

Discrete Mathematics for Computer Science

Lecture 14: Counting

Dr. Ming Tang

Department of Computer Science and Engineering
Southern University of Science and Technology (SUSTech)
Email: tangm3@sustech.edu.cn

Generating Function and Counting

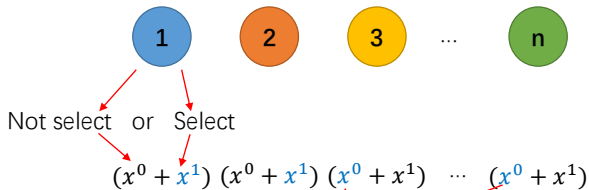
Use generating functions to find the number of r -combinations of a set with n elements.

Solution: The answer is the coefficient of x^r in generating function

$$(1 + x)^n$$

Why $(1 + x)^n$?

r -combinations: select r balls from the following n balls



Product rule:

balls 1 and 2 are selected

$$x^1 x^1 x^0 \cdots x^0 = \boxed{x^2} \text{ Two balls are selected}$$

Generating Function and Counting

Use generating functions to find the number of r -combinations of a set with n elements.

Solution: The answer is the coefficient of x^r in generating function

$$(1 + x)^n$$

Why the coefficient of x^r ?

r -combinations: select $r = 2$ balls from the following $n = 3$ balls



$$\begin{aligned}(x^0 + x^1)(x^0 + x^1)(x^0 + x^1) &= x^0 x^0 x^0 \\ &\quad + x^1 x^0 x^0 + x^0 x^1 x^0 + x^0 x^0 x^1 \\ &\quad + x^1 x^1 x^0 + x^0 x^1 x^1 + x^1 x^0 x^1 \\ &\quad + x^1 x^1 x^1\end{aligned}$$

Sum rule:

$3x^2$ Two balls are selected

Three possibilities

Example

Use generating functions to find the number of r -combinations of a set with n elements.

Solution: The answer is the coefficient of x^r in generating function

$$(1 + x)^n$$

But by the binomial theorem, we have

$$f(x) = \sum_{r=0}^n \binom{n}{r} x^r.$$

Thus, $\binom{n}{r}$ is the answer.

Example

Use **generating functions** to determine the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in the cases

- Case 1: when the order **does not matter**

E.g., three \$1 tokens; one \$1 token and a \$2 token

- Case 2: when the order **does matter**

E.g., three \$1 tokens; a \$1 token and then a \$2 token; a \$2 token and then a \$1 token

Example

Case 1: when the order **does not matter**

The answer is the coefficient of x^r in the generating function

$$(1 + x + x^2 + x^3 + \dots)(1 + x^{1 \times 2} + x^{2 \times 2} + x^{3 \times 2} + \dots) \\ (1 + x^{1 \times 5} + x^{2 \times 5} + x^{3 \times 5} + \dots).$$

For example,

- two \$1; three \$2; one \$5
- $x^2 x^{3 \times 2} x^{1 \times 5} = x^{13} \rightarrow 13$ dollars

Example

Case 2: when the order does **matter**

The number of ways to insert exactly n tokens to produce a total of r dollars is the coefficient of x^r in

$$(x + x^2 + x^5)^n$$


For example,

- Insert two tokens: \$1, then \$2
- $xx^2 = x^3 \rightarrow 3$ dollars

Because any number of tokens may be inserted,

$$1 + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + \cdots = \frac{1}{1 - (x + x^2 + x^5)}$$

Extended Binomial Theorem:

$$\frac{1}{1 - (x + x^2 + x^5)} = \sum_{k=0}^{\infty} \binom{-1}{k} (-1)^k (x + x^2 + x^5)^k$$


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Example 5

Use generating functions to find the number of r -combinations from a set with n elements when **repetition** of elements is allowed.

Solution: The answer is the coefficient of x^r in generating function

$$G(x) = (1 + x + x^2 + \cdots)^n.$$

As long as $|x| < 1$, we have $1 + x + x^2 + \cdots = 1/(1 - x)$, so

$$G(x) = 1/(1 - x)^n = (1 - x)^{-n}.$$

Applying the extended binomial theorem

$$(1 - x)^{-n} = (1 + (-x))^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r.$$

Hence, the coefficient of x^r equals $\binom{-n}{r}(-1)^r = C(n+r-1, r)$.



