

Efficient post-quantum commutative group actions from orientations of large discriminant

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Elliptic Curve Diffie–Hellman (ECDH)

Private

Public

Private

$$P \in E$$

$$a \in \mathbb{Z}$$

Alice

$$abP$$

$$aP$$

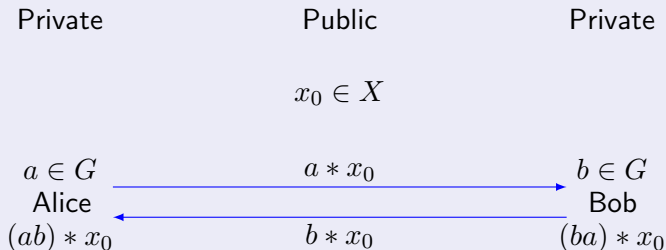
$$b \in \mathbb{Z}$$

Bob

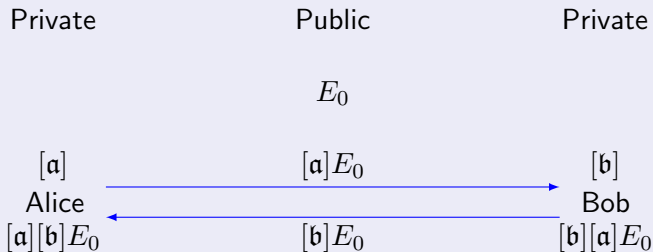
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$$bP$$

Key exchange from a group action $G \rightarrow \text{Sym}(X)$



Class group action on elliptic curves



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- Convenient building block for advanced protocols, such as: threshold schemes, public key encryption, (advanced) signatures, oblivious transfer, ID protocols, (verifiable) pseudorandom functions, zero-knowledge proofs, quantum money, password authenticated key exchange, updatable encryption.
- Subject to subexponential quantum attacks (Kuperberg's algorithm).

In this work

- Class group actions, how do they work?
- A new representation for orientations
- Mitigating subexponential attacks

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- Every $\sigma \in \text{End}(E)$ is either an integer or

$$\sigma^2 - t\sigma + d = 0,$$

where $\text{Disc}(\sigma) = t^2 - 4d < 0$.

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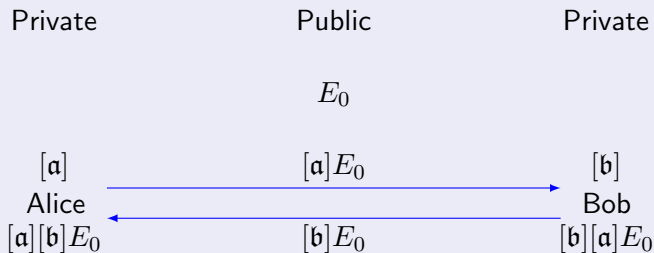
$$\ker \varphi_{\mathfrak{a}} = E[\mathfrak{a}] = \bigcap_{\alpha \in \mathfrak{a}} \ker \iota(\alpha).$$

Theorem

If the \mathcal{O} -orientation is primitive, this gives a free action

$$\text{Cl}(\mathcal{O}) \curvearrowright \{(E, \iota)\} / \cong.$$

Class group action on elliptic curves



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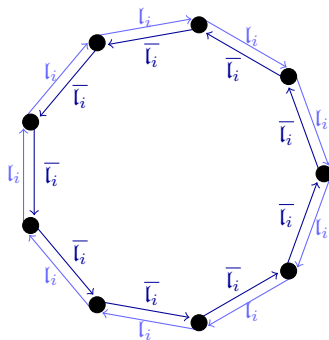
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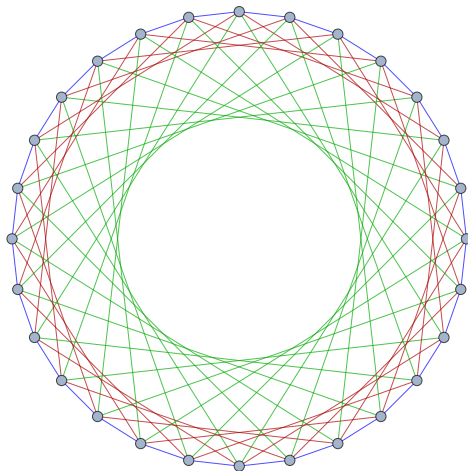
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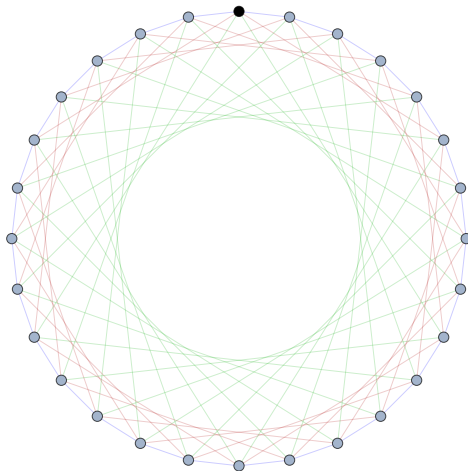
(Connected component of) a supersingular ℓ_i -isogeny graph over \mathbb{F}_p .

