# The Note of RB system

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## 1 Definition

Eigenvalue Equation:

$$\left(q^2 - a^2\right)^3 = -a^2 R$$

Two types of solutions are

$$\begin{array}{rclcrcr} W_e &=& A_0 \cos\left(q_0 z\right) &+& A_1 \cosh\left(q_\mathrm{r} z\right)\cos\left(q_\mathrm{i} z\right) &-& A_2 \sinh\left(q_\mathrm{r} z\right)\sin\left(q_\mathrm{i} z\right); \\ W_o &=& A_0 \sin\left(q_0 z\right) &+& A_1 \sinh\left(q_\mathrm{r} z\right)\cos\left(q_\mathrm{i} z\right) &+& A_2 \cosh\left(q_\mathrm{r} z\right)\sin\left(q_\mathrm{i} z\right). \end{array}$$

Thus, the even modes yield

$$\begin{array}{ccccccc} \mathrm{D}W_e &=& -q_0 A_0 \sin\left(q_0 z\right) &+& \left(q_\mathrm{r} A_1 - q_\mathrm{i} A_2\right)\sinh\left(q_\mathrm{r} z\right)\cos\left(q_\mathrm{i} z\right) &-& \left(q_\mathrm{r} A_2 + q_\mathrm{i} A_1\right)\cosh\left(q_\mathrm{r} z\right)\sin\left(q_\mathrm{i} z\right); \\ \mathrm{D}^2 W_e &=& -q_0^2 A_0 \cos\left(q_0 z\right) &+& \left[\left(q_\mathrm{r}^2 - q_\mathrm{i}^2\right) A_1 - 2 q_\mathrm{r} q_\mathrm{i} A_2\right]\cosh\left(q_\mathrm{r} z\right)\cos\left(q_\mathrm{i} z\right) &-& \left[\left(q_\mathrm{r}^2 - q_\mathrm{i}^2\right) A_2 + 2 q_\mathrm{r} q_\mathrm{i} A_1\right]\sinh\left(q_\mathrm{r} z\right)\sin\left(q_\mathrm{i} z\right); \\ \frac{\left(\mathrm{D}^2 - a^2\right) W_e}{\left(q_0^2 + a^2\right)} &=& -A_0 \cos\left(q_0 z\right) &+& \left(\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right)\cosh\left(q_\mathrm{r} z\right)\cos\left(q_\mathrm{i} z\right) &-& \left(\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right)\sinh\left(q_\mathrm{r} z\right)\sin\left(q_\mathrm{i} z\right); \\ \frac{\mathrm{D}\left(\mathrm{D}^2 - a^2\right) W_e}{\left(q_0^2 + a^2\right)} &=& q_0 A_0 \sin\left(q_0 z\right) &+& \left[q_\mathrm{r}\left(\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right) - q_\mathrm{i}\left(\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right)\right]\sinh\left(q_\mathrm{r} z\right)\cos\left(q_\mathrm{i} z\right) &-& \left[q_\mathrm{i}\left(\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right) + q_\mathrm{r}\left(\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right)\right]\cosh\left(q_\mathrm{r} z\right)\sin\left(q_\mathrm{i} z\right); \\ \frac{\left(\mathrm{D}^2 - a^2\right)^2 W_e}{\left(q_0^2 + a^2\right)^2} &=& A_0 \cos\left(q_0 z\right) &+& \left(-\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right)\cosh\left(q_\mathrm{r} z\right)\cos\left(q_\mathrm{i} z\right) &-& \left(-\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right)\sinh\left(q_\mathrm{r} z\right)\sin\left(q_\mathrm{i} z\right). \\ \frac{\mathrm{D}\left(\mathrm{D}^2 - a^2\right)^2 W_e}{\left(q_0^2 + a^2\right)^2} &=& -q_0 A_0 \sin\left(q_0 z\right) &+& \left[q_\mathrm{r}\left(-\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right) - q_\mathrm{i}\left(-\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right)\right]\sinh\left(q_\mathrm{r} z\right)\cos\left(q_\mathrm{i} z\right) &-& \left[q_\mathrm{i}\left(-\frac{1}{2} A_1 - \frac{\sqrt{3}}{2} A_2\right) + q_\mathrm{r}\left(-\frac{1}{2} A_2 + \frac{\sqrt{3}}{2} A_1\right)\right]\cosh\left(q_\mathrm{r} z\right)\sin\left(q_\mathrm{i} z\right). \end{array}$$

