走马观花看莫比乌斯变换

Mobius transformation. (分式线性变换)

$$f(z) = \frac{az+b}{cz+d}.$$

$a, b, c, d \in \mathbb{C}$.

约定. $ad-bc \neq 0$ $\left[\begin{array}{l} \text{if } ad=bc. \\ f(z) = \text{const} \end{array}\right]$

Property 1. $f(z)$ is 1-1.

$$\frac{az+b}{cz+d} = w \qquad z = \frac{dw-b}{a-cw} \leftarrow \left(w \neq \frac{a}{c}\right)$$

Note. if $c \neq 0$. $f(z)$ is not defined at $z = -\frac{d}{c}$.

Property 2. $f(z)$ can be extended to a function on $S^2 = \mathbb{C} \cup \{\infty\}$, which establishes a 1-1 correspondence between Riemann sphere

Case 1. If $c \neq 0$.
$$f\left(-\frac{d}{c}\right) = \infty$$
$$f(\infty) = \frac{a}{c}$$

Case 2 If $c = 0$ $\quad f(z) = \dfrac{az+b}{d} \quad f(\infty) = \infty$.

Property 3. Mobius transform maps circles to circles. $|z-p| = R$

(line is a special circle)

Property 4. Mobius transform is a conformal map. (共形的映射)

$f(z) = \dfrac{az+b}{cz+d}$ is a composition of $\to$

(i) $z \mapsto z + \frac{d}{c}$, 这是一个平移;

(ii) $z \mapsto (1/z)$;

(iii) $z \mapsto -\frac{(ad-bc)}{c^2}z$, 这是一个伸缩和一个旋转;

(iv) $z \mapsto z + \frac{a}{c}$, 这是另一个平移.

basic transformations.

$z \to 1/z$ is called a 复反演 (Inversion).

$re^{i\theta} \to \frac{1}{r}e^{i\theta}$, 关于单位圆周的反演.

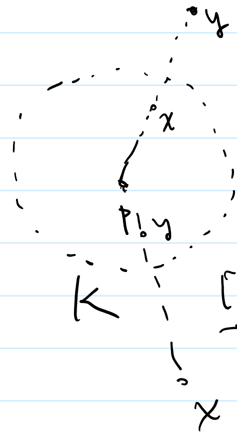
$\frac{1}{r}e^{i\theta} \to \frac{1}{r}e^{-i\theta}$, 及 共轭 关于x轴的反射.

(把圆周映成圆周
反共形.

$\frac{1}{r}e^{i\theta} \to \frac{1}{r}e^{-i\theta}$，取 共轭 关于x轴的反射. ⌊反射.

$\bar{z} = x - yi$

关于反演. 以p为圆心, R为半径的反演：保持K不动，交换圆的内外.



把x变为 px连线上的一点y 满足 $[px]\cdot[py]=R^2$

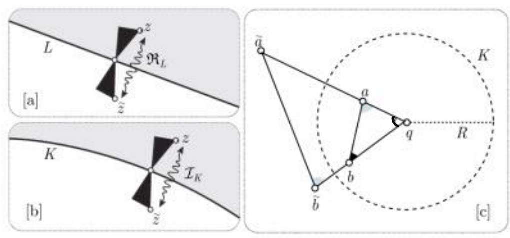


图 3-2

再加上公共角 $\angle aqb = \angle \tilde{a}q\tilde{b}$, 即得 $R^2=[pa]\cdot[p\tilde{a}]=[pb]\cdot[p\tilde{b}]$

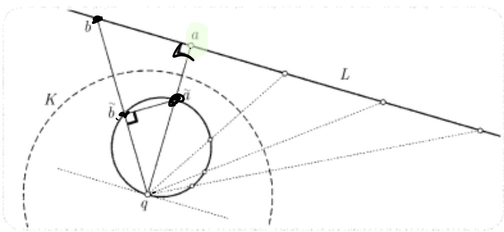
若对一个以 q 为中心的圆周的反演，将其两个点 a 和 b 映为 $\tilde{a}$ 和 $\tilde{b}$，则三角形 aqb 与 $\tilde{b}q\tilde{a}$ 相似.

