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The Dynamics of Rayleigh-Bénard Convection

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Contents

Chapter 1

The Introduction to Rayleigh-Bénard Convection

Rayleigh-Bénard convection is the type of convection considered most frequently. It has been well described by the following statement: following statement:

> *"**Rayleigh-Bénard convection** (RBC) is the buoyancy-driven flow of a fluid heated from below and cooled from above."*

The motion of the system is driven by the temperature difference between the top and bottom layers. Due to the heat expansion of matter, liquid near the top layer (colder), shall be "heavier" than liquid near the bottom layer (warmer), so the cold liquid will "fall" down from the top to the bottom by forming a flow. However, once the flow approaches to the bottom layer, the circumstance will heat it up, which can make it "light" again and "rise" up since gravity always reallocate the "lighter" to top while the "heavier" to bottom. Consequently, a convection cycle is accomplished. A typical Rayleigh-Bénard convection is illustrated in Figure 1.1.

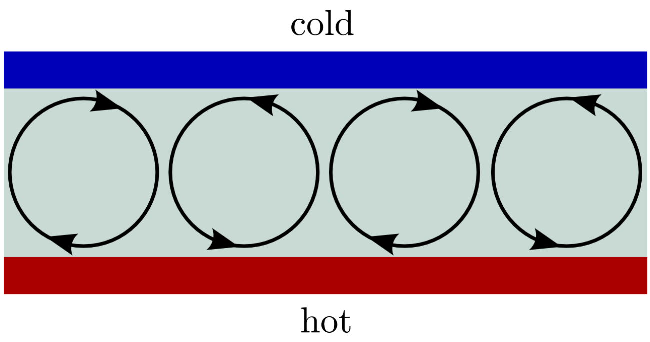

Figure 1.1: Convection Rolls: each circle represent a convection going between the top and bottom layers.

This kind of motion can only maintain in a small area: If the path of the loop is too long, the frictions coming from the viscosity will damp the oscillation heavily, ending up with the deformation of the loop. Hence, for a complanate system, whose horizontal sizes are much larger than the vertical height, if being observed from a large scale under a suitable temperature difference, then the pattern on the horizontal plane seem to be an aggregation of many tiny convection cells [1], as it is shown in Figure 1.2.

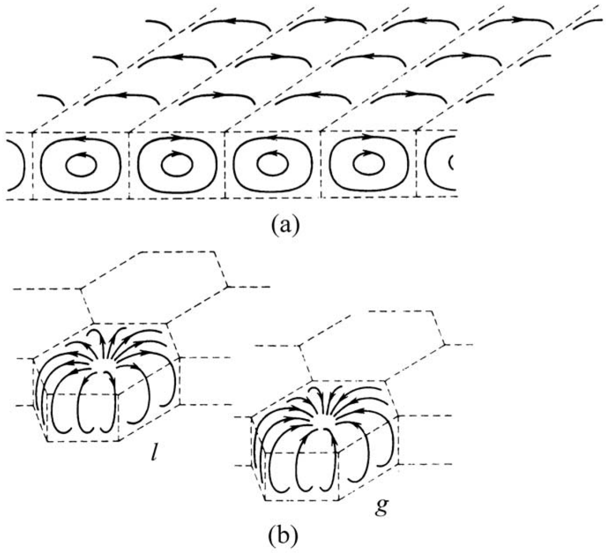

Figure 1.2: Schematic of two types of convection cells for RBC: (a) rolls; (b) hexagonal convection cells of the l and g types.

Convection cells are the fundamental elements of RBC. Two typical types of them are rolls and hexagonal. If the system is with low temperature difference, then only few kinds of motions is allowed in the cell. These permitted motions are identifies as normal modes determined by the boundary conditions. The normal modes analyze is to find out the possible normal modes of a system.

And convection can persistently transport heat from bottom to top, whose efficiency is considered to be useful. The way to get it is based on the normal modes analyze, who is thought to be stony. In this report, we established systemically models for the convection occurring under different boundary conditions, and calculated the Nusselt Number, defined as the transporting ratio of the heat flux .

