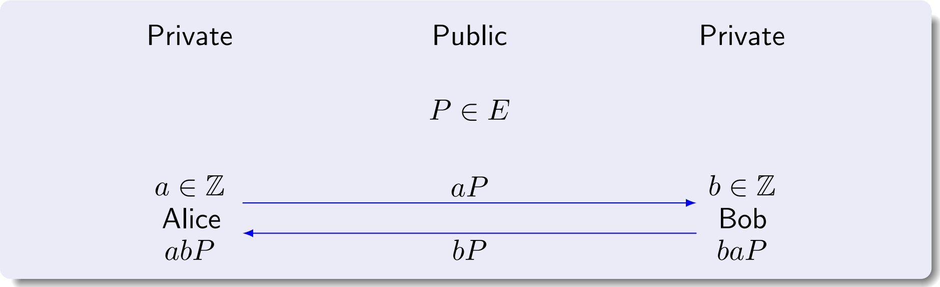
Efficient post-quantum commutative group actions from orientations of large discriminant

Marc Houben

Inria Bordeaux

10 December 2025

Elliptic Curve Diffie–Hellman (ECDH)



Private Public Private

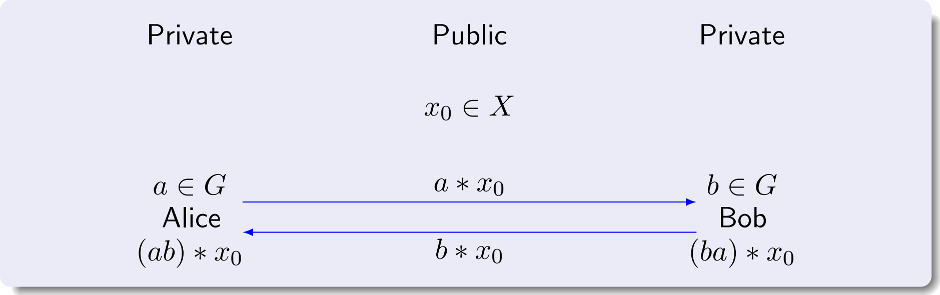
$P \in E$

$a \in \mathbb{Z}$ aP $b \in \mathbb{Z}$

Alice Bob

abP bP baP

Key exchange from a group action $G \to \mathrm{Sym}(X)$



Private

Public

Private

$x_0 \in X$

$a \in G$

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$(ab) * x_0$

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$b * x_0$

$b \in G$

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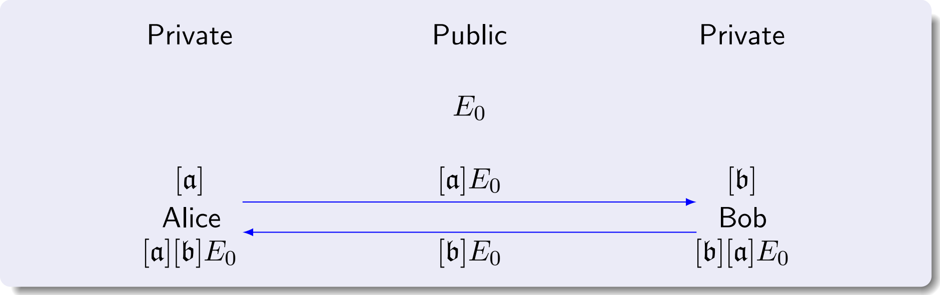
Class group action on elliptic curves



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$[\mathfrak{a}]$

$[\mathfrak{a}]E_0$

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- Convenient building block for advanced protocols, such as: threshold schemes, public key encryption, (advanced) signatures, oblivious transfer, ID protocols, (verifiable) pseudorandom functions, zero-knowledge proofs, quantum money, password authenticated key exchange, updatable encryption.
- Subject to subexponential quantum attacks (Kuperberg's algorithm).

In this work

- Class group actions, how do they work?
- A new representation for orientations
- Mitigating subexponential attacks

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- Every $\sigma \in \mathrm{End}(E)$ is either an integer or

$$\sigma^2 - t\sigma + d = 0,$$

where $\mathrm{Disc}(\sigma) = t^2 - 4d < 0$.

