Discrete Mathematics for Computer Science

Lecture 4: Set and Function

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Argument

Argument: A sequence of propositions that end with a conclusion.

Premises:

- "If you have a current password, then you can log onto the network."
- "You have a current password."

Conclusion: "You can log onto the network."

An argument form in propositional logic is a sequence of compound propositions involving propositional variables.

- p: "You have a current password"
- q: "You can log onto the network."

$$\begin{array}{rl} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$$

Validity

Validity of Argument Form: The argument form with premises $p_1, p_2, ..., p_n$ and conclusion q is valid, if

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q \text{ is a tautology.}$$

Note: According to the definition of $p \rightarrow q$, we do not worry about the case where $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ is false.

Thus, equivalently, an argument form is valid no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

Validity of Argument: The validity of an argument follows from the validity of the form of the argument.

Validity

Premises:

- "If you have a current password, then you can log onto the network."
- "You have a current password."

Conclusion: "You can log onto the network."

To prove the validity of an argument:

- Translate the argument to an argument form

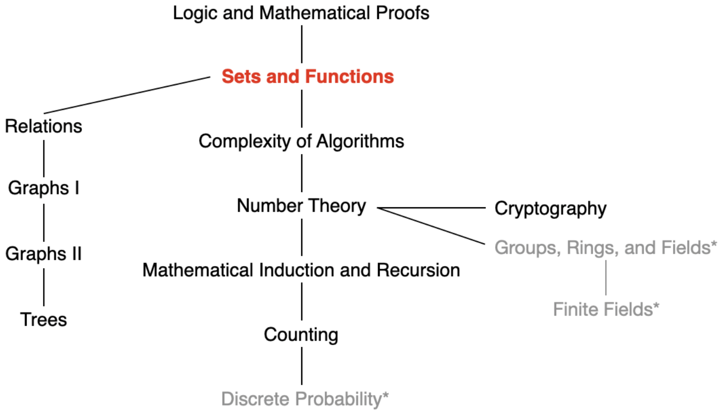
$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

- Prove this argument form is valid
  - Prove $(p \to q) \land p \to q$ is a tautology
  - Or use rules of inference

Summary of Logic and Proof

- **Logic**:
  - Proposition, applications, equivalence $\equiv$, ...
  - Predicates $P(x)$
  - Quantifiers $\forall x P(x)$, $\exists x P(x)$
- **Mathematical Proofs**:
  - Argument: premises, conclusion
  - Rules of inference
  - Proofs

This Lecture



Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets

Sets

A set is an unordered collection of objects. These objects are called elements or members.

- $A = \{1, 2, 3, 4\}$
- $B = \{a, b, c, d\}$
- $C = \{a, 2, 1, \text{Mary}\}$

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Many discrete structures are built with sets:

- combinations
- relations
- graphs

Set Representation

Examples:

- $A = \{2, 3, 5, 7\}$
- $B = \{1, 2, 3, ..., 100\}$
- $C = \{a \mid a \geq 2, a \text{ is a prime}\}$
- $D = \{2n \mid n = 0, 1, 2, ..., \}$

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Representing a set by:

- listing (enumerating) the elements
- if enumeration is hard, use ellipses (...)
- definition by property, using the set builder

$$\{x \mid x \text{ has property } P \text{ or property } P(x))\}$$

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Notation:

- $a \in A$: a is an element of set A
- $a \notin A$: a is not an element of set A

Important sets

- Natural numbers:
  - $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$
- Integers:
  - $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Positive integers:
  - $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$
- Rational numbers:
  - $\mathbf{Q} = \{\frac{p}{q} \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$
- Real numbers:
  - $\mathbf{R}$
- Complex numbers:
  - $\mathbf{C}$

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Important sets

- $[a, b] = \{x \mid a \leq x \leq b\}$
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- Two sets A, B are *equal* if and only if $\forall x\ (x \in A \leftrightarrow x \in B)$.

