Efficient post-quantum commutative group actions from orientations of large discriminant

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Elliptic Curve Diffie–Hellman (ECDH)

Private	Public	Private
	$P \in E$	
$a \in \mathbb{Z}$	$aP \longrightarrow$	$b \in \mathbb{Z}$
Alice	$\longleftarrow$	Bob
abP	bP	baP

Key exchange from a group action $G \to \mathrm{Sym}(X)$

Private	Public	Private
	$x_0 \in X$	
$a \in G$	$a * x_0$	$b \in G$
Alice		Bob
$(ab) * x_0$	$b * x_0$	$(ba) * x_0$

Class group action on elliptic curves

Private	Public	Private
	E_0	
$[\mathfrak{a}]$	$[\mathfrak{a}]E_0$	$[\mathfrak{b}]$
Alice		Bob
$[\mathfrak{a}][\mathfrak{b}]E_0$	$[\mathfrak{b}]E_0$	$[\mathfrak{b}][\mathfrak{a}]E_0$

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- Convenient building block for advanced protocols, such as: threshold schemes, public key encryption, (advanced) signatures, oblivious transfer, ID protocols, (verifiable) pseudorandom functions, zero-knowledge proofs, quantum money, password authenticated key exchange, updatable encryption.
- Subject to subexponential quantum attacks (Kuperberg's algorithm).

In this work

- Class group actions, how do they work?
- A new representation method for orientations
- Mitigating subexponential attacks

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- The set of endomorphisms forms a ring $\mathrm{End}(E)$.
- Every $\sigma \in \mathrm{End}(E)$ is either an integer or

$$\sigma^2 - t\sigma + d = 0,$$

where $\mathrm{Disc}(\sigma) = t^2 - 4d < 0$.

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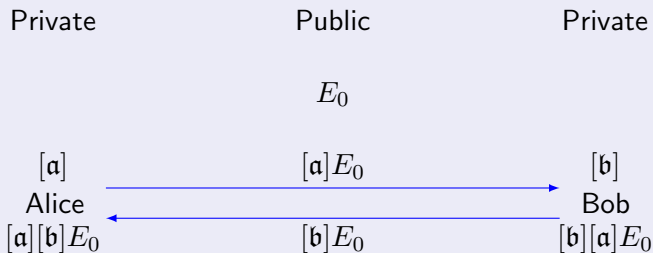
$$\ker \varphi_{\mathfrak{a}} = E[\mathfrak{a}] = \bigcap_{\alpha \in \mathfrak{a}} \ker \iota(\alpha).$$

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Theorem

If the $\mathcal{O}$-orientation is primitive, this gives a free action

$$\mathrm{Cl}(\mathcal{O}) \curvearrowright \{(E, \iota)\}/\cong .$$

Class group action on elliptic curves

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Suppose $\#E(\mathbb{F}_p) = p + 1 = 4 \cdot \prod_{i=1}^{n} \ell_i$.

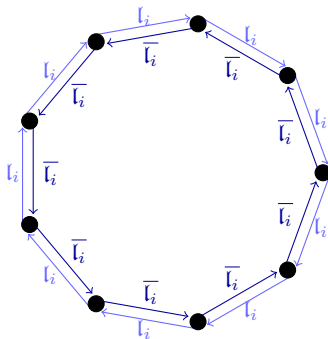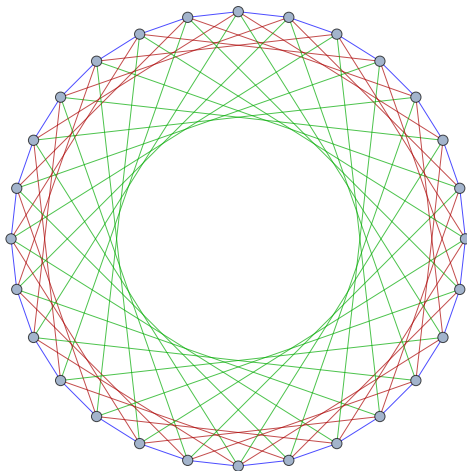Let $E/\mathbb{F}_p$ be an elliptic curve, $\mathcal{O} = \mathbb{Z}[\pi] \hookrightarrow \mathrm{End}(E)$.