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Theorem

If the $\mathcal{O}$-orientation is primitive, this gives a free action

$$\mathrm{Cl}(\mathcal{O}) \circlearrowright \{(E, \iota)\}/\cong .$$

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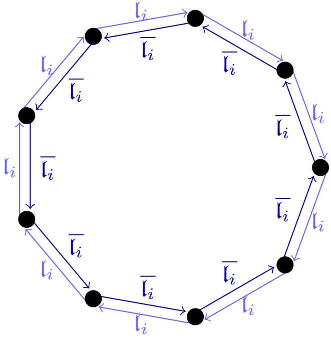
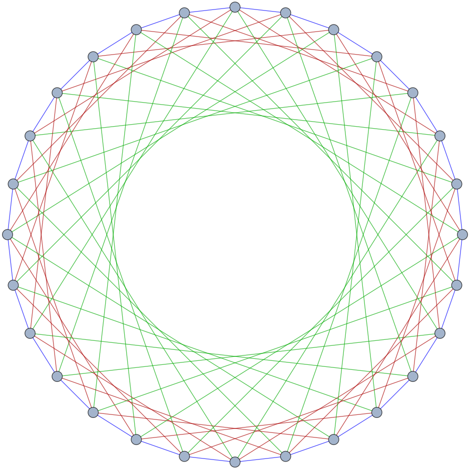


(Connected component of) a supersingular ℓ_i-isogeny graph over $\mathbb{F}_p$.

CSIDH



(Connected component of) a union of supersingular 3-, 5-, and 7-isogeny graphs over $\mathbb{F}_p$.

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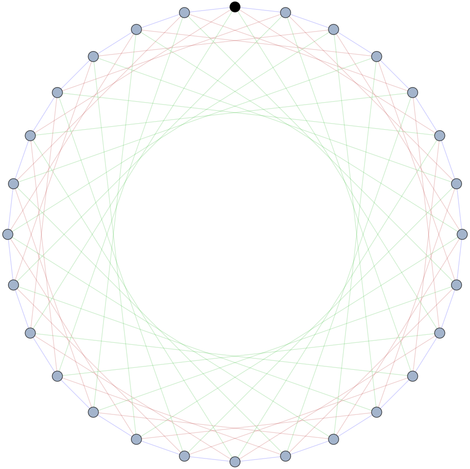

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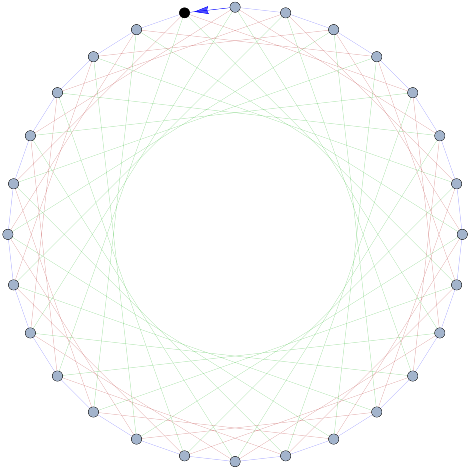

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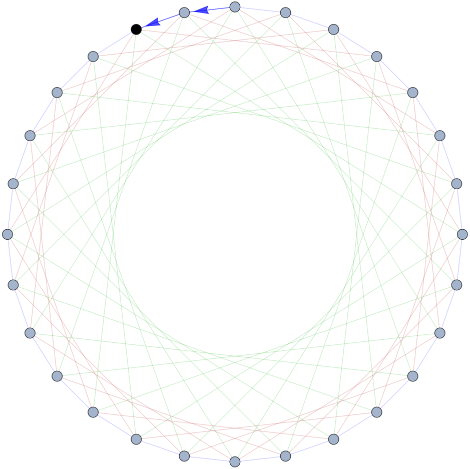

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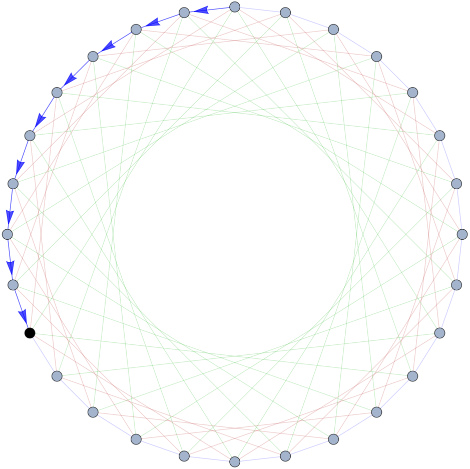

