

1. Review.

$$z = x + yi = re^{i\theta}, \quad \text{约} z, \quad \operatorname{Arg}(z) \in (-\pi, \pi]$$

复角的值.

复数.

$$f(z) = z^{\frac{1}{n}} = \left\{ r^{\frac{1}{n}} e^{i\frac{\theta + 2k\pi}{n}} \mid k=0, 1, \dots, n-1 \right\}$$

$$f(z) = \log(z) = w = \left\{ \log|z| + i(\operatorname{Arg}(z) + 2k\pi) \mid k \in \mathbb{Z} \right\},$$

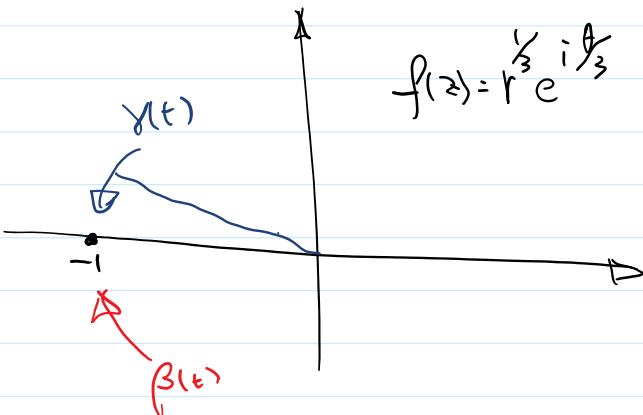
$$e^w = z$$

单位分支.

$$f(z) = z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta}{n}} \quad \theta \in (-\pi, \pi]$$

Problem: $f(z)$ defined on \mathbb{C} is not a continuous function!

$$n=3$$



$$f(z) = r^{\frac{1}{3}} e^{i\frac{\theta}{3}}$$

$$(-1)^{\frac{1}{3}} = 1 \cdot e^{i\frac{\pi}{3}} = e^{i\frac{\pi}{3}}$$

$$\operatorname{Arg}(\gamma(t)) \rightarrow \pi = \operatorname{Arg}(-1)$$

$$\gamma^{\frac{1}{3}}(t) \rightarrow (-1)^{\frac{1}{3}}$$

$$\underbrace{\operatorname{Arg}(\beta(t))}_{\downarrow \pi} \rightarrow \operatorname{Arg}(-1) = \pi$$

$$\beta^{\frac{1}{3}}(t) \rightarrow e^{-i\frac{\pi}{3}} \neq (-1)^{\frac{1}{3}}$$

$$w = t^{\frac{1}{3}}$$

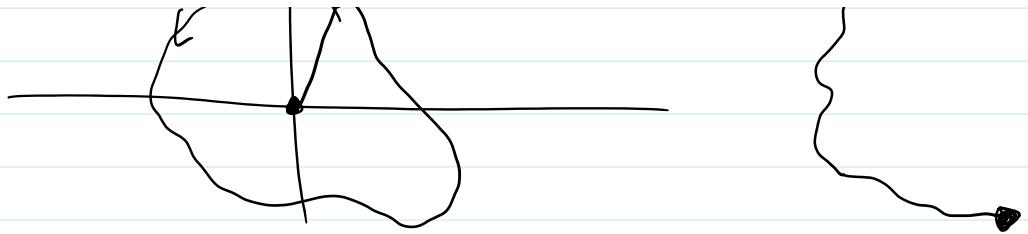
造成间断的原因：支点.

(如果在 \mathbb{C} 中绕支点一周).

相应的复数之数的逆(至单叶)为; 支点的 $(1) - \frac{1}{4}i$

$$z^{\frac{1}{3}}$$





为了给出一个单值分支：从支上画一个射线到无穷远并从之延伸
以除！

对于 $z^{\frac{1}{n}}$ ， θ 是一个常数，将 θ 和实数轴的交点。

Defn . $f(z) = z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta}{n}}$, $\theta \in (-\pi, \pi)$

defined on $\mathbb{C} \setminus \{0 \cup \mathbb{R}\}$ is called $z^{\frac{1}{n}}$ 的主值分支。

满足 $f(1) = 1$

Defn . $\log(z) = \log|z| + i \operatorname{Arg}(z)$ defined on $\mathbb{C} \setminus \{0 \cup \mathbb{R}\}$ is called $\log(z)$ 的主值分支。