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Example 6

Use generating functions to find the number of ways to select r objects of n different kinds (with repetition) if we must select **at least one** object of each kind.

Solution: The answer is the coefficient of x^r in generating function

$$G(x) = (x + x^2 + x^3 + \cdots)^n = x^n(1 + x + x^2 + \cdots)^n = x^n/(1 - x)^n.$$

$$\begin{aligned} G(x) &= x^n/(1 - x)^n \\ &= x^n \cdot (1 - x)^{-n} \\ &= x^n \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r \\ &= x^n \sum_{r=0}^{\infty} (-1)^r C(n + r - 1, r) (-1)^r x^r \\ &= \sum_{r=n}^{\infty} C(r - 1, r - n) x^r. \end{aligned}$$

Hence, there are $C(r - 1, r - n)$ ways to select r objects of n different kinds if we must select at least one object of each kind.



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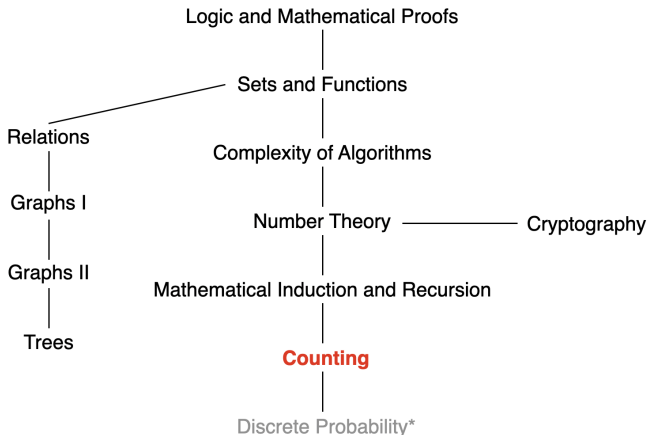
Generating Function and Combinations with Repetitions

- Based on the combination problem, transfer the problem as finding the coefficient of x^r of a generating function, e.g.,

$$G(x) = (1 + x + x^2 + x^3 + \cdots)^n$$

- Find the coefficient of x^r
 - ▶ Enumerate all possibilities or
 - ▶ \rightarrow generating functions without summation \rightarrow binomial theorem

This Lecture



Counting basis, Permutations and Combinations, Binomial Coefficients, The Birthday Paradox, Generalized Permutations and Combinations, Generating Function, **Solving Linear Recurrence Relations**, ...



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Solving Linear Recurrence Relations

Solve the closed-form:

- Linear Homogeneous Recurrence Relations
- Linear Nonhomogeneous Recurrence Relations

Solving Linear Recurrence Relations

One important class of recurrence relations can be explicitly solved in a systematic way.

Definition: A **linear homogeneous relation** of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

where c_1, c_2, \dots, c_k are real numbers, and $c_k \neq 0$.

- **linear**: it is a linear combination of previous terms
- **homogeneous**: all terms are multiples of a_j 's
- **degree k** : a_n is expressed by the previous k terms
- **constant coefficients**: coefficients are constants

Example:

- $a_n = a_{n-1} + (a_{n-2})^2$: not linear.
- $H_n = 2H_{n-1} + 1$: not homogeneous.
- $B_n = nB_{n-1}$: not constant coefficients.
- $P_n = 1.11 \cdot P_{n-1}$: **Yes; of degree 1**
- $f_n = f_{n-1} + f_{n-2}$: **Yes; of degree 2**

Solving Linear Recurrence Relations

By induction, such a recurrence relation is **uniquely** determined by this **recurrence relation** and **k initial conditions** a_0, a_1, \dots, a_{k-1} .

Example: Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2},$$

Which of the following are solutions?

- $a_n = 3n$ **Yes**
- $a_n = 2^n$ **No**
- $a_n = 5$ **Yes**

Question: Why not unique?

Question: Any systematic way?

Solving Linear Recurrence Relations

Definition: A linear homogeneous relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

where c_1, c_2, \dots, c_k are real numbers, and $c_k \neq 0$.

Basic idea: Look for solutions of the form $a_n = r^n$, where r is a constant.

