Discrete Mathematics for Computer Science

Lecture 12: Counting

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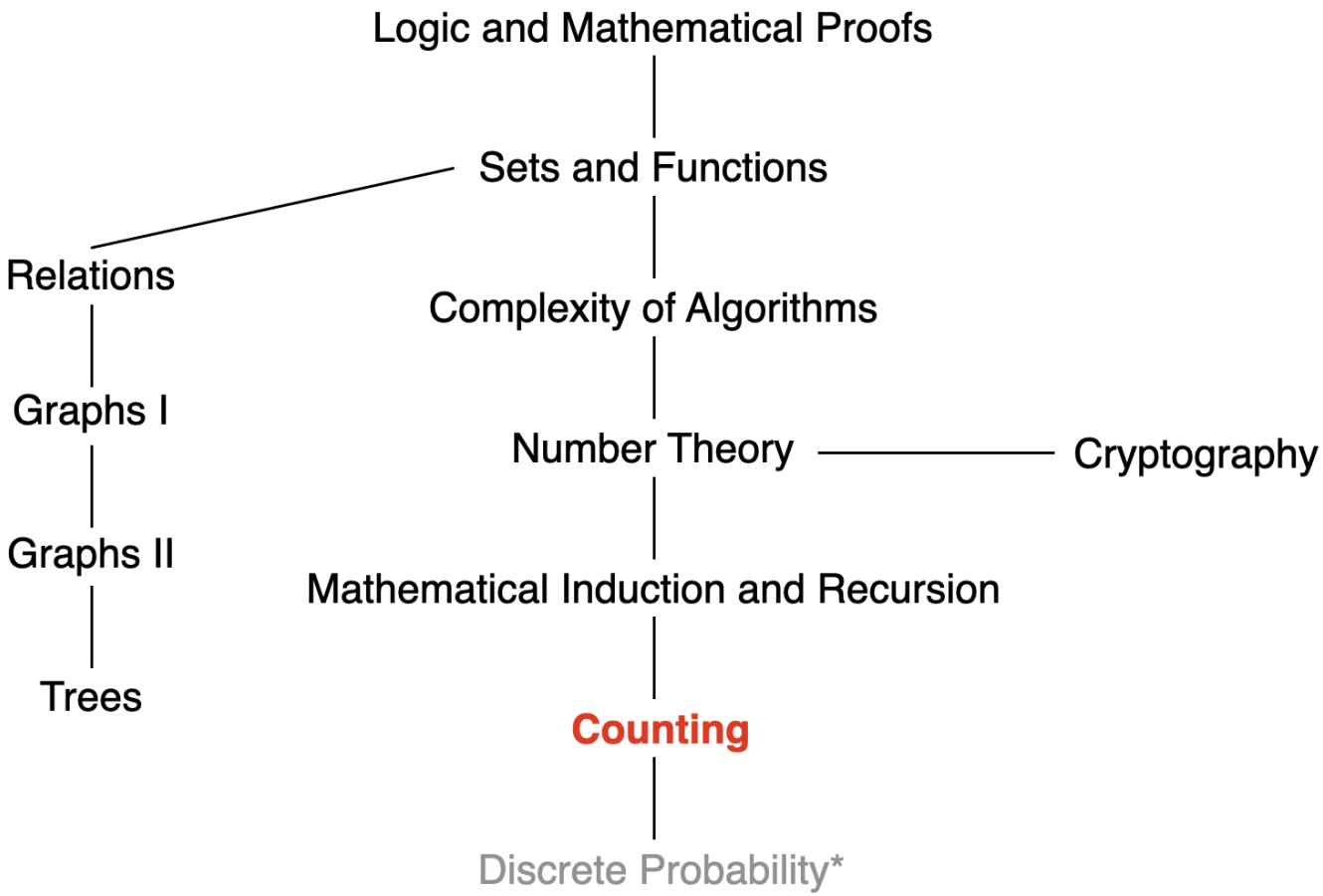

Counting basis, Permutations, ...

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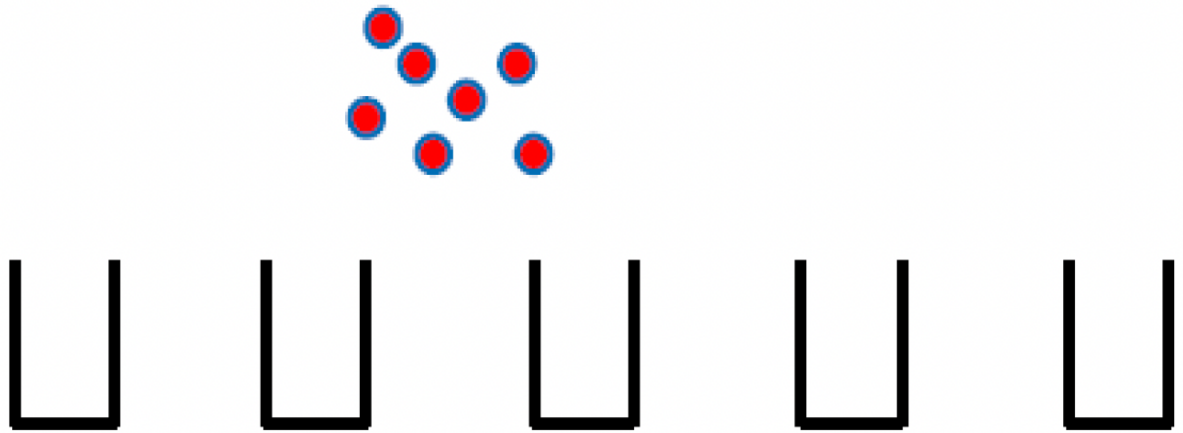
Pigeonhole Principle

The Pigeonhole Principle: If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.



