The Note of RB system

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1 Definition

Eigenvalue Equation:

$$\left(q^2 - a^2\right)^3 = -a^2 R$$

Two types of solutions are

$$\begin{array}{rclclcl} W_e & = & A_0 \cos\left(q_0 z\right) & + & A_1 \cosh\left(q_\mathrm{r} z\right) \cos\left(q_\mathrm{i} z\right) & - & A_2 \sinh\left(q_\mathrm{r} z\right) \sin\left(q_\mathrm{i} z\right); \\ W_o & = & A_0 \sin\left(q_0 z\right) & + & A_1 \sinh\left(q_\mathrm{r} z\right) \cos\left(q_\mathrm{i} z\right) & + & A_2 \cosh\left(q_\mathrm{r} z\right) \sin\left(q_\mathrm{i} z\right). \end{array}$$

Thus, the even modes yield

$$\begin{array}{ccccccc} \mathrm{D} W_e & = & -q_0 A_0 \sin\left(q_0 z\right) & + & \left(q_\mathrm{r} A_1 - q_\mathrm{i} A_2\right) \sinh\left(q_\mathrm{r} z\right) \cos\left(q_\mathrm{i} z\right) & - & \left(q_\mathrm{r} A_2 + q_\mathrm{i} A_1\right) \cosh\left(q_\mathrm{r} z\right) \sin\left(q_\mathrm{i} z\right); \\ \mathrm{D}^2 W_e & = & -q_0^2 A_0 \cos\left(q_0 z\right) & + & \left[\left(q_\mathrm{r}^2 - q_\mathrm{i}^2\right) A_1 - 2 q_\mathrm{r} q_\mathrm{i} A_2\right] \cosh\left(q_\mathrm{r} z\right) \cos\left(q_\mathrm{i} z\right) & - & \left[\left(q_\mathrm{r}^2 - q_\mathrm{i}^2\right) A_2 + 2 q_\mathrm{r} q_\mathrm{i} A_1\right] \sinh\left(q_\mathrm{r} z\right) \sin\left(q_\mathrm{i} z\right); \\ \frac{\left(\mathrm{D}^2 - a^2\right) W_e}{\left(q_0^2 + a^2\right)} & = & -A_0 \cos\left(q_0 z\right) & + & \left(\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right) \cosh\left(q_\mathrm{r} z\right) \cos\left(q_\mathrm{i} z\right) & - & \left(\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right) \sinh\left(q_\mathrm{r} z\right) \sin\left(q_\mathrm{i} z\right); \\ \frac{\mathrm{D}\left(\mathrm{D}^2 - a^2\right) W_e}{\left(q_0^2 + a^2\right)} & = & q_0 A_0 \sin\left(q_0 z\right) & + & \left[q_\mathrm{r} \left(\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right) - q_\mathrm{i} \left(\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right)\right] \sinh\left(q_\mathrm{r} z\right) \cos\left(q_\mathrm{i} z\right) & - & \left[q_\mathrm{i} \left(\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right) + q_\mathrm{r} \left(\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right)\right] \cosh\left(q_\mathrm{r} z\right) \sin\left(q_\mathrm{i} z\right); \\ \frac{\left(\mathrm{D}^2 - a^2\right)^2 W_e}{\left(q_0^2 + a^2\right)^2} & = & A_0 \cos\left(q_0 z\right) & + & \left(-\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right) \cosh\left(q_\mathrm{r} z\right) \cos\left(q_\mathrm{i} z\right) & - & \left(-\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right) \sinh\left(q_\mathrm{r} z\right) \sin\left(q_\mathrm{i} z\right). \\ \frac{\mathrm{D}\left(\mathrm{D}^2 - a^2\right)^2 W_e}{\left(q_0^2 + a^2\right)^2} & = & -q_0 A_0 \cos\left(q_0 z\right) & + & \left[q_\mathrm{r} \left(-\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right) - q_\mathrm{i} \left(-\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right)\right] \sinh\left(q_\mathrm{r} z\right) \cos\left(q_\mathrm{i} z\right) & - & \left[q_\mathrm{i} \left(-\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right) + q_\mathrm{r} \left(-\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right)\right] \cosh\left(q_\mathrm{r} z\right) \sin\left(q_\mathrm{i} z\right). \end{array}$$

