CS201: Discrete Math for Computer Science
Written Assignment 6
Spring 2023

Due: June 10th, 2023; Please submit through Sakai in ONE PDF file.
The assignment needs to be written in English. Assignments in any other language will get zero point.
Any plagiarism behavior will lead to zero point.
Does not accept late submissions. No exception!

Q. 1. Suppose that G is an undirected graph on a finite set of n vertices. Prove the following

(a) If every vertex of G has degree 2, then G contains a cycle.

(b) If G is disconnected, then its complement is connected.

Solution:

(a) Assume for contradiction that G has no cycle, and consider the longest path P in G (one must exist, since the graph is finite). Let v be the final vertex in P – since v has degree 2, it must have two edges e_1 and e_2 incident on it, of which one, say e_1, is the last edge of the path P. Then e_2 cannot be incident on any other vertex of P since that would create a cycle $(v, e_2, [\text{section of } P \text{ ending in } e_1], v)$. So e_2 an its other endpoint are not part of P, and can be appended to P to give a strictly longer path, which contradicts our choice of P. Hence, G must contain a cycle.

(b) Let $\overline{G}$ denote the complement of G. Consider any two vertices u, v in G. If u and v are in different connected components in G, then no edge of G connects them, so there will be an edge $\{u, v\}$ in $\overline{G}$. If u and v are in the same connected component in G, then consider any vertex w in a different connected component (since G is disconnected, there must be at least one other connected component). By our first argument, the edges $\{u, w\}$ and $\{v, w\}$ exist in $\overline{G}$, so u and v are connected by the path (u, w, v). Hence, any two vertices are connected in $\overline{G}$, so the whole graph is connected.

Q. 2. Show that if G is an undirected bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

Solution: Suppose that the parts are of sizes k and $v-k$, respectively. Then the maximum number of edges of the graph may have is $k(v-k)$. By algebra, we know that the function $f(k) = k(v-k)$ achieves its maximum value when $k = v/2$, giving $f(k) = v^2/4$. Thus there are at most $v^2/4$ edges.

Q. 3. An undirected simple graph G is called <u>self-complementary</u> if G and $\overline{G}$ are isomorphic. Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

Solution: If G is self-complementary, then the number of edges of G must equal the number of edges of $\overline{G}$. But the sum of these two numbers is $n(n-1)/2$, where n is the number of vertices of G, since the union of the two graphs is K_n. Therefore, the number of G must be $n(n-1)/4$. Since this number must be an integer, a look at the four cases shows that n may be congruent to either 0 or 1, but not congruent to either 2 or 3, modulo 4.