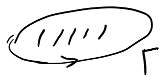
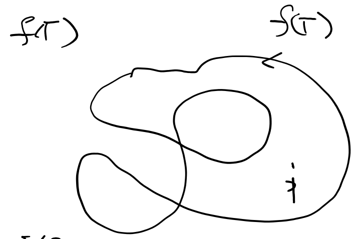
本质：连续性.

辐角原理

约定：定义域中的简单闭曲线默认是逆时针方向！



$\xrightarrow{f}$



p-point $= f^{-1}(p)$

$\# f^{-1}(p) = n(f(\Gamma), p)$

1. 解析函数的零点

定理.

现在我们来证明, 如果 $f(z)$ 在一个以原点为中心、以 R 为半径的圆周上以及内部为解析, 则 $f(z)$ 可以展开为在此圆盘中收敛的幂级数:

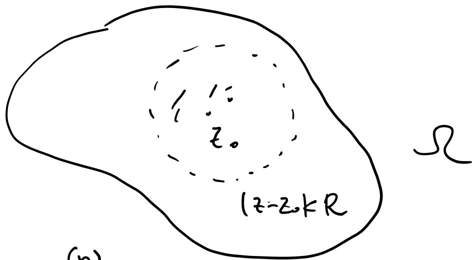
$$f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots.$$

解析函数可表为幂级数.

$f: \Omega^{\text{open}} \to \mathbb{C}$ analytic function. $\forall z_0 \in \Omega$. $\exists R > 0$. s.t.

$$(*) \quad f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots$$

on $|z-z_0| < R$.



Defn. z_0 is called a zero of order n for f, if

$$f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0 \text{ and } f^{(n)}(z_0) \neq 0.$$

Prop. Suppose z_0 is a zero of order n for f, $\exists R > 0$ and an analytic function $g: B(z_0, R) = \{|z-z_0| < R\} \to \mathbb{C}$ satisfying $g(z_0) \neq 0$, such that

$$f(z) = (z-z_0)^n \cdot g(z) \quad \text{for } z \in B(z_0, R). \quad (**)$$

Proof: using $(*)$. $\exists R > 0$. s.t.

$$f(z) = \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \frac{f^{(n+1)}(z_0)}{(n+1)!}(z-z_0)^{n+1} + \cdots \quad \text{on } B(z_0, R)$$

$$= (z-z_0)^n \Big[\underbrace{\frac{f^{(n)}(z_0)}{n!} + \frac{f^{(n+1)}(z_0)}{(n+1)!}(z-z_0) + \cdots}_{} \Big]$$

$$= (z-z_0)^n \cdot g(z)$$

Clearly $g(z)$ is an analytic function on $B(z_0, R)$. and $g(z_0) = \frac{f^{(n)}(z_0)}{n!} \neq 0$

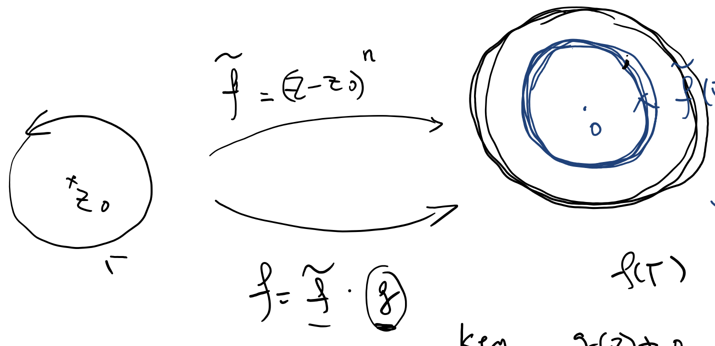
Observation:

1. f analytic 2. z_0 is a zero of order n.

3. Γ a simple closed curve surrounds z_0 with no other zero.

then $\nu(f(\Gamma), 0) = n$.