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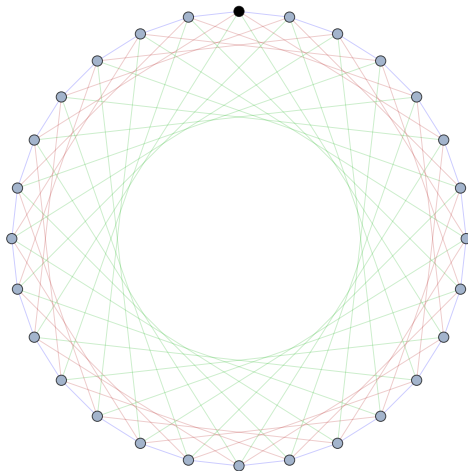$$(\ell_i) = (\ell_i, \pi - 1)(\ell_i, \pi + 1) = \mathfrak{l}_i \overline{\mathfrak{l}_i}.$$

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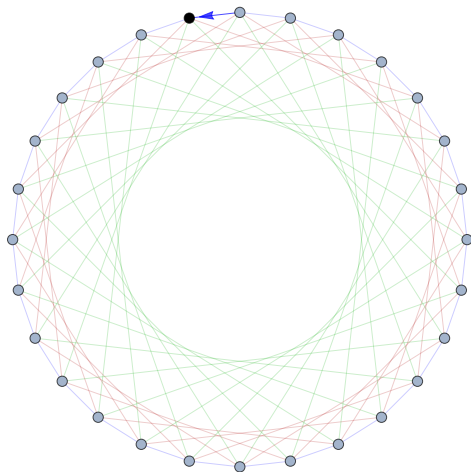(Connected component of) a supersingular ℓ_i-isogeny graph over $\mathbb{F}_p$.

CSIDH

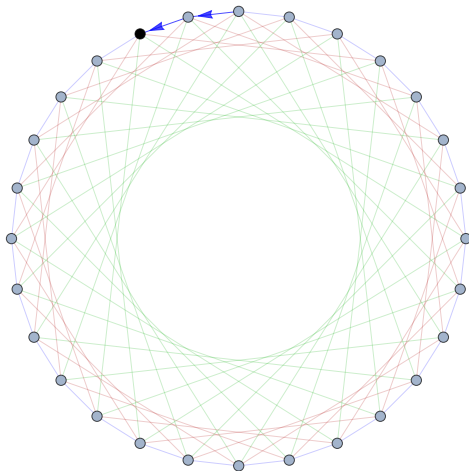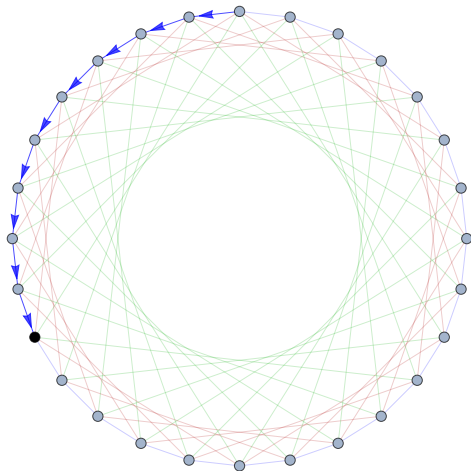(Connected component of) a union of supersingular 3-, 5-, and 7-isogeny graphs over $\mathbb{F}_p$.
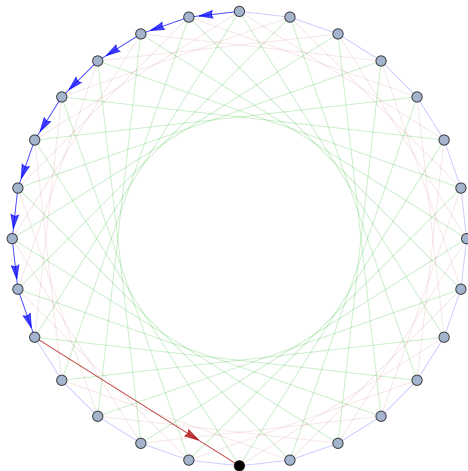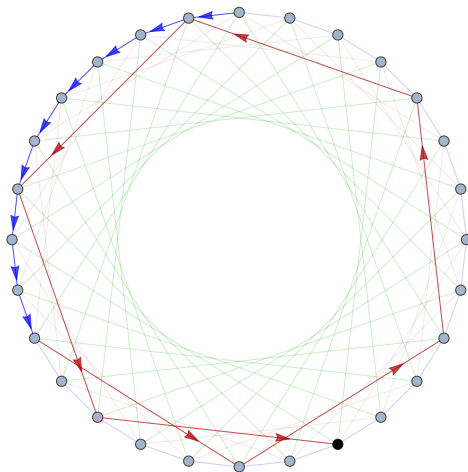
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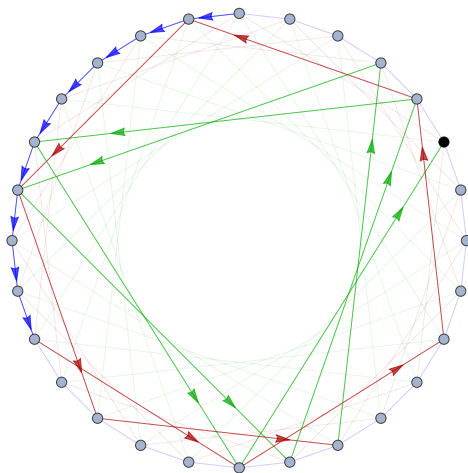(i) $\mathsf{sk_A} = (a_1, \ldots, a_{74}) \in \{-5, \ldots, 5\}^{74}$; $\mathsf{pk_A} = E_A = \prod_i [\mathfrak{l}_i]^{a_i} E_0$.