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Universal and Empty Set

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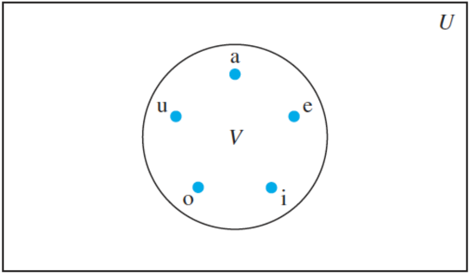
Venn Diagrams

A set can be visualized using Venn diagrams



Subset

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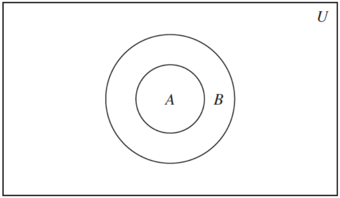

Proof of Subset

Proof:

- Showing $A \subseteq B$: if x belongs to A, then x also belongs to B.
- Showing $A \nsubseteq B$: find a single $x \in A$ such that $x \notin B$.

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Prove that $\emptyset \subseteq S$.

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Note: two sets are equal if and only if each is a subset of the other:

$$\forall x(x \in A \leftrightarrow x \in B)$$

The Size of a Set – Cardinality

Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say that S is a finite set and n is the cardinality of S, denoted by $|S|$.

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Examples:

- $A = \{1, 2, 3, ..., 20\}$, where $|A| = 20$
- $B = \{1, 2, 3, ...\}$, which is infinite
- $|\emptyset| = 0$
- $|\{\emptyset\}| = 1$

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If S is a set with $|S| = n$, then $|\mathcal{P}(S)| = 2^n$. Why?

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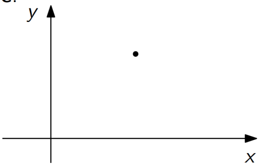

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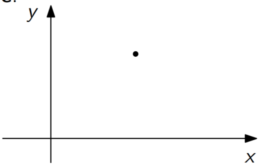

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Two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal. That is, $(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$ if and only if $a_i = b_i$ for $i = 1, 2, ..., n$.

Cartesian Product

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$:

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The Cartesian product of the sets $A_1, A_2, ..., A_n$, denoted by $A_1 \times A_2 \times ... \times A_n$, is the set of ordered n-tuples $(a_1, a_2, ..., a_n)$ where $a_i \in A_i$ for $i = 1, ..., n$:

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Example:

$A = \{0, 1\},\ B = \{1, 2\},\ C = \{0, 1, 2\}$

$A \times B \times C =$

$\{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2),$
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Let A be a set. A^n denotes $A \times A \times ... \times A$ with n sets:

$$A^n = \{(a_1, a_2, ..., a_n) \mid a_i \in A \text{ for } i = 1, 2, ..., n\}$$

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Example: What are the ordered pairs in the less than or equal to relation, which contains (a, b) if $a \leq b$, on the set $\{0, 1, 2, 3\}$?

The ordered pair (a, b) belongs to R if and only if both a and b belong to $\{0, 1, 2, 3\}$ and $a \leq b$. Consequently,

$$R = \{(0,0),(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$$

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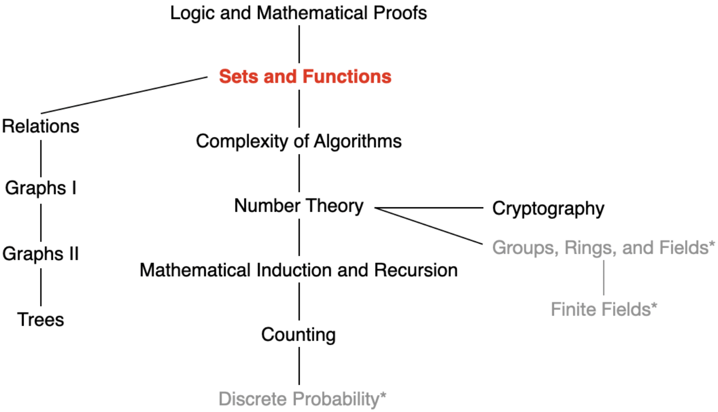
Summary of Set

- Set: unordered collection of objects
- Subset $A \subseteq B$
- Cardinality: size of set
- Power of set $\mathcal{P}(A)$
- Tuple: (a, b)
- Cartesian Product $A \times B$
- Relation: a subset of $A \times B$

This Lecture



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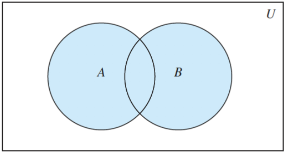
Set Operations

Union: Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set $\{x \mid x \in A \vee x \in B\}$.