CSIDH

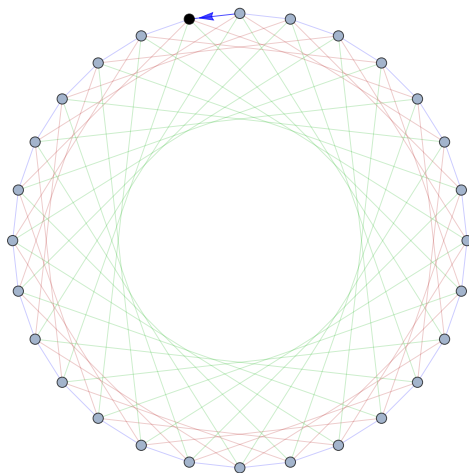


(Connected component of) a union of supersingular **3**-, **5**-, and **7**-isogeny graphs over \mathbb{F}_p .

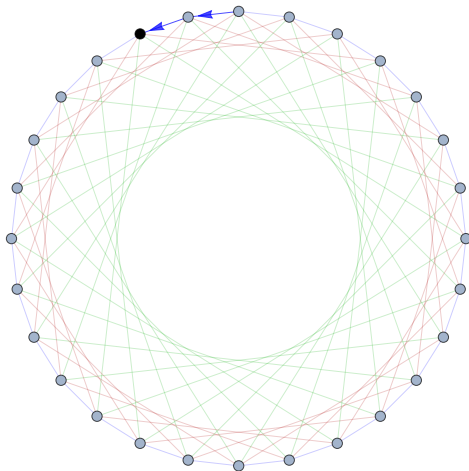
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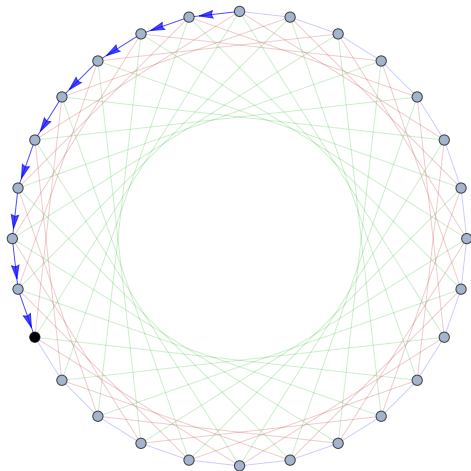
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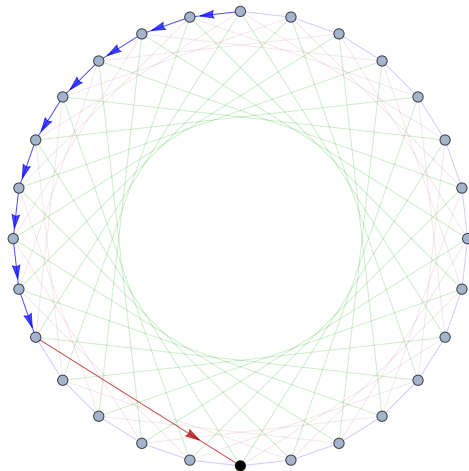
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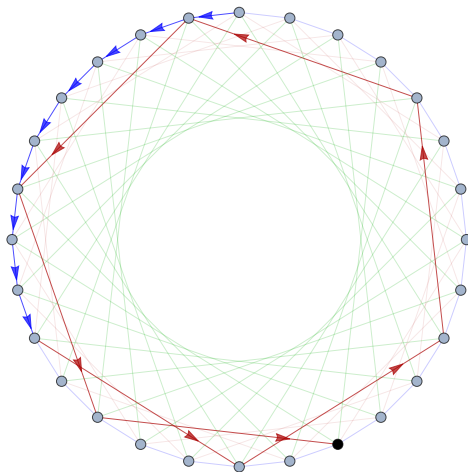
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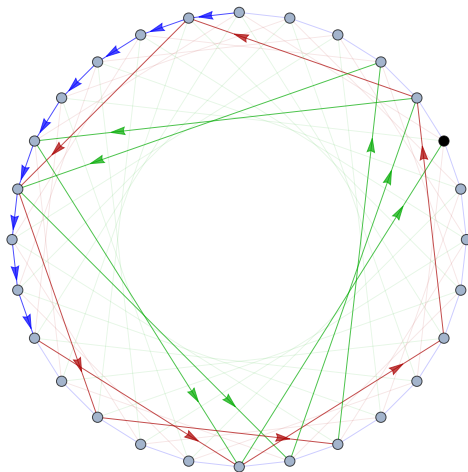
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Prime bits	f	n	Excluded	Included	Key Space	NIST level
p2048	2^{64}	226	{1361}	—	2^{221}	1 (aggressive)
p4096	2^{1728}	262	{347}	{1699}	2^{256}	1 (conservative)
p5120	2^{2944}	244	{227}	{1601}	2^{234}	2 (aggressive)
p6144	2^{3776}	262	{283}	{1693, 1697, 1741}	2^{256}	2 (conservative)
p8192	2^{4992}	338	{401}	{2287, 2377}	2^{332}	3 (aggressive)
p9216	2^{5440}	389	{179}	{2689, 2719}	2^{384}	3 (conservative)