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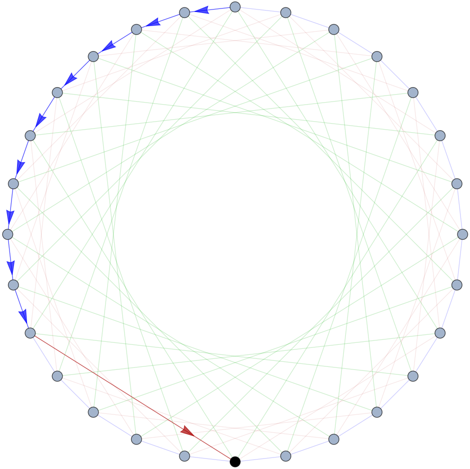

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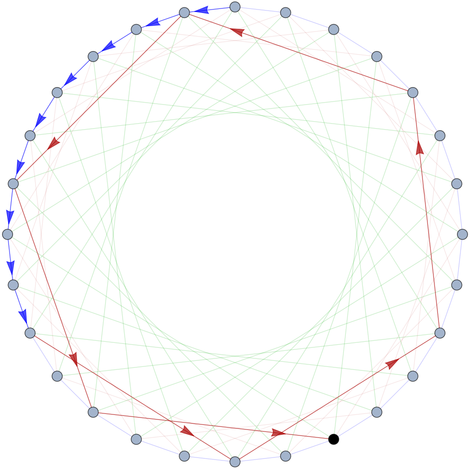

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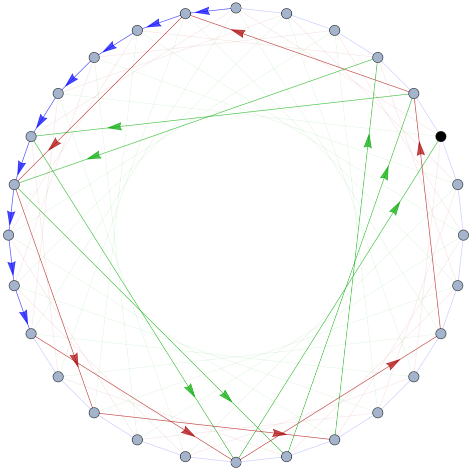

CSIDH-512

$$p = 4 \cdot \underbrace{(3 \cdot 5 \cdot \ldots \cdot 373)}_{73 \text{ consecutive primes}} \cdot 587 - 1 \approx 2^{511}.$$

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Prime bits	f	n	Excluded	Included	Key Space	NIST level
p2048	2^{64}	226	$\{1361\}$	–	2^{221}	1 (aggressive)
p4096	2^{1728}	262	$\{347\}$	$\{1699\}$	2^{256}	1 (conservative)
p5120	2^{2944}	244	$\{227\}$	$\{1601\}$	2^{234}	2 (aggressive)
p6144	2^{3776}	262	$\{283\}$	$\{1693, 1697, 1741\}$	2^{256}	2 (conservative)
p8192	2^{4992}	338	$\{401\}$	$\{2287, 2377\}$	2^{332}	3 (aggressive)
p9216	2^{5440}	389	$\{179\}$	$\{2689, 2719\}$	2^{384}	3 (conservative)

Recent estimates[1] of CSIDH's p for various NIST levels.

[1] Campos, F., Chávez-Saab, J., Chi-Domínguez, J.J., Meyer, M., Reijnders, K., Rodríguez-Henríquez, F., Schwabe, P., Wiggers, T.: Optimizations and practicality of high-security CSIDH. CiC (2024).

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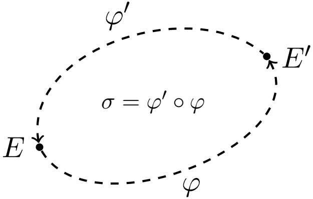
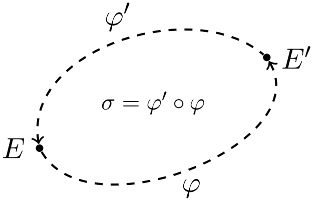


Evaluating a class group action $\equiv$ factoring an endomorphism.