Note that $a_n = r^n$ is a solution of the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}.$$

Divide both sides by r^{n-k} ,

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_{k-1} r - c_k = 0.$$



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Characteristic equation of the recurrence relation.

Solving Linear Recurrence Relations

Why of the form of $a_n = r^n$?

Short answer 1: we know that exponentials satisfy linear recurrences. So we try to fit the linear recurrence to an exponential. We check the result to justify our work a posteriori.

Short answer 2: someone has worked this method out in the past, so we use it.

Medium answer: we can write a linear recurrence relation as a matrix equation

$$v_{n+1} = Av_n$$

where v_n is a vector whose components are consecutive terms of the linear recurrence. This is easy to solve:

$$v_n = A^n v_0$$

If A is diagonalizable, we can obtain a closed form for v_n by diagonalizing and computing the exponential.

Long answer: there is a theory of *difference equations* that is quite analogous to the theory of differential equations. Given a function f , we define the (forward) difference Δf to be the function $(\Delta f)(n) = f(n+1) - f(n)$. Many things in the theory of differential calculus have analogs in this difference calculus.

[https://math.stackexchange.com/questions/926112/](https://math.stackexchange.com/questions/926112/what-is-the-intuitive-idea-behind-looking-for-a-solution-of-the-form-an-rn-for)

what-is-the-intuitive-idea-behind-looking-for-a-solution-of-the-form-an-rn-for



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Solving Linear Recurrence Relations: Degree Two

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2 .

The sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ with the initial condition $a_0 = C_0$ and $a_1 = C_1$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n = 0, 1, 2, \dots,$$

where α_1 and α_2 are constants.

Proof: $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is the solutions of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ and satisfy the initial conditions.

- $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a solution of the recurrence relation for any arbitrary constants α_1 and α_2 .
- For every recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$, there exist α_1 and α_2 that satisfy the initial conditions.

Since the solution is unique, the if and only if statement holds.

Solving Linear Recurrence Relations: Degree Two

Proof: $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a **solution** of the recurrence relation.

Since r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, it follows that $r_1^2 = c_1 r_1 + c_2$ and $r_2^2 = c_1 r_2 + c_2$. Suppose $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$:

$$\begin{aligned}c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\&= \alpha_1 r_1^n + \alpha_2 r_2^n \\&= a_n.\end{aligned}$$

Solving Linear Recurrence Relations: Degree Two

For every recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, there exist α_1 and α_2 that satisfy the **initial conditions**.

Suppose that $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold.

$$a_0 = C_0 = \alpha_1 + \alpha_2, \quad a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2.$$

Thus,

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}, \quad \alpha_2 = C_0 - \alpha_1 = \frac{C_0 r_1 - C_1}{r_1 - r_2}.$$

α_1 and α_2 exist since $r_1 \neq r_2$.

Solving Linear Recurrence Relations: Degree Two

Proof: $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is the solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and satisfy the initial conditions.

- $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a solution of the recurrence relation.
- For every recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, there exist α_1 and α_2 that satisfy the initial conditions.

Note that there is a **unique solution** of a linear homogeneous recurrence relation of degree two with two initial conditions, so the if and only if statement holds.

Solving Linear Recurrence Relations: Degree Two

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2 .

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n = 0, 1, 2, \dots,$$

where α_1 and α_2 are constants.

Solve Linear Recurrence Relations:

- Solve r_1 and r_2 with $r^2 - c_1r - c_2 = 0$.
- Solve α_1 and α_2 with the initial conditions.

Example 1: Fibonacci number

Fibonacci number: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

What is the closed-form expression of F_n ?

To solve r_1 and r_2 , consider $r^n = r^{n-1} + r^{n-2}$, i.e., $r^2 - r - 1 = 0$. There are two different roots:

$$r_1 = \frac{1 + \sqrt{5}}{2}, \quad r_2 = \frac{1 - \sqrt{5}}{2}$$

Consider the form of $F_n = \alpha_1 r_1^n + \alpha_2 r_2^n$. To solve α_1 and α_2 , by $F_0 = 0$ and $F_1 = 1$, we have $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 r_1 + \alpha_2 r_2 = 1$.