Prop. 平面上关于任意圆周K的反演把圆周变为圆周。

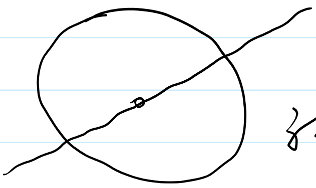
若直线 L 不经过 K 的中心 q，则它对 K 的反演把 L 映为一个经过 q 的圆周.

$L \to$ 以 $q\tilde{a}$ 为直径的圆周

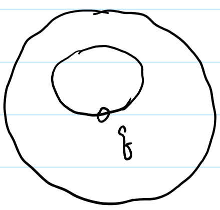
$\forall b \in L$



$\measuredangle p\tilde{b}\tilde{a} = \pi/2 \Rightarrow \tilde{b}$ 落在以 $q\tilde{a}$ 为直径的圆周上.

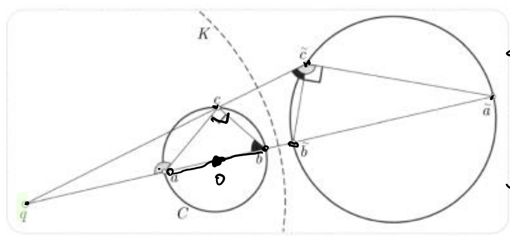


经过q的直线 映成自己.



把过q的圆周映成一条不过q的直线.

若圆周 C 不经过 K 的中心 q，则对 K 的反演将 C 映为另一个也不过 q 的圆周.



$\triangle qac = \triangle q\tilde{c}\tilde{a}$

$\measuredangle qbc = \angle q\tilde{c}\tilde{b}$

$\measuredangle \tilde{b}\tilde{c}\tilde{a} = \measuredangle q\tilde{c}\tilde{a} - \measuredangle q\tilde{c}\tilde{b} = \measuredangle qac - \measuredangle qbc = \measuredangle acb = \pi/2$

$C \to$ 以 $\tilde{a}\tilde{b}$ 为直径的圆周上

key: $\measuredangle \tilde{b}\tilde{c}\tilde{a} = \pi/2$

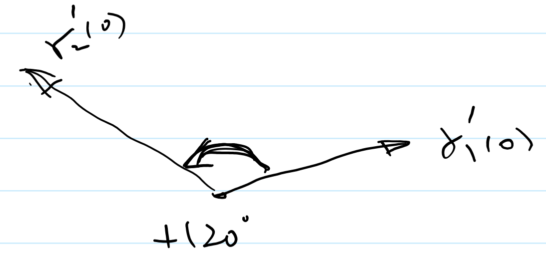
$M: \mathbb{C} \to \mathbb{C}$ is called a conformal map if it preserves angles. i.e. $\forall p \in \mathbb{C}$. $\forall \gamma_1, \gamma_2$ two smooth pathes passing through p. $\gamma_1(0)=\gamma_2(0)=p$.

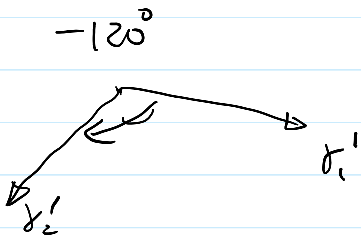
iff it preserves angles. i.e. $\forall p \in \mathbb{C}$, $\forall \gamma_1, \gamma_2$ two smooth pathes passing through p. $\gamma_1(0) = \gamma_2(0) = p$.

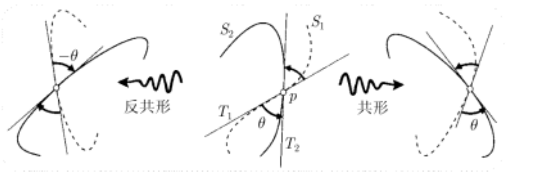
$$\measuredangle_p \gamma_1 \gamma_2 = \measuredangle_{M(p)} M(\gamma_1) M(\gamma_2)$$

