CS201: Discrete Math for Computer Science
Written Assignment on Logic
Spring 2023

Q.1 **(5 points)** Let p, q be the propositions
p: You get 100 marks on the final.
q: You get an A in this course.
Write these propositions using p and q and logical connectives (including negations).

(a) You will get an A in this course if you get 100 marks on the final.

(b) If you do not get 100 marks on the final, then you will not get an A in this course.

(c) Getting 100 marks on the final is sufficient for getting an A in this course.

(d) You get an A in this course, but you do not get 100 marks on the final.

(e) Whenever you get an A in this course, you got 100 marks on the final.

Solution:

(a) $p \to q$

(b) $\neg p \to \neg q$

(c) $p \to q$

(d) $q \wedge \neg p$

(e) $q \to p$

□

Q.2 **(6 points)** Use truth tables to decide whether or not the following two propositions are equivalent.

(a) $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$

(b) $(\neg q \wedge \neg(p \to q))$ and $\neg p$

(c) $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$

Solution:

(a) The combined truth table is:

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \wedge q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
F	F	T	T	T	T	F	T
F	T	F	T	F	F	F	F
T	F	F	F	T	F	F	F
T	T	T	F	F	F	T	T

By comparing the third and last columns, we have that they are equivalent.

(b) The combined truth table is:

p	q	$p \to q$	$\neg q$	$\neg(p \to q)$	$\neg q \wedge \neg(p \to q)$	$\neg p$
F	F	T	T	F	F	T
F	T	T	F	F	F	T
T	F	F	T	T	T	F
T	T	T	F	F	F	F

By comparing the last two columns, we have that they are not equivalent.

(c) The truth table for $(p \vee q) \to r$ is :

p	q	r	$p \vee q$	$(p \vee q) \to r$
F	F	F	F	T
F	F	T	F	T
F	T	F	T	F
F	T	T	T	T
T	F	F	T	F
T	F	T	T	T
T	T	F	T	F
T	T	T	T	T

The truth table for $(p \to r) \wedge (q \to r)$ is

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	F	T	F
T	F	T	T	T	T
T	T	F	F	F	F
T	T	T	T	T	T

Since the final columns are the same in both truth tables, we know that these two propositions are equivalent.

$\square$

Q.3 **(12 points)** Use logical equivalences to prove the following statements.

(a) $\neg(p \to q) \to \neg q$ is a tautology.

(b) $\neg p \to (q \to r)$ and $q \to (p \vee r)$ are equivalent.

(c) $\neg(p \oplus q)$ and $p \leftrightarrow q$ are equivalent.

(d) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

Solution:

(a) We have

$$\begin{aligned}
&\neg(p \to q) \to \neg q \\
&\equiv \neg\neg(p \to q) \vee \neg q \quad \text{Useful} \\
&\equiv (p \to q) \vee \neg q \quad \text{Double negation} \\
&\equiv (\neg p \vee q) \vee \neg q \quad \text{Useful} \\
&\equiv \neg p \vee (q \vee \neg q) \quad \text{Associative} \\
&\equiv T \quad \text{Domination}
\end{aligned}$$

Therefore, it is a tautology.

(b) We have

$$
\begin{aligned}
&\neg p \to (q \to r) \\
&\equiv \; p \vee (q \to r) \quad \text{Useful and double negation} \\
&\equiv \; p \vee (\neg q \vee r) \quad \text{Useful} \\
&\equiv \; (p \vee \neg q) \vee r \quad \text{Associative} \\
&\equiv \; (\neg q \vee p) \vee r \quad \text{Commutative} \\
&\equiv \; \neg q \vee (p \vee r) \quad \text{Associative} \\
&\equiv \; q \to (p \vee r) \quad \text{Useful}
\end{aligned}
$$

(c) We have

$$
\begin{aligned}
&\neg(p \oplus q) \\
&\equiv \; \neg((p \wedge \neg q) \vee (\neg p \wedge q)) \quad \text{Definition} \\
&\equiv \; \neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q) \quad \text{De Morgan} \\
&\equiv \; (\neg p \vee q) \wedge (p \vee \neg q) \quad \text{De Morgan} \\
&\equiv \; (p \to q) \wedge (q \to p) \quad \text{Useful} \\
&\equiv \; p \leftrightarrow q \quad \text{Definition}
\end{aligned}
$$

Thus, they are equivalent.

