CS201: Discrete Math for Computer Science
Written Assignment 4
Spring 2023

Due: May 5th, 2023; Please submit through Sakai in ONE PDF file.
The assignment needs to be written in English. Assignments in any other language will get zero point.
Any plagiarism behavior will lead to zero point.
Does not accept late submissions. No exception!

Q. 1. (5 points) Use induction to prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Solution: Base case: $n = 1$, $n^3 + 2n = 3$, which is divisible by 3.

Inductive hypothesis: Suppose that 3 divides $n^3 + 2n$.

Inductive step: We now prove that 3 divides $(n+1)^3 + 2(n+1)$. We have

$$\begin{aligned}(n+1)^3 + 2(n+1) &= (n^3 + 2n) + (3n^2 + 3n + 3) \\ &= (n^3 + 2n) + 3(n^2 + n + 1).\end{aligned}$$

Since $n^3 + 2n$ is divisible by 3 by i.h., and also $3(n^2 + n + 1)$ is divisible by 3. it follows that $(n+1)^3 + 2(n+1)$ is divisible by 3.

Conclusion: By mathematical induction, we prove the result.

$\square$

Q. 2. (5 points) Let $x \in \mathbb{R}$ and $x \neq 1$. Using mathematical induction, prove that for all integers $n \geq 0$,

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}.$$

Solution:

Base case: $n = 0$, we have $\sum_{i=0}^{0} x^i = 1 = \frac{x^{0+1}-1}{x-1} = 1$.

Inductive hypothesis: Suppose that $\sum_{i=0}^{k} x^i = \frac{x^{k+1}-1}{x-1}$.

Inductive step: For $n = k + 1$, we have

$$\begin{aligned}
\sum_{i=0}^{k+1} x^i &= x^{k+1} + \sum_{i=0}^{k} x^i \\
&= x^{k+1} + \frac{x^{k+1}-1}{x-1} \qquad \text{by i.h.} \\
&= \frac{x^{k+1}(x-1)}{x-1} + \frac{x^{k+1}-1}{x-1} \\
&= \frac{x^{k+2}-1}{x-1}.
\end{aligned}$$

Conclusion: By mathematical induction, we prove the result.

$\square$

Q. 3. (5 points) Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$\begin{aligned}
&(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \\
&= (A_1 \cap A_2 \cap \cdots \cap A_n) - B.
\end{aligned}$$

Solution:

If $n = 1$, there is nothing to prove, and then $n = 2$, this says that $(A_1 \cap \bar{B}) \cap (A_2 \cap \bar{B}) = (A_1 \cap A_2) \cap \bar{B})$, which is the distributive law. For the inductive step, assume that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B;$$

we must show that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) = (A_1 \cap A_2 \cap \cdots A_n \cap A_{n+1}) - B.$$

We have

$$\begin{aligned}
&(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) \\
&= ((A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B)) \cap (A_{n+1} - B) \\
&= ((A_1 \cap A_2 \cap \cdots \cap A_n) - B) \cap (A_{n+1} - B) \\
&= (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B.
\end{aligned}$$

The third line follows from the inductive hypothesis, and the fourth line follows from the $n = 2$ case.

□

Q. 4. (10 points) Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

(a) Show statements $P(18)$, $P(19)$, $P(20)$ and $P(21)$ are true, completing the basis step of the proof.

(b) What is the inductive hypothesis of the proof?

(c) What do you need to prove in the inductive step?

(d) Complete the inductive step for $k \geq 21$.

(e) Explain why these steps show that this statement is true whenever $n \geq 18$.

Solution:

(a) $P(18)$ is true, because we can form 18 cents of postage with one 4-cent stamp and two 7-cent stamps. $P(19)$ is true, because we can form 19 cents of postage with three 4-cent stamps and one 7-cent stamp. $P(20)$ is true, because we can form 20 cents of postage with five 4-cent stamps. $P(21)$ is true, because we can form 20 cents of postage with three 7-cent stamps.

(b) The inductive hypothesis is the statement that using just 4-cent and 7-cent stamps we can form j cents postage for all j with $18 \leq j \leq k$, where we assume that $k \geq 21$.

(c) In the inductive step we must show, assuming the inductive hypothesis, that we can form $k+1$ cents postage using just 4-cent and 7-cent stamps.