Thus, $\alpha_1 = 1/\sqrt{5}$ and $\alpha_2 = -\alpha_1$. Hence,

$$F_n = \alpha_1 r_1^n + \alpha_2 r_2^n = \frac{r_1^n - r_2^n}{\sqrt{5}}.$$



Example 2

$a_n = a_{n-1} + 2a_{n-2}$, with $a_0 = 2$, $a_1 = 7$.

The **characteristic equation** is

$$r^2 - r - 2 = 0.$$

The two roots are 2 and -1 . So, assume that

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n.$$

By the two **initial conditions**, we have

$$a_0 = \alpha_1 + \alpha_2 = 2, \quad a_1 = 2\alpha_1 - \alpha_2 = 7.$$

We get $\alpha_1 = 3$ and $\alpha_2 = -1$. Thus,

$$a_n = 3 \cdot 2^n - (-1)^n.$$

Example 3

$$a_n = 7a_{n-1} - 10a_{n-2}, \text{ with } a_0 = 2, a_1 = 1$$

The characteristic equation is

$$r^2 - 7r + 10 = 0.$$

Two roots are 2 and 5. So, assume that

$$a_n = \alpha_1 2^n + \alpha_2 5^n.$$

By the two initial conditions, we have

$$a_0 = \alpha_1 + \alpha_2 = 2, \quad a_1 = 2\alpha_1 + 5\alpha_2 = 1.$$

We get $\alpha_1 = 3$ and $\alpha_2 = -1$. Thus,

$$a_n = 3 \cdot 2^n - 5^n.$$

Solving Linear Recurrence Relations of Degree k

Consider an arbitrary linear homogeneous relation of degree k with constant coefficients:

$$a_n = \sum_{i=1}^k c_i a_{n-i}.$$

The characteristic equation (CE) is:

$$r^k - \sum_{i=1}^k c_i r^{k-i} = 0.$$

Theorem: If this CE has k distinct roots r_i , then the solutions to the recurrence are of the form

$$a_n = \sum_{i=1}^k \alpha_i r_i^n$$

for all $n \geq 0$, where the α_i 's are constants.

Example

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

The characteristic equation is

$$r^3 - 6r^2 + 11r - 6 = 0.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3)$. So, assume that

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

By the three initial conditions, we have $a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$, $a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3$, $a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9$.

We get $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Thus,

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

The Case of Degenerate Roots: Degree Two

Theorem: If the $r^2 - c_1r - c_2 = 0$ has **only 1 root** r_0 , then

$$a_n = (\alpha_1 + \alpha_2 n)r_0^n,$$

for all $n \geq 0$ and two constants α_1 and α_2 .

Example

$$a_n = 4a_{n-1} - 4a_{n-2} \text{ with } a_0 = 1 \text{ and } a_1 = 0$$

The characteristic equation is

$$r^2 - 4r + 4 = 0.$$

The **only root** is 2. So, assume that

$$a_n = (\alpha_1 + \alpha_2 n)2^n.$$

By the three initial conditions, we have

$$a_0 = \alpha_1 = 1, \quad a_1 = 2 \cdot (\alpha_1 + \alpha_2) = 0.$$

We get $\alpha_1 = 1$, $\alpha_2 = -1$. Thus,

$$a_n = (1 - n)2^n.$$

The Case of Degenerate Roots: Degree k

Theorem: Suppose that there are t roots r_1, \dots, r_t with multiplicities m_1, \dots, m_t . Then,

$$a_n = \sum_{i=1}^t \left(\sum_{j=0}^{m_i-1} \alpha_{i,j} n^j \right) r_i^n$$

for all $n \geq 0$ and constants $\alpha_{i,j}$.

$$\begin{aligned} a_n = & (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n \\ & + \dots + (\alpha_{t,0} + \alpha_{t,1}n + \dots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n \end{aligned}$$

Example

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3} \text{ with } a_0 = 1, a_1 = -2, a_2 = -1.$$

The characteristic equation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

There is a single root $r = -1$ of **multiplicity three** of the characteristic equation. Thus, assume that

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)(-1)^n.$$

By the three initial conditions, we have ...

We get $\alpha_{1,0} = 1$, $\alpha_{1,1} = 3$, $\alpha_{1,2} = -2$. Thus, $a_n = (1 + 3n - 2n^2)(-1)^n$.