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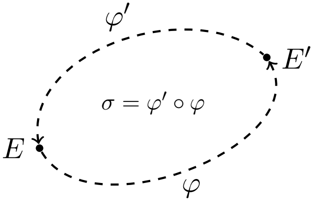

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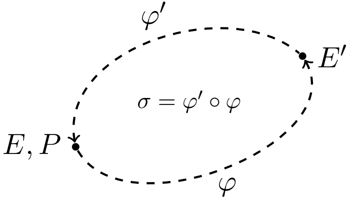

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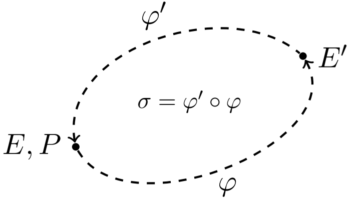

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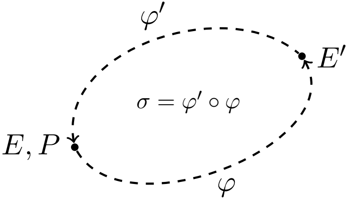

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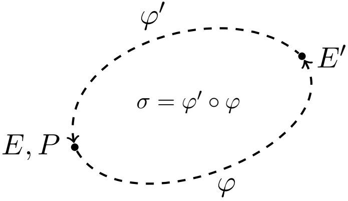

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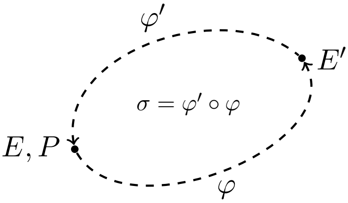

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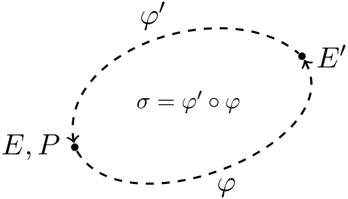

Problems

- What is $P' \in E'[\sigma]$?
- If $P \in E(\mathbb{F}_q)$ then $|\operatorname{Disc}(\mathcal{O})| = 4N(\sigma) - \operatorname{tr}(\sigma)^2 \leq 4N(\sigma) \lesssim 4q$.

CSIDH

already has $|\operatorname{Disc}(\mathcal{O})| = 4p$.

A new orientation representation

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$$\begin{tikzcd}
E_4 \arrow[d, dashed, "\varphi_4"] & E_3 \arrow[l, dashed, "\varphi_3"] \\
E_1 \arrow[r, dashed, "\varphi_1"] & E_2 \arrow[u, dashed, "\varphi_2"']
\end{tikzcd}$$

Splitting σ into four isogenies of degree $\deg \varphi_j = M$.

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$\ker \varphi_j \subseteq E[M] \subseteq E_j(\mathbb{F}_{p^2})$.

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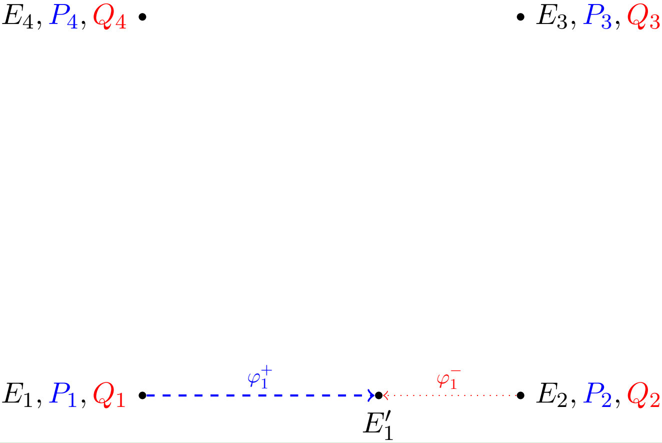
E_1, P_1, Q_1 \arrow[r, shift left, dashed, "\varphi_1"] \arrow[u, shift left, dotted, "\widehat{\varphi_4}"] & E_2, P_2, Q_2 \arrow[l, shift left, dotted, "\widehat{\varphi_1}"] \arrow[u, shift left, dashed, "\varphi_2"]
\end{tikzcd}$$

$$\ker \varphi_j = \langle P_j \rangle \leftrightarrow (1, \ldots, 1), \qquad \ker \widehat{\varphi_j} = \langle Q_{j+1} \rangle \leftrightarrow (-1, \ldots, -1).$$