(d) We have

$$
\begin{aligned}
&(p \to q) \to ((r \to p) \to (r \to q)) \\
&\equiv \; \neg(\neg p \vee q) \vee (\neg(\neg r \vee p) \vee (\neg r \vee q)) \quad \text{Useful} \\
&\equiv \; \neg(\neg p \vee q) \vee ((r \wedge \neg p) \vee (\neg r \vee q)) \quad \text{De Morgan} \\
&\equiv \; \neg(\neg p \vee q) \vee ((r \vee (\neg r \vee q)) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Distributive} \\
&\equiv \; \neg(\neg p \vee q) \vee (((r \vee \neg r) \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Associative} \\
&\equiv \; \neg(\neg p \vee q) \vee ((T \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Complement} \\
&\equiv \; \neg(\neg p \vee q) \vee (T \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Identity} \\
&\equiv \; \neg(\neg p \vee q) \vee (\neg p \vee (\neg r \vee q)) \quad \text{Identity} \\
&\equiv \; \neg(\neg p \vee q) \vee ((\neg p \vee q) \vee \neg r) \quad \text{Associative} \\
&\equiv \; (\neg(\neg p \vee q) \vee (\neg p \vee q)) \vee \neg r \quad \text{Associative} \\
&\equiv \; T \vee \neg r \quad \text{Complement} \\
&\equiv \; T \quad \text{Identity.}
\end{aligned}
$$

Thus, it is a tautology.

$\square$

Q.4 **(5 points)** Show that the following argument form is valid:

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

Solution: This is equivalent to showing that $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology. We may use a truth table to show this is a tautology, since the last column is all T's.

Alternatively, if this is not a tautology, this means that $p \to r$ is false and $p \to q$ and $q \to r$ are both true. $p \to r$ is false, implies that p is true and r is false. However, p true and r false cannot make both $p \to q$ and $q \to r$ true. Thus, the statement cannot be false, and is a tautology.

$\square$

Q.5 **(5 points)** Based on the following premises, using rules of inference to draw the conclusion:

- Premises: $p \to q$, $\neg p \to r$, $r \to s$
- Conclusion: $\neg q \to s$

Solution:

(1)	$p \to q$	Premise
(2)	$\neg q \to \neg p$	Contrapositive of (1)
(3)	$\neg p \to r$	Premise
(4)	$\neg q \to r$	Hypothetical syllogism (2) (3)
(5)	$r \to s$	Premise
(6)	$\neg q \to s$	Hypothetical syllogism (4) (5)

$\square$

Q.6 **(5 points)** Let $C(x)$ be the statement "x has a cat", let $D(x)$ be the statement "x has a dog" and let $F(x)$ be the statement "x has a ferret." Express each of these sentences in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

(a) A student in your class has a cat, a dog, and a ferret.

(b) All students in your class have a cat, a dog, or a ferret.

(c) Some student in your class has a cat and a ferret, but not a dog.

(d) No student in your class has a cat, a dog, and a ferret.

(e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Solution:

(a) $\exists x(C(x) \wedge D(x) \wedge F(x))$

(b) $\forall x(C(x) \vee D(x) \vee F(x))$

(c) $\exists x(C(x) \wedge F(x) \wedge \neg D(x))$

(d) $\neg \exists x(C(x) \wedge D(x) \wedge F(x))$

(e) $(\exists x C(x)) \wedge (\exists x D(x)) \wedge (\exists x F(x))$

$\square$

Q.7 **(9 points)** Express the negations of each of these statements so that all negation symbols immediately precede predicates.

(a) $\exists x \exists y(Q(x, y) \leftrightarrow Q(y, x))$

(b) $\forall y \exists x \exists z(T(x, y, z) \vee Q(x, y))$

(c) $\forall x \exists y(P(x, y) \wedge \exists z R(x, y, z))$

Solution:

(a)

$$\begin{aligned}
\neg\exists x\exists y(Q(x,y)\leftrightarrow Q(y,x)) &\equiv \forall x\neg\exists y(Q(x,y)\leftrightarrow Q(y,x))\\
&\equiv \forall x\forall y\neg(Q(x,y)\leftrightarrow Q(y,x))\\
&\equiv \forall x\forall y(\neg Q(x,y)\leftrightarrow Q(y,x))
\end{aligned}$$

(b)

$$\begin{aligned}
\neg\forall y\exists x\exists z(T(x,y,z)\vee Q(x,y)) &\equiv \exists y\neg\exists x\exists z(T(x,y,z)\vee Q(x,y))\\
&\equiv \exists y\forall x\neg\exists z(T(x,y,z)\vee Q(x,y))\\
&\equiv \exists y\forall x\forall z\neg(T(x,y,z)\vee Q(x,y))\\
&\equiv \exists y\forall x\forall z(\neg T(x,y,z)\wedge\neg Q(x,y))
\end{aligned}$$

(c)

$$\begin{aligned}
\neg\forall x\exists y(P(x,y)\wedge\exists zR(x,y,z)) &\equiv \exists x\neg\exists y(P(x,y)\wedge\exists zR(x,y,z))\\
&\equiv \exists x\forall y(\neg P(x,y)\vee\forall z\neg R(x,y,z))
\end{aligned}$$

□

Q.8 **(5 points)** Let $P(x,y)$ be a propositional function. Prove or disprove that $\exists x\forall yP(x,y)\to\forall y\exists xP(x,y)$ is a tautology.