Solving Linear Recurrence Relations

- Linear Homogeneous Recurrence Relations
- Linear Nonhomogeneous Recurrence Relations

Linear Nonhomogeneous Recurrence Relations

Definition: A **linear nonhomogeneous relation** with constant coefficients may contain some terms $F(n)$ that depend only on n

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n).$$

The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ is called the **associated homogeneous recurrence relation**.

Example:

- $a_n = a_{n-1} + 2^n$ $a_n = a_{n-1}$
- $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$ $a_n = a_{n-1} + a_{n-2}$
- $a_n = 3a_{n-1} + n3^n$ $a_n = 3a_{n-1}$
- $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$ $a_n = a_{n-1} + a_{n-2} + a_{n-3}$



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Linear Nonhomogeneous Recurrence Relations

Every solution is the **sum** of a **particular solution** and a **solution of the associated** linear homogeneous recurrence relation.

Theorem: If $\{a_n^{(p)}\}$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)},$$

where $\{a_n^{(h)}\}$ is any solution to the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$.

Note: $a_n^{(p)}$ does not need to satisfy the initial conditions



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Linear Nonhomogeneous Recurrence Relations

Proof: Suppose $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous recurrence relation,

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \dots + c_k a_{n-k}^{(p)} + F(n).$$

Now suppose that $\{b_n\}$ is a **second solution** of the nonhomogeneous recurrence relation,

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \dots + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} = c_1 (b_{n-1} - a_{n-1}^{(p)}) + \dots + c_k (b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\{b_n - a_n^{(p)}\}$ is a solution of the associated homogeneous linear recurrence, say, $\{a_n^{(h)}\}$.

Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ for all n .

Linear Nonhomogeneous Recurrence Relations

Theorem: If $\{a_n^{(p)}\}$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)},$$

where $\{a_n^{(h)}\}$ is any solution to the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$.

The key is to find the particular solution to the linear nonhomogeneous relation. However, there is no general method for finding such a solution.

Example 1

There are techniques that work for certain types of functions $F(n)$, such as **polynomials** and **powers of constants**.

Example 1: $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

- Compute $a_n^{(h)}$
- Compute $a_n^{(p)}$
- Initial condition

Example 1

Example 1: $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

To compute $a_n^{(h)}$:

The characteristic equation is

$$r^2 - 3r = 0.$$

The roots are $r_1 = 3$ and $r_2 = 0$. By So, assume that

$$a_n^{(h)} = \alpha 3^n.$$

To compute $a_n^{(p)}$: Try $a_n^{(p)} = cn + d$. Thus,

$$cn + d = 3(c(n-1) + d) + 2n.$$

We get $c = -1$ and $d = -3/2$. Thus, $a_n^{(p)} = -n - 3/2$.



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Example 1

Example 1: $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

To compute $a_n^{(h)}$: $a_n^{(h)} = \alpha 3^n$.

To compute $a_n^{(p)}$: $a_n^{(p)} = -n - 3/2$.

Initial condition:

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha 3^n - n - 3/2.$$

Base on the initial condition $a_1 = 3$. We have $3 = -1 - 3/2 + 3\alpha$, which implies $\alpha = 11/6$. Thus, $a_n = -n - 3/2 + (11/6)3^n$.

Example 2

Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.
(Since we do not provide the initial conditions, obtain the general form would be sufficient.)

Solution:

- $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$
- Try $a_n^{(p)} = C \cdot 7^n$:

$$C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n.$$

Thus, $C = 49/20$, and $a_n^{(p)} = (49/20)7^n$.

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$

Linear Nonhomogeneous Recurrence Relations

For the previous two examples, we **made a guess** that there are solutions of a particular form. **This was not an accident.**

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where c_1, c_2, \dots, c_k are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where b_0, b_1, \dots, b_t and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m , there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Taken care when $s = 1$!

Example 1

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n) \text{ with } F(n) = n^2 2^n \text{ and } F(n) = (n^2 + 1)3^n.$$

To compute $a_n^{(h)}$:

The characteristic equation is

$$r^2 - 6r + 9 = 0.$$

This characteristic equation has a single root $r = 3$ of multiplicity $m = 2$.

$$a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n.$$

Example 1

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ with $F(n) = n^2 2^n$ and $F(n) = (n^2 + 1)3^n$.