The boundary conditions at the top and bottom layers are classified into two classes: boundary conditions for velocity field, and for thermal field. Each of the types has two cases: free surface without tangential stresses or rigid surface without slips, for velocity fields, prescribed temperature or prescribed heat flux, for thermal field. The combination of them will give four cases:

[1]The minimum volume containing a intact convectional loop is called a “cell”.

a. free surface for velocity field and prescribed temperature for thermal field;

b. free surface for velocity field and prescribed heat flux for thermal field;

c. rigid surface for velocity field and prescribed temperature for thermal field;

d. rigid surface for velocity field and prescribed heat flux for thermal field.

Saltzman derived a group of nonlinear equations from Naiver-Stokes equation of RBC. And by substituting Fourier modes into them, a numerical solution for case (a) was achieved. Lorenz simplified them to a dynamic system and discovered chaotic behaviors. On the other hand, Chandrasekhar and Hurle has already characterized the normal modes for rigid boundaries of RBC, we apply them in approach of Saltzman to derive a dynamical system of Lorenz type for other three cases. Then, its dynamics can be analyzed.

As is well known to all, a state of a fluid system needs at least three quantities to represent. They are the three components of the velocity $u\left(x, y, z, t\right)$, $v\left(x, y, z, t\right)$, $w\left(x, y, z, t\right)$ along three directions x,y,z correspondingly, the density distribution $\rho\left(x, y, z, t\right)$ and the temperature profile $T\left(x, y, z, t\right)$. In addition, the height H and temperature difference ΔT_0 between the layers and the viscosity ν, heat transparency κ, heat expansion coefficient α are also required. They can be combined into two dimensionless numbers: Prandtl Number $\sigma = \nu/\kappa$ and Rayleigh number $R = g\alpha H^3 \Delta T_0/\left(\kappa\nu\right)$ whose importance would be further discussed in below.

The situation we are interested in is not the extremely turbulent system so that the temperature difference ΔT_0 should be assumed not too high, which represents the motion won't go far away from the static state. Under this limits, the Boussinesq approximation can be introduced which has been precisely defined by the following sentence:

> *"In fluid dynamics, the **Boussinesq approximation** (pronounced, named for Joseph Valentin Boussinesq) is used in the field of buoyancy-driven flow (also known as natural convection). It ignores density differences except where they appear in terms multiplied by g, the acceleration due to gravity. The essence of the Boussinesq approximation is that the difference in inertia is negligible but gravity is sufficiently strong to make the specific weight appreciably different between the two fluids. Sound waves are impossible/neglected when the Boussinesq approximation is used since sound waves move via density variations."*

With the help of it, the transit from static to turbulent is simplified to a group of linear equations which can be analytically solved to obtain the normal modes.

The report is organized as follow: we first introduce the works of Saltzman and Lorenz in Chapter.

Chapter 2

The Characterization for The Normal Modes of RBC

The basic hydrodynamic problem is to solve a group of partial differential equations (PDE) under specified boundary conditions. Although the existence of solutions for Naiver-Stokes equation is still unsolved, scientist has to face the challenge without rigorous mathematical guarantee. One of the approach is to use linear approximation to ignore the nonlinear terms, by which people can jump across the uniqueness problem, since the nonlinear term always break the superposition law of states, resulting in the ruin of uniqueness.

The advantage to use linear approximation is that, the PDE without nonlinear term can have exact eigenvalues and eigenstates. Regard of the superposition law, any states of Linear PDE can be decomposed to the sum of many eigenstates. Here we promote this principle to more general cases.

> **An arbitrary state, even for turbulence with nonlinear effects, can be expanded to a series of solutions whose spatial components is corresponding to the PDE regardless of the nonlinearity, but with a different time dependent factor.**

In this chapter, we will introduce Chandrasekhar's linear approach. The symbols being used are defined in Table 2.1.