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Prime bits	f	n	Excluded	Included	Key Space	NIST level
p2048	2^{64}	226	$\{1361\}$	$-$	2^{221}	1 (aggressive)
p4096	2^{1728}	262	$\{347\}$	$\{1699\}$	2^{256}	1 (conservative)
p5120	2^{2944}	244	$\{227\}$	$\{1601\}$	2^{234}	2 (aggressive)
p6144	2^{3776}	262	$\{283\}$	$\{1693, 1697, 1741\}$	2^{256}	2 (conservative)
p8192	2^{4992}	338	$\{401\}$	$\{2287, 2377\}$	2^{332}	3 (aggressive)
p9216	2^{5440}	389	$\{179\}$	$\{2689, 2719\}$	2^{384}	3 (conservative)

Recent estimates[1] of CSIDH's p for various NIST levels.

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[1] Campos, F., Chávez-Saab, J., Chi-Domínguez, J.J., Meyer, M., Reijnders, K., Rodríguez-Henríquez, F., Schwabe, P., Wiggers, T.: Optimizations and practicality of high-security CSIDH. CiC (2024).

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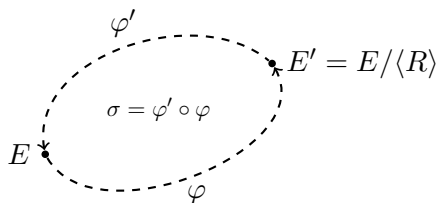$$\sigma(K) \subseteq K \implies \sigma(R) = \lambda R \implies R \in \ker(\sigma - \lambda).$$

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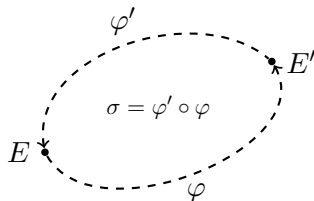 $\implies$ we have that φ is a *factor* of σ.

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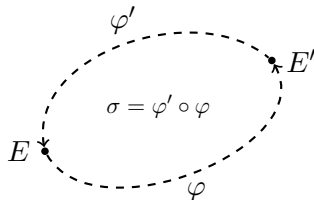 $\implies$ we have that φ is a *factor* of σ.

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In CSIDH

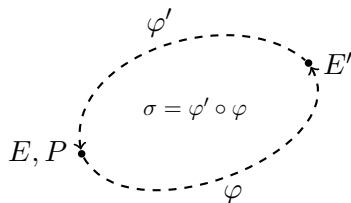we have $N(\pi - 1) = p + 1 = 4 \cdot \prod_{i=1}^{n} \ell_i$.

Representing the orientation

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Give $P \in E(\mathbb{F}_q)$ (of smooth order) such that $E[\sigma] = \langle P \rangle$.
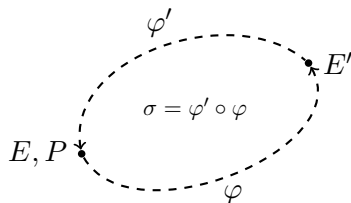
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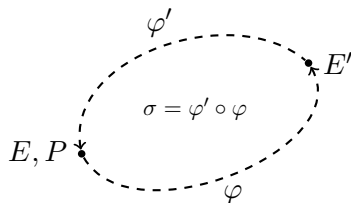$$E'$$
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- What is $P' \in E'[\sigma]$?