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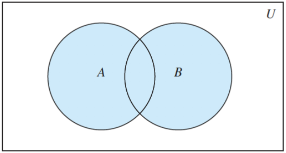

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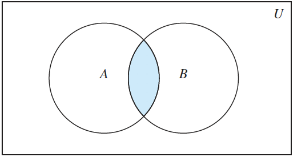
Intersection: The intersection of the sets A and B, denoted by $A \cap B$, is the set $\{x \mid x \in A \wedge x \in B\}$.



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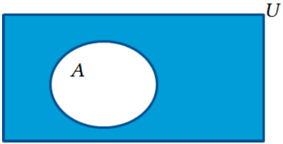
Set Operations

Complement: If A is a set, then the complement of the set A (with respect to U), denoted by $\bar{A}$ is the set $U - A$, $\bar{A} = \{x \in U \mid x \notin A\}$



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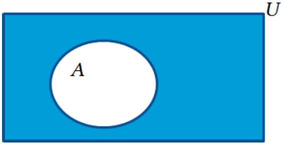
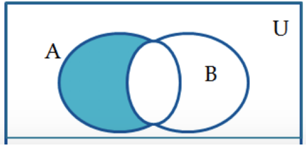


Difference: Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B.
$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \bar{B}$.



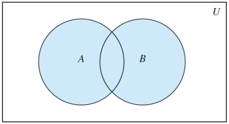
Disjoint Sets

Two sets A and B are called disjoint if their intersection is empty, i.e., $A \cap B = \emptyset$.

Example: $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$ are disjoint, because $A \cap B = \emptyset$.

Cardinality of the Union

What is the cardinality of $A \cup B$?



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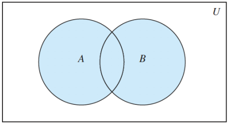

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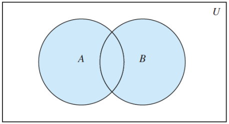

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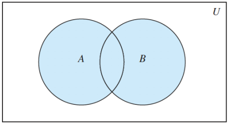
The generalization of this result to unions of an arbitrary number of sets is called the principle of inclusion–exclusion

Cardinality of the Union

What is the cardinality of $A \cup B$?



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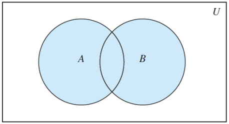

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THE PRINCIPLE OF INCLUSION–EXCLUSION Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Exercises

- $U = \{0, 1, 2, \ldots, 10\}, A = \{1, 2, 3, 4, 5\}, B = \{4, 5, 6, 7, 8\}$

1. $A \cup B$
2. $A \cap B$
3. $\bar{A}$
4. $\bar{B}$
5. $A - B$
6. $B - A$

Exercises

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1. $A \cup B$ {1,2,3,4,5,6,7,8}
2. $A \cap B$ {4,5}
3. $\bar{A}$ {0,6,7,8,9,10}
4. $\bar{B}$ {0,1,2,3,9,10}
5. $A - B$ {1,2,3}
6. $B - A$ {6,7,8}

Set Identities

The properties and laws of sets that help us demonstrate and prove set operations, subsets and equivalence.

