**CS201: Discrete Math for Computer Science**
**Quiz 2, Spring 2023**

The quiz needs to be written in English. Quiz in any other language will get zero point. Any plagiarism behavior will lead to zero point.

**Does not accept late submissions. No exception!**

**Q. 1** (Mathematical Induction, 25 points)**.** Use strong induction to prove that $\sqrt{2}$ is irrational.

(a) To prove the above, we need to first consider the statement $P(n)$ that we need to prove. Write this statement $P(n)$: ________________

(b) Prove the above statement using strong induction. Suppose we know $\sqrt{2} > 1$.

**Solution**:

(a) $P(n)$: there is no positive integer $b$ such that $\sqrt{2} = n/b$.

(b) Basic step: $P(1)$ is true, because $\sqrt{2} > 1 > 1/b$.

Inductive step: Suppose $P(j)$ is true for all $j \leq k$, where $k$ is an arbitrary positive integer. We prove that $P(k+1)$ is true by contradiction.

Suppose $P(k+1)$ is false. Then, there exists a $b$ such that $\sqrt{2} = (k+1)/b$. Thus, $2b^2 = (k+1)^2$, so $(k+1)^2$ must be even, and hence $k+1$ must be even. This implies $k+1 = 2t$ for some positive integer $t$. Substituting $k+1$, we have $2b^2 = 4t^2$, so $b = 2s$ for some positive integer $s$. This implies that $\sqrt{2} = (k+1)/b = 2t/(2s) = t/s$. That is, when $n = t$, there exists an $s$ such that $\sqrt{2} = t/s$. However, since $t < k+1$, this statement is contradict with $P(t)$ is true.

**Q. 2** (Counting and Permutation, 25 points)**.** Consider 10 identical balloons (i.e., non-distinguishable balloons). We aim to give these balloons to four children, and each child should receive at least one balloons.

(a) How many ways to give these balloons to the children? ________________

(b) The answer to the above question is the coefficient of term ____________ of generating function ________________.

(c) Among all the four children, Alice has the most balloons. The number of balloons that Alice has is at least ________________.

**Solution**:

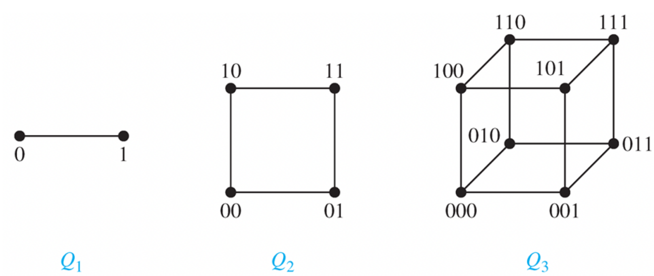
(a) $C(9,6)$

(b) $x^{10}$; $(x^1+x^2+x^3+\cdots)^4$ or $(x^1+x^2+x^3+\cdots+x^{10})^4$ or $(x^1+x^2+x^3+\cdots+x^7)^4$ (any of these functions is ok)

Or alternatively, $x^6$; $(1+x^1+\cdots)^4$ or $(1+x^1+\cdots+x^6)^4$ (any of these functions is ok)

(c) The correct answer is 3. However, the question itself is ambiguous. If some students wrote 4, he or she can get the point.

**Q. 3** (Linear Recurrence Relation, 25 points)**.** An $n$-dimensional hypercube, or $n$-cube, $Q_n$ is a graph with $2^n$ vertices representing all bit strings of length $n$, where there is an edge between two vertices that differ in exactly one bit position. Let $l(n)$ denote the number of edges of $Q_n$.




(a) What is the initial condition of $l(n)$? ____________________

(b) What is the recursive function of $l(n)$?____________________

(c) Derive the closed-form of $l(n)$ using the general approach we have learned for solving linear recurrence relation. Please provide the derivation details. Please do NOT use mathematical induction.

**Solution**:

(a) $l(1) = 1$;

(b) $l(n) = 2l(n-1) + 2^{n-1}$;

(c) First, compute the solution to the associated homogeneous recurrence relation $l^{(h)}(n)$. Since the characteristic equation is $r - 2 = 0$. Thus, $l^{(h)}(n) = \alpha 2^n$. Second, compute the particular solution $l^{(p)}(n)$. According to the formulation of $l(n)$, the particular solution is in the following form: $l^{(p)}(n) = pn2^n$. Substituting $l^{(p)}(n)$ into the recurrence relation of $l(n)$, we have

$$pn2^n = 2p(n-1)2^{n-1} + 2^{n-1}.$$

Thus, $p = 1/2$. Thus, the closed-form solution

$$l(n) = l^{(h)}(n) + l^{(p)}(n) = \alpha 2^n + \frac{1}{2}n2^n.$$

By substituting the initial condition,

$$l(1) = 2\alpha + 1 = 1.$$

Thus, $\alpha = 0$. As a result $l(n) = n2^{n-1}$.

**Q. 4** (Relation, 25 points)**.** Consider the following relations defined on the set of positive integers $\boldsymbol{Z}^+$. Note that $f(n)$ is an arbitrary function on $\boldsymbol{Z}^+$.

(i) $R_1 = \{(a, b) \mid a \text{ divides } b\}$

(ii) $R_2 = \{(a, b) \mid a \equiv b \pmod 3\}$

(iii) $R_3 = \{(a, b) \mid f(a) = f(b)\}$

(iv) $R_4 = \{(a, b) \mid f(a) \leq f(b)\}$

(v) $R_5 = \{(a, b) \mid f(a)f(b) = 1\}$, where $R_5$ is nonempty.

Answer the following questions:

(a) Which of above is/are partial ordering? (Please list all answers) ________

(b) Which of above is/are equivalence relation? (Please list all answers) ____________

(c) Prove whether $R_3$ defined in (iii) is a partial ordering or an equivalence relation.

(d) Draw the Hasse Diagram of relation $R_1$ defined in (ii) on set $\{1, 2, 3, 6, 12\}$. Draw the final diagram would be sufficient. No need to draw the detailed process.

**Solution**:

(a) The correct answer is "(i)";

If a student wrote (iv) as well <u>and</u> gave a condition for $f(\cdot)$ that leads to partial ordering, he and she can get the point.

(b) (ii) (iii)

(c) $R_3$ is an equivalence relation.

- Reflexive: $f(a) = f(a)$ always hold for any $a \in \boldsymbol{Z}^+$
- Symmetric: if $f(a) = f(b)$, then $f(b) = f(a)$
- Transitive: if $f(a) = f(b)$ and $f(b) = f(c)$, then $f(a) = f(c)$

(d)