The Note of RB system

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1 Definition

Eigenvalue Equation:

$$\left(q^2-a^2\right)^3=-a^2R$$

Two types of solutions are

$$\begin{array}{rcccccc} W_e & = & A_0\cos\left(q_0z\right) & + & A_1\cosh\left(q_\mathrm{r}z\right)\cos\left(q_\mathrm{i}z\right) & - & A_2\sinh\left(q_\mathrm{r}z\right)\sin\left(q_\mathrm{i}z\right); \\ W_o & = & A_0\sin\left(q_0z\right) & + & A_1\sinh\left(q_\mathrm{r}z\right)\cos\left(q_\mathrm{i}z\right) & + & A_2\cosh\left(q_\mathrm{r}z\right)\sin\left(q_\mathrm{i}z\right). \end{array}$$

Thus, the even modes yield

$$\begin{array}{ccccccc} \mathrm{D}W_e & = & -q_0A_0\sin\left(q_0z\right) & + & \left(q_\mathrm{r}A_1-q_\mathrm{i}A_2\right)\sinh\left(q_\mathrm{r}z\right)\cos\left(q_\mathrm{i}z\right) & - & \left(q_\mathrm{r}A_2+q_\mathrm{i}A_1\right)\cosh\left(q_\mathrm{r}z\right)\sin\left(q_\mathrm{i}z\right); \\ \mathrm{D}^2W_e & = & -q_0^2A_0\cos\left(q_0z\right) & + & \left[\left(q_\mathrm{r}^2-q_\mathrm{i}^2\right)A_1-2q_\mathrm{r}q_\mathrm{i}A_2\right]\cosh\left(q_\mathrm{r}z\right)\cos\left(q_\mathrm{i}z\right) & - & \left[\left(q_\mathrm{r}^2-q_\mathrm{i}^2\right)A_2+2q_\mathrm{r}q_\mathrm{i}A_1\right]\sinh\left(q_\mathrm{r}z\right)\sin\left(q_\mathrm{i}z\right); \\ \frac{\left(\mathrm{D}^2-a^2\right)W_e}{\left(q_0^2+a^2\right)} & = & -A_0\cos\left(q_0z\right) & + & \left(\frac{1}{2}A_1-\frac{\sqrt{3}}{2}A_2\right)\cosh\left(q_\mathrm{r}z\right)\cos\left(q_\mathrm{i}z\right) & - & \left(\frac{1}{2}A_2+\frac{\sqrt{3}}{2}A_1\right)\sinh\left(q_\mathrm{r}z\right)\sin\left(q_\mathrm{i}z\right); \\ \frac{\mathrm{D}\left(\mathrm{D}^2-a^2\right)W_e}{\left(q_0^2+a^2\right)} & = & q_0A_0\sin\left(q_0z\right) & + & \left[q_\mathrm{r}\left(\frac{1}{2}A_1-\frac{\sqrt{3}}{2}A_2\right)-q_\mathrm{i}\left(\frac{1}{2}A_2+\frac{\sqrt{3}}{2}A_1\right)\right]\sinh\left(q_\mathrm{r}z\right)\cos\left(q_\mathrm{i}z\right) & - & \left[q_\mathrm{i}\left(\frac{1}{2}A_1-\frac{\sqrt{3}}{2}A_2\right)+q_\mathrm{r}\left(\frac{1}{2}A_2+\frac{\sqrt{3}}{2}A_1\right)\right]\cosh\left(q_\mathrm{r}z\right)\sin\left(q_\mathrm{i}z\right); \\ \frac{\left(\mathrm{D}^2-a^2\right)^2W_e}{\left(q_0^2+a^2\right)^2} & = & A_0\cos\left(q_0z\right) & + & \left(-\frac{1}{2}A_1-\frac{\sqrt{3}}{2}A_2\right)\cosh\left(q_\mathrm{r}z\right)\cos\left(q_\mathrm{i}z\right) & - & \left(-\frac{1}{2}A_2+\frac{\sqrt{3}}{2}A_1\right)\sinh\left(q_\mathrm{r}z\right)\sin\left(q_\mathrm{i}z\right). \\ \frac{\mathrm{D}\left(\mathrm{D}^2-a^2\right)^2W_e}{\left(q_0^2+a^2\right)^2} & = & -q_0A_0\cos\left(q_0z\right) & + & \left[q_\mathrm{r}\left(-\frac{1}{2}A_1-\frac{\sqrt{3}}{2}A_2\right)-q_\mathrm{i}\left(-\frac{1}{2}A_2+\frac{\sqrt{3}}{2}A_1\right)\right]\sinh\left(q_\mathrm{r}z\right)\cos\left(q_\mathrm{i}z\right) & - & \left[q_\mathrm{i}\left(-\frac{1}{2}A_1-\frac{\sqrt{3}}{2}A_2\right)+q_\mathrm{r}\left(-\frac{1}{2}A_2+\frac{\sqrt{3}}{2}A_1\right)\right]\cosh\left(q_\mathrm{r}z\right)\sin\left(q_\mathrm{i}z\right). \end{array}$$