- Identity laws
  - $A \cup \emptyset = A$
  - $A \cap U = A$
- Domination laws
  - $A \cup U = U$
  - $A \cap \emptyset = \emptyset$
- Idempotent laws
  - $A \cup A = A$
  - $A \cap A = A$
- Complementation laws
  - $\bar{\bar{A}} = A$

Set Identities

- Commutative laws
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- Associative laws
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive laws
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's laws
  - $\overline{A \cap B} = \bar{A} \cup \bar{B}$
  - $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Set Identities

- Absorbtion laws
  - $A \cup (A \cap B) = A$
  - $A \cap (A \cup B) = A$
- Complement laws
  - $A \cup \bar{A} = U$
  - $A \cap \bar{A} = \emptyset$

Proof of Set Identities

Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Proof of Set Identities

Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Proof 1: Using membership tables. Consider an arbitrary element x: 1, x is in A; 0, x is not in A.

A	B	$\overline{A}$	$\overline{B}$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	0	0
1	0	0	1	1	1
0	1	1	0	1	1
0	0	1	1	1	1

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Proof 2: by showing that $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

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    -
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$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Generalized Unions and Intersections

- The *union of a collection of sets* is the set that contains those elements that are members of at least one set in the collection $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$.

- The *intersection of a collection of sets* is the set that contains those elements that are members of all sets in the collection $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$.

Computer Representation of Sets

Question: How to represent sets in a computer?

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- A better solution: assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is in the set.
  - Universal set U is finite and with n elements
  - Represent a subset A of U with n bits, where the i-th bit is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Computer Representation of Sets

Example: $U = \{1, 2, 3, 4, 5\}$
$A = \{2, 5\}$. Thus, A is represented by 01001
$B = \{1; 5\}$. Thus, B is represented by 10001

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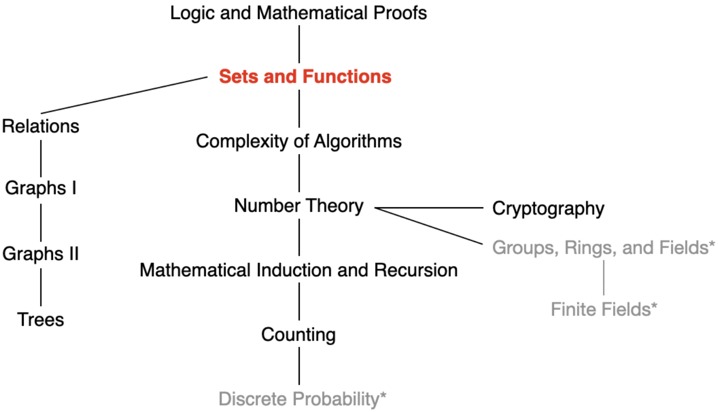
- Union: $A \vee B = 11001$, i.e., $\{1, 2, 5\}$
- Intersection: $A \wedge B = 00001$, i.e., $\{5\}$
- Complement: $\bar{A} = 10110$, i.e., $\{1, 3, 4\}$

Summary of Set Operations

- Union $A \cup B$, cardinality (principle of inclusion-exclusion)
- Intersection $A \cap B$
- Complement $\bar{A}$
- Difference $A - B$
- Disjoint set
- Set identities
- Proof of set identities
  - membership table, subset, set build and logical equivalences
- Computer representations

This Lecture



Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets

Function

Let A and B be two sets. A function from A to B, denoted by $f : A \to B$, is an assignment of exactly one element of B to each element of A.

- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

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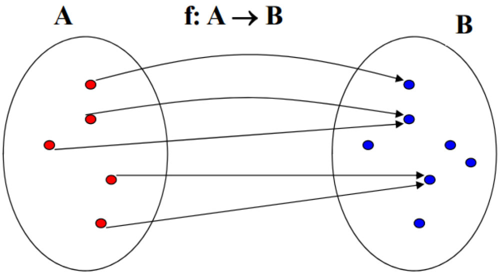

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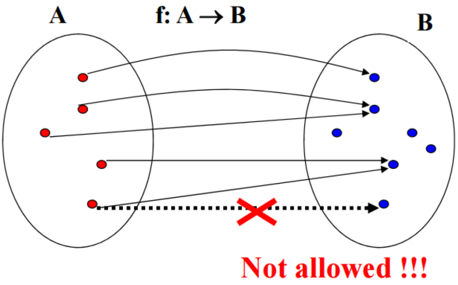
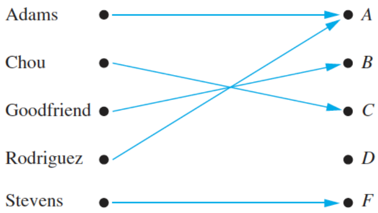


Representing Functions

1 Explicitly state the assignments between elements of the two sets



Note: Admas $\mapsto A$, Chou $\mapsto C$, ...