Solution: We assume that $\exists x\forall yP(x,y)$ holds. This means that there is some x_0 such that $P(x_0,y)$ holds for all y. Then it is certainly that for all y there exists an x such that $P(x,y)$ is true, since in each case we can at least take $x=x_0$. Note that the converse is not always true, since x depends on y in $\forall y\exists xP(x,y)$.

□

Q.9 **(4 points)** Each of the two below contains a pair of statements, (i) and (ii). For each pair, say whether (i) is equivalent to (ii), i.e., for all $P(x)$ and

$Q(x)$, (i) is true if and only if (ii) is true. Here $\mathbb{R}$ denotes the set of all real numbers.

If they are equivalent, all you have to do is to say that they are equivalent. If they are not equivalent, give a counterexample. A counterexample should involve a specification of $P(x)$ and $Q(x)$ and an explanation as to why the resulting statement is false.

(1) (i) $(\forall x \in \mathbb{R}\ P(x)) \vee (\forall x \in \mathbb{R}\ Q(x))$
(ii) $\forall x \in \mathbb{R}\,(P(x) \vee Q(x))$

(2) (i) $(\forall x \in \mathbb{R}\ P(x)) \wedge (\forall x \in \mathbb{R}\ Q(x))$
(ii) $\forall x \in \mathbb{R}\,(P(x) \wedge Q(x))$

(3) (i) $(\forall x \in \mathbb{R}\ P(x)) \wedge (\exists y \in \mathbb{R}\ Q(y))$
(ii) $\forall x \in \mathbb{R}\,(\exists y \in \mathbb{R}\ (P(x) \wedge Q(y)))$

(4) (i) $(\forall x \in \mathbb{R}\ P(x)) \vee (\exists y \in \mathbb{R}\ Q(y))$
(ii) $\forall x \in \mathbb{R}\,(\exists y \in \mathbb{R}\ (P(x) \vee Q(y)))$

Solution:

(1) Not equivalent. Let $P(x)$ be "$x \geq 0$" and $Q(x)$ be "$x < 0$". (i) is false but (ii) is true.

(2) Equivalent.

(3) Equivalent.

(4) Equivalent.

$\square$

Q.10 **(12 points)** Let P be a proposition in atomic propositions p and q.

(a) If $P = \neg(p \leftrightarrow (q \vee \neg p))$, prove that P is equivalent to $\neg p \vee \neg q$.

(b) Suppose P is of any length, using any of the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. Prove that P is logically equivalent to a proposition of the from

$$A \Box B,$$

where $\Box$ is one of $\wedge$, $\vee$, $\leftrightarrow$, and A and B are chosen from $\{p, \neg p, q, \neg q\}$.

Solution:

(a) This can be proved by truth table.

Alternatively, we prove it using logical equivalences as follows.

$$
\begin{aligned}
P &= \neg(p \leftrightarrow (q \vee \neg p)) \\
&\equiv \neg((p \rightarrow (q \vee \neg p)) \wedge ((q \vee \neg p) \rightarrow p)) \quad \text{Definition} \\
&\equiv \neg((\neg p \vee (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee p)) \quad \text{Useful} \\
&\equiv \neg((\neg p \vee q) \wedge (\neg(q \vee \neg p) \vee p)) \quad \text{Idempotent}
\end{aligned}
$$

For simplicity, let $r = \neg p \vee q$, then we have

$$
\begin{aligned}
P &\equiv \neg(r \wedge (\neg r \vee p)) \\
&\equiv \neg r \vee \neg(\neg r \vee p) \quad \text{De Morgan} \\
&\equiv \neg r \vee (r \wedge \neg p) \quad \text{De Morgan and double negation} \\
&\equiv (\neg r \vee r) \wedge (\neg r \vee \neg p) \quad \text{Distributive} \\
&\equiv T \wedge (\neg r \vee \neg p) \quad \text{Negation} \\
&\equiv \neg r \vee \neg p \quad \text{Identity} \\
&\equiv (p \wedge \neg q) \vee \neg p \quad \text{De Morgan} \\
&\equiv (p \vee \neg p) \wedge (\neg q \vee \neg p) \quad \text{Distributive} \\
&\equiv T \wedge (\neg q \vee \neg p) \quad \text{Negation} \\
&\equiv \neg p \vee \neg q \quad \text{Identity.}
\end{aligned}
$$

(b) For the proposition P, since the two involved atomic propositions p and q can have at most 4 combinations of truth tables, P has at most 2^4 different forms in terms of truth tables up to logical equivalence. It then suffices to prove that a proposition of the form $A \square B$ has also 2^4 different forms in terms of truth tables up to logical equivalence.

