零、恒等定理的应用:实恒等式的复推广

In real analysis and complex analysis, branches of mathematics, the **identity theorem** for analytic functions states: given functions f and g analytic on a domain D (open and connected subset of $\mathbb{R}$ or $\mathbb{C}$), if $f = g$ on some $S \subseteq D$, where S has an accumulation point, then $f = g$ on D.

解析（无穷）⇒ 刚性地生长.

例1: $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \cdots$$

identity theorem ⇒ e^z is the unique analytic function defined on $\mathbb{C}$ whose restriction on $\mathbb{R}$ is e^x

例2: $\sin^2(x) + \cos^2(x) = 1$.

⇒ $\sin^2(z) + \cos^2(z) = 1$.

$\begin{cases} ① \ \sin^2(z) + \cos^2(z) - 1 = f(z) \text{ is analytic on } \mathbb{C} \\ ② \ f(z)\big|_{\mathbb{R}} \equiv 0 \end{cases}$

⇒ $f(z) \equiv 0$ on $\mathbb{C}$.

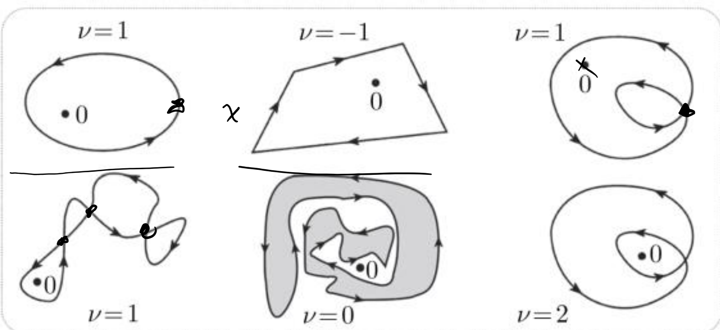
一、环绕数 winding number.

Defn: let γ be a closed curve in $\mathbb{C}$, with a given orientation.

$\forall p \in \mathbb{C}$.

$$\nu(\gamma, p) := \frac{\{\text{Set } p \text{ as origin. net change of } \gamma\text{'s argument}\}}{2\pi}$$

$$= \left[\arg(\gamma)\Big|_{\text{initial}}^{\text{terminal}}\right] \Big/ 2\pi .$$



简单 simple closed curve.

p 外面 . $\nu(\gamma, p) = 0$

一个拓扑插曲：若当曲线定理

A *Jordan curve* or a *simple closed curve* in the plane $\mathbf{R}^2$ is the image C of an injective continuous map of a circle into the plane, $\varphi: S^1 \to \mathbf{R}^2$.

Theorem — Let C be a Jordan curve in the plane $\mathbf{R}^2$. Then its complement, $\mathbf{R}^2 \setminus C$, consists of exactly two connected components. One of these components is bounded (the **interior**) and the other is unbounded (the **exterior**), and the curve C is the boundary of each component.

FACT: let γ be a simple closed curve. then

$$\begin{cases} \nu(\gamma, p) = 0 & \text{if } p \text{ is in the exterior of } \gamma. \\ \nu(\gamma, p) = \pm 1 & \text{if } p \text{ is in the interior of } \gamma \end{cases}$$

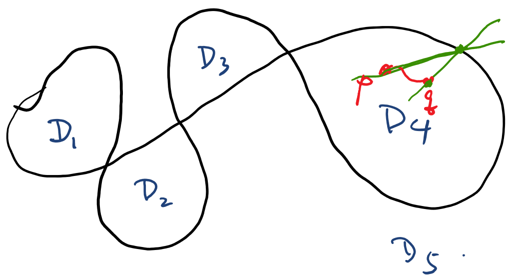
环绕数的主要性质

1. 局部常数: If p, q belongs to the same connected component of $\mathbb{C} \setminus \gamma$, then $\nu(\gamma,p) = \nu(\gamma, q)$.
2. 穿越法则

locally constant means

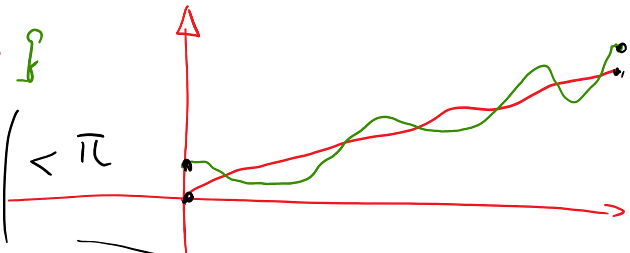
$\forall\, p, q \in D_i$ for same i.

then $\nu(\gamma, p) = \nu(\gamma, q)$.