2 By a formula

3 By a relation from A to B

Representing Functions

1. Explicitly state the assignments between elements of the two sets
2. By a formula: $f(x) = x + 1$
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3. By a relation from A to B: (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22).

Important Sets of Functions

Let f be a function from A to B.

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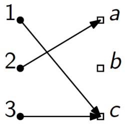
Example:

$A = \{1, 2, 3\}$, $B = \{a, b, c\}$



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Example:

$A = \{1, 2, 3\}$, $B = \{a, b, c\}$

– c is the image of 1
– 2 is a preimage of a
– the domain of f is $\{1, 2, 3\}$
– the codomain of f is $\{a, b, c\}$
– the range of f is $\{a, c\}$

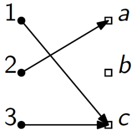

Image of a Subset

For a function $f : A \to B$ and $S \subseteq A$, the image of S is a subset of B that consists of the images of the elements of S, denoted by $f(S)$, where $f(S) = \{f(s) | s \in S\}$

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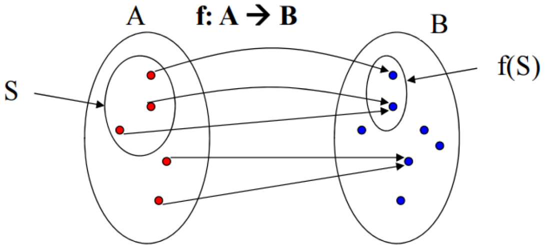

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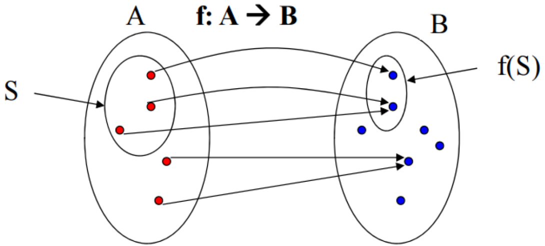
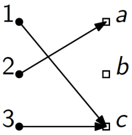




Let $S = \{1, 3\}$, what is $f(S)$?

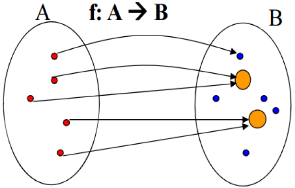
One-to-One and Onto Functions

- **One-to-one function**
  - never assign the same value to two different domain elements.
- **Onto function**
  - every member of the codomain is the image of some element of the domain.
- **One-to-one correspondence**
  - One-to-one and onto

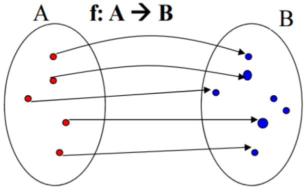
One-to-One (Injective) Function

A function f is called one-to-one or injective if and only if $f(x) = f(y)$ implies $x = y$ for all x, y in the domain of f. Also called an injunction.

Alternatively: A function is one-to-one if and only if $x \neq y$ implies $f(x) \neq f(y)$. (contrapositive!)



Not injective



Injective function

How about:

- $f(x) \neq f(y)$ implies $x \neq y$?
- $x = y$ implies $f(x) = f(y)$?

One-to-One (Injective) Function

Example 1:
Whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one? Yes.

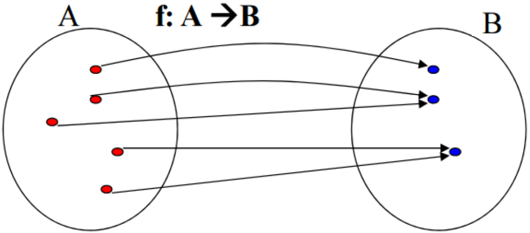
Example 2:
Whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one? No, $f(-1) = f(1)$

What if it is from the set of positive integers to the set of integers? Yes.