Note: $\log(1) = 0$

$$\alpha = zk\pi \quad (1+z)^{\alpha} =$$

Euler's way

$$B(z, \alpha) := \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!} \quad \alpha \in \mathbb{R}, n \in \mathbb{N}, z \in \mathbb{C}.$$

Observation 1: $R=1$

Observation 2: $\alpha \in \mathbb{N}$ $B(z, \alpha) = (1+z)^{\alpha} = zh. k. 展开$.

$$\text{let } \sum \binom{\alpha}{j} \binom{\beta}{j} := C(\alpha, \beta, k)$$

$$\text{Claim: } C(\alpha, \beta, k) = \binom{\alpha+\beta}{k}$$

Proof of the claim. Note if $\alpha, \beta \in \mathbb{N}$.

$$\begin{aligned} \text{两边同时} \quad B(z, \alpha) B(z, \beta) &= (1+z)^\alpha (1+z)^\beta = (1+z)^{\alpha+\beta} = B(z, \alpha+\beta) \\ (\text{等号左边}) \quad z^{i+j} \text{ 的系数} &\rightarrow \sum_{i+j=k} \binom{\alpha}{i} \binom{\beta}{j} = \binom{\alpha+\beta}{k} \quad (*) \end{aligned}$$

for k fixed. both sides of $(*)$ are polynomials on α, β of degree k .

$\Rightarrow (*)$ holds for all $\alpha, \beta \in \mathbb{R}$

$\Rightarrow \forall \alpha, \beta \in \mathbb{R}$

$$\begin{aligned} B(z, \alpha) B(z, \beta) &= \left(\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \right) \left(\sum_{n=0}^{\infty} \binom{\beta}{n} z^n \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i+j=k} \binom{\alpha}{i} \binom{\beta}{j} \right) z^k = \sum_{k=0}^{\infty} \binom{\alpha+\beta}{k} z^k \\ B(z, \alpha) B(z, \beta) &= B(z, \alpha+\beta) \quad \forall \alpha, \beta \in \mathbb{R} \end{aligned}$$

$\forall \alpha = n, \beta = -n$.

$$B(z, n) \cdot B(z, -n) = 1 \quad \text{Note } B(z, n) = (1+z)^n$$

$$\Rightarrow B(z, -n) = (1+z)^{-n}$$

$$\forall \alpha = \frac{1}{p}$$

$$B(z, \frac{1}{p}) B(z, \frac{1}{p}) \cdots B(z, \frac{1}{p}) = B(z, 1) = (1+z)$$

$$\Rightarrow B(z, \frac{1}{p}) = (1+z)^{\frac{1}{p}}$$

$$B(0, \frac{1}{p}) = 1 \quad (1+0)^{\frac{1}{p}} = 1 \rightarrow z^{\frac{1}{p}} \text{ 有意义}$$

$$(1+z)^{\frac{1}{p}} = B(z, \frac{1}{p}) = \sum_{n=0}^{\infty} \left(\frac{1}{p} \right)_n z^n, \quad |z| < 1$$

18. 下面的论证的基本思想来自欧拉. 令 n 为任意实数 (可以是无理数), 定义

$$B(z, n) \equiv \sum_{r=0}^{\infty} \binom{n}{r} z^r, \quad \text{其中 } \binom{n}{r} \equiv \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

而 $\binom{n}{r} \equiv 1$. 由初等代数知道, 若 n 为正整数, 则 $B(z, n) = (1+z)^n$. 为了对有理幂证明二项式定理 (2.14), 必须要证明, 若 p, q 为整数, 则 $B\left(z, \frac{p}{q}\right)$ 是 $(1+z)^{\frac{p}{q}}$ 的主支.

- (i) 对固定的 n , 用比值判别法证明 $B(z, n)$ 在单位圆盘 $|z| < 1$ 内收敛.
- (ii) 将两个幂级数相乘, 导出

$$B(z, n)B(z, m) = \sum_{r=0}^{\infty} C_r(n, m)z^r, \quad \text{其中 } C_r(n, m) = \sum_{j=0}^r \binom{n}{j} \binom{m}{r-j}.$$

- (iii) 若 m, n 均为正整数, 证明

$$B(z, n)B(z, m) = B(z, n+m) \quad (2.43)$$

由此得知 $C_r(n, m) = \binom{n+m}{r}$. 但 $C_r(n, m)$ 和 $\binom{n+m}{r}$ 对于 n 和 m 恰为多项式, 因此, 它们二者对无穷多个 [正整数值] m 和 n 相等, 意味着它们必对 m, n 的所有实值相等, 所以关键性的 (2.43) 对所有实值 m 与 n 均成立.