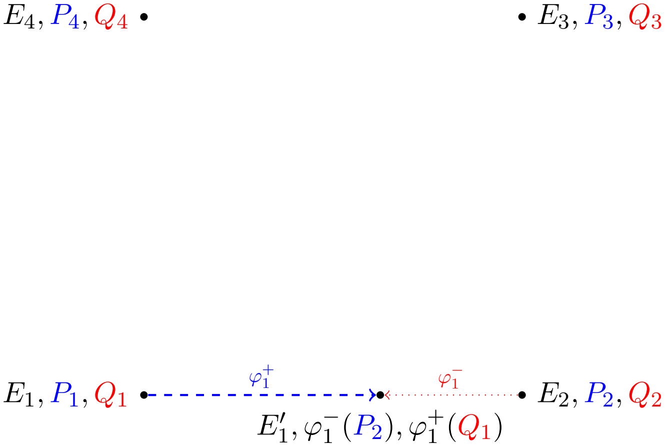
Acting by a non-trivial ideal class



Example

$$\varphi_1^+ \leftrightarrow (1,0,1,1,0,\ldots), \qquad \varphi_1^- \leftrightarrow (0,-1,0,0,-1,\ldots).$$

Acting by a non-trivial ideal class



E_4, P_4, Q_4

E_3, P_3, Q_3

E_1, P_1, Q_1

φ_1^+

φ_1^-

E_2, P_2, Q_2

$E_1', \varphi_1^-(P_2), \varphi_1^+(Q_1)$

Example

$$\varphi_1^+ \leftrightarrow (1, 0, 1, 1, 0, \ldots), \qquad \varphi_1^- \leftrightarrow (0, -1, 0, 0, -1, \ldots).$$

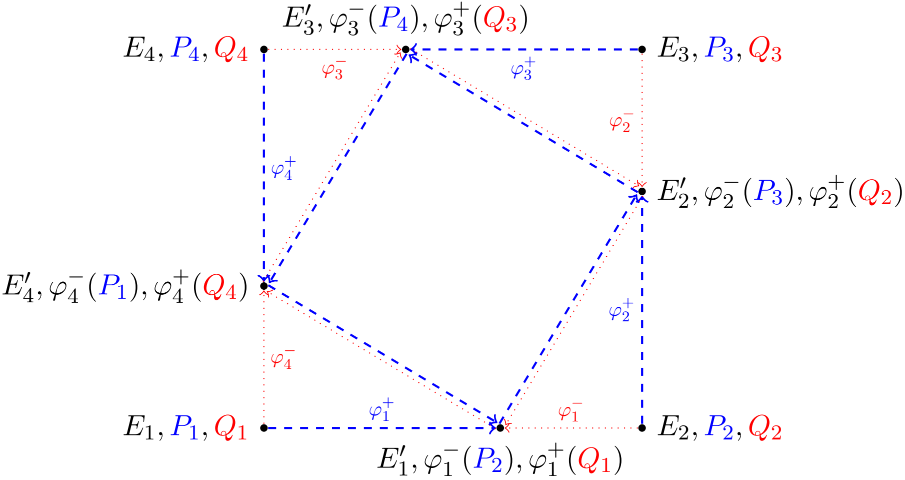
Acting by a non-trivial ideal class

$$\begin{tikzcd}
E_4, P_4, Q_4 \arrow[r, dotted, "\varphi_3^-"'] \arrow[dd, dashed, "\varphi_4^+"] & E_3', \varphi_3^-(P_4), \varphi_3^+(Q_3) & E_3, P_3, Q_3 \arrow[l, dashed, "\varphi_3^+"] \arrow[d, dotted, "\varphi_2^-"'] \\
 & & E_2', \varphi_2^-(P_3), \varphi_2^+(Q_2) \\
E_4', \varphi_4^-(P_1), \varphi_4^+(Q_4) & & \\
E_1, P_1, Q_1 \arrow[r, dashed, "\varphi_1^+"] \arrow[u, dotted, "\varphi_4^-"] & E_1', \varphi_1^-(P_2), \varphi_1^+(Q_1) & E_2, P_2, Q_2 \arrow[l, dotted, "\varphi_1^-"'] \arrow[uu, dashed, "\varphi_2^+"]
\end{tikzcd}$$

