CS201: Discrete Math for Computer Science
Written Assignment 5
Spring 2023

Due: May 22th, 2023; Please submit through Sakai in ONE PDF file.
The assignment needs to be written in English. Assignments in any other language will get zero point.
Any plagiarism behavior will lead to zero point.
Does not accept late submissions. No exception!

Q. 1. Let S be the set of all strings of English letters. Determine whether these relations are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive. You do not need to explain the reason.

a) $R_1 = \{(a, b)|a$ and b have no letters in common$\}$

b) $R_2 = \{(a, b)|a$ and b are not the same length$\}$

c) $R_3 = \{(a, b)|a$ is longer than $b\}$

Solution:

a) Irreflexive, symmetric

b) Irreflexive, symmetric

c) Irreflexive, antisymmetric, transitive

□

Q. 2. Show that a subset of an antisymmetric relation is also antisymmetric.

Solution: Suppose that $R_1 \subseteq R_2$ and that R_2 is antisymmetric. We must show that R_1 is also antisymmetric. Let $(a, b) \in R_1$ and $(b, a) \in R_1$. Since these two pairs are also both in R_2, we know that $a = b$, as desired.

□

Q. 3. How many relations are there on a set with n elements that are

a) antisymmetric?

b) irreflexive?

c) neither reflexive nor irreflexive?

d) symmetric, antisymmetric and transitive?

Please explain your answer.

Solution:

a) $2^n 3^{n(n-1)/2}$

b) $2^{n(n-1)}$

c) $2^{n^2} - 2 \cdot 2^{n(n-1)}$

d) 2^n

$\square$

Q. 4. Suppose that R_1 and R_2 are both reflexive relations on a set A.

a) Show that $R_1 \oplus R_2$ is irreflexive.

b) Is $R_1 \cap R_2$ also reflexive? Explain your answer.

c) Is $R_1 \cup R_2$ also reflexive? Explain your answer.

Solution:

a) Since $(a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$, it follows that $(a, a) \notin R_1 \oplus R_2$ for all $a \in A$. Thus, $R_1 \oplus R_2$ is irreflexive.

b) Yes. Since $(a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$, it follows that $(a, a) \notin R_1 \cap R_2$

c) Yes. Since $(a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$, it follows that $(a, a) \notin R_1 \cup R_2$

$\square$

Q. 5. Suppose that R is a symmetric relation on a set A. Is $\overline{R}$ also symmetric? Explain your answer.

Solution: Under this hypothesis, $\overline{R}$ must also be symmetric. If $(a, b) \in \overline{R}$, then $(a, b) \notin R$, whence (b, a) cannot be in R since R is symmetric. In other words, (b, a) is also contained in $\overline{R}$. Thus, $\overline{R}$ is symmetric.

$\square$

Q. 6. Show that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Solution: Suppose that (a, b) is in the symmetric closure of the transitive closure of R. We must show that (a, b) is in the transitive closure of the symmetric closure of R. We know that at least one of (a, b) and (b, a) is in the transitive closure of R. Hence, there is either a path from a to b in R or a path from b to a in R (or both). In the former case, there is a path from a to b in the symmetric closure of R. In the latter case, we can form a path from a to b in the symmetric closure of R by reversing the directions of all the edges in a path from b to a, going backward.

Hence, (a, b) is in the transitive closure of the symmetric closure of R.

$\square$

Q. 7. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$.

a) Show that R is an equivalence relation.

b) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation R?

c) Give an interpretation of the equivalence classes for the equivalence relation R.

Solution:

a) For reflexivity, $((a, b), (a, b)) \in R$ because $a \cdot b = b \cdot a$. If $((a, b), (c, d)) \in R$ then $ad = bc$, which also means that $cb = da$, so $((c, d), (a, b)) \in R$; this tells us that R is symmetric. Finally, if $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$ then $ad = bc$ and $cf = de$. Multiplying these equations gives $acdf = bcde$, and since all these numbers are nonzero, we have $af = be$, so $((a, b), (e, f)) \in R$; this tells us that R is transitive.

b) The equivalence classes of $(1, 2)$ is the set of all pairs (a, b) such that the fraction a/b equals $1/2$.

c) The equivalence classes are the positive rational numbers.

□

Q. 8. How many different equivalence relations with exactly three different equivalence classes are there on a set with five elements? Please explain your answer.

