

# 1. Review.

$$z = x + yi = re^{i\theta}$$

$$\text{Arg}(z) \in (-\pi, \pi]$$

辐角主值.

$$\left\{ \begin{array}{l} \text{多值函数} \\ f(z) = z^{1/n} = \left\{ r^{1/n} e^{i \frac{\theta + 2k\pi}{n}} \right. \end{array} \right. \quad k=0, 1, \dots, n-1$$

$$f(z) = \log(z) = w = \left\{ \log|z| + i(\text{Arg}(z) + 2k\pi) \right. \quad k \in \mathbb{Z}$$

$$\updownarrow$$

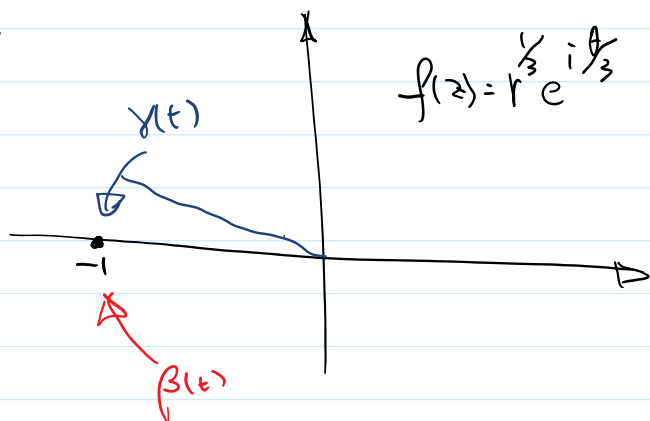
$$e^w = z$$

单值分支.

$$f(z) = z^{1/n} = r^{1/n} e^{i \frac{\theta}{n}} \quad \theta \in (-\pi, \pi]$$

Problem:  $f(z)$  defined on  $\mathbb{C}$  is not a continuous function!

$n=3$



$$f(z) = r^{1/3} e^{i\theta/3}$$

$$(-1)^{1/3} = 1 \cdot e^{i\pi/3} = e^{i\pi/3}$$

$$\text{Arg}(\gamma(t)) \rightarrow \pi = \text{Arg}(-1)$$

$$\gamma(t)^{1/3} \rightarrow (-1)^{1/3}$$

$$\text{Arg}(\beta(t)) \rightarrow \text{Arg}(-1) = \pi$$

$$\downarrow (-\pi)$$

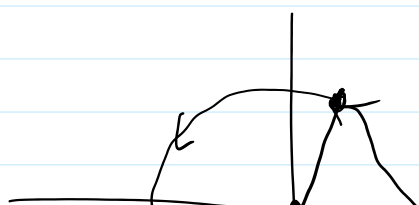
$$\gamma(t)^{1/3} \rightarrow e^{-i\pi/3} \neq (-1)^{1/3}$$

造成问题的根源: 支点.

(如果在  $\mathbb{C}$  中绕支点一圈).

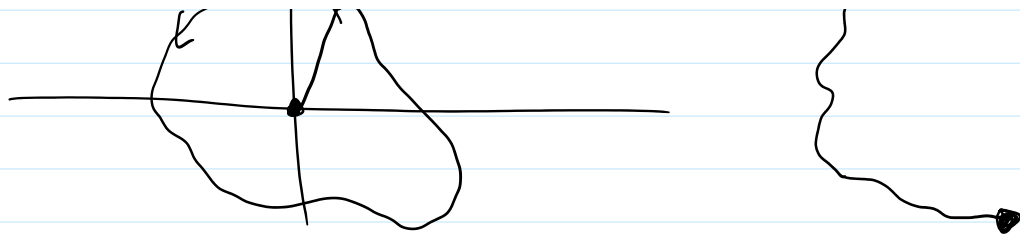
(相应的多值函数的连续分支; 无法回到同一值)

$$z^{1/3}$$



$$w = z^{1/3}$$





为了给出一个单值分支: 从支点上画一条射线到无穷远并从此区域中删除!

对于  $z^{1/n}$ :  $0$  是一个支点, 将  $0$  和实负半轴删除.