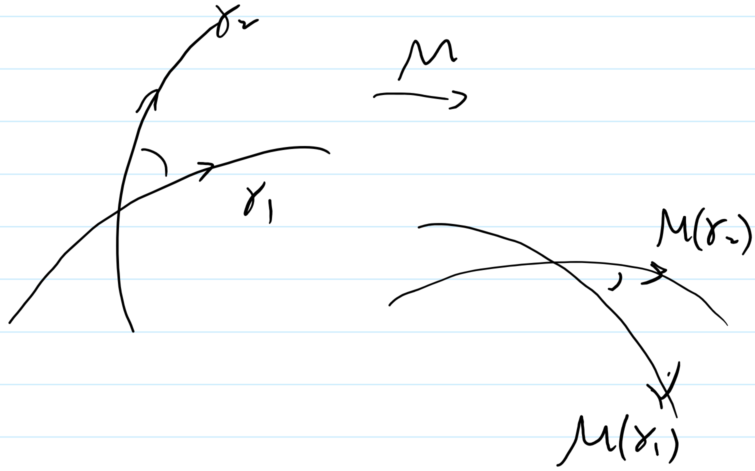
[$\forall p \in \mathbb{C}$ let γ_1, γ_2 be two smooth pathes passing through p.

$$\measuredangle_p \gamma_1\gamma_2 := \measuredangle \gamma_1'(0)\,\gamma_2'(0) \in [-\pi, \pi] \begin{cases} \text{positive if } \gamma_1'(0) \to \gamma_2'(0) \text{ counterclockwise} \\ \text{Negative if } \gamma_1'(0) \to \gamma_2'(0) \text{ clockwise} \end{cases}$$]









在对 K 的反演下, 每个正交于 K 的圆必映为自己.

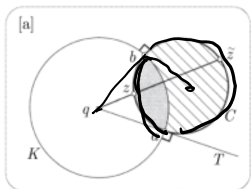
$q \to q$
$b \to b$
$qa \to qa$

$C \to$ 圆周

C 和 qa 相切

↓

$\tilde{C}$ 和 qa 相切. $\tilde{C}$ 和 qb 相切.



$\Rightarrow \tilde{C} = C$

对圆周的反演是反共形映射.

$\forall p \neq q$ $\forall v$ p点处的切向量 $\exists$唯一的圆周经过 p 和 v 相切. 且和 K 正交.

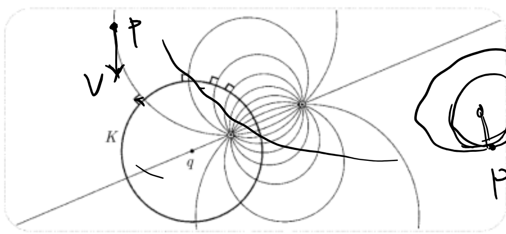
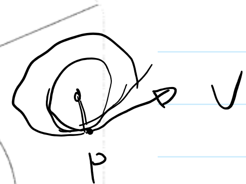
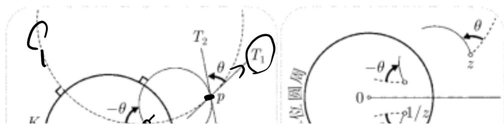


图 3-9



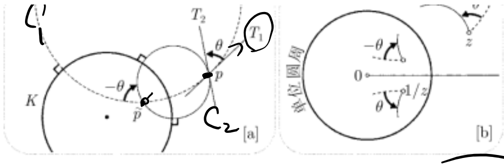
$p \in C_1$ $\quad \tilde{p} \in \tilde{C_1} = C_1$
$p \in C_2$ $\quad \tilde{p} \in \tilde{C_2} = C_2$



$p \in C_1 \qquad \tilde{p} \in \tilde{C}_1 = C_1$

$p \in C_2 \qquad \tilde{p} \in \tilde{C}_2 = C_2$

$\Rightarrow p, \tilde{p} \in C_1 \cap C_2.$



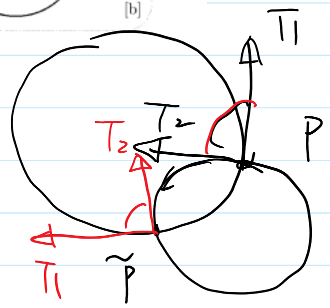
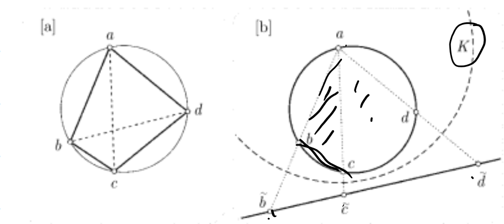