Solution: 25. There are two possibilities to form exactly three different equivalence classes with 5 elements. One is 3, 1, 1 elements for each equivalence class, and the other is 2, 2, 1 elements for each equivalence class. By counting techniques, there are $\binom{5}{3} + \binom{5}{1} \cdot \binom{4}{2}/2 = 25$.

□

Q. 9. Which of these are posets? Please explain your answer.

a) $(\mathbf{R}, =)$

b) $(\mathbf{R}, <)$

c) $(\mathbf{R}, \leq)$

d) $(\mathbf{R}, \neq)$

Solution:

a) Yes. (It is the smallest partial order: reflexivity ensures that very partial order contains at least all pairs (a, b).)

b) No. It is not reflexive.

c) Yes.

d) No. The relations is not reflexive, not antisymmetric, not transitive.

□

Q. 10. Let $\sim$ be a relation defined on $\mathbb{N}$ by the rule that $x \sim y$ if $x = 2^k y$ or $y = 2^k x$ for some $k \in \mathbb{N}$. Show that $\sim$ is an equivalence relation.

Solution: We first show the following lemma.

Lemma For any $x, y \in \mathbb{N}$, $x \sim y$ if and only if there exists some $k \in \mathbb{Z}$ such that $x = 2^k y$ in $\mathbb{Q}$.

Proof. Suppose that $x \sim y$. Then either $x = 2^k y$ for some $k \in \mathbb{N} \subseteq \mathbb{Z}$ and we are done, or $y = 2^{k'} x$ for some $k' \in \mathbb{N}$. In the latter case, solve for $x = 2^{-k'} y$ and let $k = -k'$. In the other direction, if $x = 2^k y$, and $k \geq 0$, then $x = 2^k y$ for some $k \in \mathbb{N}$, giving $x \sim y$. If instead $k < 0$, then $y = 2^{-k}$, again giving $x \sim y$.

To show $\sim$ is an equivalence relation, we show the following three properties.

Reflexive For any $x \in \mathbb{N}$, $x = 2^0 x$ so $x \sim x$.

Symmetric If $x \sim y$, then from **Lemma** there exists $k \in \mathbb{Z}$ such that $x = 2^k y$. But then $y = 2^{-k} x$, so applying the lemma again, gives $y \sim x$.

Transitive If $x \sim y \sim z$, then $x = 2^k y$ and $y = 2^\ell z$ for some $k, \ell \in \mathbb{Z}$ by **Lemma**. Solve to get $x = 2^{k+\ell} z$, which gives $x \sim z$.

$\square$

Q. 11. Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, f is **dominated** by g if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Write $f \preceq g$ if f is dominated by g.

a) Prove that $\preceq$ is a partial ordering.

b) Prove or disprove: $\preceq$ is a total ordering.

Solution:

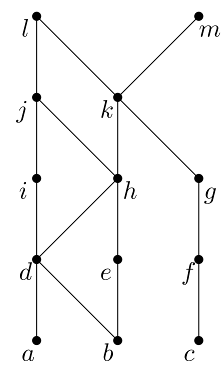
a) **Reflexive** For all $x \in \mathbb{R}$, $f(x) \leq f(x)$, so $f \preceq f$.

   Antisymmetric Let $f \preceq g$ and $g \preceq f$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq f(x)$ and thus $f(x) = g(x)$. Since this holds for all x, we have $f = g$.

   Transitive Let $f \preceq g \preceq h$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq h(x)$, giving $f(x) \leq h(x)$. So, $f \preceq h$.

b) It is not a total ordering. Let $f(x) = x$ and $g(x) = -x$. Then $f(1) = 1 \not\leq -1 = g(1)$ and $g(-1) = 1 \not\leq -1 = f(-1)$. So it is not the case that for all x, $f(x) \leq g(x)$, and it is not the case that for all x, $g(x) \leq f(x)$. That is, these two functions are incomparable.

$\square$



Q. 12. Answer these questions for the partial order represented by this Hasse diagram (see the next page).

a) Find the maximal elements.

b) Is there a greatest element?

c) Find all upper bounds of $\{a, b, c\}$.

d) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

Solution:

a) The maximal elements are the ones with no other elements above them, namely l and m.

b) There is no greatest element, since neither l nor m is greater than the other.

c) We need to find elements from which we can find downward paths to all of a, b, and c. It is clear that k, l and m are the elements fitting this description.

d) Since there is no lower bound, there cannot be greatest lower bound.

$\square$