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Pigeonhole Principle: Example 2

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Suppose that $a_1, a_2, \ldots, a_N$ is a sequence of real numbers:

- A subsequence of this sequence is a sequence of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$, where $1 \leq i_1 < i_2 < \ldots < i_m \leq N$.
- A sequence is called strictly increasing if each term is larger than the one that precedes it.

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Pigeonhole Principle: Example 2

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Example: The sequence 8, 11, 9, 1, 4, 6, 12, 10, 5, 7 contains 10 terms. Note that $10 = 3^2 + 1$.

There are four strictly increasing subsequences of length four:

$$1, 4, 6, 12 \quad 1, 4, 6, 7$$

$$1, 4, 6, 10 \quad 1, 4, 5, 7$$

There is also a strictly decreasing subsequence of length four:

$$11, 9, 6, 5$$

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Pigeonhole Principle: Example 2

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Proof: Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers. Associate (i_k, d_k) to the term a_k:

- i_k: the length of the longest increasing subsequence starting at a_k
- d_k: the length of the longest decreasing subsequence starting at a_k.

Suppose that there are no increasing or decreasing subsequences of length $n + 1$. I.e., $i_k \leq n$ and $d_k \leq n$ for $k = 1, 2, ..., n^2 + 1$.

By the product rule there are n^2 possible ordered pairs for (i_k, d_k). By the pigeonhole principle, two of these $n^2 + 1$ ordered pairs are equal.

That is, there exist terms a_s and a_t with $s < t$ such that $i_s = i_t$ and $d_s = d_t$.

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Pigeonhole Principle: Example 2

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Proof: There exist terms a_s and a_t with $s < t$ such that $i_s = i_t$ and $d_s = d_t$. We will show that this is impossible.

The terms of the sequence are distinct, either $a_s < a_t$ or $a_s > a_t$:

- $a_s < a_t$: an increasing subsequence of length $i_t + 1$ can be built, i.e., a_s, a_t, ... (followed by an increasing subsequence of length i_t beginning at a_t); thus, $i_s > i_t$
- $a_s > a_t$: $d_s = d_t$; an decreasing sequence of length $d_t + 1$ can be built, i.e., a_s, a_t, ...; thus, $d_s > d_t$;

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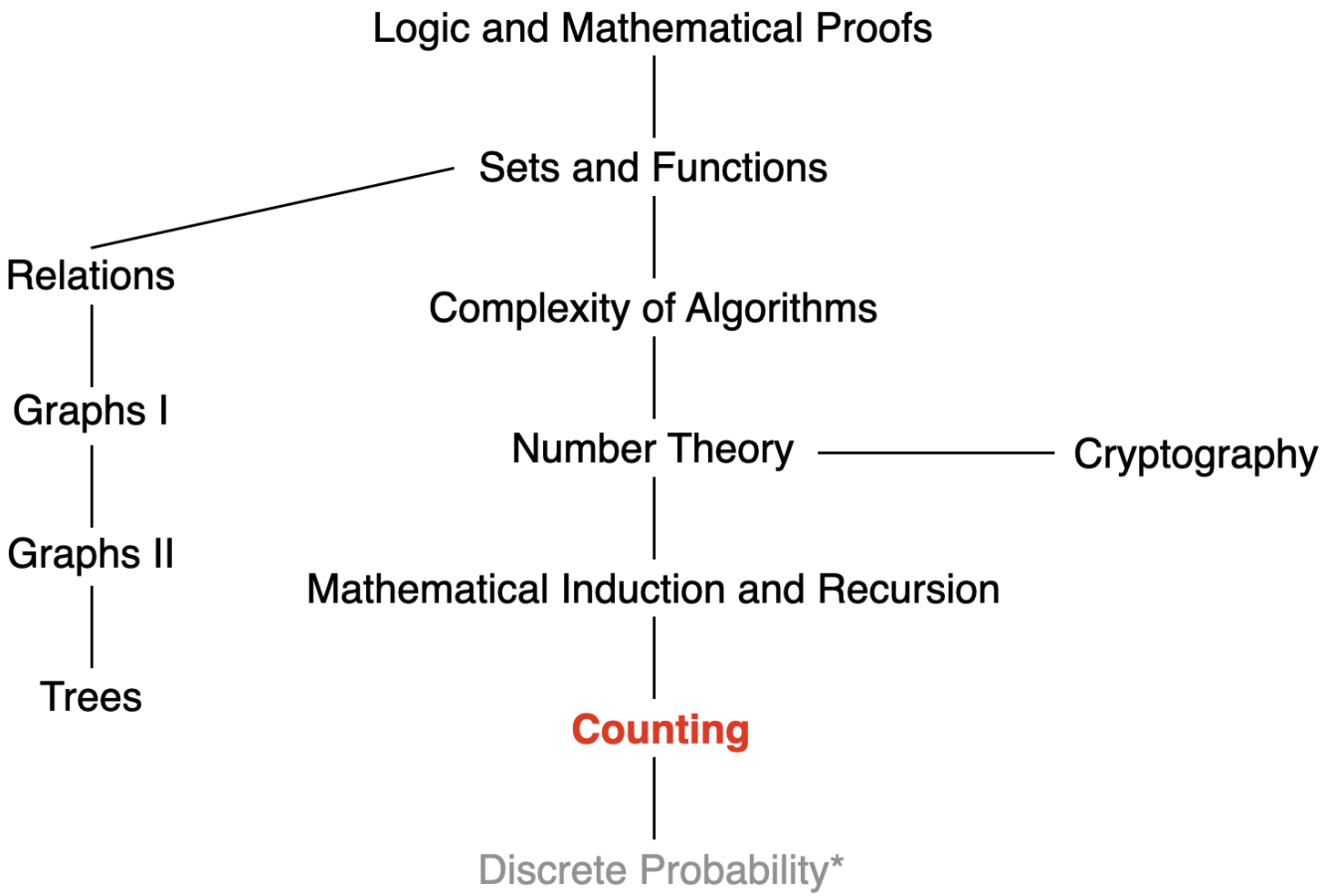

Counting basis, Permutations, ...

