CS201: Discrete Math for Computer Science
Written Assignment on Set and Functions
Spring 2023

Due: Mar. 26th, 2023; Please submit through Sakai in ONE PDF file.
The assignment needs to be written in English. **Assignments in any other language will get zero point.**
Any plagiarism behavior will lead to zero point.

Q. 1. (8 points) Suppose that A, B and C are three finite sets. For each of the following, determine whether or not it is true. Explain your answers.

(a) $(A \cap B \neq \emptyset) \rightarrow ((A - B) \subset A)$

(b) $(A \subseteq B) \rightarrow (|A \cup B| \geq 2|A|)$

(c) $(A \cap B \cap C) \subseteq (A \cup B)$

(d) $\overline{(A - B)} \cap (B - A) = B$

Solution:

(a) True. $A \cap B \neq \emptyset$ means that an element of the intersection will not be in $A - B$. So, a nonempty intersection means $A - B$ is missing at least one element of A.

(b) False. Let $A = B = \{1\}$. Then, $A \subseteq B$ is true, but $|A \cup B| = 1 < 2 = 2|A|$, which is false.

(c) True. $A \cap B \cap C \subseteq A \cap B \subseteq A \cup B$.

(d) False. Let $A = B = \{1\}$. Then, $\overline{A - B} \cap (B - A) = U \cap \emptyset \neq B = \{1\}$.

$\square$

Q. 2. (5 points) Let A, B and C be sets. Prove the following using set identities.

(1) $(B - A) \cup (C - A) = (B \cup C) - A$

(2) $(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C) = \emptyset$

Solution:

(1) We have

$$\begin{aligned}(B - A) \cup (C - A) &= (B \cap \overline{A}) \cup (C \cap \overline{A}) \quad \text{by definition}\\ &= \overline{A} \cap (B \cup C) \quad \text{ditributive law}\\ &= (B \cup C) - A \quad \text{by definition}\end{aligned}$$

(2) We have

$$\begin{aligned}&(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C)\\ &= (A \cap B) \cap (A \cap C) \cap \overline{(B \cap C)} \quad \text{commutative law}\\ &= (A \cap B \cap C) \cap \overline{(B \cap C)} \quad \text{associative law}\\ &= (A \cap B \cap C) \cap (\overline{B} \cup \overline{C}) \quad \text{De Morgan}\\ &= ((A \cap B \cap C) \cap \overline{B}) \cup ((A \cap B \cap C) \cap \overline{C}) \quad \text{distributive law}\\ &= \emptyset \cup \emptyset \quad \text{Complement}\\ &= \emptyset.\end{aligned}$$

□

Q. 3. (9 points) For each set defined below, determine whether the set is countable or uncountable. Explain your answers. Recall that $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers.

(a) The set of all subsets of students in CS201

(b) $\{(a, b) | a,\ b \in \mathbb{N}\}$

(c) $\{(a, b) | a \in \mathbb{N},\ b \in \mathbb{R}\}$

Solution:

(a) Countable. The number of students in CS201 is finite, so the size of its power set is also finite. All finite sets are countable.

(b) Countable. The set is the same as $\mathbb{N} \times \mathbb{N}$. We now show a bijections between $\mathbb{Z}^+$ and the set:

$$(0, 0), (0, 1), (1, 0), (1, 1), (1, 2), \ldots.$$

(c) Uncountable. Since $\mathbb{R}$ is uncountable, any sequence that includes an element from $\mathbb{R}$ must also be uncountable.

$\square$

Q. 4. (5 points) Give an example of two uncountable sets A and B such that the difference $A - B$ is

(a) finite,

(b) countably infinite,

(c) uncountable.

Solution: In each case, let A be the set of real numbers.

(a) Let B be the set of real numbers as well, then $A - B = \emptyset$, which is finite.

(b) Let B be the set of real numbers that are not positive integers, then $A - B = \mathbf{Z}^+$, which is countably infinite.

(c) Let B be the set of positive real numbers. Then $A - B$ is the set of negative real numbers, which is uncountable.

$\square$

Q. 5. (5 points) For each of the following mappings, indicate what type of function they are (if any). Use the following options to describe them, and explain your answers.
i. Not a function.
ii. A function which is neither one-to-one nor onto.
iii. A function which is onto but not one-to-one.
iv. A function which is one-to-one but not onto.
v. A function which is both one-to-one and onto.

(a) The mapping f from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = |2x|$.

(b) The mapping f from $\{1, 3\}$ to $\{2, 4\}$ defined by $f(x) = 2x$.

(c) The mapping f from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 8 - 2x$.

(d) The mapping f from $\mathbb{R}$ to $\mathbb{Z}$ defined by $f(x) = \lfloor x + 1 \rfloor$.