While the odd modes yield

$$
\begin{aligned}
\mathrm{D}W_o &= q_0 A_0 \cos(q_0 z) &+& \left(q_\mathrm{r} A_1 + q_\mathrm{i} A_2\right) \cosh(q_\mathrm{r} z) \cos(q_\mathrm{i} z) &+& \left(q_\mathrm{r} A_2 - q_\mathrm{i} A_1\right) \sinh(q_\mathrm{r} z) \sin(q_\mathrm{i} z)\,; \\
\mathrm{D}^2 W_o &= -q_0^2 A_0 \sin(q_0 z) &+& \left[\left(q_\mathrm{r}^2 - q_\mathrm{i}^2\right) A_1 + 2 q_\mathrm{r} q_\mathrm{i} A_2\right] \sinh(q_\mathrm{r} z) \cos(q_\mathrm{i} z) &+& \left[\left(q_\mathrm{r}^2 - q_\mathrm{i}^2\right) A_2 - 2 q_\mathrm{r} q_\mathrm{i} A_1\right] \cosh(q_\mathrm{r} z) \sin(q_\mathrm{i} z)\,; \\
\frac{\left(\mathrm{D}^2 - a^2\right) W_o}{\left(q_0^2 + a^2\right)} &= -A_0 \sin(q_0 z) &+& \left(\tfrac{1}{2} A_1 + \tfrac{\sqrt{3}}{2} A_2\right) \sinh(q_\mathrm{r} z) \cos(q_\mathrm{i} z) &+& \left(\tfrac{1}{2} A_2 - \tfrac{\sqrt{3}}{2} A_1\right) \cosh(q_\mathrm{r} z) \sin(q_\mathrm{i} z)\,; \\
\frac{\mathrm{D}\left(\mathrm{D}^2 - a^2\right) W_o}{\left(q_0^2 + a^2\right)} &= -q_0 A_0 \cos(q_0 z) &+& \left[q_\mathrm{r} \left(\tfrac{1}{2} A_1 + \tfrac{\sqrt{3}}{2} A_2\right) + q_\mathrm{i} \left(\tfrac{1}{2} A_2 - \tfrac{\sqrt{3}}{2} A_1\right)\right] \cosh(q_\mathrm{r} z) \cos(q_\mathrm{i} z) &+& \left[-q_\mathrm{i} \left(\tfrac{1}{2} A_1 + \tfrac{\sqrt{3}}{2} A_2\right) + q_\mathrm{r} \left(\tfrac{1}{2} A_2 - \tfrac{\sqrt{3}}{2} A_1\right)\right] \sinh(q_\mathrm{r} z) \sin(q_\mathrm{i} z)\,; \\
\frac{\left(\mathrm{D}^2 - a^2\right)^2 W_o}{\left(q_0^2 + a^2\right)^2} &= A_0 \sin(q_0 z) &+& \left(-\tfrac{1}{2} A_1 + \tfrac{\sqrt{3}}{2} A_2\right) \sinh(q_\mathrm{r} z) \cos(q_\mathrm{i} z) &+& \left(-\tfrac{1}{2} A_2 - \tfrac{\sqrt{3}}{2} A_1\right) \cosh(q_\mathrm{r} z) \sin(q_\mathrm{i} z)\,. \\
\frac{\mathrm{D}\left(\mathrm{D}^2 - a^2\right)^2 W_o}{\left(q_0^2 + a^2\right)^2} &= q_0 A_0 \cos(q_0 z) &+& \left[q_\mathrm{r} \left(-\tfrac{1}{2} A_1 + \tfrac{\sqrt{3}}{2} A_2\right) + q_\mathrm{i} \left(-\tfrac{1}{2} A_2 - \tfrac{\sqrt{3}}{2} A_1\right)\right] \cosh(q_\mathrm{r} z) \cos(q_\mathrm{i} z) &+& \left[-q_\mathrm{i} \left(-\tfrac{1}{2} A_1 + \tfrac{\sqrt{3}}{2} A_2\right) + q_\mathrm{r} \left(-\tfrac{1}{2} A_2 - \tfrac{\sqrt{3}}{2} A_1\right)\right] \sinh(q_\mathrm{r} z) \sin(q_\mathrm{i} z)\,.
\end{aligned}
$$

And the corresponding boundary conditions are

- Rigid Layers / Fixed Temperature
- Rigid Layers / Prescribed Heat
- Free Layers / Fixed Temperature
- Free Layers / Prescribed Heat

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2 Boundary Conditions

2.1 Rigid Layers / Fixed Temperature

$$
\begin{cases}
W\big|_{z=\pm\frac{1}{2}} &= 0 \\
\mathrm{D}W\big|_{z=\pm\frac{1}{2}} &= 0 \\
\left(\mathrm{D}^2 - a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} &= 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
\theta\big|_{z=\pm\frac{1}{2}} &= 0 \\
\left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} &= 0 \\
\mathrm{D}\left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} &= 0
\end{cases}.
$$

2.2 Rigid Layers / Prescribed Heat

$$
\begin{cases}
W\big|_{z=\pm\frac{1}{2}} &= 0 \\
\mathrm{D}W\big|_{z=\pm\frac{1}{2}} &= 0 \\
\mathrm{D}\left(\mathrm{D}^2 - a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} &= 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
\mathrm{D}\theta\big|_{z=\pm\frac{1}{2}} &= 0 \\
\left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} &= 0 \\
\mathrm{D}\left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} &= 0
\end{cases}.
$$

2.3 Free Layers / Fixed Temperature

$$\begin{cases} W\big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}^2 W\Big|_{z=\pm\frac{1}{2}} & = 0 \\ \left(\mathrm{D}^2 - a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} & = 0 \end{cases} \quad \text{and} \quad \begin{cases} \theta\big|_{z=\pm\frac{1}{2}} & = 0 \\ \left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}\left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} & = 0 \end{cases} .$$

2.4 Free Layers / Prescribed Heat

$$\begin{cases} W\big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}^2 W\Big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}\left(\mathrm{D}^2 - a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} & = 0 \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{D}\theta\big|_{z=\pm\frac{1}{2}} & = 0 \\ \left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}\left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} & = 0 \end{cases} .$$