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Permutations

A **permutation** of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of r elements of a set is called an **r-permutation**.

Example: Let $S = \{a, b, c\}$. The 2-permutations of S are the ordered arrangements (a, b), (a, c), (b, a) (b, c), (c, a). (c, b).

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Permutations

How many 3-permutations of $\{1, 2, ..., n\}$ are there?

Based on product rule:

- n choices for first number.
- For each way of choosing first number, there are $n-1$ choices for the second.
- For each way of choosing first two numbers, there are $n-2$ choices for the third number.

By product rule, there are $n(n-1)(n-2)$ ways to choose the permutation.

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Permutations

Theorem: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Proof by the Product Rule: The first element of the permutation can be chosen in n ways, because there are n elements in the set.

There are $n-1$ ways to choose the second element of the permutation.

...

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Permutations

Theorem: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Corollary: If n and r are integers with $0 \leq r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!}.$$

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Permutations: Example

Example 1: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

$P(100, 3) = 100 \times 99 \times 98 = 970,200.$

Example 2: How many permutations of the letters ABCDEFGH contain the string ABC?

The letters ABC must occur as a block. Thus, it is equivalent to finding the number of permutations of six objects:

ABC, D, E, F, G, H.

Thus, there are $P(6, 6) = 6! = 720$ permutations.

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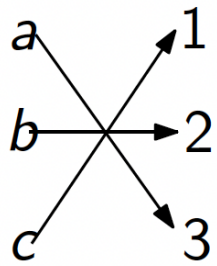
Bijections and Permutations

A function that is both one-to-one and onto is called a bijection, or a one-to-one correspondence.

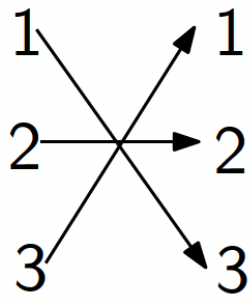
How many bijections are there?

$f : \{a, b, c\} \to \{1, 2, 3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1$ is a bijection.



A bijection from a set onto itself is called a permutation.

$f : \{1, 2, 3\} \to \{1, 2, 3\}$ defined by $f(1) = 3, f(2) = 2, f(3) = 1$ is a bijection.



Bijections and Permutations

A function that is both one-to-one and onto is called a bijection, or a one-to-one correspondence.
A bijection from a set onto itself is called a permutation.

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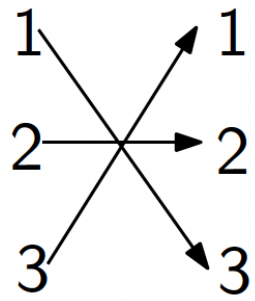

In a bijection, exactly one arrow leaves each item on the left and exactly one arrow arrives at each item on the right.

Thus, the left and right sides must have the same size.

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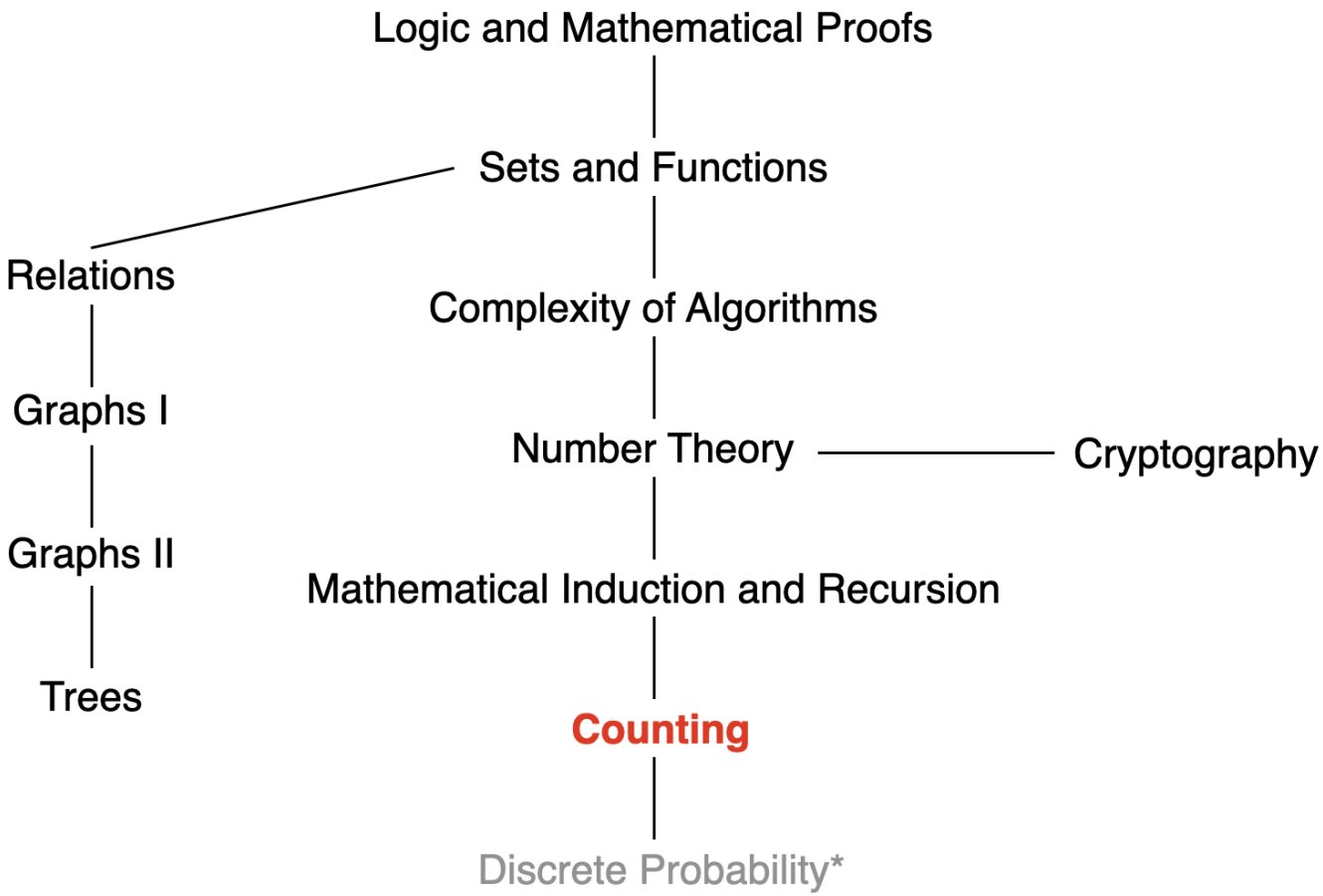