2.1 Derivation of The Perturbation Equations

The key point of solving RBC system is first to establish the equations of motion for an incompressible fluid. Chandrasekhar provided a path starting from the basic conservation laws.

Table 2.1: The meaning of the symbols required.

symbols	meaning
t	time
ρ	the density field
u_i	the velocity field at the i'th direction
μ	the viscosity coefficient
X_i	the external force at the i'th direction
T	the temperature field
p	pressure
c_V	the constant volume heat capacity
k	the heat conductivity
κ	the thermometric conductivity
α	the expansion coefficient
ν	the kinematic viscosity coefficient
g	gravitational acceleration

Momentum law:

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = \rho X_i - \frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i. \tag{2.1}$$

Jour's law:

$$\rho\frac{\partial}{\partial t}\left(c_V T\right) + \rho u_j\frac{\partial}{\partial x_j}\left(c_V T\right) = \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right). \tag{2.2}$$

Since the influence of the high order term is quite small, the Boussinesq approximation can be applied here. Consequently, Equation 2.1 and Equation 2.2 are reformed to Equation 2.3 and Equation 2.4.

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \left(1 + \frac{\delta\rho}{\rho_0}\right)X_i + \nu\nabla^2 u_i. \tag{2.3}$$

$$\frac{\partial T}{\partial t} + u_j\frac{\partial T}{\partial x_j} = \kappa\nabla^2 T. \tag{2.4}$$

ρ here is related to T due to thermal expansion

$$\rho = \rho_0\left[1 - \alpha\left(T - T_0\right)\right]. \tag{2.5}$$

$\delta\rho$ is defined as the difference of density varies from the middle plane

$$\delta\rho = -\rho_0\alpha\left(T - T_0\right). \tag{2.6}$$

T_0 here is the temperature at middle plane, ρ_0 is the density when temperature equals to T_0.

Obviously, the gravitational force only influence on the vertical direction. Thus, we can simplify the representation of the external force as $X_i = -g\lambda_i$, while $\boldsymbol{\lambda}$ here is a unit vector $(0, 0, 1)^{\mathrm{T}}$.

Then the form of temperature, density, pressure at the stable state is obtained.

$$T = T_0 - \beta\lambda_j x_j. \tag{2.7}$$

$$\rho = \rho_0 \left(1 + \alpha\beta\lambda_j x_j\right). \tag{2.8}$$

$$p = p_0 - g\rho_0 \left(\lambda_i x_i + \frac{1}{2}\alpha\beta\lambda_i\lambda_j x_i x_j\right). \tag{2.9}$$

β here is the slope of temperature profile.

Evidently, it is not easy to solve T out. An alternative way is to determine the perturbation differing from static state: it is because the situation we want to deal with is not under extremely high temperature difference between the top and bottom layers, which means the system won't go far away from the static case.

We assume that the actual temperature differs from the static state with a small quantity θ, which lead to Equation 2.10 and Equation 2.11.

$$T' = T_0 - \beta\lambda_j x_j + \theta. \tag{2.10}$$

$$\delta\rho = -\alpha\rho\theta = -\alpha\rho_0 \left(1 + \alpha\beta\lambda_j x_j\right). \tag{2.11}$$

Then Equation 2.3 and Equation 2.4 are transformed to Equation 2.12 and Equation 2.13.

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i}\left(\frac{\delta p}{\rho_0}\right) + g\alpha\theta\lambda_i + \nu\nabla^2 u_i. \tag{2.12}$$

$$\frac{\partial \theta}{\partial t} = \beta\lambda_j u_j + \kappa\nabla^2\theta. \tag{2.13}$$

The high order term in Equation 2.3 is ignored.