Representing the orientation

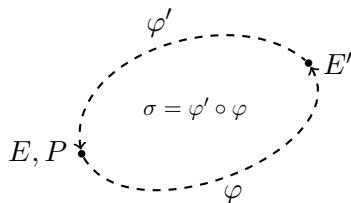Give $P \in E(\mathbb{F}_q)$ (of smooth order) such that $E[\sigma] = \langle P \rangle$.

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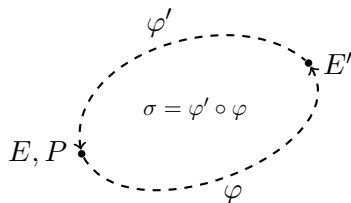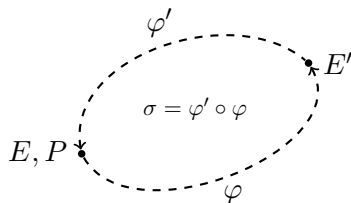
Problems

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Problems

- What is $P' \in E'[\sigma]$?
- If $P \in E(\mathbb{F}_q)$ then $|\operatorname{Disc}(\mathcal{O})| = 4N(\sigma) - \operatorname{tr}(\sigma)^2 \leq 4N(\sigma) \lesssim 4q$.
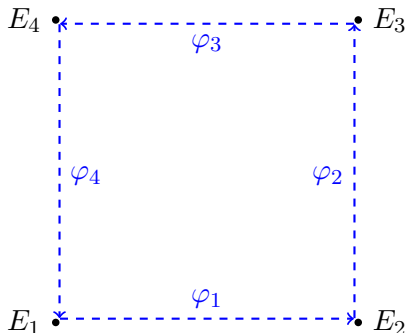
CSIDH

already has $|\operatorname{Disc}(\mathcal{O})| = 4p$.

A new isogeny representation

$$p + 1 = 4 \cdot \prod \ell_i, \qquad N(\sigma) = \prod \ell_i^4 = M^4$$
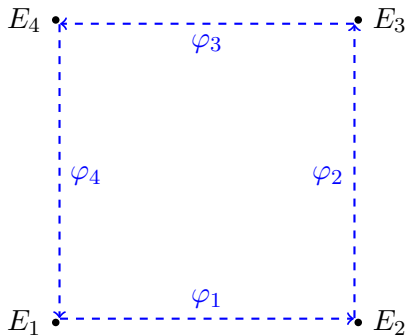
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Splitting σ into 4 isogenies of degree $\deg \varphi_j = M$.

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$\ker \varphi_j \subseteq E[M] \subseteq E_j(\mathbb{F}_{p^2}).$

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E_4, P_4 •- - - - - - - - - - - - φ_3 - - - - - - - - - - - - • E_3, P_3

φ_4

φ_2

E_1, P_1 •- - - - - - - - - - - - φ_1 - - - - - - - - - - - - • E_2, P_2

$$\ker \varphi_j = \langle P_j \rangle \leftrightarrow (1, \ldots, 1),$$

A new isogeny representation

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E_4, P_4, Q_4 $\qquad \widehat{\varphi_3} \qquad$ E_3, P_3, Q_3

φ_3

$\widehat{\varphi_4}$ φ_4 $\qquad\qquad\qquad$ φ_2 $\widehat{\varphi_2}$

φ_1

E_1, P_1, Q_1 $\qquad \widehat{\varphi_1} \qquad$ E_2, P_2, Q_2

$$\ker \varphi_j = \langle P_j \rangle \leftrightarrow (1, \ldots, 1), \qquad \ker \widehat{\varphi_j} = \langle Q_{j+1} \rangle \leftrightarrow (-1, \ldots, -1).$$

Acting by a non-trivial ideal class

E_4, P_4, Q_4 •

• E_3, P_3, Q_3

E_1, P_1, Q_1 • $\xleftarrow{\hspace{1.2cm} \varphi_1^+ \hspace{1.2cm}}$ •$\xrightarrow{\hspace{0.6cm} \varphi_1^- \hspace{0.6cm}}$ • E_2, P_2, Q_2

E_1'