1. By (**), we see that z_0 is isolated. (非常值的解析函数零点一定孤立)

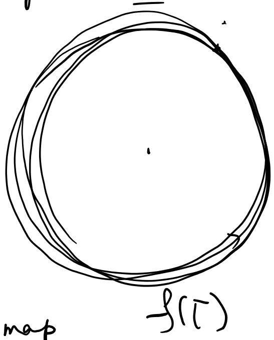


$\tilde{f}(\Gamma)$ rotates around 0 n-time counterclockwise.

$\nu(\tilde{f}(\Gamma), 0) = n$.

key. $g(z) \neq 0 \quad \forall z \in \Gamma$

$|g(z) - g(z_0)| < \varepsilon$



$\nu(\tilde{f}(\Gamma) \cdot g(z_0), 0) = n$

$\nu(f(\Gamma), 0) = n$

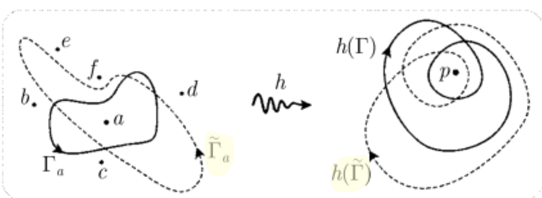
1. 连续映射孤立零点的重数

Defn. let $h: \mathbb{R}^2 \to \mathbb{R}^2$ continuous map.

(拓扑度) suppose $h(a) = p$ and a is an isolated p-point $\in h^{-1}(p)$.

then h's order at a is defined as

$\mathcal{J}(a) := \nu(h(\Gamma_a), p)$, where Γ_a is a simple closed curve surround a, and a is the only p-point in the interior of Γ_a and $\tilde{\Gamma}_a$



Remark: The definition is well-defined, i.e. is independent of choice of Γ_a.

环路 K 可以连续变形为另一环路 L 而不穿过点 p, 当且仅当 K 与 L 有同样的绕 p 的环绕数.

Γ_a continuous deform to $\tilde{\Gamma}_a$ without passing any p-point.

$\Rightarrow$ $h(\Gamma_a)$ continuous deform to $h(\tilde{\Gamma}_a)$ without passing p.

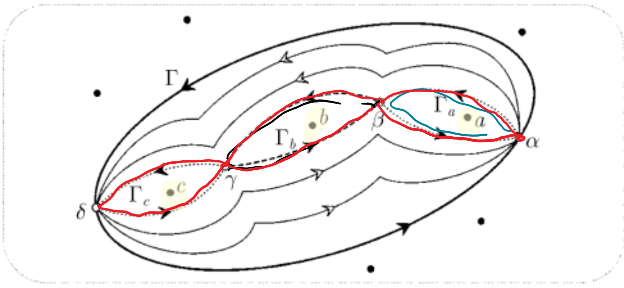
Remark: It is possible that $\mathcal{J}(a) \leq 0$.

1. 拓扑辐角原理及其证明

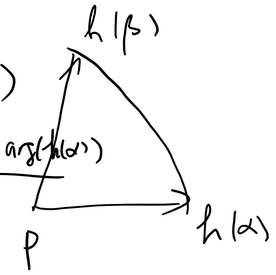
大前提: assume $\# f^{-1}(p) < +\infty$ in the interior of Γ.

在 Γ 的内域的 p-点的总数 (每个 p-点均按其拓扑重数计算) 等于 $h(\Gamma)$ 绕 p 的环绕数.

约定: Γ simple closed curve, counterclockwise.



$$\nu(h(\alpha\beta),p) := \frac{\arg(h(\beta)) - \arg(h(\alpha))}{2\pi}$$



$\Gamma \sim \tilde{\Gamma} = \{\alpha\beta\gamma\delta\gamma\beta\alpha\}$

$h(\Gamma) \sim h(\tilde{\Gamma})$ $\quad \nu(h(\Gamma),p) = \nu(h(\tilde{\Gamma}),p)$

$= \nu(h(\alpha\beta),p) + \nu(h(\beta\gamma),p) + \cdots + \nu(h(\beta\alpha),p).$ ← 关于p 的辐角 有限定

$= \nu(h(\Gamma_a),p) + \nu(h(\Gamma_b),p) + \nu(h(\Gamma_c),p) = \nu(a)+\nu(b)+\nu(c)$

解析辐角原理

若 $f(z)$ 在一个简单环路 Γ 的内域和在 Γ 上均为解析, 而 N 为 Γ 内域中的 p-点数目 [按重数计], 则 $N = \nu[f(\Gamma), p]$.