Counting basis, Permutations, Combinations, ...

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Combinations

An r-combination of elements of a set is an unordered selection of r elements from the set.

The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.

Note that $C(n, r)$ is also denoted by $\binom{n}{r}$ and is called a binomial coefficient.

Example: The 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$. Thus, $C(4, 2) = 6$.

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Binomial Coefficient

Theorem: For integers n and r with $0 \leq r \leq n$, the number of r-element subsets of an n-element set is

$$\binom{n}{r} = C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

Proof: The $P(n,r)$ r-permutations of the set can be obtained by

- forming the $C(n,r)$ r-combinations of the set.
- ordering the elements in each r-combination, which can be done in $P(r,r)$ ways.

By the product rule,

$$P(n,r) = C(n,r)P(r,r).$$

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Some Properties of Binomial Coefficients

- $C(n, 0) = 1$: one set of size 0.
- $C(n, n) = 1$: one set of size n.
- $C(n, r) = C(n, n - r)$

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

$$C(n, n-r) = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!}$$

Any other ideas to prove?

We will address this later.

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Combinations: Example

Example 1: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

$C(10,5) = \frac{10!}{5!5!} = 252.$

Example 2: There are 9 faculty members in the mathematics department and 11 in the computer science department.

How many ways are there to form a committee with 3 faculty members from the mathematics department and 4 from the computer science department?

$C(9,3) \cdot C(11,4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 27,720.$

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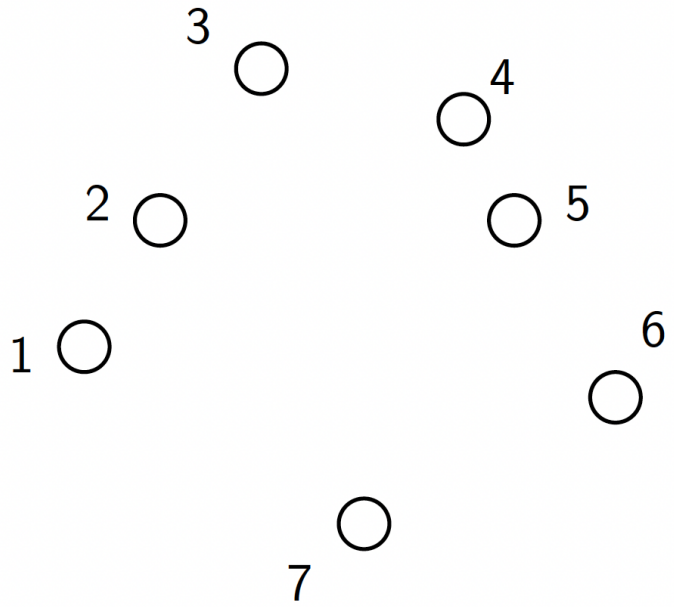
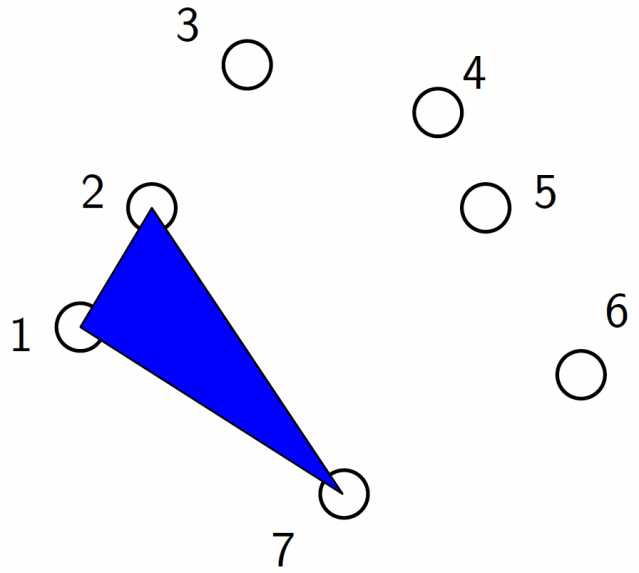
Example: Counting Triangle