Recent estimates¹ of CSIDH's p for various NIST levels.

¹Campos, F., Chávez-Saab, J., Chi-Domínguez, J.J., Meyer, M., Reijnders, K., Rodríguez-Henríquez, F., Schwabe, P., Wiggers, T.: Optimizations and practicality of high-security CSIDH. CiC (2024).

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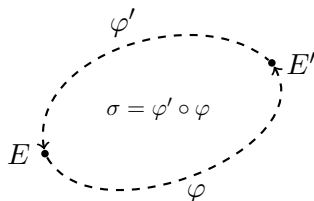
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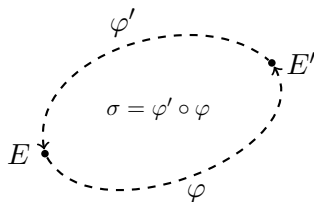
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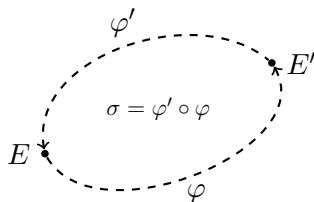


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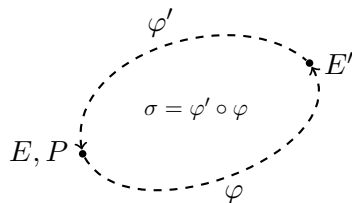
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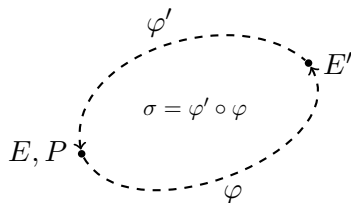
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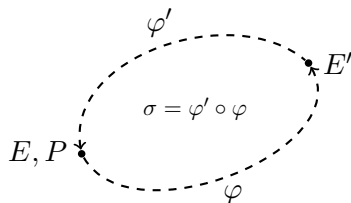


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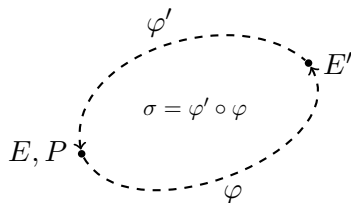