Defn.  $f(z) = z^{1/n} = r^{1/n} e^{i\theta/n}$ ,  $\theta \in (-\pi, \pi)$

defined on  $\mathbb{C} \setminus \{0 \cup \mathbb{R}^-\}$  is called  $z^{1/n}$  的主值分支.

满足  $f(1) = 1$

Defn.  $\text{Log}(z) = \log|z| + i \text{Arg}(z)$  defined on  $\mathbb{C} \setminus \{0 \cup \mathbb{R}^-\}$  is called  $\log(z)$  的主值分支.

Note:  $\text{Log}(1) = 0$

$\alpha$  为实数或复数  $(1+z)^\alpha =$

Euler's way

$$B(z, \alpha) := \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!}$$

$\alpha \in \mathbb{R}, n \in \mathbb{N}, z \in \mathbb{C}$ .

Observation 1:  $R=1$

Observation 2:  $\alpha \in \mathbb{N}$ .  $B(z, \alpha) = (1+z)^\alpha =$  二项式展开.

let  $\sum \binom{\alpha}{i} \binom{\beta}{j} = C(\alpha, \beta, k)$

$$\text{Claim: } C(\alpha, \beta, k) = \binom{\alpha+\beta}{k} \quad i+j=k$$

Proof of the claim. Note if  $\alpha, \beta \in \mathbb{N}$ .

$$\begin{aligned} B(z, \alpha) B(z, \beta) &= (1+z)^\alpha (1+z)^\beta = (1+z)^{\alpha+\beta} = B(z, \alpha+\beta) \\ \text{比较 } z^{i+j} \text{ 的系数} \Rightarrow \sum_{i+j=k} \binom{\alpha}{i} \binom{\beta}{j} &= \binom{\alpha+\beta}{k} \quad (*) \end{aligned}$$

for  $k$  fixed. both sides of  $(*)$  are polynomials on  $\alpha, \beta$  of degree  $k$ .

$\Rightarrow (*)$  holds for all  $\alpha, \beta \in \mathbb{R}$ .

$\Rightarrow \forall \alpha, \beta \in \mathbb{R}$

$$\begin{aligned} B(z, \alpha) B(z, \beta) &= \left( \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \right) \left( \sum_{n=0}^{\infty} \binom{\beta}{n} z^n \right) \\ &= \sum_{k=0}^{\infty} \left( \sum_{i+j=k} \binom{\alpha}{i} \binom{\beta}{j} \right) z^k = \sum_{k=0}^{\infty} \binom{\alpha+\beta}{k} z^k \\ &= B(z, \alpha+\beta). \end{aligned}$$

$$\text{Take } \alpha = n, \beta = -n.$$

$$B(z, n) \cdot B(z, -n) = 1 \quad \text{Note } B(z, n) = (1+z)^n$$

$$\Rightarrow B(z, -n) = (1+z)^{-n}$$

$$\text{Take } \alpha = \frac{1}{p}.$$

$$B(z, \frac{1}{p}) B(z, \frac{1}{p}) \cdots B(z, \frac{1}{p}) B(z, 1) = (1+z)$$

$$\Rightarrow B(z, \frac{1}{p}) = (1+z)^{\frac{1}{p}}$$

$$B(0, \frac{1}{p}) = 1. \quad (1+0)^{\frac{1}{p}} = 1 \rightarrow z^{\frac{1}{p}} \text{ 是分支}$$

$$(1+z)^{\frac{1}{p}} = B(z, \frac{1}{p}) = \sum_{n=0}^{\infty} \binom{\frac{1}{p}}{n} z^n, \quad |z| < 1.$$

18. 下面的论证的基本思想来自欧拉. 令  $n$  为任意实数 (可以是无理数), 定义

$$B(z, n) \equiv \sum_{r=0}^{\infty} \binom{n}{r} z^r, \quad \text{其中} \quad \binom{n}{r} \equiv \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

而  $\binom{n}{0} \equiv 1$ . 由初等代数知道, 若  $n$  为正整数, 则  $B(z, n) = (1+z)^n$ . 为了对有理数证明二项式定理 (2.14), 必须要证明, 若  $p, q$  为整数, 则  $B(z, \frac{p}{q})$  是  $(1+z)^{\frac{p}{q}}$  的主支.