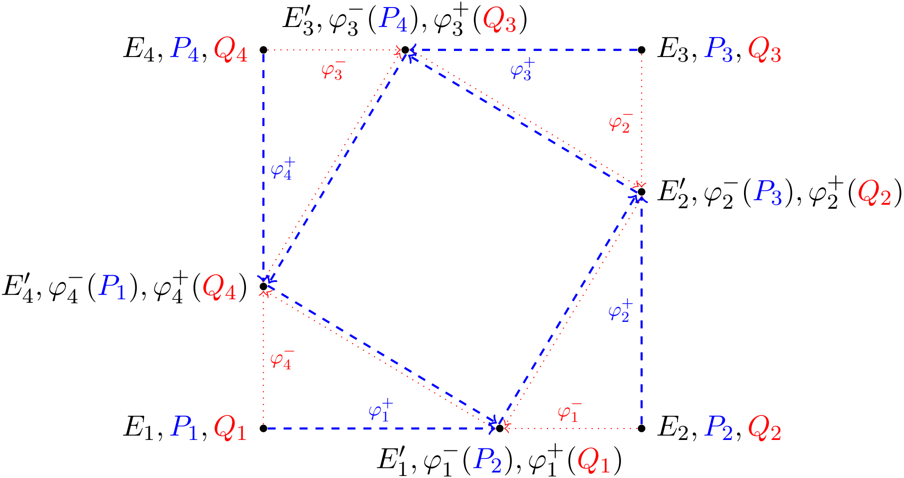
Example

$$\varphi_1^+ \leftrightarrow (1, 0, 1, 1, 0, \ldots), \qquad \varphi_1^- \leftrightarrow (0, -1, 0, 0, -1, \ldots).$$

Acting by a non-trivial ideal class


$E'_3, \varphi_3^-(P_4), \varphi_3^+(Q_3)$
E_4, P_4, Q_4
E_3, P_3, Q_3
φ_3^-
φ_3^+
φ_2^-
φ_4^+
$E'_2, \varphi_2^-(P_3), \varphi_2^+(Q_2)$
$E'_4, \varphi_4^-(P_1), \varphi_4^+(Q_4)$
φ_2^+
φ_4^-
φ_1^+
φ_1^-
E_1, P_1, Q_1
E_2, P_2, Q_2
$E'_1, \varphi_1^-(P_2), \varphi_1^+(Q_1)$

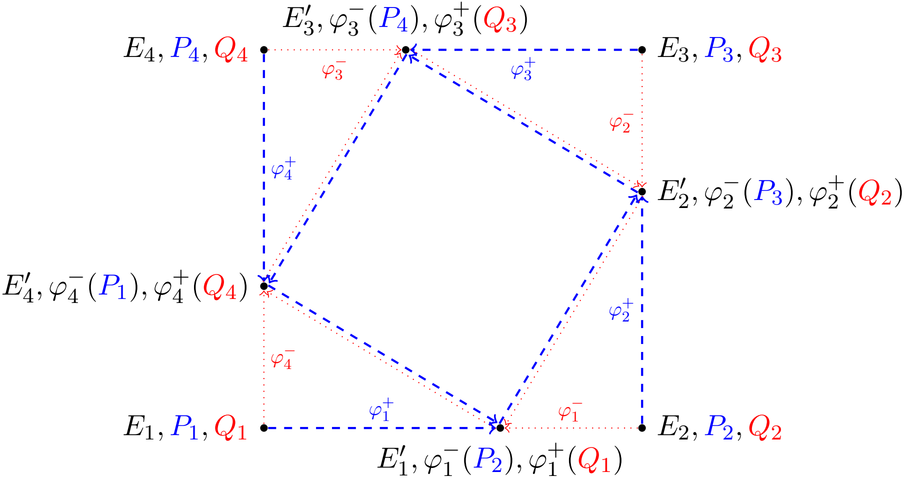
Acting by a non-trivial ideal class


$E'_3, \varphi_3^-(P_4), \varphi_3^+(Q_3)$
E_4, P_4, Q_4
E_3, P_3, Q_3
φ_3^-
φ_3^+
φ_2^-
φ_4^+
$E'_2, \varphi_2^-(P_3), \varphi_2^+(Q_2)$
$E'_4, \varphi_4^-(P_1), \varphi_4^+(Q_4)$
φ_2^+
φ_4^-
φ_1^+
φ_1^-
E_1, P_1, Q_1
E_2, P_2, Q_2
$E'_1, \varphi_1^-(P_2), \varphi_1^+(Q_1)$

Result

Orientation data on E'_1.