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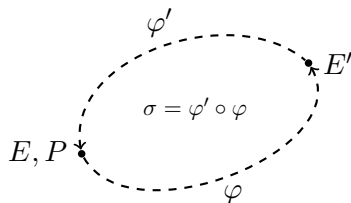
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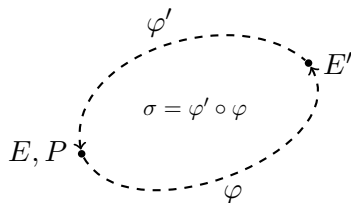
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Problems

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- If $P \in E(\mathbb{F}_q)$ then $|\text{Disc}(\mathcal{O})| = 4N(\sigma) - \text{tr}(\sigma)^2 \leq 4N(\sigma) \lesssim 4q$.

CSIDH

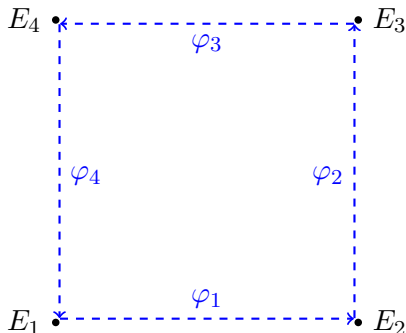
already has $|\text{Disc}(\mathcal{O})| = 4p$.

A new orientation representation

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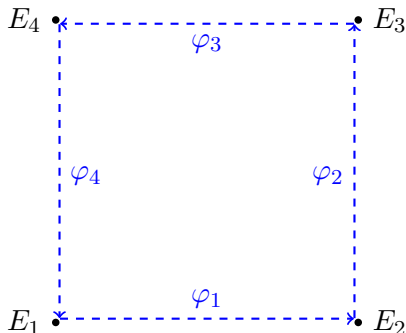
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Splitting σ into four isogenies of degree $\deg \varphi_j = M$.

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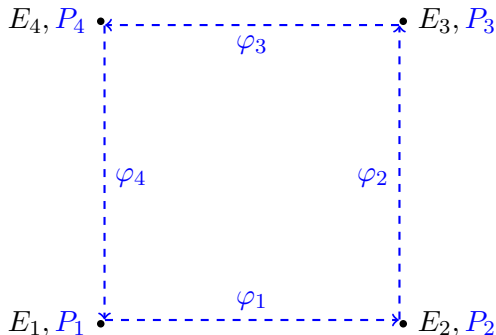
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$$\ker \varphi_j \subseteq E[M] \subseteq E_j(\mathbb{F}_{p^2}).$$

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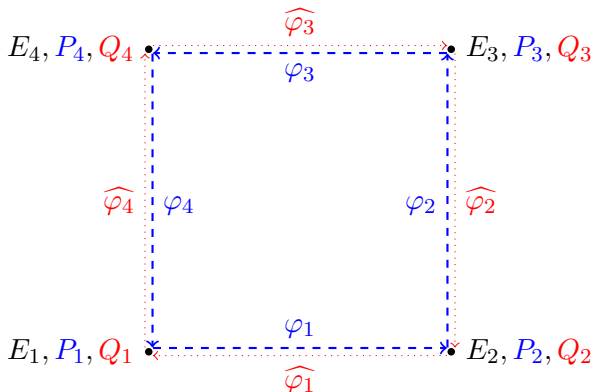
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$$\ker \varphi_j = \langle P_j \rangle \leftrightarrow (1, \dots, 1), \quad \ker \widehat{\varphi}_j = \langle Q_{j+1} \rangle \leftrightarrow (-1, \dots, -1).$$

Acting by a non-trivial ideal class

$$E_4, P_4, Q_4 \bullet$$

$$\bullet E_3, P_3, Q_3$$

$$E_1, P_1, Q_1 \bullet \overset{\varphi_1^+}{\text{---}} \bullet \overset{E'_1}{\text{---}} \overset{\varphi_1^-}{\text{---}} \bullet E_2, P_2, Q_2$$