图 3-18a 画出了一个内接于圆的四边形 *abcd*. 托勒密[①]给出了一个美丽的事实：两对对边乘积之和等于两对角线之乘积. 这就是著名的**托勒密定理**. 用符号来写就是

$$[ad][bc] + [ab][cd] = [ac][bd].$$



以 a 为圆心作反演

$$[\tilde{b}\tilde{c}] + [\tilde{c}\tilde{d}] = [\tilde{b}\tilde{d}]$$

$$\frac{[bc]}{[\tilde{b}\tilde{c}]} = \frac{[ab]}{[ac]} \qquad \frac{[cd]}{[\tilde{c}\tilde{d}]} = \frac{[ad]}{[ac]} \qquad \frac{[bd]}{[\tilde{b}\tilde{d}]} = \frac{[ab]}{[ad]}$$

$$\Rightarrow [ab][cd] + [ad][bc] = [ac][bd]$$

Thm $\forall\, q, r, s \in \mathbb{C}\cup\{\infty\}$, $\forall\, q', r', s' \in \mathbb{C}\cup\{\infty\}$ $\exists!$ (唯一) Mobius transformation M s.t. $M(q) = q'$, $M(r) = r'$, $M(s) = s'$.

Proof: On Uniqueness (不证存在)

$f(z) = z$

Observation: A nontrivial Mobius transformation has at most two fixed-points.

$$\frac{az+b}{cz+d} = z \quad \to \quad cz^2 + (d-a)z - b = 0$$

$\Rightarrow$ 该方程至多两个解.

Suppose $\exists\, M_1 \neq M_2$ s.t.

$$M_i(q) = q', \quad M_i(r) = r', \quad M_i(s) = s'. \qquad i = 1, 2.$$

$$\Rightarrow \quad M_1^{-1} \circ M_2(q) = q$$

Note. $M_1^{-1} \circ M_2$ is also

$\Rightarrow \quad M_1 \circ M_2(q) = q$

$M_1^{-1} \circ M_2(r) = r$

$M_1^{-1} \circ M_2(s) = s$

Note. $M_1^{-1} \circ M_2$ is also a Mobius transformation which fixes q, r, s.

$\Rightarrow \quad M_1^{-1} \circ M_2 = z$

$\Rightarrow \quad M_1 = M_2$. contradiction.

$\Rightarrow$ Uniqueness.

On existence. 之:记 (cross ratio)

$$[z, q, r, s] := \frac{(z-q)(r-s)}{(z-s)(r-q)} = f(z)$$

$q \to 0 \quad r \to 1 \quad s \to \infty$

Note. $f(z)$ is a Mobius transform which maps $q \to 0$, $r \to 1$, $s \to \infty$

目标: $q \to q' \quad r \to r' \quad s \to s'$

$q \to 0 \quad r \to 1 \quad s \to \infty$

$q' \to 0 \quad r' \to 1 \quad s' \to \infty$

$(q, r, s) \longrightarrow (0, 1, \infty) \longleftrightarrow (q', r', s')$

$z \longrightarrow w$

If w satisfies

$$[w, q', r', s'] = [z, q, r, s]$$

then $w = w(z)$ is the Mobius transform which maps (q, r, s) to (q', r', s')

Mobius transformation $\to$ group $\longrightarrow$ Matrix.

$$f(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad g \circ f(z) = \frac{a_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) + b_2}{c_2\left(\quad\right) + d_2} = \frac{\boxed{}\, z + \boxed{}}{\boxed{}\, z + \boxed{}}$$

$$g(z) = \frac{a_2 z + b_2}{\ldots}$$

$$g(z)=\frac{a_2 z+b_2}{c_2 z+d_2} \qquad g\circ f(z)=\frac{\qquad}{c_2(\qquad)+d_2}=\frac{\qquad}{\boxed{\ }z+\boxed{\ }}$$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \begin{pmatrix} \Box & \Box \\ \Box & \Box \end{pmatrix} \overset{?}{=} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbb{C}^2$.