- (iv) 在 (2.43) 中令 $n = -m$, 并对 n 是负整数值的情况导出二项式定理.

- (v) 用 (2.43) 证明, 若 q 为整数, 则 $[B\left(z, \frac{1}{q}\right)]^q = (1+z)$, 由此导出 $B\left(z, \frac{1}{q}\right)$ 是 $(1+z)^{\frac{1}{q}}$ 的主支.
- (vi) 最后证明, 若 p, q 均为整数, 则 $B\left(z, \frac{p}{q}\right)$ 确为 $(1+z)^{\frac{p}{q}}$ 的主支.

32. 下面是对数幂级数的另一个处理方法. 和前面一样, 令 $L(z) = \log(1+z)$, 因为 $L(0) = 0$, $L(z)$ 的幂级数必有以下形式: $L(z) = az + bz^2 + cz^3 + dz^4 + \dots$. 把它代入方程

$$1+z = e^L = 1 + L + \frac{1}{2!}L^2 + \frac{1}{3!}L^3 + \frac{1}{4!}L^4 + \dots,$$

令 z 的同次幂系数相等即可得出 a, b, c, d . [从历史上看是先有对数幂级数——麦卡托^①和牛顿都用了上题的方法得出了它——然后牛顿把本题的方法倒过来应用得出 e^x 的级数式. 详见 Stillwell [1989, 第 108 页].

$$\text{3. } \underbrace{\log(1+z)}_{=} = a\underbrace{z + bz^2 + cz^3 + \dots}_{=} = L(z)$$

$$e^{\circled{L(z)}} = 1+z.$$

$$e^z = 1+z + \frac{z^2}{2!} + \dots, \quad z \in \mathbb{C}$$

$$(1+L(z)) + \frac{L(z)}{2!} + \frac{L(z)}{3!} + \dots = 1+z.$$

$$\Rightarrow a, b, c, d = :$$

$$\underbrace{\log(1+z)}_{=} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

$$|z| < 1$$

by Product. Claim: Above series converges on $|z|=1$ except $z=-1$.

Dirichlet Product. $\sum_{n=1}^{\infty} a_n b_n < +\infty$ provided 1. $a_n \rightarrow 0$
2. $|\sum_{k=1}^n b_k| \leq M, \forall n$

$$\text{Take } a_n = \frac{1}{n}, \quad b_n = (-1)^{n-1} z^n.$$

$$|b_1 + \dots + b_n| = |z| \leq M, \quad \forall n$$

$$\text{take } z=i, \quad \log(1+i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} \dots$$

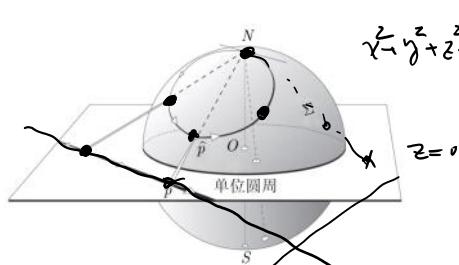
$$\log(1+i) = \log\sqrt{2} + i \cdot \frac{\pi}{4}$$

$$\text{右边} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} = \frac{\pi}{4}$$

$$\text{左边} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} = \log\sqrt{2}.$$