In order to get rid of the term containing δp, we apply the curl operator Equation 2.14 on Equation 2.12.

$$\mathrm{curl}_k = \epsilon_{ijk}\frac{\partial}{\partial x_j}. \tag{2.14}$$

And by using $\boldsymbol{\omega}$ to represent the vorticity of the velocity field as Equation 2.15 showing,

$$\omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}. \tag{2.15}$$

we have the equation

$$\frac{\partial \omega_i}{\partial t} = g\alpha\epsilon_{ijk}\frac{\partial \theta}{\partial x_j}\lambda_k + \nu\nabla^2\omega_i. \tag{2.16}$$

And again take the curl of the equation, one can get

$$\frac{\partial}{\partial t}\epsilon_{ijk}\frac{\partial \omega_k}{\partial x_j} = g\alpha\epsilon_{ijk}\epsilon_{ijk}\frac{\partial^2\theta}{\partial x_l \partial x_j}\lambda_m + \nu\nabla^2\epsilon_{ijk}\frac{\partial \omega_k}{\partial x_j}. \tag{2.17}$$

This equation can be simplified to

$$\frac{\partial}{\partial t}\nabla^2 u_i = g\alpha\left(\lambda_i\nabla^2\theta - \lambda_j\frac{\partial^2\theta}{\partial x_l \partial x_j}\right) + \nu\nabla^4 u_i. \tag{2.18}$$

In the end, the perturbation equation can be obtained, which is given as Equation 2.19, Equation 2.20 and Equation 2.21.

$$\frac{\partial \zeta}{\partial t} = \nu\nabla^2\zeta. \tag{2.19}$$

$$\frac{\partial}{\partial t}\nabla^2 w = g\alpha\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) + \nu\nabla^4 w. \tag{2.20}$$

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa\nabla^2\theta. \tag{2.21}$$

where $\zeta = \lambda_j\omega_j$ and $w = \lambda_j u_j$.

2.2 The Boundary Conditions

As it has been mentioned before, there are 4 cases of boundary conditions in RBC. In this section, we will translate them from linguistic interpretation to mathematical formalism.

We confined the fluid between the planes $z = 0$ and $z = d$, where the following relation must be satisfied

$$w|_{z=0,d} = 0$$

since the vertical velocity field shall vanish at the boundary layers. Despite of that, two extra condition can also be conducted here

- rigid surface on which no slip occurs: $\frac{\partial u}{\partial x}\Big|_{z=0,d} = \frac{\partial v}{\partial y}\Big|_{z=0,d} = 0$.
- free surface on which no tangential stresses act: $\frac{\partial u}{\partial z}\Big|_{z=0,d} = \frac{\partial v}{\partial z}\Big|_{z=0,d} = 0$.

According to the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.22}$$

the rigid surface condition will have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big|_{z=0,d} = \frac{\partial w}{\partial z}\Big|_{z=0,d} = 0$$

while the free surface condition will have

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\bigg|_{z=0,d}=\frac{\partial^2 w}{\partial z^2}\bigg|_{z=0,d}=0$$

In spite of the velocity fields, two kinds of the thermal field are

- prescribed temperature: $T|_{z=0,d}=\text{const}$
- prescribed heat flux: $\frac{\partial T}{\partial z}\big|_{z=0,d}=\text{const}$

Since only the perturbation term effects, the prescribed temperature has

$$\theta|_{z=0,d}=0$$

while the prescribed heat flux has

$$\frac{\partial \theta}{\partial z}\bigg|_{z=0,d}=0$$

So far, the boundary conditions have already been translated to their corresponding forms. In the next section, We will prove that with different combinations of boundary conditions, the normal modes of the Equation 2.19-2.21 will perform typically different.

2.3 The Normal modes

The normal modes of Equation 2.19-2.21 can be written as

$$\zeta=Z\left(z\right)\exp\left[\mathrm{i}\left(k_x x+k_y y\right)+pt\right]. \tag{2.23}$$

$$w=W\left(z\right)\exp\left[\mathrm{i}\left(k_x x+k_y y\right)+pt\right]. \tag{2.24}$$

$$\theta=\Theta\left(z\right)\exp\left[\mathrm{i}\left(k_x x+k_y y\right)+pt\right]. \tag{2.25}$$

where $k=\sqrt{k_x^2+k_y^2}$. The differential operators under the normal modes become variables:

$$\frac{\partial}{\partial t}=p,\quad \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=-k^2,\quad \nabla^2=\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2. \tag{2.26}$$