(e) The mapping f from $\mathbb{R}^+$ to $\mathbb{R}^+$ defined by $f(x) = x - 1$.

(f) The mapping f from $\mathbb{Z}^+$ to $\mathbb{Z}^+$ defined by $f(x) = x + 1$.

Solution:

(a) ii.

(b) i.

(c) v.

(d) iii.

(e) i.

(f) iv.

$\square$

Q. 6. (8 points) Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Solution: For the "if" part, given $A \subseteq B$, we want to show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, i.e., if $C \subseteq A$, then $C \subseteq B$. Since $A \subseteq B$, $A \subseteq C$ directly follows.

For the "only if" part, given that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we want to show that $A \subseteq B$. Suppose that $a \in A$. Then $\{a\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, it follows that $\{a\} \in \mathcal{P}(B)$, which means that $\{a\} \subseteq B$. This implies that $a \in B$, and completes the proof.

$\square$

Q. 7. (5 points) The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

(a) Determine whether the symmetric difference is associative; that is, if A, B and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

(b) Suppose that A, B and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

Solution:

(a) Using membership table, one can show that each side consists of the elements that are in an odd number of the sets A, B and C. Thus, it follows.

(b) Yes. We prove that for every element $x \in A$, we have $x \in B$ and vice versa.

First, for elements $x \in A$ and $x \notin C$, since $A \oplus C = B \oplus C$, we know that $x \in A \oplus C$ and thus $x \in B \oplus C$. Since $x \notin C$, we must have $x \in B$. For elements $x \in A$ and $x \in C$, we have $x \notin A \oplus C$. Thus, $x \notin B \oplus C$. Since $x \in C$, we must have $x \in B$.

The proof of the other way around is similar.

□

Q. 8. (5 points) Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Solution: This follows directly from the definition. We want to show that

$$\begin{aligned} & ((f \circ g) \circ (g^{-1} \circ f^{-1}))(z) \\ = \ & (f \circ g)((g^{-1} \circ f^{-1})(z)) \\ = \ & (f \circ g)(g^{-1}(f^{-1}(z))) \\ = \ & f(g(g^{-1}(f^{-1}(z)))) \\ = \ & f(f^{-1}(z)) \\ = \ & z. \end{aligned}$$

The second equality is similar.

□

Q. 9. (5 points) Suppose that two functions $g : A \to B$ and $f : B \to C$ and $f \circ g$ denotes the composition function.

(a) If $f \circ g$ is one-to-one and g is one-to-one, must f be one-to-one? Explain your answer.

(b) If $f \circ g$ is one-to-one, must g be one-to-one? Explain your answer.

(c) If $f \circ g$ is onto, must f be onto? Explain your answer.

Solution:

(a) No. We prove this by giving a counterexample. Let $A = \{1,2\}$, $B = \{a,b,c\}$, and $C = A$. Define the function g by $g(1) = a$ and $g(2) = b$, and define the function f by $f(a) = 1$, and $f(b) = f(c) = 2$. Then it is easily verified that $f \circ g$ is one-to-one and g is one-to-one. But f is not one-to-one.

(b) Yes. Similar to (ii), the condition that f is one-to-one is in fact not used.

(c) Yes. Since $f \circ g$ is onto, we know that $f \circ g(A) = C$, which means that $f(g(A)) = C$. Note that $g(A)$ is a subset of B, thus, $f(B)$ must also be C. This means that f is also onto.

$\square$

Q. 10. (5 points) Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Solution:

Certainly every real number x lies in an interval $[n, n+1)$ for some integer n; indeed $n = \lfloor x \rfloor$.

- if $x \in [n, n + \frac{1}{3})$, then $3x$ lies in the interval $[3n, 3n+1)$, so $\lfloor 3x \rfloor = 3n$. Moreover in this case $x + \frac{1}{3}$ is still less than $n+1$, and $x + \frac{2}{3}$ is still less than $n+1$. So, $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = n + n + n = 3n$ as well.
- if $x \in [n + \frac{1}{3}, n + \frac{2}{3})$, then $3x \in [3n+1, 3n+2)$, so $\lfloor 3x \rfloor = 3n+1$. Moreover in this case $x + \frac{1}{3}$ is in $[n + \frac{2}{3}, n+1)$, and $x + \frac{2}{3}$ is in $[n+1, n+\frac{4}{3})$, so $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = n + n + (n+1) = 3n+1$ as well.
- if $x \in [n + \frac{2}{3}, n+1)$, similar and both sides equal $3n+2$.