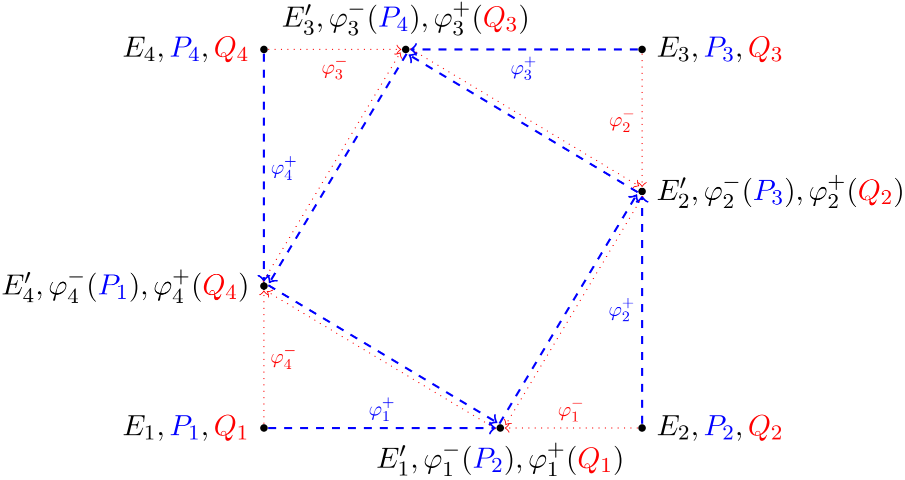
Acting by a non-trivial ideal class


$E_3', \varphi_3^-(P_4), \varphi_3^+(Q_3)$
E_4, P_4, Q_4
E_3, P_3, Q_3
φ_3^-
φ_3^+
φ_2^-
φ_4^+
$E_2', \varphi_2^-(P_3), \varphi_2^+(Q_2)$
$E_4', \varphi_4^-(P_1), \varphi_4^+(Q_4)$
φ_2^+
φ_4^-
φ_1^+
φ_1^-
E_1, P_1, Q_1
E_2, P_2, Q_2
$E_1', \varphi_1^-(P_2), \varphi_1^+(Q_1)$

Result

Orientation data on E_1'. Iterate to act by any exponent vector $\in \mathbb{Z}^n$.

Acting by a non-trivial ideal class

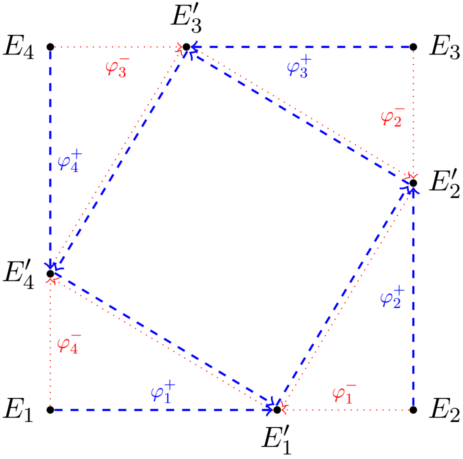

$E_3', \varphi_3^-(P_4), \varphi_3^+(Q_3)$
E_4, P_4, Q_4
E_3, P_3, Q_3
φ_3^-
φ_3^+
φ_2^-
φ_4^+
$E_2', \varphi_2^-(P_3), \varphi_2^+(Q_2)$
$E_4', \varphi_4^-(P_1), \varphi_4^+(Q_4)$
φ_2^+
φ_4^-
φ_1^+
φ_1^-
E_1, P_1, Q_1
E_2, P_2, Q_2
$E_1', \varphi_1^-(P_2), \varphi_1^+(Q_1)$

Cost of one iteration

Four ℓ_i-isogenies for every i, i.e. one evaluation of σ.

Properties of the algorithm

- Constant time
- Deterministic
- Dummy free
- Branchless
- Perfectly parallelizable



Public key compression

$$\begin{tikzcd}
E_4, P_4, Q_4 \arrow[r, dotted, shift left, "\widehat{\varphi_3}"] \arrow[d, dashed, shift left, "\varphi_4"] & E_3, P_3, Q_3 \arrow[l, dashed, shift left, "\varphi_3"] \arrow[d, dotted, shift left, "\widehat{\varphi_2}"] \\
E_1, P_1, Q_1 \arrow[r, dashed, shift left, "\varphi_1"] \arrow[u, dotted, shift left, "\widehat{\varphi_4}"] & E_2, P_2, Q_2 \arrow[l, dotted, shift left, "\widehat{\varphi_1}"] \arrow[u, dashed, shift left, "\varphi_2"]
\end{tikzcd}$$

Public key compression

$$\begin{tikzcd}
E_4, P_4, Q_4 \arrow[r, dotted, bend left, "\widehat{\varphi_3}"] \arrow[d, dashed, "\varphi_4"] & E_3, P_3, Q_3 \arrow[l, dashed, "\varphi_3"] \arrow[d, dotted, bend left, "\widehat{\varphi_2}"] \\
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\end{tikzcd}$$

Compressed orientation data

$(E_1, \iota) \leftrightarrow (E_1, \langle P_1 \rangle, \ldots, \langle P_r \rangle)$.