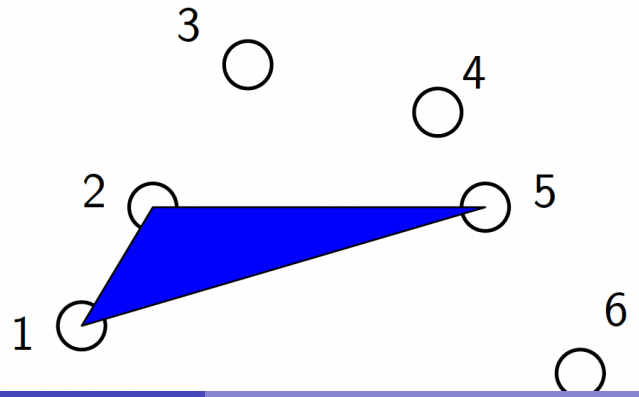
Design an algorithm to count the number of triangles formed by n points in the plane:

- 3 points form a triangle if and only if they are non-collinear





$1-2-7$: yes



$1-2-7$: yes

$1-2-5$: yes

Example: Counting Triangle

The following loop is a part of the program to determine the number of triangles formed by n points in the plane:

```
(1) trianglecount = 0
(2)   for i = 1 to n
(3)     for j = i+1 to n
(4)       for k = j+1 to n
(5)         if points i, j, k are not collinear
(6)           trianglecount = trianglecount + 1
```

Question: Among all iterations of line 5, what is the total number of times this line checks three points to see if they are collinear?

This corresponds to the total number of combinations. Why?

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Example: Counting Triangle

```
(1) trianglecount = 0
(2)   for i = 1 to n
(3)     for j = i+1 to n
(4)       for k = j+1 to n
(5)         if points i, j, k are not collinear
(6)           trianglecount = trianglecount + 1
```

- First loop begins with $i = 1$ and i increases up to n.
- Second loop begins with $j = i + 1$ and j increases up to n.
- Third loop begins with $k = j + 1$ and k increases up to n.

Thus each triple i, j, k with $i < j < k$ is examined exactly once.

For example, if $n = 4$, then triples (i, j, k) used by algorithm are $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$, and $(2, 3, 4)$.

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Example: Counting Triangle

Want to compute the number of increasing triples (i, j, k) with $1 \leq i < j < k \leq n$.

Claim: The number of increasing triples is exactly the same as the number of 3-combinations from $\{1, 2, ..., n\}$. Why?

- X: set of increasing triples
- Y: set of 3-combinations from $\{1, 2, ..., n\}$

Define: $f : X \to Y$ by $f((i, j, k)) = \{i, j, k\}$

Claim: f is a bijection, so $|X| = |Y|$.

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Example: Counting Triangle

- X: set of increasing triples
- Y: set of 3-combinations from $\{1, 2, ..., n\}$

Define: $f : X \to Y$ by $f((i, j, k)) = \{i, j, k\}$

Claim: f is a bijection, so $|X| = |Y|$.

- One-to-one: if $f((i, j, k)) = f((i', j', k'))$, then $(i, j, k) = (i', j', k')$.
- Onto: for any $\{i, j, k\}$, there exists a (i, j, k) such that $f((i, j, k)) = \{i, j, k\}$.

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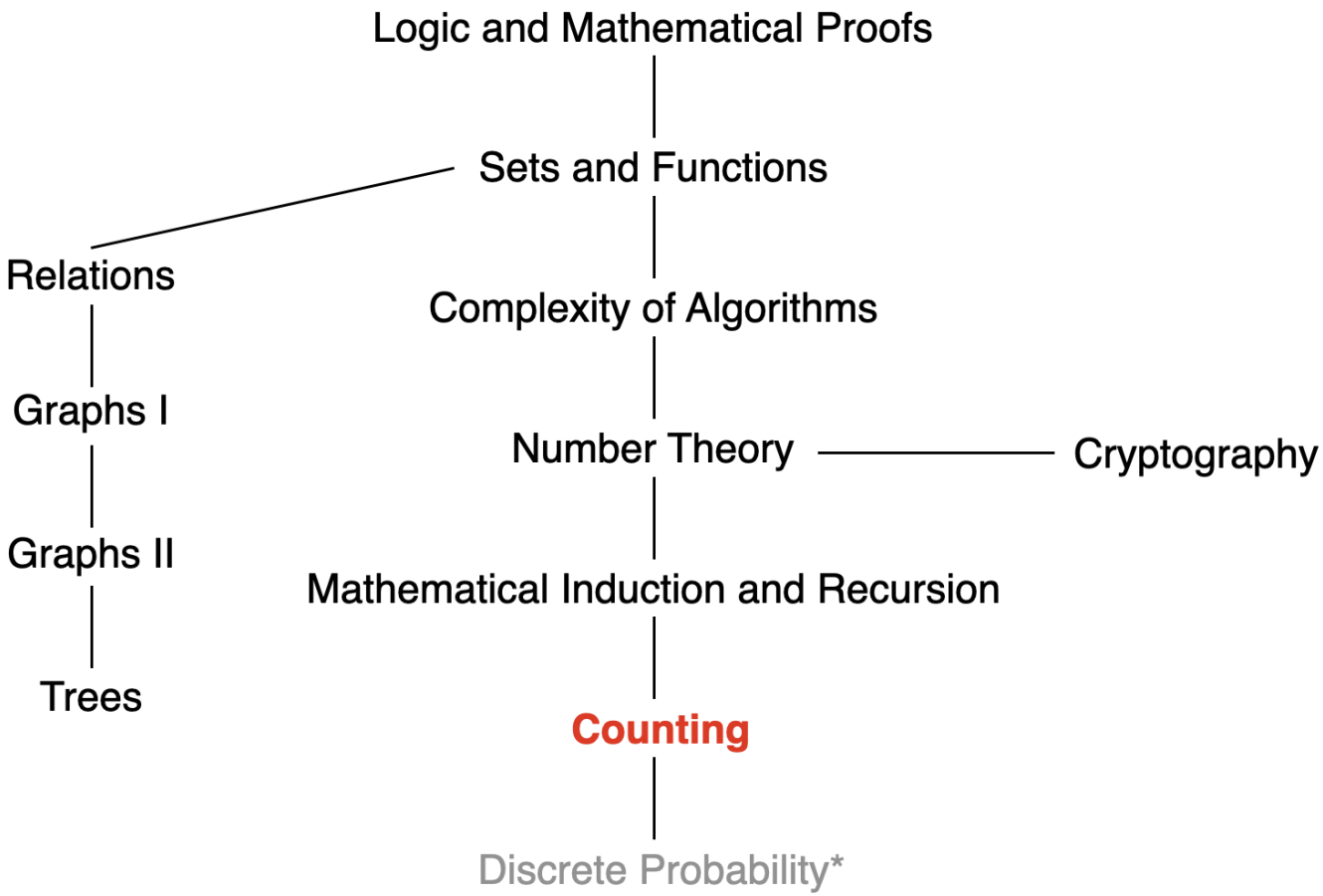