Example

$$\varphi_1^+ \leftrightarrow (1, 0, 1, 1, 0, \dots), \qquad \varphi_1^- \leftrightarrow (0, -1, 0, 0, -1, \dots).$$

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$E_1', \varphi_1^-(P_2), \varphi_1^+(Q_1)$

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Acting by a non-trivial ideal class

E_4, P_4, Q_4

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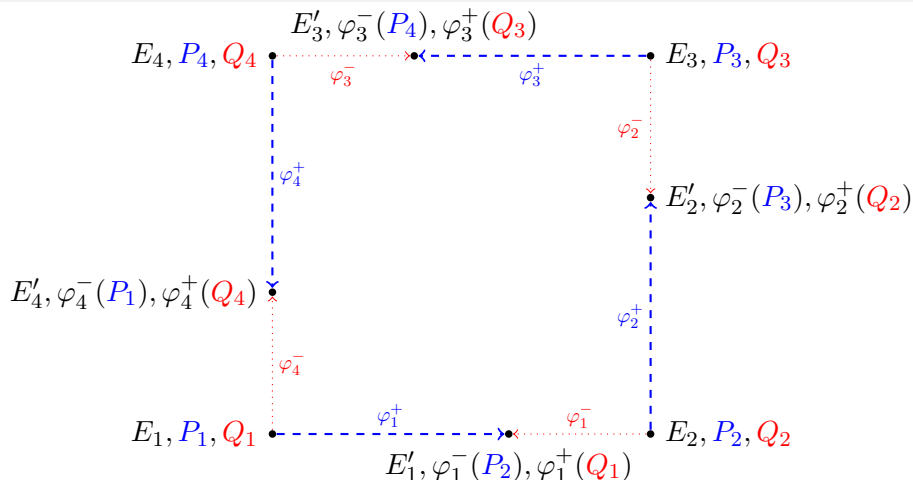
φ_3^-

φ_3^+

E_3, P_3, Q_3

φ_2^-

φ_4^+

$E'_2, \varphi_2^-(P_3), \varphi_2^+(Q_2)$

$E'_4, \varphi_4^-(P_1), \varphi_4^+(Q_4)$

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$E'_1, \varphi_1^-(P_2), \varphi_1^+(Q_1)$

Example

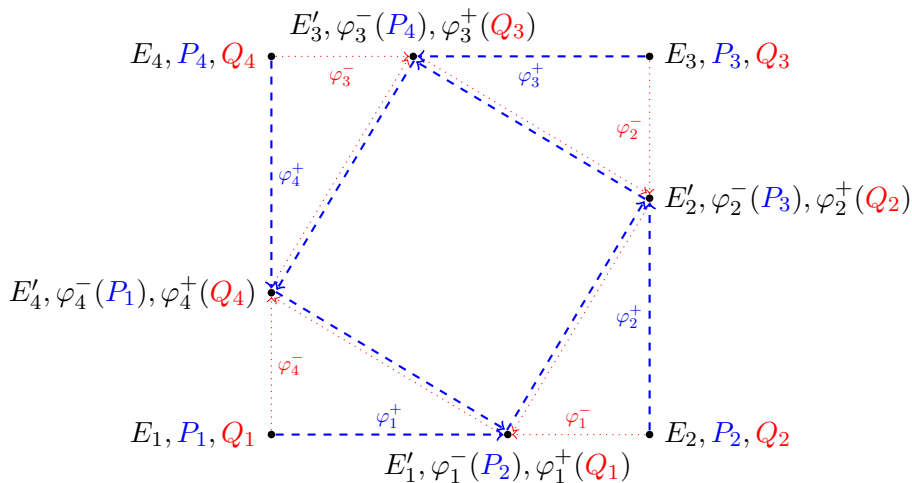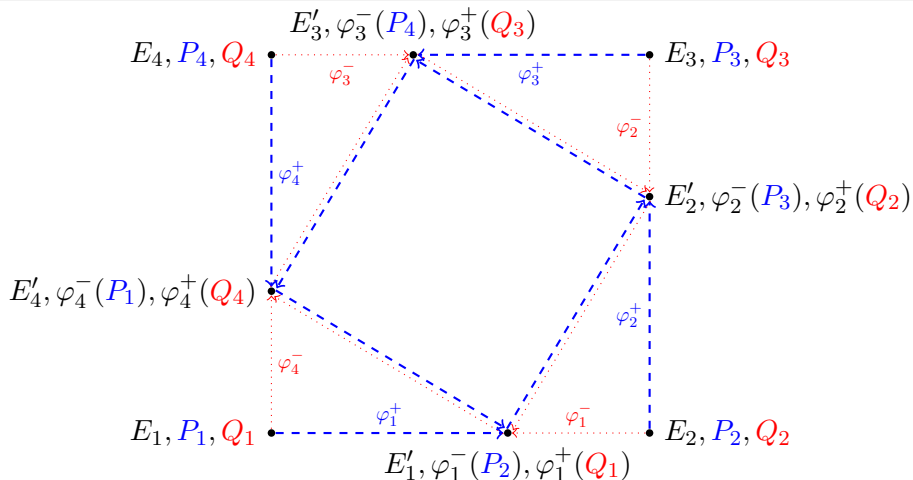$$\varphi_1^+ \leftrightarrow (1, 0, 1, 1, 0, \ldots), \qquad \varphi_1^- \leftrightarrow (0, -1, 0, 0, -1, \ldots).$$