(d) We want to form $k + 1$ cents of postage. Since $k \geq 21$, we know that $P(k - 3)$ is true, that is, we can form $k - 3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k + 1$ cents of postage, as desired.

(e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 18.

□

Q. 5. (5 points) Find $f(n)$ when $n = 4^k$, where f satisfies the recurrence relation $f(n) = 5f(n/4) + 6n$, with $f(1) = 1$.

Solution: $f(n) = 25n^{\log_4 5} - 24n$.

□

Q. 6. (10 points) The running time of an algorithm A is described by the following recurrence relation:

$$S(n) = \begin{cases} b & n = 1 \\ 9S(n/2) + n^2 & n > 1 \end{cases}$$

where b is a positive constant and n is a power of 2. The running time of a competing algorithm B is described by the following recurrence relation:

$$T(n) = \begin{cases} c & n = 1 \\ aT(n/4) + n^2 & n > 1 \end{cases}$$

where a and c are positive constants and n is a power of 4. For the rest of this problem, you may assume that n is always a power of 4. You should also assume that $a > 16$. (Hint: you may use the equation $a^{\log_2 n} = n^{\log_2 a}$)

(a) Find a solution for $S(n)$. Your solution should be in closed form (in terms of b if necessary) and should not use summation.

(b) Find a solution for $T(n)$. Your solution should be in closed form (in terms of a and c if necessary) and should not use summation.

(c) For what range of values of $a > 16$ is Algorithm B at least as efficient as Algorithm A asymptotically ($T(n) = O(S(n))$)?

Solution:

(a) By repeated substitution, we get

$$
\begin{aligned}
S(n) &= 9S(n/2) + n^2 \\
&= 9\left[9S\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2\right] + n^2 \\
&= 9^2 S\left(\frac{n}{2^2}\right) + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= 9^2\left[9S\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2\right] + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= 9^3 S\left(\frac{n}{2^3}\right) + \left(\frac{9}{4}\right)^2 n^2 + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= \cdots \\
&= 9^{\log_2 n} S(1) + n^2 \sum_{i=0}^{\log_2 n - 1} \left(\frac{9}{4}\right)^i \\
&= bn^{\log_2 9} + \frac{4}{5} n^{\log_2 9} - \frac{4}{5} n^2 \\
&= \left(b + \frac{4}{5}\right) n^{\log_2 9} - \frac{4}{5} n^2,
\end{aligned}
$$

where we are using the fact that

$$
\left(\frac{9}{4}\right)^{\log_2 n} = \frac{9^{\log_2 n}}{n^2} = \frac{n^{\log_2 9}}{n^2}.
$$

(b) Similar to (a), we get

$$\begin{aligned}
T(n) &= aT\left(\frac{n}{4}\right) + n^2 \\
&= a\left[aT\left(\frac{n}{4^2}\right) + \left(\frac{n}{4}\right)^2\right] + n^2 \\
&= a^2T\left(\frac{n}{4^2}\right) + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= a^2\left[aT\left(\frac{n}{4^3}\right) + \left(\frac{n}{4^2}\right)^2\right] + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= a^3T\left(\frac{n}{4^3}\right) + \left(\frac{a}{16}\right)^2 n^2 + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= \cdots \\
&= a^{\log_4 n}T(1) + n^2 \sum_{i=0}^{\log_4 n - 1} \left(\frac{a}{16}\right)^i \\
&= cn^{\log_4 a} + \frac{16}{a-16}n^{\log_4 a} - \frac{16}{a-16}n^2 \\
&= \left(c + \frac{16}{a-16}\right)n^{\log_4 a} - \frac{16}{a-16}n^2,
\end{aligned}$$

where we are using the fact that

$$\left(\frac{a}{16}\right)^{\log_4 n} = \frac{a^{\log_4 n}}{n^2} = \frac{n^{\log_4 a}}{n^2}.$$

(c) For $T(n) = O(S(n))$, we should have

$$\begin{aligned}
n^{\log_4 a} &\leq n^{\log_2 9} \\
\log_4 a &\leq \log_2 9 \\
a &\leq 9^2 = 81.
\end{aligned}$$

So the range of values is $16 < a \leq 81$.

$\square$

Q. 7. (5 points) Suppose that $n \geq 1$ is an integer.

(a) How many functions are there from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, 3\}$?

(b) How many of the functions in part (a) are one-to-one functions?

