

2. 47 证: $P_0 = (x_0, y_1^0, \dots, y_n^0) \in G$, 则上述微分系统在 P_0 某个邻域 $G_0 \subset G$ 内有几个相互独立的首次积分.

• 考虑初值条件: $y_i(x_0) = c_i \ (i=1, 2, \dots, n)$

其中 (x_0, c_1, \dots, c_n) 在 P_0 某个邻域 G^* 中

由初值问题连续可微性定理: $y_i = \varphi(x, c_1, \dots, c_n), i=1, \dots, n$ 对所有变量连续可微
而 $\frac{\partial \varphi_i}{\partial c_i} \Big|_{x=x_0} = 1$, 由隐函数定理,

可取解 $c_i = \phi_i(x, y_1, \dots, y_n) \ (i=1, 2, \dots, n)$

其中 ϕ_i 在 P_0 某个邻域 G_0 连续可微, 且 $\frac{\partial(\phi_1, \dots, \phi_n)}{\partial(y_1, \dots, y_n)} \neq 0$.

故 $G_0 = \phi(x, y_1, \dots, y_n)$ 为 n 个相互独立的首次积分

47 证: ϕ_1, \dots, ϕ_n 为 G_0 区域内 n 个相互独立的首次积分,

则该区域内任何一个首次积分都可表示为 $V = H(\phi_1, \dots, \phi_n)$

其中 $H: \mathbb{R}^n \rightarrow \mathbb{R}$ 连续可微

• 由于 $\frac{\partial(\phi_1, \dots, \phi_n)}{\partial(y_1, \dots, y_n)} \neq 0$, 由隐函数定理可得: 有解

$y_i = y_i(x, \phi_1, \dots, \phi_n), i=1, 2, \dots, n$

设 V 是任意一个首次积分, 定义函数 $H: H(x, \phi_1, \dots, \phi_n) = V(x, y_1, \dots, y_n)$

下证 H 与 x 无关:

$\frac{\partial H}{\partial x} = \frac{\partial V}{\partial x} + \sum_{i=1}^n \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial x}$, 由隐函数定理: $\frac{\partial y_i}{\partial x} = -\frac{1}{J} \frac{\partial(\phi_1, \dots, \phi_n)}{\partial(y_1, \dots, y_n)}$

$\Rightarrow \frac{\partial H}{\partial x} = \frac{\partial V}{\partial x} - \frac{1}{J} \sum_{i=1}^n \frac{\partial V}{\partial y_i} \frac{\partial(\phi_1, \dots, \phi_n)}{\partial(y_1, \dots, y_n)} = \frac{1}{J} \frac{\partial(V, \phi_1, \dots, \phi_n)}{\partial(x, y_1, \dots, y_n)}$

由首次积分充要条件: $\frac{\partial V}{\partial x} + \sum_{i=1}^n \frac{\partial V}{\partial y_i} f_i = 0$

可得 $\frac{\partial(V, \phi_1, \dots, \phi_n)}{\partial(x, y_1, \dots, y_n)} = 0 = \frac{\partial H}{\partial x} \Rightarrow V = H(\phi_1, \dots, \phi_n) \quad \square$

综上: n 维微分系统在 G 内最多有 n 个函数独立的首次积分



2. (1) 必要性: 已知具为首次积分, 由首次积分定义

该式任一解 $x_i = \varphi_i(t)$ 代入 $F(t, x_1, \dots, x_n) = C$

~~(2) 充分性:~~ $0 \equiv \frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^n \frac{\partial F}{\partial x_i} f_i(t, \varphi_1, \dots, \varphi_n)$

$$= \frac{\partial F}{\partial t} + \sum_{i=1}^n \frac{\partial F}{\partial x_i} f_i(t, x_1, \dots, x_n)$$

(2) 充分性: 由于 $\forall x_i \in G$, $\frac{\partial F}{\partial t} + \sum_{i=1}^n \frac{\partial F}{\partial x_i} f_i(t, x_1, \dots, x_n) = 0$

F 任意方向偏导恒为 0, 故 $\frac{dF}{dt} \equiv 0$, $F(t, x_1, \dots, x_n) = C$

4. (1) $S(t) + I(t) + R(t) = C_1$ (三式做和)

(2) $I(t) = \frac{1}{\beta} \ln(S(t)) - S(t) + C_2$ (②除以①)

5. $H(x, y) = C_1$ (①式乘 $\frac{dy}{dx}$, ②式乘 $\frac{dx}{dy}$)



6.1(1) $y' = \frac{2y-x}{2x-y}$, 令 $y=ux$, 其中 u 为关于 x 的函数

$x \frac{du}{dx} + u = \frac{2u-x}{2-u}$, 即 $\frac{2-u}{u^2-1} du = \frac{dx}{x}$, 积分得: $\ln|\frac{1-u}{1+u}| - \frac{1}{2}\ln|u^2-1| = \ln|x| + C$