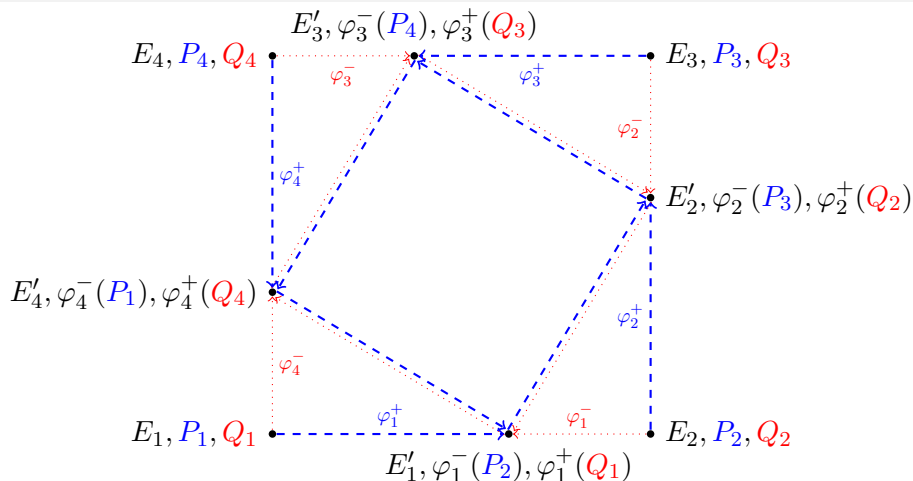# Acting by a non-trivial ideal class

Acting by a non-trivial ideal class

Result

Orientation data on E_1'.

Acting by a non-trivial ideal class
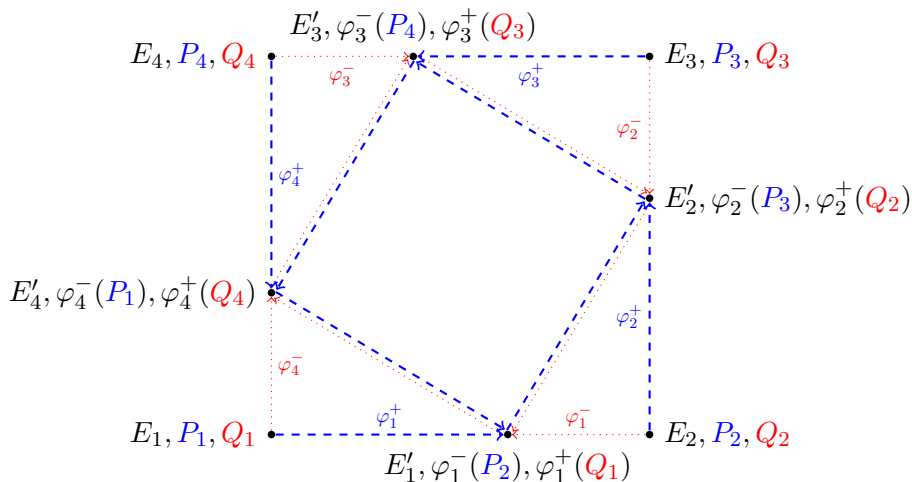

Result

Orientation data on E_1'. Iterate to act by any exponent vector $\in \mathbb{Z}^n$.