Public key compression

$$\begin{tikzcd}
E_4, \textcolor{blue}{P_4}, \textcolor{red}{Q_4} \arrow[rr, dotted, red, bend left, "\widehat{\varphi_3}"] \arrow[dd, dashed, blue, "\varphi_4"] & & E_3, \textcolor{blue}{P_3}, \textcolor{red}{Q_3} \arrow[ll, dashed, blue, "\varphi_3"] \arrow[dd, dotted, red, bend left, "\widehat{\varphi_2}"] \\
& & \\
E_1, \textcolor{blue}{P_1}, \textcolor{red}{Q_1} \arrow[rr, dashed, blue, "\varphi_1"] \arrow[uu, dotted, red, bend left, "\widehat{\varphi_4}"] & & E_2, \textcolor{blue}{P_2}, \textcolor{red}{Q_2} \arrow[ll, dotted, red, bend left, "\widehat{\varphi_1}"] \arrow[uu, dashed, blue, "\varphi_2"]
\end{tikzcd}$$

Compressed orientation data

$(E_1, \iota) \leftrightarrow (E_1, \langle P_1 \rangle, \ldots, \langle P_r \rangle)$.

$\implies$ public keys of size $\approx 2\log_2(p) + \log_2(\mathrm{Disc}(\mathcal{O}))$.

Numbers

Let $q = p^2$, where

$$p = 2^{12} \cdot 3^6 \cdot 5^4 \cdot \underbrace{(7 \cdot 11 \cdot \ldots \cdot 281)}_{57 \text{ consecutive primes}} -1 \approx 2^{409.2}.$$

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Then $E/\mathbb{F}_q : y^2 = x^3 + x$ can be oriented by $\mathcal{O} = \mathbb{Z}[\sigma]$, where

$$N(\sigma) = \prod_i \ell_i^{5e_i}, \qquad \mathrm{tr}(\sigma) = 1800301,$$

such that

$$|\operatorname{Disc}(\sigma)| = 4N(\sigma) - \mathrm{tr}(\sigma) \approx 2^{2048} \text{ is prime.}$$

More numbers

Let $p \approx 2^{255.45}$ such that

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$$\begin{aligned}
p+1 &= 2^5 \cdot 7^2 \cdot 11 \cdot 23 \cdot 29 \cdot 37 \cdot 41 \cdot 43 \cdot 53 \cdot 61 \cdot 67 \cdot 73 \cdot 79 \cdot 83 \\
&\quad \cdot 103 \cdot 107 \cdot 131 \cdot 137 \cdot 149 \cdot 173 \cdot 199 \cdot 211 \cdot 277 \cdot 307 \\
&\quad \cdot 5370594787 \cdot 1039866451667097907655 9; \\
p-1 &= 2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 47 \cdot 59 \cdot 71 \cdot 89 \cdot 97 \\
&\quad \cdot 101 \cdot 109 \cdot 113 \cdot 127 \cdot 139 \cdot 151 \cdot 157 \cdot 163 \\
&\quad \cdot 167 \cdot 179 \cdot 181 \cdot 191 \cdot 193 \cdot 223 \cdot 233 \cdot 269 \\
&\quad \cdot 16793651481272952227055481.
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$\implies$ class group action 7× faster than dCSIDH-4096 (excluding random point sampling), unoptimized in SageMath.

Open problem

Is there an efficient algorithm to validate public keys?

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Equivalently...

can we efficiently *verify* the value of $\mathrm{tr}(\sigma)$ (given an efficient representation of σ)?

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(iii) We can increase $\log(|\operatorname{Disc}(\mathcal{O})|)$ by a factor r for a (parallelizable) cost factor r.

(iv) In particular, there exist families of class group actions more efficient than CSIDH.

Thank you!

https://ia.cr/2025/1098