CS201: Discrete Math for Computer Science
Quiz 1, Spring 2023

The quiz needs to be accomplished in English. Closed-book, no cheating sheet, no discussion. Any plagiarism behavior will lead to zero point.

Q. 1. (50 points) For each of the following questions, determine whether the following statements are correct or incorrect. Explain your answer.

(1) $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$ are equivalent.

(2) Under the domain of all real numbers, the truth value of $\exists x \forall y (y \neq 0 \to xy = 1)$ is T.

(3) Consider an argument form with premise $p \vee q$, premise $\neg p \vee r$, and conclusion $q \vee r$. To prove that this argument form is valid, based on the definition, we need to show that ______ is a tautology. Tautology is a proposition that is _____________.

Solution:

(1) Correct. This can be proven as follows:

$$\begin{aligned} (p \vee q) \to r &\equiv \neg(p \vee q) \vee r && \text{(Useful Law)} \\ &\equiv (\neg p \wedge \neg q) \vee r && \text{(De Moegan's Law)} \\ &\equiv (\neg p \vee r) \wedge (\neg q \vee r) && \text{(Distributive Law)} \\ &\equiv (p \to r) \wedge (q \to r) && \text{(Useful Law)} \end{aligned}$$

Any proof that shows the equivalence is acceptable.

(2) Incorrect. This proposition means that there is a real number x for which $y \neq 0 \to xy = 1$ for every real number y. Consider an arbitrary x. Suppose $y_1 \neq 0$ and $xy_1 = 1$. Let $y_2 = 2y_1$. Then, $xy_2 = 2$, i.e., $y \neq 0 \to xy = 1$ does not hold for every y.

(3) $((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$; always true given all possibile truth values of the proposition variables.

Q. 2. (25 points) Prove or disprove that $\sqrt[3]{2}$ is a rational number.

Solution: Through proving with contradiction, we have $\sqrt[3]{2}$ is a not rational number. Suppose $\sqrt[3]{2}$ is a rational number. Then, there exists m and n such that $\sqrt[3]{2} = m/n$, where m and n are two integers which have no common divisors. Thus, $m^3 = 2n^3$. Since m is an integer, m must be divisble by 2 and can be represented by $m = 2k$, where k is an integer. By substituting m into $m^3 = 2n^3$, we have $4k^3 = n^3$. This implies that n must be divisble by 2 as well. As a result, m and n have a common divisor 2, which leads to a contradiction.

Q. 3. (25 points + Bonus 20 points) Consider the following functions:

$$2^n, n^{20}, n^2(\log n)^{20}, (n!)^5, (\log n)^{\log\log n}, \log(n^n),$$

where the base of the logarithm is 2.

(1) Which function has the highest growth rates?

(2) Which functions have the lowest and second lowest growth rates? List these two functions would be sufficient. You do not need to order these two functions.

(3) [Bonus 20 points] Order the two functions you listed in question (2) by their grow rates, and prove it. That is, suppose the two functions you listed are $f_1(n)$ and $f_2(n)$. You need to indicate whether it is $f_1(n) = O(f_2(n))$ or $f_2(n) = O(f_1(n))$, and then prove it.

Solution:

(1) $(n!)^5$

(2) $(\log n)^{\log\log n}$ and $\log(n^n)$

(3) $(\log n)^{\log\log n} = O(\log(n^n))$. To prove $(\log n)^{\log\log n} = O(\log(n^n))$, let $n = 2^{2^k}$, then we need to show:

$$(\log 2^{2^k})^{\log\log 2^{2^k}} = O(2^{2^k}\log(2^{2^k})).$$

As a result, we need to show $(2^k)^k = O(2^{2^k}2^k)$, i.e., $2^{k^2} = O(2^{2^k+k})$. Equation $2^{k^2} = O(2^{2^k+k})$ is true, since $2^{k^2} \leq 2^{2^k+k}$ for all $k \geq 0$.