(c) How many of the functions in part (a) are onto functions?

Solution:

(a) There are 3^n functions.

(b) If $n \leq 3$, there are $P(3,n)$ one-to-one functions. Hence, there are 3 when $n = 1$, 6 when $n = 2$, and 6 when $n = 3$. If $n > 3$, then there are 0 injective functions; there cannot be a one-to-one function from A to B if $|A| > |B|$.

(c) By the Inclusion-Exclusion Principle, we have

$$\begin{aligned} \# &= \#\{f : f(A) \subseteq \{1,2,3\}\} - \#\{f : f(A) \subseteq \{1,2\}\} - \#\{f : f(A) \subseteq \{1,3\}\} \\ &\quad -\#\{f : f(A) \subseteq \{2,3\}\} + \{f : f(A) \subseteq \{1\}\} + \{f : f(A) \subseteq \{2\}\} \\ &\quad \#\{f : f(A) \subseteq \{3\}\} \\ &= 3^n - 2^n - 2^n - 2^n + 1 + 1 + 1 \\ &= 3^n - 3 \cdot 2^n + 3. \end{aligned}$$

□

Q. 8. (5 points) How many 6-card poker hands consist of exactly 2 pairs? That is two of one rank of card, two of another rank of card, one of a third rank, and one of a fourth rank of card? Recall that a deck of cards consists of 4 suits each with one card of each of the 13 ranks.

You should leave your answer as an equation.

Solution: First, we choose the ranks of the 2 pairs, noting that the order we pick these two ranks does not matter, so there are $\binom{13}{2}$ options here. Next we pick the 2 suits for the first pair, $\binom{4}{2}$ and the suits for the second pair $\binom{4}{2}$. Then we decide which 2 ranks of the remaining 11 to use for the other cards, $\binom{11}{2}$, and finally choose each of their suits $\binom{4}{1}\binom{4}{1}$. Altogether, by the product rule, this gives $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{2}\binom{4}{1}\binom{4}{1}$ hands.

□

Q. 9. (5 points) How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Solution: First we count the number of bit strings of length 10 that contain five consecutive 0s. We will count based on where the string of five or more consecutive 0s starts. If it starts in the first bit, then the first five bits are all 0s, but there is free choice for the last five bits, therefore there are $2^5 = 32$ such strings. If it starts in the second bit, then the first bit must be a 1, the next five bits are all 0s, but there is free choice for the last four bits; therefore there are $2^4 = 16$ such strings. If it starts in the third bit, then the second bit must be a 1 but the first bit and the last three bits are arbitrary; therefore there are $2^4 = 16$ such strings. Similarly, there are 16 such strings that have the consecutive 0s starting in each of positions four, five, and six. This gives us a total of $32 + 5 \cdot 16 = 112$ strings that contain five consecutive 0s. Symmetrically there are 112 strings that contain five consecutive 1s. Clearly there are exactly two strings that contain both (0000011111 and 1111100000). Therefore by the inclusion-exclusion principle, the answer is $112 + 112 - 2 = 222$.

□

Q. 10. (5 points) Suppose that p and q are prime numbers and that $n = pq$. Use the principle of inclusion-exclusion to find the number of positive integers not exceeding n that are relatively prime to n, i.e., the Euler function $\phi(n)$.

Solution: Let P be the set of numbers in $\{1, 2, 3, \ldots, n\}$ that are divisible by p, and similarly define the set Q. We want to count the numbers not divisible by either p or q, so we want $n - |P \cup Q|$. By the principle of inclusion-exclusion, $|P \cup Q| = |P| + |Q| - |P \cap Q|$. Every pth number is divisible by p, so $|P| = \lfloor n/p \rfloor = q$. Similarly $|Q| = \lfloor n/q \rfloor = q$. Clearly, n is the only positive integer not exceeding n that is divisible by both p and q, so $|P \cap Q| = 1$. Therefore, the number of positive integers not exceeding n that are relatively prime to n is $n - p - q + 1$.

□

Q. 11. (5 points) How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

Solution:

Working modulo 5 there are 25 pairs: $(0, 0), (0, 1), \ldots, (4, 4)$. Thus, we could have 25 ordered pairs of integers (a, b) such that no two of them were equal when reduced modulo 5. The pigeonhole principle, however, guarantees that if we have 26 such pairs, then at least two of them will have the same coordinates, modulo 5.