Onto (Surjective) Function

A function f is called onto or surjective if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$. Also called a surjection.

Alternatively: A function is onto if and only if all codomain elements are covered, i.e., $f(A) = B$.



Onto (Surjective) Function: Example

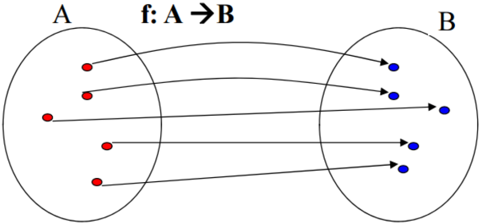
Example 1:
Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function? Yes.

What if the codomain were $\{1, 2, 3, 4\}$? No.

Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto? No, as there is no integer x with $x^2 = -1$.

One-to-One Correspondence (Bijective Function)

A function f is called one-to-one correspondence or bijective, if and only if it is both one-to-one and onto. Also called bijection.



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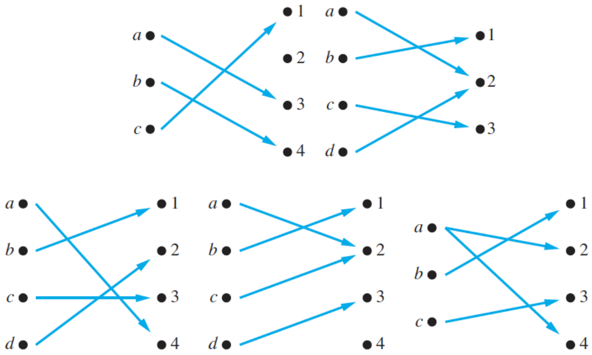
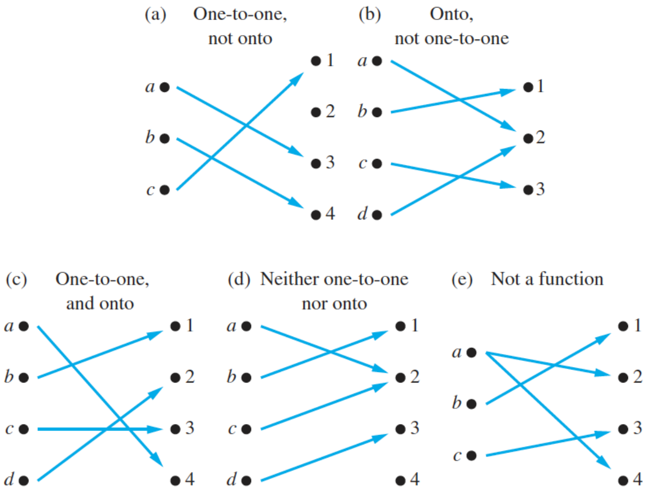
Example 2: Consider an identity function on A, i.e., $\iota : A \to A$, where $\iota_A(x) = x$. Is this function a one-to-one correspondence? Yes.

Are These Functions Injective, Surjective, Bijective?



Are These Functions Injective, Surjective, Bijective?



Proof for One-to-One and Onto

Example

$f : \boldsymbol{Z} \rightarrow \boldsymbol{Z}$, where $f(x) = x + 1$. Is f injective? Surjective? Bijective?

Proof:

- Injective (one-to-one function): If $f(x) = f(x')$ for any arbitrary x and x', then $x = x'$.
- Surjective (onto function): For every integer y, these exists an integer x such that $f(x) = y$.
- Bijective (one-to-one correspondence): injective and surjective

One-to-One and Onto

Prove that "for a function $f : A \to B$ with $|A| = |B| = n$, f is one-to-one if and only if f is onto."

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- If f is onto, then f is one-to-one (contradiction): Suppose that f is onto. Suppose that f is not one-to-one. Thus, $f(x_i) = f(x_j)$ for some $i \neq j$. Then, $|\{f(x_1), ..., f(x_n)\}| \leq n - 1$. Note that $|f(A)| = |B| = n$, which leads to a contradiction.