4. Riemann Sphere = $C \cup \{\infty\}$
 球面 + 在 ∞ 处
 元素 ∞

$$\text{球面: } \frac{1}{0} = \infty, \quad \frac{1}{\infty} = 0$$



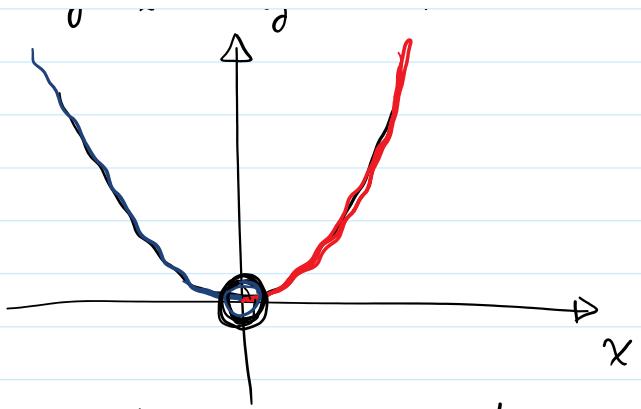
球面映射

$$S^2 \setminus N \xrightarrow{1-1} C$$

$$N \leftrightarrow \infty$$

$$S^2 \xleftarrow{1-1} C \cup \{\infty\}$$

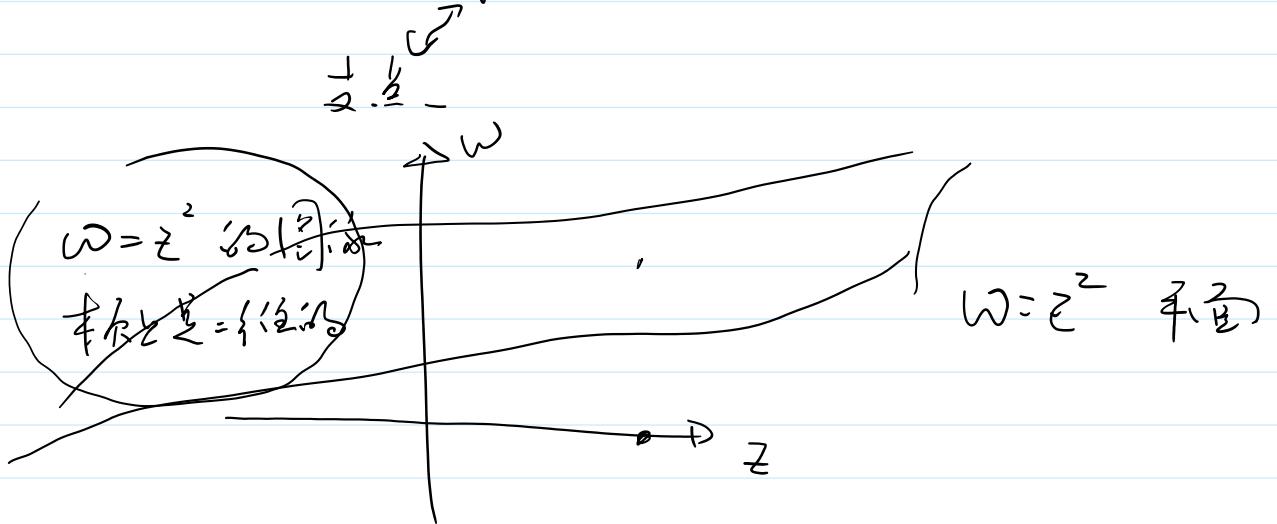
$$y = x^2, \quad y \in \mathbb{R}, \quad x \in \mathbb{R}$$



$$x = y^{\frac{1}{2}}$$

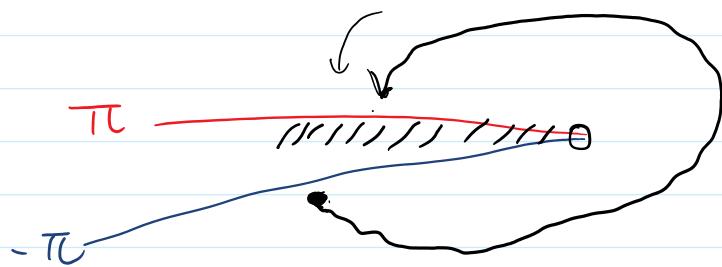
$$x = -y^{\frac{1}{2}}$$

Not differentiable at $y=0$.