By substituting them into Equation 2.19-2.21, we get

$$pZ=\nu\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2\right)Z. \tag{2.27}$$

$$p\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2\right)W=-g\alpha k^2\Theta+\nu\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2\right)^2 W. \tag{2.28}$$

$$p\Theta=\beta W+\kappa\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2\right)\Theta. \tag{2.29}$$

Then we introduce dimensionless units here to let

$$a = kd, \quad \sigma = pd^2/\nu. \tag{2.30}$$

Thus, the dimensionless version of Equation 2.24-2.25 come out

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \sigma\right) W = \left(\frac{g\alpha}{\nu} d^2\right) a^2 \Theta. \tag{2.31}$$

$$\left(D^2 - a^2 - \mathrm{Pr}\,\sigma\right) \Theta = -\left(\frac{\beta}{\kappa} d^2\right) W. \tag{2.32}$$

Pr is the Prandtl number. By combining those two equations together to eliminate Θ, we obtained the eigenvalues equation

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \sigma\right)\left(D^2 - a^2 - \mathrm{Pr}\,\sigma\right) W = -Ra^2 W. \tag{2.33}$$

where

$$R = \frac{g\alpha\beta}{\kappa\nu} d^4. \tag{2.34}$$

is the Rayleigh number.

While for marginal states where the it has

$$\left(D^2 - a^2\right)^3 W = -Ra^2 W. \tag{2.35}$$

since the time factor $p = 0$.

Then we determined the eigenvalue of the normal modes from Equation 2.35. By assuming that the normal modes with an eigenvalue q:

$$W = Ae^{qz}. \tag{2.36}$$

The eigenvalue equation is obtained:

$$\left(q^2 - a^2\right)^3 = -Ra^2. \tag{2.37}$$

The common solutions for W now are given by the roots $q = \mathrm{i}q_0, q_1, q_2$ of Equation 2.37.

Chapter 3

The Derivation of Lorenz System

The equations of motion are used in the cases come from the following three principles: the incompressibility, the momentum law and the Juor diffusion law.

The incompressibility

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{3.1}$$

The momentum law (Naiver-Stokes equations):

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}=-\boldsymbol{\nabla}p+\mu\nabla^2\boldsymbol{u}+\rho\boldsymbol{g} \tag{3.2}$$

Juor diffusion law:

$$\frac{\partial T}{\partial t}+\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)T=\kappa\nabla^2 T. \tag{3.3}$$

For two dimensional "rolls" in the x-z plane, the v-component vanishes everywhere. The equations become

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0. \tag{3.4}$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\mu\nabla^2 u. \tag{3.5}$$

$$\rho\frac{\partial w}{\partial t}+\rho u\frac{\partial w}{\partial x}+\rho w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\mu\nabla^2 w-g\rho. \tag{3.6}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial z}=\kappa\nabla^2 T. \tag{3.7}$$

Equation 3.4 can be satisfied if ψ is represented by a stream function

$$u=-\frac{\partial \psi}{\partial z},\quad w=\frac{\partial \psi}{\partial x}. \tag{3.8}$$

As for the temperature field, we split it into the static temperature profile and a perturbation $\theta\left(x,z,t\right)$, i.e.

$$T\left(t,x,z\right)=\left(T_0-\beta z\right)+\theta\left(x,z,t\right). \tag{3.9}$$