Counting basis, Permutations and Combinations, Binomial Coefficients.

Combinatorial Proof

Theorem: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Definition: A combinatorial proof of an identity is

- a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways
- or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity.

These two types of proofs are called double counting proofs and bijective proofs, respectively.

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Combinatorial Proof: Bijective Proof

Theorem: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Bijective Proof: Suppose that S is a set with n elements.

The function that maps a subset A of S to $\bar{A}$ is a bijection between subsets of S with r elements and subsets with $n - r$ elements.

- X: the set of all possible A, where $|X| = C(n, r)$
- Y: the set of all possible $\bar{A}$, where $|Y| = C(n, n - r)$
- $f : X \to Y$ is defined as $f(A) = \bar{A}$
  - One-to-one: if $f(A_1) = f(A_2)$, then $A_1 = A_2$.
  - Onto: for any $\bar{A}$, there exists an A such that $f(A) = \bar{A}$.

Since there is a bijection between two finite sets X and Y, they must have the same number of elements. Thus, $C(n, r) = C(n, n - r)$.

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Combinatorial Proof: Double Counting Proof

Theorem: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Double Counting Proof:

- Left-hand side $C(n, r)$: The number of subsets A of S with r elements.
- Right-hand side $C(n, n - r)$: The number of subsets $\bar{A}$ (i.e., the complement of A) of S with $n - r$ elements.

Each subset A of S is also determined by specifying which elements are not in A, so are in $\bar{A}$. Thus, both sides count the same thing

It follows that $C(n, r) = C(n, n - r)$.

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The Binomial Theorem

Let x and y be variables, and let n be a nonnegative integer:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Proof: The terms in the product when it is expanded are of the form $x^{n-j}y^j$ for $j = 0, 1, 2, \ldots, n$.

To count the number of terms of the form $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums (so that the other j terms in the product are ys).

Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$, which is $\binom{n}{j}$.

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The Binomial Theorem: Example

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Example 1: What is the expansion of $(x+y)^4$?

$$\begin{aligned}
(x+y)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j \\
&= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 \\
&= x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4.
\end{aligned}$$

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The Binomial Theorem: Example

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Example 2: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

First, note that this expression equals $(2x+(-3y))^{25}$

$$(2x+(-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j.$$

The coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$:

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13!\,12!} 2^{12} 3^{13}.$$

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The Binomial Theorem

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Corollary: Let n be a nonnegative integer,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

This is proven by substituting $x = 1$ and $y = 1$.
Any other ideas to prove?

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The Binomial Theorem

Corollary: Let n be a nonnegative integer,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Double Counting Proof: Let P denote the set of all subsets of $\{1, 2, ..., n\}$.

- Left-hand side: Let S_k be the set of all subsets of $\{1, 2, ..., n\}$ with k elements.

$$|P| = \sum_{k=0}^{n} |S_k| = \sum_{k=0}^{n} \binom{n}{k}$$

- Right-hand side: A set with n elements has a total of 2^n different subsets, i.e., $|P| = 2^n$.

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The Binomial Theorem

Corollary: Let n be a nonnegative integer,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Bijective Proof: Let P denote the set of all subsets of $\{1, 2, ..., n\}$. Let S_k be the set of all subsets of $\{1, 2, ..., n\}$ with k elements.

$$|P| = \sum_{i=0}^{n} |S_k| = \sum_{i=0}^{n} \binom{n}{k}$$

Consider $L = L_1 L_2 ... L_n$ be a list of size n from $\{0, 1\}$. Let $\mathcal{L}$ be the set of all such lists, we have $|\mathcal{L}| = 2^n$.

Objective: there is a bijection between $\mathcal{L}$ and P, so $|P| = |\mathcal{L}| = 2^n$.

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The Binomial Theorem

Define the following function $f : \mathcal{L} \to P$

- If $L \in \mathcal{L}$, then $f(L)$ is the set $S \subset \{1, 2, ..., n\}$ defined by

$$i \in S, \text{for } L_i = 1$$

f is a bijection between $\mathcal{L}$ and P.

- one-to-one: If $f(L_1) = f(L_2)$, then $L_1 = L_2$
- onto: for any S, there exists an L such that $f(L) = S$

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The Binomial Theorem

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Corollary: Let n be a positive integer.

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

This is proven by substituting $x = -1$ and $y = 1$.

Corollary: Let n be a nonnegative integer.

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

This is proven by substituting $x = 1$ and $y = 2$.

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Pascal's Identity

Theorem: Let n and k be positive integers with $n \geq k$. Then,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof: Suppose that T is a set containing $n+1$ elements.

- let a be an element in T
- let $S = T - a$.

Left-hand side counts the number of subsets of T containing k elements, i.e., $\binom{n+1}{k}$.