If $A \in \{p, \neg p\}$, $B \in \{q, \neg q\}$, and $\square \in \{\wedge, \vee\}$, then $A \square B$ has $2 \times 2 \times 2 = 8$ different possible forms. If $A \in \{p, \neg p\}$, $B \in \{q, \neg q\}$ and $\square = \leftrightarrow$, then there are two extra different possibilities: $p \leftrightarrow q$ and $p \leftrightarrow \neg q$. Together with $p \vee p \equiv p$, $p \vee \neg p \equiv T$, $q \vee q \equiv q$, $p \wedge \neg p \equiv F$ and similarly $\neg p$, $\neg q$, we will have the $2^4 = 16$ different forms by $A \square B$. This proves the statement.

$\square$

Q.11 **(12 points)** For each of these arguments, explain which rules of inference are used for each step.

(a) "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."

(b) "Each of five roommates, A, B, C, D, and E, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."

Solution:

(a) Let $c(x)$ denote "x is in this class", $w(x)$ denote "x enjoys whale watching", and $p(x)$ denote "x cares about ocean pollution." The premises are $\exists x(c(x) \land w(x))$ and $\forall x(w(x) \rightarrow p(x))$. From the first premise, $c(y) \land w(y)$ for a particular person y. Using simplification, $w(y)$ follows. Using the second premise and universal instantiation, $w(y) \rightarrow p(y)$ follows. Using modus ponens, $p(y)$ follows, and by conjunction, $c(y) \land p(y)$ follows. Finally, by existential generalization, the desired conclusion, $\exists x(c(x) \land p(x))$ follows.

(b) Let $r(x)$ be "x is one of the five roommates listed", let $d(x)$ be "x has taken a course in discrete mathematics", and let $a(x)$ be "x can take a course in algorithms". We are given premises $\forall x(r(x) \rightarrow d(x)$, $\forall x(d(x) \rightarrow a(x)$, and we want to conclude $\forall x(r(x) \land a(x))$.

Step	**Reason**
1. $\forall x(r(x) \rightarrow d(x))$	Hypothesis
2. $r(y) \rightarrow d(y)$	Universal Instantiation using 1.
3. $\forall x(d(x) \rightarrow a(x))$	Hypothesis
4. $d(y) \rightarrow a(y)$	Universal instantiation using 3.
5. $r(y) \rightarrow a(y)$	Hypothetical syllogism using 2. and 4.
6. $\forall x(r(x) \rightarrow a(x))$	Universal generalization using 5.

□

Q.12 **(5 points)** Prove the **triangle inequality**, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$.

Solution: We prove by four cases.

Case 1: $x \geq 0$ and $y \geq 0$. Then $|x| + |y| = x + y = |x + y|$.

Case 2: $x < 0$ and $y < 0$. Then $|x| + |y| = -x + (-y) = -(x + y) = |x + y|$.

Case 3: $x \geq 0$ and $y < 0$. Then $|x| + |y| = x + (-y)$. If $x \geq -y$, then $|x + y| = x + y$. But because $y < 0$, $-y > y$, so $|x| + |y| = x + (-y) > x + y = |x + y|$. If $x < -y$, then $|x + y| = -(x + y)$. But because $x < 0$, $x \geq -x$, so $|x| + |y| = x + (-y) \geq -x + (-y) = |x + y|$.

Case 4: $x < 0$ and $y \geq 0$. Similar to Case 3.

$\square$

Q.13 **(5 points)** Prove or disprove that there is a rational number x and an irrational number y such that x^y is irrational.

Solution: Let $x = 2$ and $y = \sqrt{2}$. If $x^y = 2^{\sqrt{2}}$ is irrational, we are done. If not, let $x = 2^{\sqrt{2}}$ and $y = \sqrt{2}/4$. Then $x^y = (2^{\sqrt{2}})^{\sqrt{2}/4} = 2^{\sqrt{2} \cdot (\sqrt{2})/4} = \sqrt{2}$.

$\square$

Q.14 **(5 points)** Prove that $\sqrt[3]{2}$ is irrational.

Solution: Suppose that $\sqrt[3]{2}$ is the rational number p/q, where p and q are positive integers with no common factors. Cubing both sides, we have $2 = p^3/q^3$, or $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd number is odd, this means that p is even, so we can write $p = 2k$ for some integer k. We then have $q^3 = 4k^3$. Since q^3 is even, q must be even. We have now seen that both p and q are even, a contradiction.

$\square$

Q.15 **(5 points)** Give a direct proof that: Let a and b be integers. If $a^2 + b^2$ is even, then $a + b$ is even.

Solution: Observe that $a^2 + b^2 = (a + b)^2 - 2ab$. Thus, $(a + b)^2$ has the same parity as $a^2 + b^2$. So $(a + b)^2$ is even. Then $a + b$ is also even.

$\square$