(i) 对固定的  $n$ , 用比值判别法证明  $B(z, n)$  在单位圆盘  $|z| < 1$  内收敛.

(ii) 将两个幂级数相乘, 导出

$$B(z, n)B(z, m) = \sum_{r=0}^{\infty} C_r(n, m) z^r, \quad \text{其中} \quad C_r(n, m) = \sum_{j=0}^r \binom{n}{j} \binom{m}{r-j}.$$

(iii) 若  $m, n$  均为正整数, 证明

$$B(z, n)B(z, m) = B(z, n+m) \quad (2.43)$$

由此得知  $C_r(n, m) = \binom{n+m}{r}$ . 但  $C_r(n, m)$  和  $\binom{n+m}{r}$  对于  $n$  和  $m$  恰为多项式, 因此, 它们二者对无穷多个 [正整数值]  $m$  和  $n$  相等, 意味着它们必对  $m, n$  的所有实值相等, 所以关键性的 (2.43) 对所有实值  $m$  与  $n$  均成立.

(iv) 在 (2.43) 中令  $n = -m$ , 并对  $n$  是负整数值的情况导出二项式定理.

(v) 用 (2.43) 证明, 若  $q$  为整数, 则  $[B(z, \frac{1}{q})]^q = (1+z)$ , 由此导出  $B(z, \frac{1}{q})$  是  $(1+z)^{\frac{1}{q}}$  的主支.

(vi) 最后证明, 若  $p, q$  均为整数, 则  $B(z, \frac{p}{q})$  确为  $(1+z)^{\frac{p}{q}}$  的主支.

32. 下面是对数幂级数的另一个处理方法. 和前面一样, 令  $L(z) = \text{Log}(1+z)$ , 因为  $L(0) = 0$ ,  $L(z)$  的幂级数必有以下形式:  $L(z) = az + bz^2 + cz^3 + dz^4 + \cdots$ . 把它代入方程

$$1+z = e^L = 1 + L + \frac{1}{2!}L^2 + \frac{1}{3!}L^3 + \frac{1}{4!}L^4 + \cdots,$$

令  $z$  的同次幂系数相等即可得出  $a, b, c, d$ . [从历史上看是先有对数幂级数——麦卡托<sup>⑩</sup>和牛顿都用了上题的方法得出了它——然后牛顿把本题的方法倒过来应用得出  $e^x$  的级数式. 详见 Stillwell [1989, 第 108 页].

$$3. \quad \text{Log}(1+z) = az + bz^2 + cz^3 + \cdots = L(z)$$

$$e^{L(z)} = 1+z.$$

$$e^z = 1+z + \frac{z^2}{2!} + \cdots, \quad z \in \mathbb{C}$$

$$1 + L(z) + \frac{L(z)^2}{2!} + \frac{L(z)^3}{3!} + \cdots = 1+z.$$

$$\Rightarrow a, b, c, d, \dots = :$$

$$\text{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}, \quad |z| < 1$$

by-Product. Claim: Above series converges on  $|z|=1$  except  $z=-1$ .

Dirichlet  $\sum_{n=1}^{\infty} a_n b_n < +\infty$  provided 1.  $a_n \rightarrow 0$   
2.  $\left| \sum_{k=1}^n b_k \right| \leq M, \forall n$

for  $a_n = \frac{1}{n}, b_n = (-1)^{n-1} z^n$ .

$$|b_1 + \dots + b_n| = \left| \sum_{k=1}^n (-1)^{k-1} z^k \right| \leq M, \forall n$$

take  $z=i$ ,  $\text{Log}(1+i) = \frac{i}{2} - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} \dots$