Observation: if p is very close to q

$$\left| \arg(\gamma, p)\Big|_{\text{initial}}^{\text{terminal}} - \arg(\gamma, q)\Big|_{\text{initial}}^{\text{terminal}} \right| < \pi$$

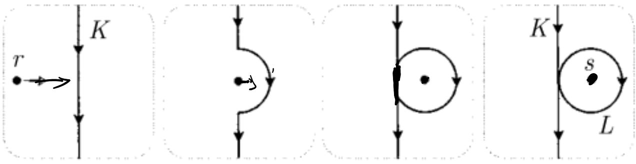


$\Longrightarrow \nu(\gamma, p) = \nu(\gamma, q)$

$\Downarrow$

locally constant.

穿越法则: 当我们即将穿越 K 时，如果 K 的方向是从我们的左侧指向我们的右侧（从我们的右侧指向我们的左侧），则环绕数增加 1（减少 1）.



$v(K,r) = v(K\sqcup L, s)$

$= v(K,s) + v(L,s)$

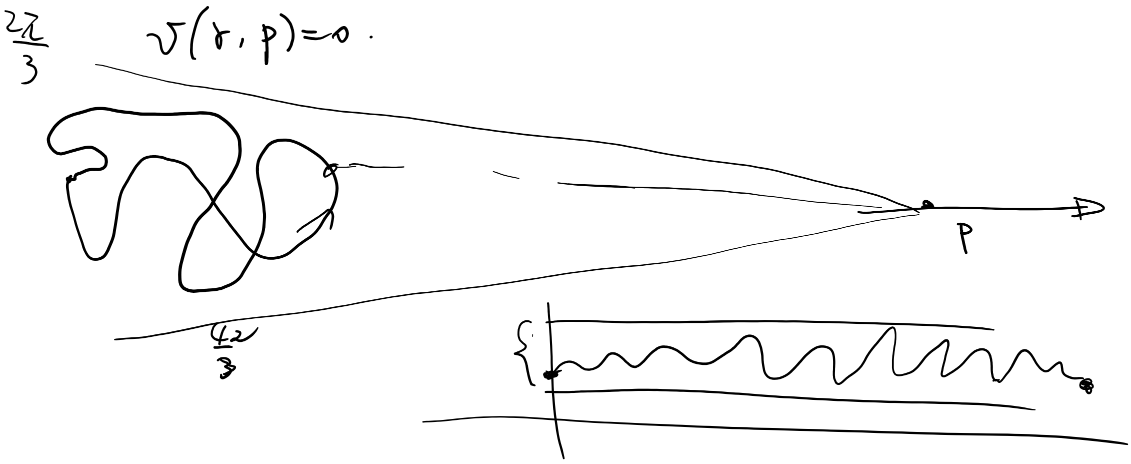
$= v(K,s) - 1.$ $\Rightarrow v(K,s) = v(K,r) + 1.$

$v(K\cup L, s) = \frac{T-I}{2\pi}$

$v(L,s) = \frac{B-A}{2\pi}$ $\quad v(K,s) = \frac{T+A-B-I}{2\pi}$ $\Rightarrow v(K\sqcup L, s) = v(K,s) + v(L,s)$