$\square$

Q. 11. (5 points) Derive the formula for $\sum_{k=1}^{n} k^2$.

Solution: First we note that $k^3 - (k-1)^3 = 3k^2 - 3k + 1$. Then we sum this equation for all values of k from 1 to n. On the left, because of telescoping, we have just n^3; on the right we have

$$3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3\sum_{k=1}^{n} k^2 - \frac{3n(n+1)}{2} + n.$$

Equating the two sides and solving for $\sum_{k=1}^{n} k^2$, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \frac{1}{3}\left(n^3 + \frac{3n(n+1)}{2} - n\right) \\
&= \frac{n}{3}\left(\frac{2n^2 + 3n + 3 - 2}{2}\right) \\
&= \frac{n}{3}\left(\frac{2n^2 + 3n + 1}{2}\right) \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

□

Q. 12. (5 points) Show that if A, B, C and D are sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.

Solution: We are given bijections f from A to B and g from C to D. Then the function from $A \times C$ to $B \times D$ that sends (a, c) to $(f(a), g(c))$ is a bijection. Thus, we have $|A \times C| = |B \times D|$.

□

Q. 13. (10 points) Suppose that A is a countable set. Show that the set B is also countable if there is an onto function from A to B.

Solution: If $A = \emptyset$, then the only way for the conditions to be met are that $B = \emptyset$ as well, and we are done. So assume that A is nonempty. Let f be the given onto function from A to B, and let $g : \mathbf{Z}^+ \to A$ be an onto function that establishes the countability of A. If A is finite rather than countably infinite, say of cardinality of k, then the function g can be simply defined so that $g(1), g(2), \ldots, g(k)$ will list the elements of A, and $g(n) = g(1)$ for $n > k$. We need to find an onto function from $\mathbf{Z}^+$ to B. The function $f \circ g$ does the trick, because the composition of two onto functions is onto.

$\square$

Q. 14. (10 points) Show that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$, and a_n are real numbers and $a_n \neq 0$, then $f(x)$ is $\Theta(x^n)$.

Solution:

We need to show inequalities in both ways. First, we show that $|f(x)| \leq Cx^n$ for all $x \geq 1$ in the following. Noting that $x^i \leq x^n$ for such values of x whenever $i < n$. We have the following inequalities, where M is the largest of the absolute values of the coefficients and $C = (n+1)M$:

$$\begin{aligned}
|f(x)| &= |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \\
&\leq |a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x + |a_0| \\
&\leq |a_n|x^n + |a_{n-1}|x^n + \cdots + |a_1|x^n + |a_0|x^n \\
&\leq Mx^n + Mx^n + \cdots + Mx^n \\
&= Cx^n.
\end{aligned}$$

For the other direction, let k be chosen larger than 1 and larger than $2nm/|a_n|$, where m is the largest of the absolute values of the a_i's for $i < n$. Then each a_{n-i}/x^i will be smaller than $|a_n|/2n$ in absolute value for all $x > k$. Now we have for all $x > k$,

$$\begin{aligned}
|f(x)| &= |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \\
&= x^n \left| a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right| \\
&\geq x^n \left(|a_n| - \left| \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right| \right) \\
&\geq x^n \left(|a_n| - \left(\left| \frac{a_{n-1}}{x} \right| + \cdots \left| \frac{a_1}{x^{n-1}} \right| + \left| \frac{a_0}{x^n} \right| \right) \right) \\
&\geq x^n |a_n/2|
\end{aligned}$$

$\square$

Q. 15. (5 points) Consider the following algorithm for evaluating the value of a polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ at $x = c$.

(a) How many multiplications and additions are used to evaluate a polynomial of degree n at $x = c$? (Do not count additions used to increment the loop variable).

Algorithm 1 polynomial ($c, a_0, a_1, \ldots, a_n$: real numbers)

$power := 1$
$y := a_0$
for $i := 1$ to n **do**
  $power := power * c$
  $y := y + a_i * power$
end for
return y $\{y = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0\}$

(b) Under the operations considered in (a), what is the time complexity with respect to n (in Big-Theta Notation)?

Solution: (a) $2n$ multiplications and n additions. (b) Given the results in (a), there are $3n$ operations. Thus, the time complexity is $\Theta(n)$.

□

Q. 16. (5 points) Answer the following questions:

(1) Show that $(\sqrt{2})^{\log n} = O(\sqrt{n})$, where the base of the logarithm is 2.

(2) Arrange the functions

$$n^n,\ (\log n)^2,\ n^{1.0001},\ (1.0001)^n,\ 2^{\sqrt{\log_2 n}},\ n(\log n)^{1001}$$

in a list such that each function is big-O of the next function. You do not need to include the proof.

Solution:

(1) We have
$$(\sqrt{2})^{\log n} = 2^{\log n \frac{1}{2}} = n^{\frac{1}{2}} = \sqrt{n}.$$
Thus, it is clear that $(\sqrt{2})^{\log n} = O(\sqrt{n})$.

(2) $(\log n)^2$, $2^{\sqrt{\log_2 n}}$, $n(\log n)^{1001}$, $n^{1.0001}$, $(1.0001)^n$, n^n.

□