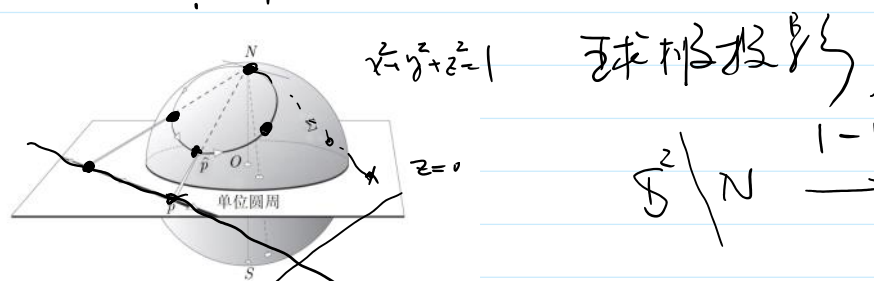
$$\text{Log}(1+i) = \log \sqrt{2} + i \cdot \frac{\pi}{4}$$

the imaginary part  $= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \dots = \frac{\pi}{4}$

the real part  $= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} \dots = \log \sqrt{2}$

4. Riemann sphere  $= \mathbb{C} \cup \{\infty\}$   
黎曼球面 (无窮遠點)

注意:  $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$

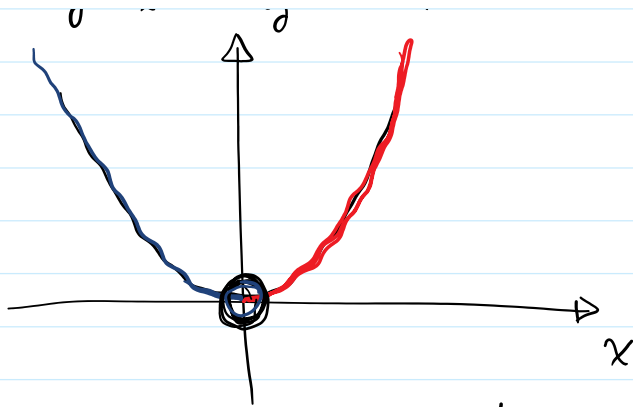


$$\mathbb{S}^2 \setminus N \xrightarrow{1-1} \mathbb{C}$$

$$N \leftrightarrow \infty$$

$$\mathbb{S}^2 \xrightarrow{1-1} \mathbb{C} \cup \{\infty\}$$

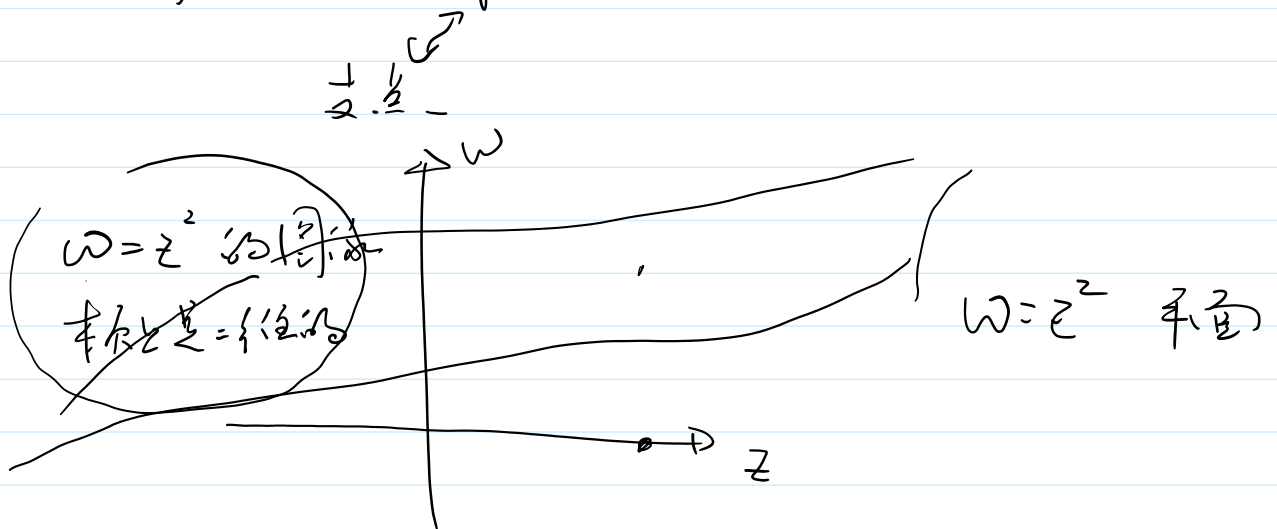
$y = x^2$   $y \in \mathbb{R}$   $x \in \mathbb{R}$