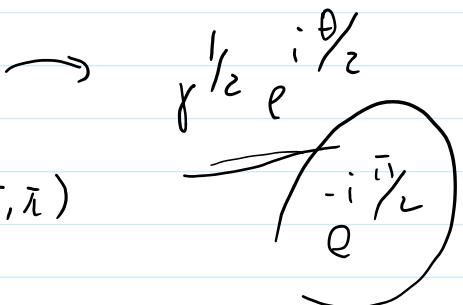


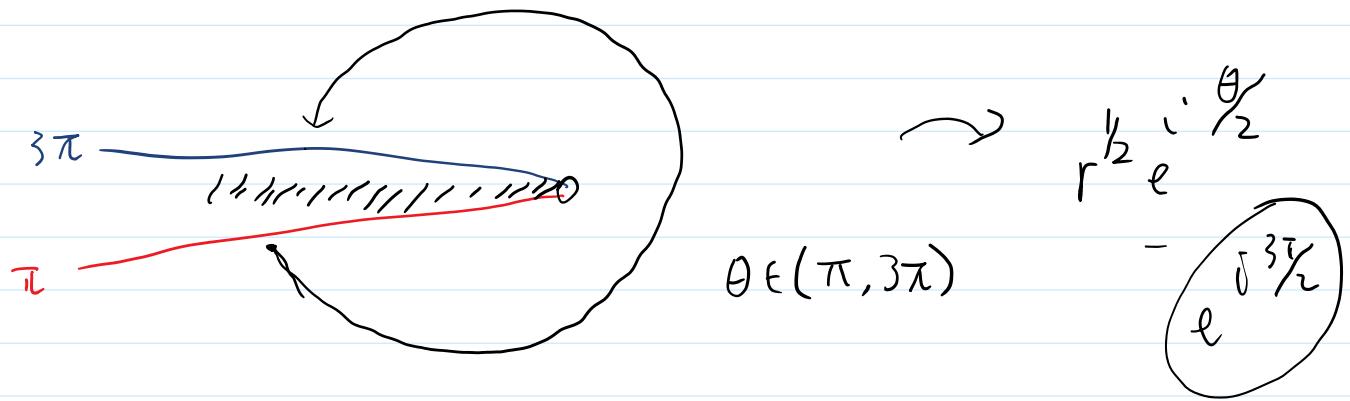
$$z = w^{\frac{1}{2}}$$

$$w \neq 0$$

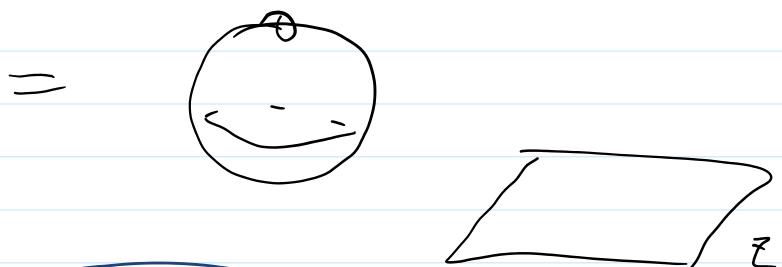
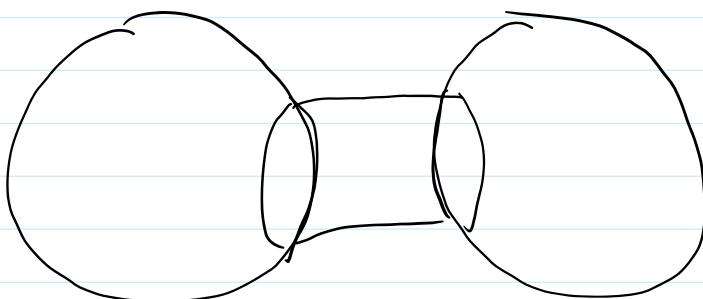
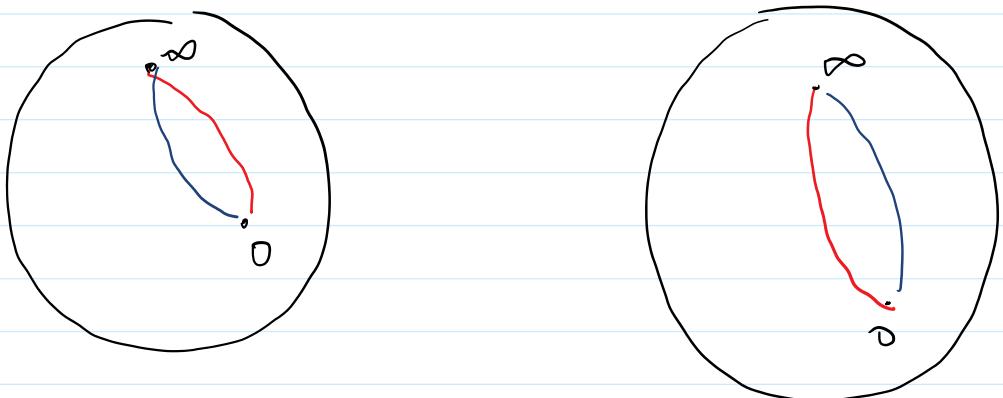


$$\theta \in (-\pi, \pi)$$





都添上 ∞



$$\omega = \sqrt{(1+z)(1+z^2)} \Rightarrow w^2 = (1+z)(1+z^2)$$

$z = -1$, $z = \pm i$ 为分支点.