While the odd modes yield

$$
\begin{aligned}
\mathrm{D}W_o &= q_0 A_0 \cos(q_0 z) + (q_\mathrm{r} A_1 + q_\mathrm{i} A_2)\cosh(q_\mathrm{r} z)\cos(q_\mathrm{i} z) + (q_\mathrm{r} A_2 - q_\mathrm{i} A_1)\sinh(q_\mathrm{r} z)\sin(q_\mathrm{i} z)\,;\\
\mathrm{D}^2 W_o &= -q_0^2 A_0 \sin(q_0 z) + \left[\left(q_\mathrm{r}^2 - q_\mathrm{i}^2\right) A_1 + 2 q_\mathrm{r} q_\mathrm{i} A_2\right]\sinh(q_\mathrm{r} z)\cos(q_\mathrm{i} z) + \left[\left(q_\mathrm{r}^2 - q_\mathrm{i}^2\right) A_2 - 2 q_\mathrm{r} q_\mathrm{i} A_1\right]\cosh(q_\mathrm{r} z)\sin(q_\mathrm{i} z)\,;\\
\frac{\left(\mathrm{D}^2-a^2\right)W_o}{\left(q_0^2+a^2\right)} &= -A_0 \sin(q_0 z) + \left(\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right)\sinh(q_\mathrm{r} z)\cos(q_\mathrm{i} z) + \left(\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\cosh(q_\mathrm{r} z)\sin(q_\mathrm{i} z)\,;\\
\frac{\mathrm{D}\left(\mathrm{D}^2-a^2\right)W_o}{\left(q_0^2+a^2\right)} &= -q_0 A_0 \cos(q_0 z) + \left[q_\mathrm{r}\left(\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right) + q_\mathrm{i}\left(\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\right]\cosh(q_\mathrm{r} z)\cos(q_\mathrm{i} z) + \left[-q_\mathrm{i}\left(\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right) + q_\mathrm{r}\left(\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\right]\sinh(q_\mathrm{r} z)\sin(q_\mathrm{i} z)\,;\\
\frac{\left(\mathrm{D}^2-a^2\right)^2 W_o}{\left(q_0^2+a^2\right)^2} &= A_0 \sin(q_0 z) + \left(-\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right)\sinh(q_\mathrm{r} z)\cos(q_\mathrm{i} z) + \left(-\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\cosh(q_\mathrm{r} z)\sin(q_\mathrm{i} z)\,.\\
\frac{\mathrm{D}\left(\mathrm{D}^2-a^2\right)^2 W_o}{\left(q_0^2+a^2\right)^2} &= q_0 A_0 \cos(q_0 z) + \left[q_\mathrm{r}\left(-\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right) + q_\mathrm{i}\left(-\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\right]\cosh(q_\mathrm{r} z)\cos(q_\mathrm{i} z) + \left[-q_\mathrm{i}\left(-\tfrac{1}{2}A_1 + \tfrac{\sqrt{3}}{2}A_2\right) + q_\mathrm{r}\left(-\tfrac{1}{2}A_2 - \tfrac{\sqrt{3}}{2}A_1\right)\right]\sinh(q_\mathrm{r} z)\sin(q_\mathrm{i} z)\,.
\end{aligned}
$$

And the corresponding boundary conditions are

$$
\begin{cases}
W\big|_{z=\pm\frac{1}{2}} &= 0\\
\mathrm{D}W\big|_{z=\pm\frac{1}{2}} &= 0\\
\left(\mathrm{D}^2-a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} &= 0
\end{cases},\quad
\begin{cases}
W\big|_{z=\pm\frac{1}{2}} &= 0\\
\mathrm{D}W\big|_{z=\pm\frac{1}{2}} &= 0\\
\mathrm{D}\left(\mathrm{D}^2-a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} &= 0
\end{cases},\quad
\begin{cases}
W\big|_{z=\pm\frac{1}{2}} &= 0\\
\mathrm{D}^2 W\big|_{z=\pm\frac{1}{2}} &= 0\\
\left(\mathrm{D}^2-a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} &= 0
\end{cases},\quad
\begin{cases}
W\big|_{z=\pm\frac{1}{2}} &= 0\\
\mathrm{D}^2 W\big|_{z=\pm\frac{1}{2}} &= 0\\
\mathrm{D}\left(\mathrm{D}^2-a^2\right)^2 W\Big|_{z=\pm\frac{1}{2}} &= 0
\end{cases}
$$