$$x = y^{1/2}$$

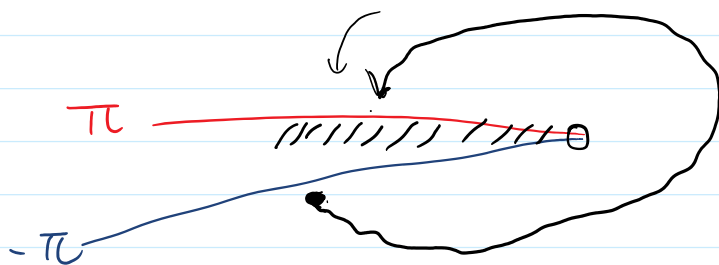
$$x = -y^{1/2}$$

Not differentiable at  $y=0$ .



$$z = w^{1/2}$$

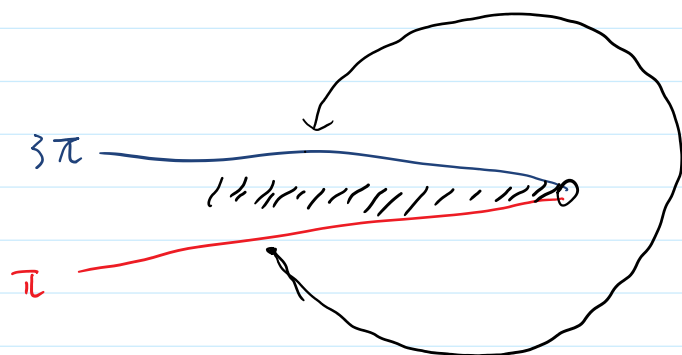
$w$  平面



$$\theta \in (-\pi, \pi)$$

$$r^{1/2} e^{i\theta/2}$$

$$-i^{1/2}$$

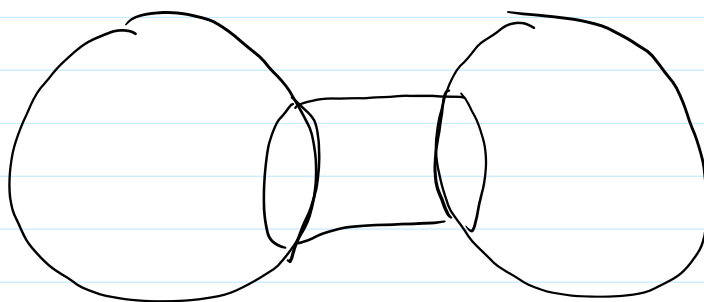
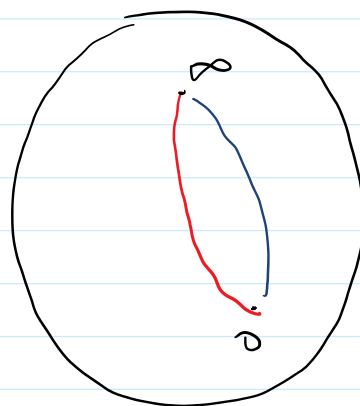
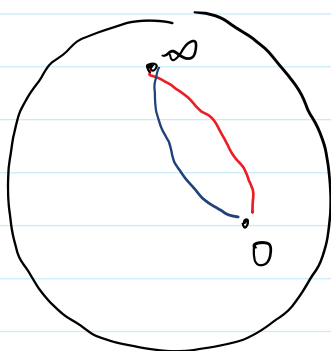


$$\theta \in (\pi, 3\pi)$$

$$r^{1/2} e^{i\theta/2}$$

$$- e^{i3\pi/2}$$

都添  $\infty$



$$\omega = \sqrt{(1+z)(1+z^2)}$$

$$\Leftrightarrow$$

$$w^2 = (1+z)(1+z^2)$$

$z = -1$ .  $z = \pm i$  为支点.