Example

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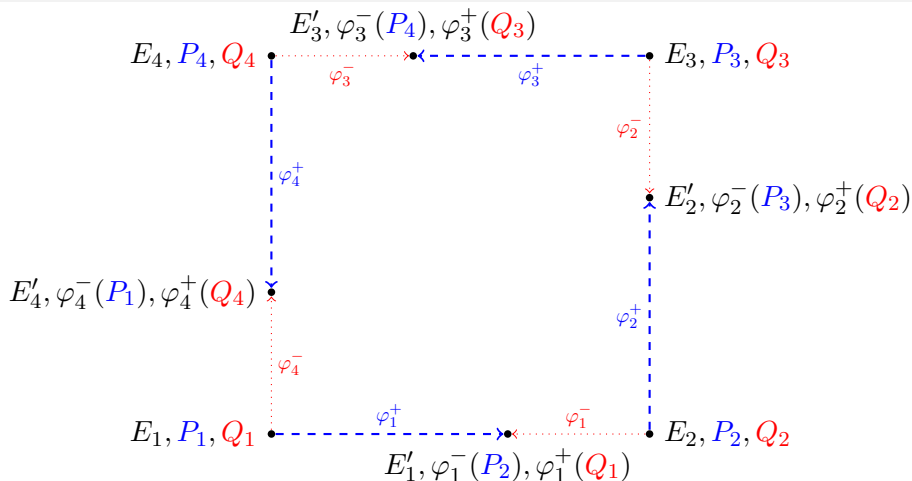
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$$E'_1, \varphi_1^-(P_2), \varphi_1^+(Q_1)$$

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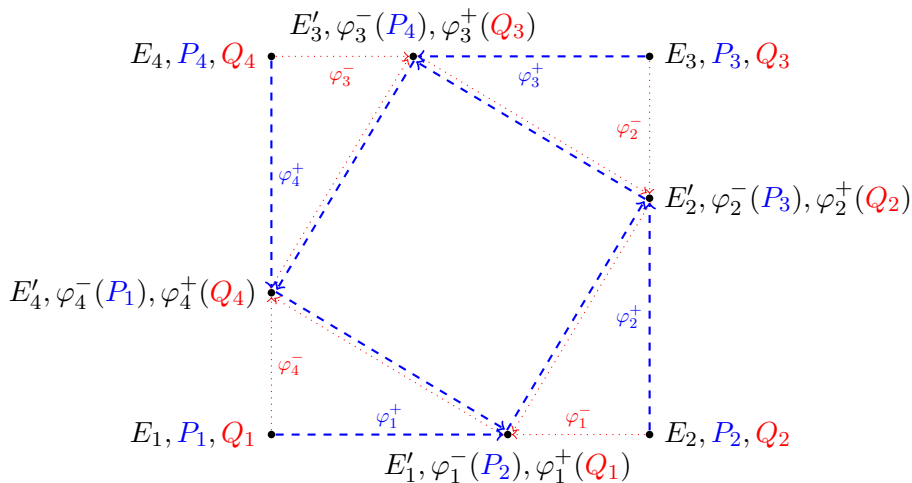
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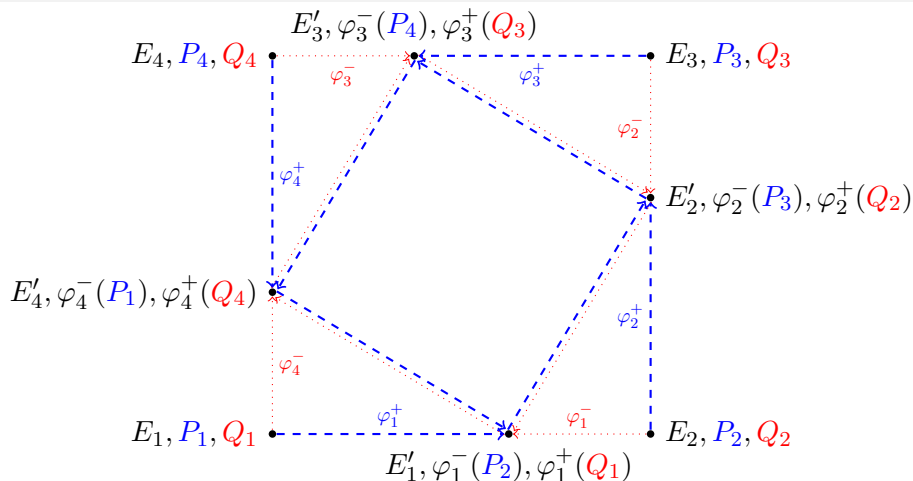
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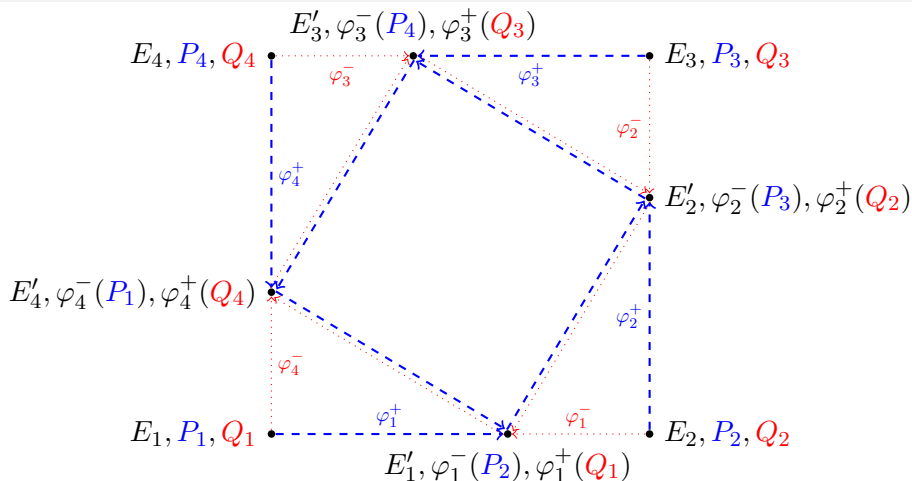
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Result