Note that a subset of T with k elements either contains a together with $k-1$ elements of S, or contains k elements of S and does not contain a.

Right-hand side counts

- the subsets of $k-1$ elements of S, i.e., $\binom{n}{k-1}$
- the subsets of k elements of T, i.e., $\binom{n}{k}$.

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Pascal's Identity

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$$

Consider $S = \{A, B, C, D, E\}$.

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

Set S_1 of 2-subsets of S can be partitioned into 2 disjoint parts:

- S_2: the 2-subsets that contain E
- S_3: the set of 2-subsets that do not contain E

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

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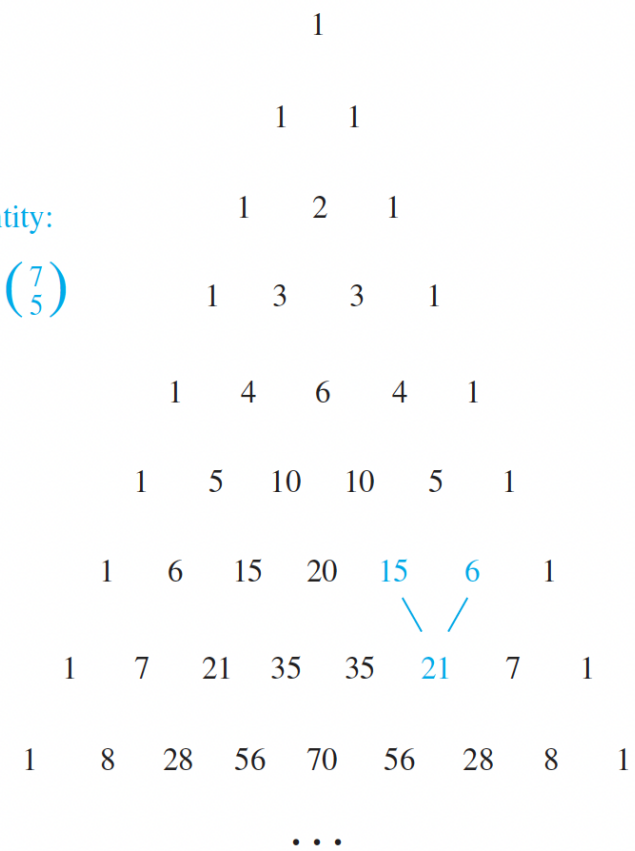
Pascal's Triangle

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$



Pascal's identity, together with the initial conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all integers n, can be used to recursively define binomial coefficients.

Other Identities Involving Binomial Coefficients

Let n and r be nonnegative integers with $r \leq n$.

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof: Consider bit strings of length $n+1$.

The left-hand side, $\binom{n+1}{r+1}$, counts the bit strings of length $n+1$ containing $r+1$ ones.
We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones.

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Other Identities Involving Binomial Coefficients

Let n and r be nonnegative integers with $r \leq n$.

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof: We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones.

- This final one must occur at position $r+1$, $r+2$, . . . , or $n+1$.
- If the last one is the k-th bit there must be r ones among the first $k-1$ positions. There are $\binom{k-1}{r}$ such bit strings.

Summing over k with $r+1 \leq k \leq n+1$, we find that there are

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}.$$

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Combinatorial Proof: Example

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Hint: Consider a set of people, and count the number of ways to select a committee and select one leader within the committee.

- Left-hand side: Suppose there are k people in a committee.
  - First select k people from the n people to form a committee.
  - Then, given the people in the committee, select a leader from the committee, i.e., k ways.

  Thus, there are a total of $\sum_{k=1}^{n} k\binom{n}{k}$ ways.
- Right-hand side:
  - There are n ways to choose a leader.
  - Then, each person other than the leader can be either in the committee or not, i.e., 2^{n-1} ways.

  Hence, using product rule, there are $n2^{n-1}$ ways.

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Labelling and Trinomial Coefficients

Suppose we have k_1 labels of one kind (e.g., red) and $k_2 = n - k_1$ labels of another (e.g., blue). How many different ways to label n distinct objects?

$$C(n, k_1) = \frac{n!}{k_1! k_2}$$

If we have k_1 labels of one kind (e.g., red), k_2 labels of a second kind (e.g., blue), and $k_3 = n - k_1 - k_2$ labels of a third kind (e.g., green). How many different ways to label n distinct objects?

- There are $\binom{n}{k_1}$ ways to choose the red items
- There are then $\binom{n-k_1}{k_2}$ ways to choose the blue items from the remaining $n - k_1$.

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Labelling and Trinomial Coefficients

How many different ways to label n distinct objects?

- There are $\binom{n}{k_1}$ ways to choose the red items
- There are then $\binom{n-k_1}{k_2}$ ways to choose the blue items from the remaining $n - k_1$.

$$\begin{aligned}\binom{n}{k_1}\binom{n-k_1}{k_2} &= \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!} \\ &= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}\end{aligned}$$

This is called a trinomial coefficient and denote it as

$$\binom{n}{k_1 \quad k_2 \quad k_3} = \frac{n!}{k_1!k_2!k_3!},$$

where $k1 + k2 + k3 = n$.

What is the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in $(x + y + z)^n$?