Here, T_0 is the temperature of middle plane. β is the gradient of the static linear temperature profile, defined as

$$\beta = \frac{\Delta T_0}{H}. \tag{3.10}$$

The pressure p can be eliminated by using the z-derivative of Equation 3.5 to subtract the x-derivative of Equation 3.6. And by substituting the stream function, we obtain

$$\frac{\partial}{\partial t}\nabla^2\psi - \left(\frac{\partial}{\partial z}\psi\right)\left(\frac{\partial}{\partial x}\nabla^2\psi\right) + \left(\frac{\partial}{\partial x}\psi\right)\left(\frac{\partial}{\partial z}\nabla^2\psi\right) - g\alpha\frac{\partial\theta}{\partial x} - \nu\nabla^4\psi = 0. \tag{3.11}$$

$$\frac{\partial}{\partial t}\theta - \left(\frac{\partial}{\partial z}\psi\right)\left(\frac{\partial}{\partial x}\theta\right) + \left(\frac{\partial}{\partial x}\psi\right)\left(\frac{\partial}{\partial z}\theta\right) - \beta\frac{\partial}{\partial x}\psi - \kappa\nabla^2\theta = 0. \tag{3.12}$$

This is Saltzmann's nonlinear equations. The foregoing equations can be translated to dimensionless form

$$\frac{\partial}{\partial t}\nabla^2\psi - \left(\frac{\partial}{\partial z}\psi\right)\left(\frac{\partial}{\partial x}\nabla^2\psi\right) + \left(\frac{\partial}{\partial x}\psi\right)\left(\frac{\partial}{\partial z}\nabla^2\psi\right) - \sigma\frac{\partial\theta}{\partial x} - \sigma\nabla^4\psi = 0. \tag{3.13}$$

$$\frac{\partial}{\partial t}\theta - \left(\frac{\partial}{\partial z}\psi\right)\left(\frac{\partial}{\partial x}\theta\right) + \left(\frac{\partial}{\partial x}\psi\right)\left(\frac{\partial}{\partial z}\theta\right) - R\frac{\partial}{\partial x}\psi - \nabla^2\theta = 0. \tag{3.14}$$

under the transformation

$$\begin{cases} x \to Hx \\ z \to Hx \\ t \to \left(H^2/\kappa\right)t \\ \psi \to \kappa\psi \\ \theta \to \left(\kappa\nu/g\alpha H^3\right)\theta \end{cases}. \tag{3.15}$$

Here, the Prandtl number and the Rayleigh number first appear

$$\begin{cases} \sigma = \frac{\nu}{\kappa} \\ R = \frac{g\alpha\beta H^4}{\kappa\nu} \end{cases}.$$

The result for case (a) is that for a critical minimum value of Rayleigh number,

$$R = R_c = \frac{27}{4}\pi^4.$$

a steady solution of the form

$$\psi = \frac{1}{a}A\sin(ax)\sin(\pi z);$$
$$\theta = \frac{\left(\pi^2 + a^2\right)^2}{a^2}B\cos(ax)\sin(\pi z).$$

can satisfy the boundary conditions. If we substitute them to Equation 3.13-3.14, one of the term will throw out

$$-\left(\frac{\partial}{\partial z}\psi\right)\left(\frac{\partial}{\partial x}\theta\right) + \left(\frac{\partial}{\partial x}\psi\right)\left(\frac{\partial}{\partial z}\theta\right) = \frac{\left(\pi^2 + a^2\right)^2\pi}{2a^2}AB\sin(2\pi z)$$

whose wave number is other than the critical mode. The appearance of this term suggests it should also be put into the solution to get more exact approach

$$
\begin{aligned}
\psi &= \frac{1}{a} A \sin(ax) \sin(\pi z); \\
\theta &= \frac{\left(\pi^2 + a^2\right)^2}{a^2} B \cos(ax) \sin(\pi z) - \frac{\left(\pi^2 + a^2\right)^2 \pi}{2a^2} C \sin(2\pi z).
\end{aligned}
$$