Orientation data on E'_1 .

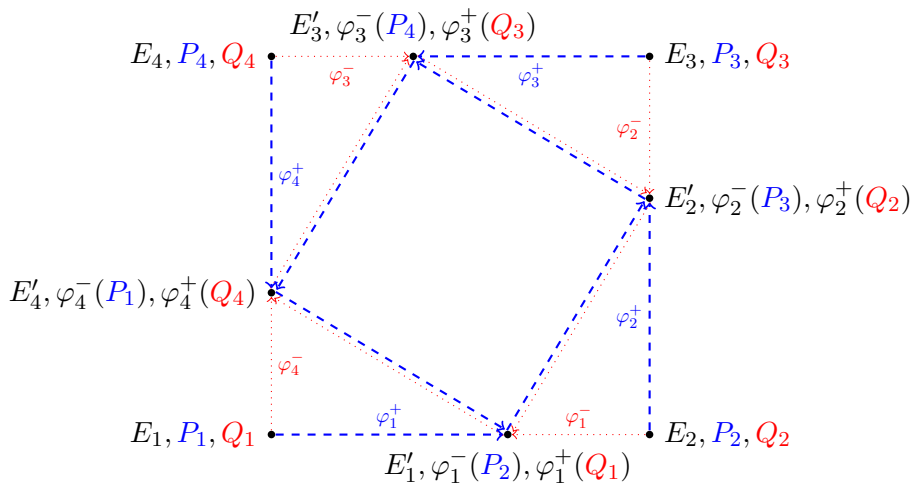
Acting by a non-trivial ideal class



Result

Orientation data on E'_1 . Iterate to act by any exponent vector $\in \mathbb{Z}^n$.