To compute $a_n^{(h)}$: $a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n$.

To compute $a_n^{(p)}$ of $F(n) = n^2 2^n$:

Since $s = 2$ is not a root of the characteristic equation, we have

$$a_n^{(p)} = (p_2 n^2 + p_1 n + p_0)2^n.$$

Substituting $a_n^{(p)}$ into $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ to derive p_2 , p_1 , and p_0 :

$$\begin{aligned}(p_2 n^2 + p_1 n + p_0)2^n &= 6(p_2(n-1)^2 + p_1(n-1) + p_0)2^{n-1} \\ &\quad - 9(p_2(n-2)^2 + p_1(n-2) + p_0)2^{n-2} + n^2 2^n.\end{aligned}$$

$$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 n)3^n + (p_2 n^2 + p_1 n + p_0)2^n$$



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Example 1

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ with $F(n) = n^2 2^n$ and $F(n) = (n^2 + 1)3^n$.

To compute $a_n^{(h)}$: $a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n$.

To compute $a_n^{(p)}$ of $F(n) = (n^2 + 1)3^n$:

Since $s = 3$ is a root of the characteristic equation
with multiplicity $m = 2$, we have

$$a_n^{(p)} = n^2(p_2 n^2 + p_1 n + p_0)3^n.$$

Substituting $a_n^{(p)}$ into $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ to derive p_2 , p_1 , and p_0 :

...

$$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 n)3^n + n^2(p_2 n^2 + p_1 n + p_0)3^n.$$

Example 2: The Term n^m

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 2^n$$

Solution:

- $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$
- $a_n^{(p)}$ should be in the form of $np_0 2^n$.
- Try $a_n^{(p)} = p_0 \cdot 2^n$ instead:

$$p_0 \cdot 2^n = 5p_0 \cdot 2^{n-1} - 6p_0 \cdot 2^{n-2} + 2^n.$$

Since $s = 2$ is a root of the characteristic equation,

$$p_0 \cdot 2^n = 5p_0 \cdot 2^{n-1} - 6p_0 \cdot 2^{n-2}$$

always holds. Thus, we obtain $0 = 4$. Contradiction.



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Generating Function

Generating function and recurrent relation ...

Example 1

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \dots$ and initial condition $a_0 = 2$.

Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$. We aim to first derive the formulation of $G(x)$.

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3 \sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k \\ &= 2, \end{aligned}$$

Thus, $G(x) - 3xG(x) = (1 - 3x)G(x) = 2$:

$$G(x) = \frac{2}{(1 - 3x)}.$$



Example 1

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \dots$ and initial condition $a_0 = 2$.

Solution: We aim to first derive the formulation of $G(x)$.

$$G(x) = \frac{2}{(1 - 3x)}.$$

Then, derive a_k using the identity $1/(1 - ax) = \sum_{k=0}^{\infty} a^k x^k$. That is,

$$G(x) = 2 \sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

Consequently, $a_k = 2 \cdot 3^k$.

Example 2

Consider the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1},$$

and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n .

Solution: We extend this sequence by setting $a_0 = 1$. We have $a_1 = 8a_0 + 10^0 = 8 + 1 = 9$. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\begin{aligned} G(x) - 1 &= \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (8a_{n-1} x^n + 10^{n-1} x^n) \\ &= 8 \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} 10^{n-1} x^n \\ &= 8x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1} x^{n-1} \\ &= 8x \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} 10^n x^n \\ &= 8xG(x) + x/(1 - 10x), \end{aligned}$$

Example 2

Consider the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1},$$

and the initial condition $a_1 = 9$.

Solution: Thus,

$$G(x) = \frac{1 - 9x}{(1 - 8x)(1 - 10x)} = G(x) = \frac{1}{2} \left(\frac{1}{1 - 8x} + \frac{1}{1 - 10x} \right).$$

$$\begin{aligned} G(x) &= \frac{1}{2} \left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{2} (8^n + 10^n) x^n. \end{aligned}$$

Thus, $a_n = \frac{1}{2}(8^n + 10^n)$.

Generating function to solve recurrence relations

Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$.

- Based on the recurrence relations, derive the formulation of $G(x)$.
- Using identities (or the useful facts of generating functions), derive sequence $\{a_k\}$.

Next Lecture