Observation. let p be far "away from" γ

$v(\gamma, p) = 0.$

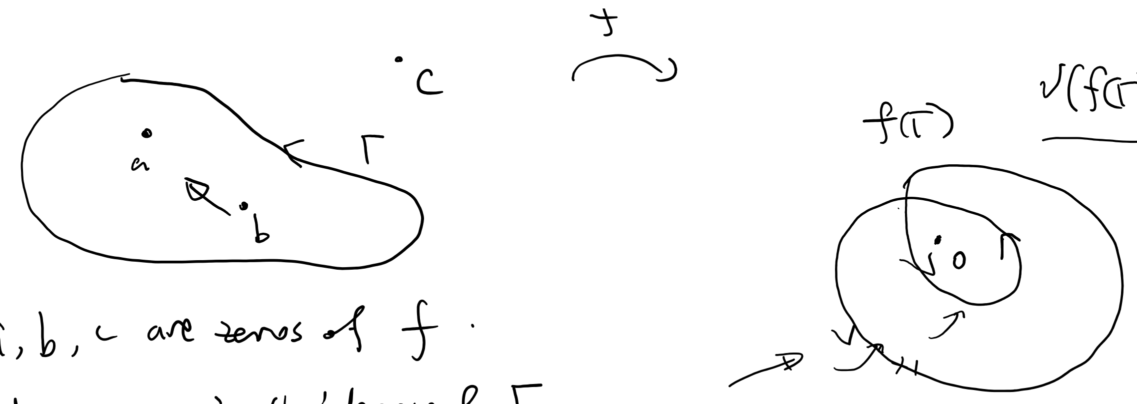


二、辐角原理

1. Motivation example

$f(z) = (z-a)(z-b)(z-c)$

$\cdot c$ $\quad \overset{f}{\curvearrowright}$ $\quad v(f(\Gamma), 0) = ?$

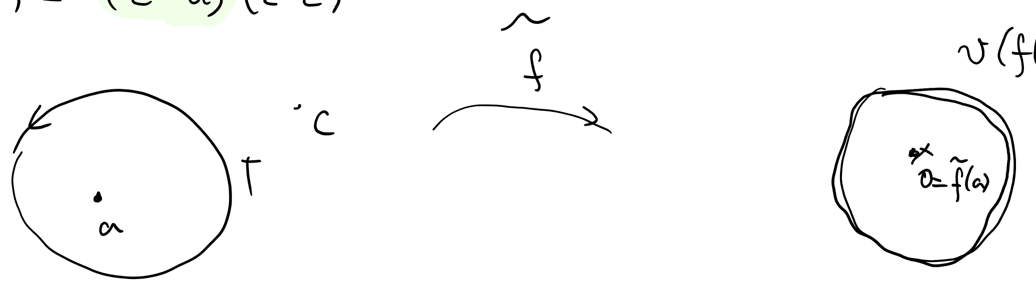


$\nu(f(\Gamma), 0) = 2$.

a, b, c are zeros of f.

and a, b are in the interior of Γ.

$$\tilde{f}(z) = (z-a)^2 (z-c)$$



$\nu(f(\Gamma), 0) = 2$

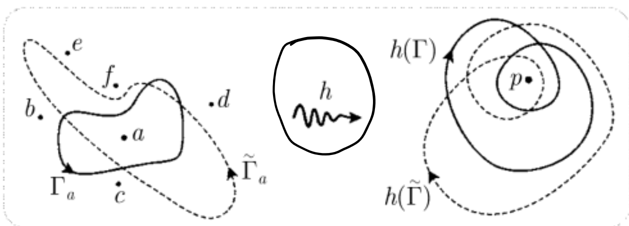
1. Statement of Argument principle

辐角原理.

若 $f(z)$ 在一个简单环路 Γ 的内域和在 Γ 上均为解析, 而 N 为 Γ 内域中的 p-点数目 [按重数计], 则 $N = \nu[f(\Gamma), p]$.

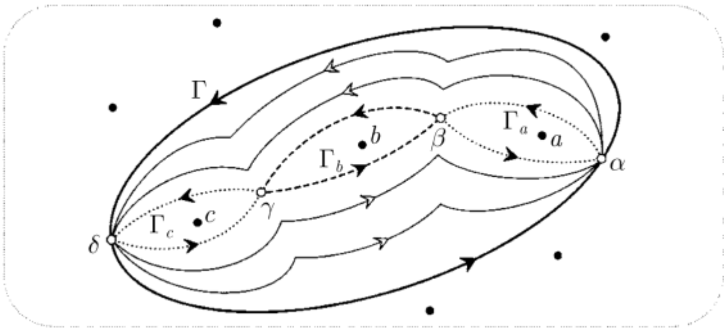
if $f(z) = p$, then z is called a p-点

1. Topological argument principle and its proof.



拓扑辐角原理. $h: \mathbb{R}^2 \to \mathbb{R}^2$ continuous map. . Γ simple closed curve.