Acting by a non-trivial ideal class
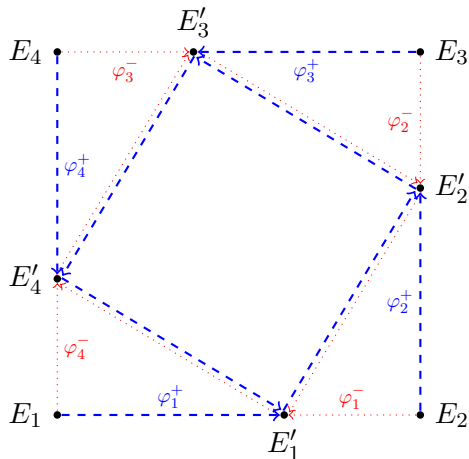

Cost of one iteration

Four ℓ_i-isogenies for every i, i.e. one evaluation of σ.

Properties of the algorithm

- Constant time
- Deterministic
- Dummy free
- Branchless
- Perfectly parallelizable

Public key compression

Public key compression

E_4, P_4, Q_4 $\quad\widehat{\varphi_3}$ $\quad E_3, P_3, Q_3$

φ_3

$\widehat{\varphi_4}$ φ_4 $\qquad\qquad\qquad \varphi_2$ $\widehat{\varphi_2}$

φ_1

E_1, P_1, Q_1 $\qquad\qquad\qquad E_2, P_2, Q_2$

$\widehat{\varphi_1}$

Compressed orientation data

$(E_1, \iota) \leftrightarrow (E_1, \langle P_1 \rangle, \dots, \langle P_r \rangle).$

Public key compression

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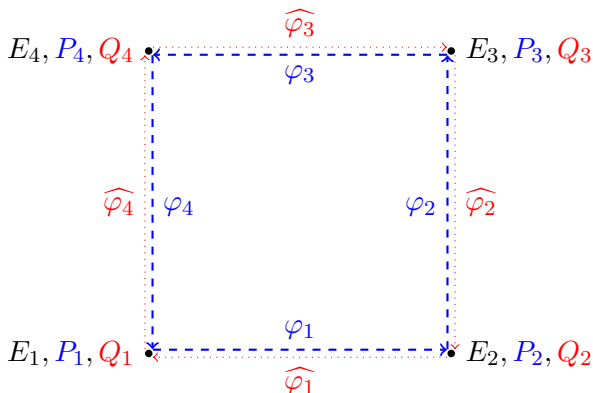
$\implies$ public keys of size $\approx 2 \log_2(p) + \log_2(\mathrm{Disc}(\mathcal{O})).$

Numbers

Let $q = p^2$, where

$$p = 2^{12} \cdot 3^6 \cdot 5^4 \cdot \underbrace{(7 \cdot 11 \cdot \ldots \cdot 281)}_{57 \text{ consecutive primes}} -1 \approx 2^{409.2}.$$

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Then $E/\mathbb{F}_q : y^2 = x^3 + x$ can be oriented by $\mathcal{O} = \mathbb{Z}[\sigma]$, where

$$N(\sigma) = \prod_i \ell_i^{5e_i}, \qquad \mathrm{tr}(\sigma) = 1800301,$$

such that

$$|\mathrm{Disc}(\sigma)| = 4N(\sigma) - \mathrm{tr}(\sigma) \approx 2^{2048} \text{ is prime.}$$

More numbers

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&\quad \cdot 103 \cdot 107 \cdot 131 \cdot 137 \cdot 149 \cdot 173 \cdot 199 \cdot 211 \cdot 277 \cdot 307 \\
&\quad \cdot 5370594787 \cdot 10398664516670979076559; \\
p - 1 &= 2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 47 \cdot 59 \cdot 71 \cdot 89 \cdot 97 \\
&\quad \cdot 101 \cdot 109 \cdot 113 \cdot 127 \cdot 139 \cdot 151 \cdot 157 \cdot 163 \\
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$\implies$ class group action $7\times$ faster than dCSIDH-4096 (excluding random point sampling), unoptimized in SageMath.

Open problem

Is there an efficient algorithm to validate public keys?

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Equivalently...

can we efficiently *verify* the value of $\mathrm{tr}(\sigma)$ (given an efficient representation of σ)?

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(iii) We can increase $\log(|\operatorname{Disc}(\mathcal{O})|)$ by a factor r for a (parallelizable) cost factor r.

(iv) In particular, there exist families of class group actions more efficient than CSIDH.

Thank you!

https://ia.cr/2025/1098