还原变量并化简得: $(y-x) = C(x+y)^3$

1(3) 设 $v = x+2y$, 则 $\frac{dv}{dx} = 1+2\frac{dy}{dx} = 1+2\frac{v-1}{2v-1}$

整理得: $\frac{dv}{dx} = \frac{4v+1}{2v-1} \Rightarrow (2v-1)dv = (4v+1)dx \Rightarrow \frac{2v-1}{4v+1} dv = dx$

$\frac{1}{2}v - \frac{3}{8}\ln|4v+1| = x+C \Rightarrow 8y-4x-3\ln|4x+8y+1|=C$

7. 2-(2) 左右同乘 u : $(3u^2v+uv^2)dv + (u^3+u^2v)du = 0$

即: $(3u^2vdu + u^3dv) + (uv^2du + u^2v dv) = 0$

$\Rightarrow u^3v + \frac{1}{2}u^2v^2 = C$

2-(3) 原方程可化为: $\frac{2ydy}{2x+3} = \frac{4y^2-2x^2}{x^2+y^2+3}$, 令 $x^2=v$, $y^2=u$

则 $\frac{du}{dv} = \frac{4u-2v}{u+v+3}$; 由 $\begin{cases} 4u-2v=0 \\ u+v+3=0 \end{cases} \Rightarrow \begin{cases} u=-1 \\ v=-2 \end{cases}$

令 $\begin{cases} m=u+1 \\ n=v+2 \end{cases}$, 则 $\frac{dm}{dn} = \frac{4m-2n}{m+n}$, 令 $\frac{m}{n}=z$, 则 $m=zn$, $\frac{dm}{dn} = \frac{dz}{dn}n + z = \frac{4z-2}{z+1}$

故有: $\frac{dz}{dn}n = \frac{(1-z)(z-2)}{z+1}$, 此方程为分离变量方程

$\frac{1}{n}dn = \frac{z+1}{(1-z)(z-2)} dz \Rightarrow \ln|\frac{z-1}{z-2}| = C + \ln n \Rightarrow (x^2-y^2+1)^2 = C(-2x^2+y^2-3)^3$

8. 3-(1) $y' = -y^2 - \frac{1}{4x^2}$, 令 $z=xy$, $\frac{dz}{dx} = \frac{1}{x}(-z^2+z-\frac{1}{4})$

① $z \neq \frac{1}{2}$ 时, 分离变量 $\frac{-1}{(z-\frac{1}{2})^2} dz = \frac{1}{x} dx \Rightarrow \frac{1}{z-\frac{1}{2}} = \ln|x| + C$

还原变量并化简得: $y = \frac{1}{2x} + \frac{1}{x \ln|x| + Cx}$

② $z = \frac{1}{2}$, $y = \frac{1}{2x}$ 为一特解

3-(2) $xy' = x^2y^2 + xy + 1$, 即为: $y' = y^2 + \frac{y}{x} + \frac{1}{x^2}$

令 $z=xy$, 则 $\frac{dz}{dx} = \frac{1}{x}(z+1)^2$ 可分离变量

$-\frac{1}{z+1} = \ln|x| + C \Rightarrow y = -\frac{1}{x} - \frac{1}{x \ln|x| + Cx}$



9-4. $y'' + p(x)y' + q(x)y = 0$

令 $y = e^{\int u dx}$, $y' = u e^{\int u dx}$, $y'' = u^2 e^{\int u dx} + u' e^{\int u dx}$

则 $y'' + p(x)y' + q(x)y = u^2 e^{\int u dx} + p(x)u e^{\int u dx} + q(x)e^{\int u dx} = 0$
 $+ u' e^{\int u dx}$

即有: $u^2 + u' + p(x)u + q(x) = 0$ \square

5. 设曲线 $y = y(x)$. 则:

$\frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{dy}{dx} \frac{y}{x}} = 1 \Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$ 齐次方程.

解得 $\arctan \frac{y}{x} - \frac{1}{2} \ln(x^2 + y^2) = C$

10-4. 设 Q 的运动轨迹为 $y = y(x)$, 则 y 满足

$\frac{dx}{dy} = \frac{y^{\frac{a}{b}}}{2} - \frac{y^{-\frac{a}{b}}}{2}$, $x(1) = 0$.

通过分离变量, 可得 $x = \frac{y^{(1+\frac{a}{b})} - 1}{2(1+\frac{a}{b})} - \frac{y^{(1-\frac{a}{b})} - 1}{2(1-\frac{a}{b})}$

P, Q 在 T 时刻相遇且 $x(aT) = 0$,

代入可得: $T = \frac{b}{2(b^2 - a^2)}$

5. $\frac{gmm}{R^2} = \frac{mv_0^2}{R} \Rightarrow v_0 = 7.94 \text{ km/s}$