While the odd modes yield

$$
\begin{aligned}
\mathrm{D}W_o &= q_0 A_0 \cos(q_0 z) + (q_{\mathrm{r}} A_1 + q_{\mathrm{i}} A_2)\cosh(q_{\mathrm{r}} z)\cos(q_{\mathrm{i}} z) + (q_{\mathrm{r}} A_2 - q_{\mathrm{i}} A_1)\sinh(q_{\mathrm{r}} z)\sin(q_{\mathrm{i}} z)\,;\\
\mathrm{D}^2 W_o &= -q_0^2 A_0 \sin(q_0 z) + \left[\left(q_{\mathrm{r}}^2 - q_{\mathrm{i}}^2\right) A_1 + 2q_{\mathrm{r}} q_{\mathrm{i}} A_2\right]\sinh(q_{\mathrm{r}} z)\cos(q_{\mathrm{i}} z) + \left[\left(q_{\mathrm{r}}^2 - q_{\mathrm{i}}^2\right) A_2 - 2q_{\mathrm{r}} q_{\mathrm{i}} A_1\right]\cosh(q_{\mathrm{r}} z)\sin(q_{\mathrm{i}} z)\,;\\
\frac{\left(\mathrm{D}^2-a^2\right)W_o}{\left(q_0^2+a^2\right)} &= -A_0 \sin(q_0 z) + \left(\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right)\sinh(q_{\mathrm{r}} z)\cos(q_{\mathrm{i}} z) + \left(\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\cosh(q_{\mathrm{r}} z)\sin(q_{\mathrm{i}} z)\,;\\
\frac{\mathrm{D}\left(\mathrm{D}^2-a^2\right)W_o}{\left(q_0^2+a^2\right)} &= -q_0 A_0 \cos(q_0 z) + \left[q_{\mathrm{r}}\left(\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right) + q_{\mathrm{i}}\left(\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\right]\cosh(q_{\mathrm{r}} z)\cos(q_{\mathrm{i}} z) + \left[-q_{\mathrm{i}}\left(\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right) + q_{\mathrm{r}}\left(\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\right]\sinh(q_{\mathrm{r}} z)\sin(q_{\mathrm{i}} z)\,;\\
\frac{\left(\mathrm{D}^2-a^2\right)^2W_o}{\left(q_0^2+a^2\right)^2} &= A_0 \sin(q_0 z) + \left(-\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right)\sinh(q_{\mathrm{r}} z)\cos(q_{\mathrm{i}} z) + \left(-\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\cosh(q_{\mathrm{r}} z)\sin(q_{\mathrm{i}} z)\,.\\
\frac{\mathrm{D}\left(\mathrm{D}^2-a^2\right)^2W_o}{\left(q_0^2+a^2\right)^2} &= q_0 A_0 \cos(q_0 z) + \left[q_{\mathrm{r}}\left(-\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right) + q_{\mathrm{i}}\left(-\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\right]\cosh(q_{\mathrm{r}} z)\cos(q_{\mathrm{i}} z) + \left[-q_{\mathrm{i}}\left(-\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right) + q_{\mathrm{r}}\left(-\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\right]\sinh(q_{\mathrm{r}} z)\sin(q_{\mathrm{i}} z)\,.
\end{aligned}
$$

And the corresponding boundary conditions are

- Rigid Layers / Fixed Temperature
- Rigid Layers / Prescribed Heat
- Free Layers / Fixed Temperature
- Free Layers / Prescribed Heat

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# 2 Boundary Conditions

## 2.1 Rigid Layers / Fixed Temperature

$$
\begin{cases}
W\big|_{z=\pm\frac{1}{2}} &= 0\\
\mathrm{D}W\big|_{z=\pm\frac{1}{2}} &= 0\\
\left(\mathrm{D}^2-a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} &= 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
\theta\big|_{z=\pm\frac{1}{2}} &= 0\\
\left(\mathrm{D}^2-a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} &= 0\\
\mathrm{D}\left(\mathrm{D}^2-a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} &= 0
\end{cases}.
$$

for even modes it has

$$
-q_0 \tan\frac{1}{2}q_0 = \frac{\left(\sqrt{3}q_i + q_r\right)\sinh q_r + \left(\sqrt{3}q_r - q_i\right)\sin q_i}{\cosh q_r + \cos q_i}.
$$

for odd modes it has

$$
q_0 \cot\frac{1}{2}q_0 = \frac{\left(\sqrt{3}q_i + q_r\right)\sinh q_r - \left(\sqrt{3}q_r - q_i\right)\sin q_i}{\cosh q_r - \cos q_i}.
$$

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## 2.2 Rigid Layers / Prescribed Heat

$$\begin{cases} W\big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}W\big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}\left(\mathrm{D}^2 - a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} & = 0 \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{D}\theta\big|_{z=\pm\frac{1}{2}} & = 0 \\ \left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}\left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} & = 0 \end{cases}.$$

for even modes it has

$$q_0 \tan\frac{q_0}{2} = \frac{\left(q_r^2 + q_i^2\right)\left(\cosh q_r - \cos q_i\right)}{\left(\sqrt{3}q_i - q_r\right)\sinh q_r + \left(\sqrt{3}q_r + q_i\right)\sin q_i}$$

for odd modes it has

$$q_0 \cot\frac{q_0}{2} = \frac{\left(q_r^2 + q_i^2\right)\left(\cosh q_r + \cos q_i\right)}{\left(-\sqrt{3}q_i + q_r\right)\sinh q_r + \left(\sqrt{3}q_r + q_i\right)\sin q_i}$$

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## 2.3 Free Layers / Fixed Temperature

$$\begin{cases} W\big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}^2W\Big|_{z=\pm\frac{1}{2}} & = 0 \\ \left(\mathrm{D}^2 - a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} & = 0 \end{cases} \quad \text{and} \quad \begin{cases} \theta\big|_{z=\pm\frac{1}{2}} & = 0 \\ \left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}\left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} & = 0 \end{cases}.$$

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## 2.4 Free Layers / Prescribed Heat

$$\begin{cases} W\big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}^2W\Big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}\left(\mathrm{D}^2 - a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} & = 0 \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{D}\theta\big|_{z=\pm\frac{1}{2}} & = 0 \\ \left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} & = 0 \\ \mathrm{D}^2\left(\mathrm{D}^2 - a^2\right)\theta\Big|_{z=\pm\frac{1}{2}} & = 0 \end{cases}.$$

for even modes it has

$$\frac{1}{2}q_0 \tan\frac{q_0}{2} = \frac{q_r \sinh q_r - q_i \sin q_i}{\cosh q_r + \cos q_i}.$$

for odd modes it has

$$-\frac{1}{2}q_0 \cot\frac{q_0}{2} = \frac{q_r \sinh q_r + q_i \sin q_i}{\cosh q_r - \cos q_i}.$$