So that the equations will be transformed to

$$
\begin{aligned}
-\frac{\pi^2 + a^2}{a} \frac{\mathrm{d}A}{\mathrm{d}t} + \frac{\left(\pi^2 + a^2\right)^2}{a} \sigma B - \frac{\left(\pi^2 + a^2\right)^2}{a} \sigma A = 0. \\
\frac{\left(\pi^2 + a^2\right)^2}{a^2} \frac{\mathrm{d}B}{\mathrm{d}t} + \frac{\left(\pi^2 + a^2\right)^2 \pi^2}{2a^2} AC - RA + \frac{\left(\pi^2 + a^2\right)^3}{a^2} B = 0. \\
-\frac{\left(\pi^2 + a^2\right)^2 \pi}{2a^2} \frac{\mathrm{d}C}{\mathrm{d}t} + \frac{\left(\pi^2 + a^2\right)^2 \pi}{2a^2} AB - \frac{4\left(\pi^2 + a^2\right)^2 \pi^3}{2a^2} C = 0.
\end{aligned}
$$

By rescale them with

$$
\begin{cases}
t = \left(\frac{3\pi^2}{2}\right)^{-1} \tau \\
A = 3X \\
B = \frac{27\pi^3}{2\sqrt{2}} Y \\
C = \frac{27\pi^3}{4} Z
\end{cases}
\tag{3.16}
$$

In the end, the Lorenz system can be obtained

$$
\frac{\mathrm{d}}{\mathrm{d}\tau} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} & & -\sigma X + \sigma Y & \\ - XZ & + rX & - Y & \\ XY & & & - bZ \end{pmatrix}
\tag{3.17}
$$

Chapter 4

The Dynamic System of The Rigid Boundaries RBC

As it has already been mentioned before, the eigenmodes of RBC for rigid boundaries are

$$\psi(x,z) = A\, a^{-1} W(z) \sin(ax) \tag{4.1}$$

$$\theta(x,z) = B\, a^{-2} \left(D^2 - a^2\right)^2 W(z) \cos(ax) \tag{4.2}$$

where $W(z)$ can be

$$W_e(z) = \alpha_0 \cos(q_0 z) + \alpha_1 \cosh\left[(q_1 - q_2 \mathrm{i}) z\right] + \alpha_2 \cosh\left[(q_1 + q_2 \mathrm{i}) z\right].$$

for the even modes,

$$W_o(z) = \alpha_0 \sin(q_0 z) + \alpha_1 \sinh\left[(q_1 - q_2 \mathrm{i}) z\right] + \alpha_2 \sinh\left[(q_1 + q_2 \mathrm{i}) z\right].$$

for the odd modes. However, the case we care about is at the low Rayleigh number. The occurrence of odd modes require higher fluctuation. In the other words, it requires larger Rayleigh number. Due to the reason, the odd modes can be simply throw away.

Since we want to know about the time evolution of the state. Some time dependent factors shall be coupling to the state

$$\begin{aligned} W_e(z,t) = &\cos(q_0 z) \exp\left[-\left(q_0^2 + a^2\right) f(t)\right] \\ &+ \alpha \cosh\left[(q_1 + q_2 \mathrm{i}) z\right] \exp\left\{\left[(q_1 + q_2 \mathrm{i})^2 - a^2\right] f(t)\right\} \\ &+ \alpha^* \cosh\left[(q_1 - q_2 \mathrm{i}) z\right] \exp\left\{\left[(q_1 - q_2 \mathrm{i})^2 - a^2\right] f(t)\right\}. \end{aligned}$$

Thus it can give

$$\frac{\partial \psi(x,z,t)}{\partial t} = \left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) \nabla^2 \psi(x,z,t)$$

whose importance will be uncovered later.

Obviously, the time factor can be divided into real and imaginary parts

$$\exp\left\{\left[(q_1 \pm q_2 \mathrm{i})^2 - a^2\right] f(t)\right\} = \exp\left[\left(q_1^2 - q_2^2 - a^2 \pm 2q_1 q_2 \mathrm{i}\right) f(t)\right].$$

According to Vieta theorem, we have

$$q_1^2 - q_2^2 - a^2 = \frac{1}{2}\left(q_0^2 + a^2\right)$$
$$2q_1 q_2 = \frac{\sqrt{3}}{2}\left(q_0^2 + a^2\right)$$

Thus, the even normal modes become

Bibliography