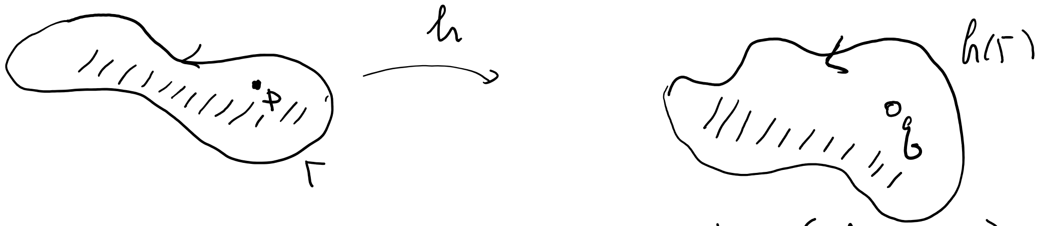
在 Γ 的内域的 p-点的总数 (每个 p-点均按其拓扑重数计算) 等于 $h(\Gamma)$ 绕 p 的环绕数.



三、辐角原理的应用

1. 达布定理（Darboux）:　Γ is a simple closed curve.

若一解析函数 $h(z)$ 把 Γ 一一地映为 $h(\Gamma)$, 则它也把 Γ 的内域一一地映到 $h(\Gamma)$ 的整个内域中.



Key: $h(\Gamma)$ is also a simple closed curve.!　(h is 1-1)

$\forall\, q \in h(\Gamma)$ 的内域

$$\nu(h(\Gamma), q) = \pm 1 = \begin{cases} 1 & \text{if } \Gamma \text{ is counterclockwise} \\ -1 & \text{if } \Gamma \text{ is clockwise} \end{cases}$$

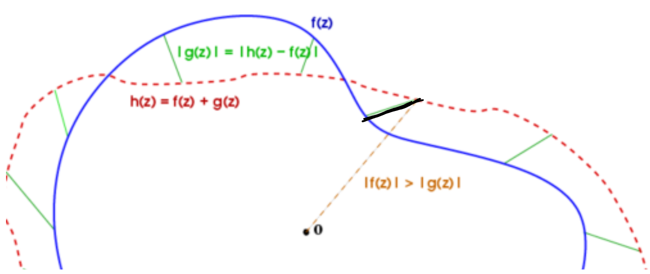
$\|$

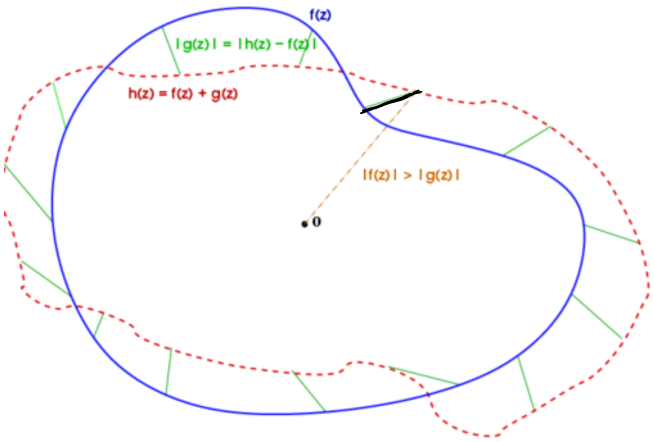
Γ 中 q 点个数 $\Rightarrow \exists!\, p \in \Gamma$ 的内域 s.t. $h(p) = q$.

2.鲁歇定理 (Rouche)　Let f, g be two analytic functions. Γ is a simple closed curve.

若在 Γ 上 $|g(z)| < |f(z)|$, 则在 Γ 内 $(f+g)$ 与 f 零点个数相同.

注: 计重数.





蓝线：溜狗的人的路线。 到原点(树) $|f(z)|$

绿：狗的绳子。 长度 $|g(z)|$

$$|f(z)| > |g(z)|$$

红色虚线：狗的轨迹 $\underline{f(z)+g(z)}$

$$\Rightarrow \quad \underbrace{\nu(f(\Gamma), 0)}_{\#\text{ zero's in }\Gamma \text{ of } f} = \underbrace{\nu((f+g)(\Gamma), 0)}_{\#\text{ zero's in }\Gamma \text{ of } f+g} \Longleftarrow \underset{z\in\Gamma}{|f(z)| > |g(z)|}$$

Corollary: (代数基本定理)

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

has n-roots. in $\mathbb{C}$.

Rouché

$$\underline{f(z) = a_n z^n}. \qquad g(z) = a_{n-1} z^{n-1} + \cdots + a_0$$

Let Γ be a circle of radius R

exercise if R is sufficiently large.

$$|a_n| R^n = |a_n z^n| > |g(z)| \quad \text{on} \quad \{|z| = R\} =: \Gamma$$

Rouché $\Rightarrow$ $P = f+g$ has same number of roots as f.