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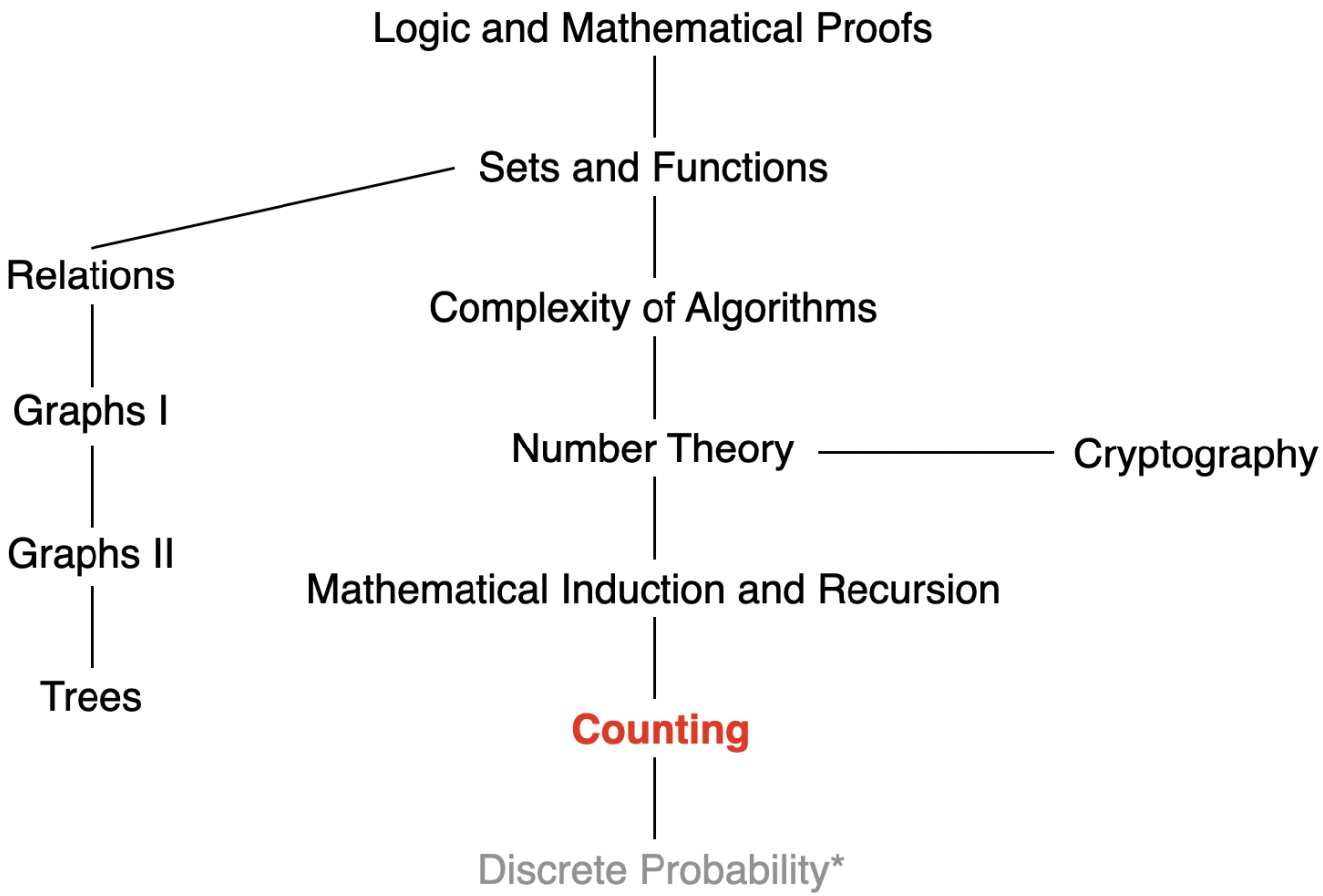
This Lecture



Counting basis, Permutations and Combinations, Binomial Coefficients, The Birthday Paradox, Solving Linear Recurrence Relations, ...

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The Birthday Paradox

Suppose that 25 students are in a room. What is the probability that at least two of them share a birthday?

It's greater than 1/2! (only need 23).

A_n - "there are n students in a room and at least two of them share a birthday."

We may assume that a year has 365 days and there are no twins in the room.

This will be very similar to the analysis of hashing n keys into a table of size 365.

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The Birthday Paradox

A_n - "there are n students in a room and at least two of them share a birthday."

Sample space: $|S| = 365^n$

B_n - "there are n students in a room and none of them share a birthday."

$$\#B_n = 365 \times 364 \times ... \times (365 - (n-1))$$

$$\#A_n + \#B_n = 365^n$$

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The Birthday Paradox

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The Birthday Paradox

Event A: at least two people in the room have the same birthday
Event B: no two people in the room have the same birthday

$$\Pr[A] = 1 - \Pr[B]$$

$$\begin{aligned} \Pr[B] &= \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdot \cdots \cdot \left(1 - \frac{n-1}{365}\right) \\ &= \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right). \end{aligned}$$

$$\Pr[A] = 1 - \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right)$$

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"Birthday" Attacks

Given a function f, the goal of the attack is to find two different inputs x_1 and x_2 such that $f(x_1) = f(x_2)$. Such a pair x_1 and x_2 is called a collision.

Collision in Hashing Functions: A good hashing function yields few collisions (i.e., which are mappings of two different keys to the same memory location).

$$p(n; H) := 1 - \prod_{i=1}^{n-1} (1 - \frac{i}{H})$$

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"Birthday" Attacks

$$p(n; H) := 1 - \prod_{i=1}^{n-1}(1 - \frac{i}{H})$$

Note that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$, for $|x| \ll 1$, $e^x \approx 1 + x$.

Thus, we have $e^{-i/H} \approx 1 - \frac{i}{H}$.

This probability can be approximated as

$$p(n; H) \approx 1 - e^{-n(n-1)/(2H)} \approx 1 - e^{-n^2/(2H)}.$$

Let $n(p; H)$ be the smallest number of values we have to choose, such that the probability for finding a collision is at least p. By inverting the expression above, we have

$$n(p; H) \approx \sqrt{2H \ln \frac{1}{1-p}}$$

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