdots+\left|\frac{z}{a}\right|^m\right) < \varepsilon$ provide $n \geq N$.

$\Rightarrow \exists M$ s.t.

$|a_k a^k| \leq M$, for $k >> 1$ $\quad$ (Since $\left|\frac{z}{a}\right| = \rho < 1$)

Prop. $P(z) = \sum_{n=0}^{\infty} a_n z^n$. then

① $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ $\quad$ 比值判别法. ~ 等比级数

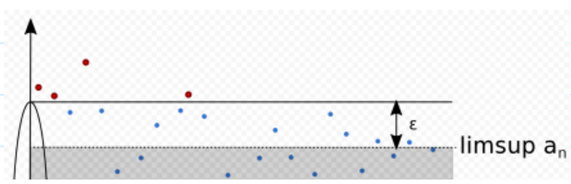
② Cauchy-Hadamard

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

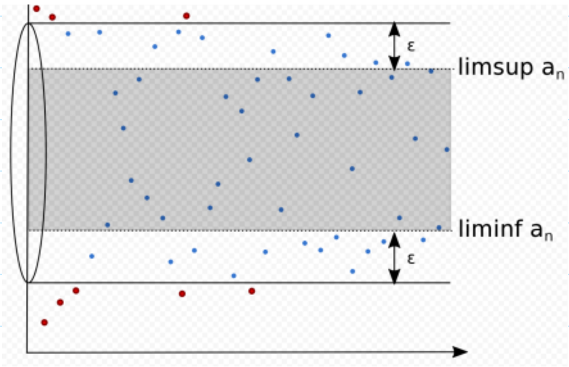
Claim: for $|z|<R$ $P(z)$ converges absolutely.

Proof: $\forall \varepsilon > 0$. $\exists$ finite many terms s.t. $\underbrace{\sqrt[n]{|a_n|}} \geq \underbrace{\limsup \sqrt[n]{|a_n|}} + \varepsilon$

$$= \frac{1}{R} + \varepsilon.$$



$\Rightarrow |a_n| \leq \left(\frac{1}{R}+\varepsilon\right)^n$ for n sufficient large.



$\rightarrow |a_n| \le (\frac{1}{R}+\varepsilon)$ for n sufficient large.

$$|a_n z^n| \le (R-\delta)^n \cdot (\frac{1}{R}+\varepsilon)^n = \rho^n$$

fix $|z|<R$ suppose $|z|=R-\delta$

$\exists\, \varepsilon$ s.t. $(R-\delta)(\frac{1}{R}+\varepsilon)=\rho<1$.

Weierstrass M-test. Suppose that (f_n) is a sequence of real- or complex-valued functions defined on a set A, and that there is a sequence of non-negative numbers (M_n) satisfying the conditions

- $|f_n(x)| \le M_n$ for all $n \ge 1$ and all $x \in A$, and
- $\sum_{n=1}^{\infty} M_n$ converges.

Then the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges absolutely and uniformly on A.

6. 指数函数

定义 $e^z := 1+z+\frac{z^2}{2!}+\cdots\cdots$ $z\in\mathbb{C}$ C-H $(R=\infty)$.

$\forall\, z, w.$ $e^z \cdot e^w = e^{z+w}$.

$(\quad)(\quad) = (\quad)$

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

定义

- $\cos(z) = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots$ $R=\infty$
- $\sin(z) = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots$ $R=\infty$

Euler 公式

$$e^{iz} = \cos(z) + i\sin(z). \qquad z \in \mathbb{C}.$$

$\cos(3+4i)$

$$e^{-iz} = \cos(-z) + i\sin(-z)$$

$$= \cos(z) - i\sin(z)$$

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz}-e^{-iz}}{2}.$$