Acting by a non-trivial ideal class

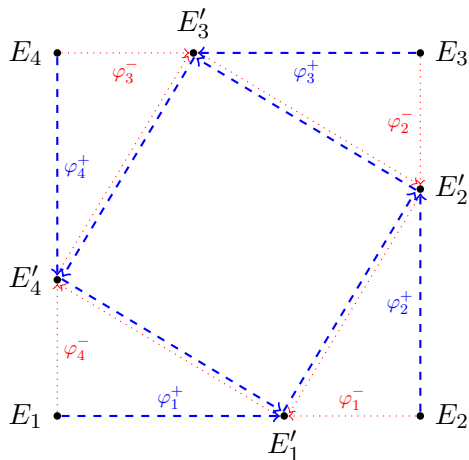


Cost of one iteration

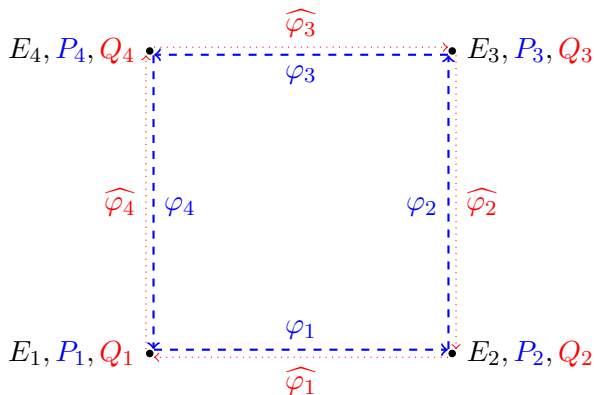
Four ℓ_i -isogenies for every i , i.e. one evaluation of σ .

Properties of the algorithm

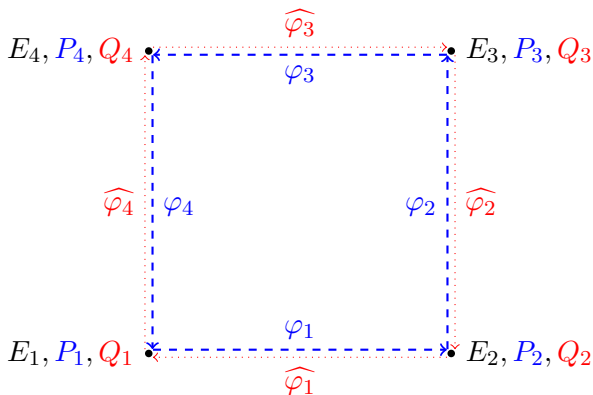
- Constant time
- Deterministic
- Dummy free
- Branchless
- Perfectly parallelizable



Public key compression



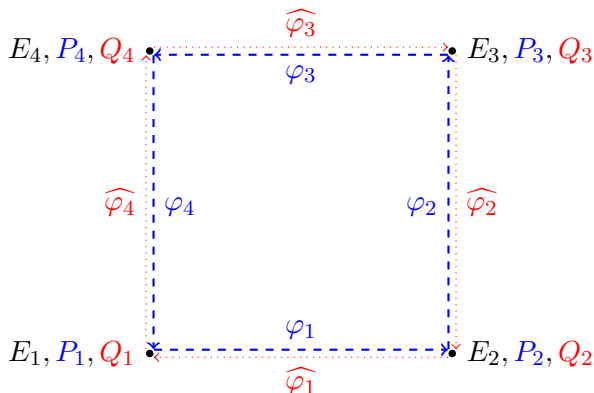
Public key compression



Compressed orientation data

$$(E_1, \iota) \leftrightarrow (E_1, \langle P_1 \rangle, \dots, \langle P_r \rangle).$$

Public key compression



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\implies public keys of size $\approx 2 \log_2(p) + \log_2(\text{Disc}(\mathcal{O}))$.

Numbers

Let $q = p^2$, where

$$p = 2^{12} \cdot 3^6 \cdot 5^4 \cdot \underbrace{(7 \cdot 11 \cdot \dots \cdot 281)}_{57 \text{ consecutive primes}} - 1 \approx 2^{409.2}.$$

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Then $E/\mathbb{F}_q : y^2 = x^3 + x$ can be oriented by $\mathcal{O} = \mathbb{Z}[\sigma]$, where

$$N(\sigma) = \prod_i \ell_i^{5e_i}, \quad \text{tr}(\sigma) = 1800301,$$

such that

$$|\text{Disc}(\sigma)| = 4N(\sigma) - \text{tr}(\sigma) \approx 2^{2048} \text{ is prime.}$$

More numbers

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\implies class group action $7\times$ faster than dCSIDH-4096 (excluding random point sampling), unoptimized in SageMath.

Open problem

Is there an efficient algorithm to validate public keys?

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Equivalently...

can we efficiently *verify* the value of $\text{tr}(\sigma)$ (given an efficient representation of σ)?

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- (ii) This can be done in constant time at the cost of one evaluation of the endomorphism (i.e. by evaluating all of the factors).
- (iii) We can increase $\log(|\text{Disc}(\mathcal{O})|)$ by a factor r for a (parallelizable) cost factor r .
- (iv) In particular, there exist families of class group actions more efficient than CSIDH.

Thank you!



<https://ia.cr/2025/1098>