□

Q. 12. (5 points) Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integers coordinates.

Solution:

The midpoint of the segment whose endpoints are (a, b) and (c, d) is $((a+c)/2, (b+d)/2)$. We are concerned only with integer values of the original coordinates. Clearly the coordinates of these fractions will be integers as well if and only if a and c have the same parity (both odd or both even) and b and d have the same parity. There are four possible pairs of parities: (odd, odd), $(odd, even)$, $(even, odd)$, $(even, even)$. Since we are given five points, the pigeonhole principle guarantees that at least two of them will have the same pair of parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

□

Q. 13. (10 points) Prove the hockeystick identity

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

(a) using a combinatorial argument

(b) using Pascal's identity.

Solution:

(a) $\binom{n+r+1}{r}$ counts the number of ways to choose a sequence of r 0s and $n+1$ 1s by choosing the positions of the 0s. Alternatively, suppose that the $(j+1)$st term is the last term equal to 1, so that $n \leq j \leq n+r$. Once we have determined where the last 1 is, we decide where the 0s are to be placed in the j spaces before the last 1. There are n 1s and $j-n$ 0s in this range. By the sum rule it follows that there are $\sum_{j=n}^{n+r} \binom{j}{j-n} = \sum_{k=0}^{r} \binom{n+k}{k}$ ways to this.

(b) Let $P(r)$ be the statement to be proved. The basis step is the equation $\binom{n}{0} = \binom{n+1}{0}$, which is just $1 = 1$. Assume that $P(r)$ is true. Then

$$\begin{aligned}
&\sum_{k=0}^{r+1} \binom{n+k}{k} \\
&= \sum_{k=0}^{r} \binom{n+k}{k} + \binom{n+r+1}{r+1} \\
&= \binom{n+r+1}{r} + \binom{n+r+1}{r+1} \\
&= \binom{n+r+2}{r+1},
\end{aligned}$$

using the inductive hypothesis and Pascal's identity.

□

Q. 14. (10 points) Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = 0$, and $a_2 = 7$.

Solution: The CE is

$$r^3 - 2r^2 - r + 2 = (r+1)(r-1)(r-2).$$

The roots are $r = -1$, $r = 1$ and $r = 2$. Hence, the solutions to this recurrence are of the form

$$a_n = \alpha_1(-1)^n + \alpha_2 1^n + \alpha_3 2^n.$$

To find the constants α_1, α_2 and α_3, we use the initial conditions. Plugging in $n = 0, n = 1$, and $n = 2$, we have

$$a_0 = 1 = \alpha_1 + \alpha_2 + \alpha_3 a_1 = 0 = -\alpha_1 + \alpha_2 + 2\alpha_3 a_2 = 7 = \alpha_1 + \alpha_2 + 4\alpha_3.$$

We then have $\alpha_1 = 3/2$, $\alpha_2 = -5/2$, and $\alpha_3 = 2$. Hence,

$$a_n = 3/2 \cdot (-1)^n - 5/2 \cdot 1^n + 2 \cdot 2^n = 2^{n+1} + (-1)^n \cdot 3/2 - 5/2.$$

$\square$

Q. 15. (10 points) Use generating functions to prove Pascal's identity: $C(n,r) = C(n-1,r) + C(n-1,r-1)$ when n and r are positive integers with $r < n$. [Hint: Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

Solution:

First we note, as the hint suggests, that $(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)^{n-1} + x(1+x)^{n-1}$. Expanding both sides of this equality using the binomial theorem, we have

$$\begin{aligned}\sum_{r=0}^{n} C(n,r)x^r &= \sum_{r=1}^{n-1} C(n-1,r)x^r + \sum_{r=0}^{n-1} C(n-1,r)x^{r+1} \\ &= \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=1}^{n} C(n-1,r-1)x^r.\end{aligned}$$

Thus,

$$1+\left(\sum_{r=1}^{n-1} C(n,r)x^r\right)+x^n = 1+\left(\sum_{r=1}^{n-1}(C(n-1,r)+C(n-1,r-1))x^r\right)+x^n.$$

Comparing these two expressions, coefficient by coefficient, we see that $C(n,r)$ must equal $C(n-1,r) + C(n-1,r-1)$ for $1 \le r \le n-1$, as desired.

$\square$