约定: Γ 上是没有 p-点

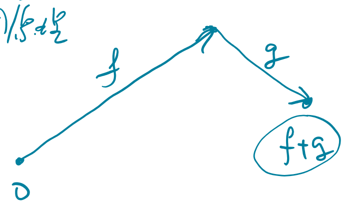
Note: for analytic function, the order of its zero ≥ 1.

4. 鲁歇定理 (Rouche) (续)

若在 Γ 上 $|g(z)| < |f(z)|$, 则在 Γ 内 $(f+g)$ 与 f 零点个数相同.

$\Downarrow$

$\nu(f(\Gamma),0) = \nu((f+g)(\Gamma),0)$ ↗ 辐角原理



布劳威尔不动点定理

闭

布劳威尔不动点定理

由一个闭圆盘到其自身的连续映射必有不动点.

Remark. 闭圆盘 = $\overline{B(0,1)} = \{|z| \le 1\}$

$h: \overline{B(0,1)} \to \overline{B(0,1)}$ has a fixed point. ($\exists z \in \overline{B(0,1)}$ s.t. $h(z)=z$)

A weak version.

$f: \overline{B(0,1)} \to B(0,1)$ an analytic function. then f has a fixed point.

Remark: $f: \overline{B(0,1)} \to \mathbb{C}$ is analytic if $\exists R>1$. and an analytic extension $\tilde{f}$ of f on $B(0,R)$.

Proof. let $\Gamma = \partial B(0,1)$ the unit circle.

$|f(z)| < 1$ for $z \in \Gamma$.

we have

$|f(z)| < 1 = |-z|$ for $z \in \Gamma$.

fixed pt of f $\Updownarrow$ zero of $f(z) - z$. $\tilde{g}$ $\Updownarrow$ $(-z) \sim f$

$\xRightarrow{\text{Rouché}}$ $-z$ has the number of zeros with $f(z)-z$ inside of Γ

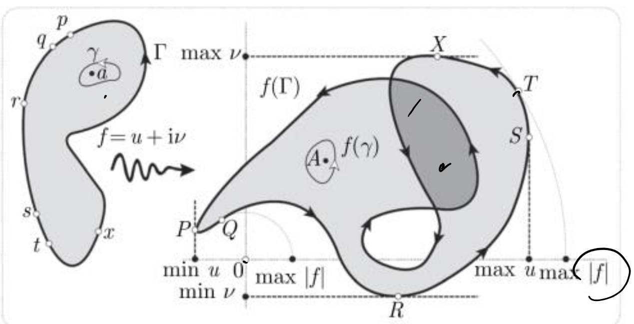
$\Rightarrow$ $f(z)-z$ has exactly one zero of order 1.

1. 最大模原理

若 f 在一区域内解析[1], 则 $|f(z)|$ 之最大值[2] 必在区域的边界点上达到, 而不可能在内点达到.

Assumption: f is analytic in Ω (open. connected, bounded)

f is continuous on $\overline{\Omega}$, $\Rightarrow$ $\overline{\Omega}$ is a bounded, closed set



$\Rightarrow$ $|f(z)|$ attains its maximum in $\overline{\Omega}$.

Proof. let $\Gamma = \partial\Omega$. $f(\Gamma)$

if p is a point such that $\nu(f(\Gamma), p) = 0$ (p is called in the exterior of $f(\Gamma)$)

if p is a point such that $\nu(f(\Gamma), p) = 0$ (p is called in the exterior of $f(\Gamma)$)

then $f^{-1}(p) \cap \Omega = 0$

$\Rightarrow f$ maps Ω to interior of $f(\Gamma)$

$\Rightarrow \operatorname{Max}|f(\overline{\Omega})|$ is achieved on $f(\Gamma)$

1.