One-to-One and Onto

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Proof (Counterexample): Consider the following $f : \boldsymbol{Z} \to \boldsymbol{Z}$, where $f(x) = 2x$. f is one-to-one but not onto:

- $f(1) = 2$
- $f(2) = 4$
- $f(3) = 6$
- ...

We can prove that 3 has no preimage.

Two Functions on Real Numbers

Let f_1 and f_2 be functions from A to $\boldsymbol{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\boldsymbol{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
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Example:

$f_1 = x - 1$ and $f_2 = x^3 + 1$

Then

$$(f_1 + f_2)(x) = x^3 + x$$
$$(f_1 f_2)(x) = x^4 - x^3 + x - 1$$

Inverse Functions

Let f be a one-to-one correspondence (bijection) from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.

The inverse function of f is denoted by f^{-1}.
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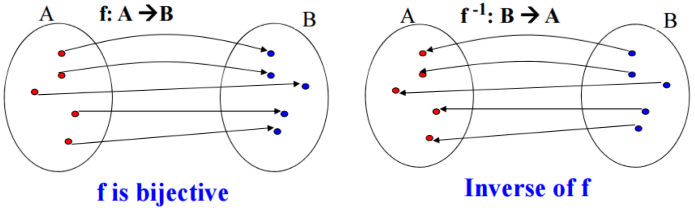

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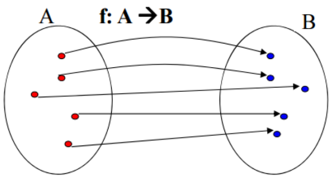
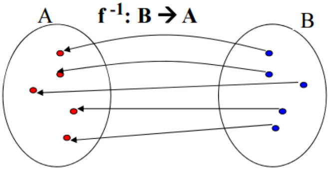


f is bijective



Inverse of f

A bijection is called invertible.

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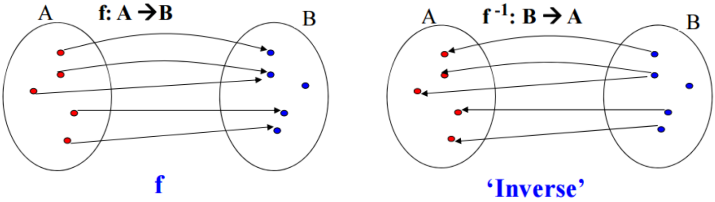

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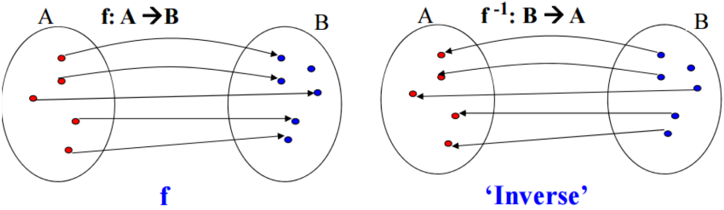

The inverse is not a function: one element of B is not assigned an element of A.

Proof for Inverse Function

1. Prove function f is a bijection: injective, surjective

To show that f is *injective*	Show that if $f(x) = f(y)$ for all $x, y \in A$, then $x = y$
To show that f is not *injective*	Find specific elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$
To show that f is *surjective*	Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$
To show that f is not *surjective*	Find a specific element $y \in B$ such that $f(x) \neq y$ for all $x \in A$

2. If f is a bijection, then it is invertible
3. Determine the inverse function

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$f : \boldsymbol{Z} \rightarrow \boldsymbol{Z}$, where $f(x) = x + 1$. Is f invertible? If yes, then what is the inverse function f^{-1}?

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To reverse the function, suppose that y is the image of x, so that $y = x + 1$. Then, $x = y - 1$. This means that $y - 1$ is the unique element of $\boldsymbol{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.

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- Surjective: Consider an arbitrary nonnegative real number y. There exists a nonnegative real number $x = \sqrt{y}$ such that $f(x) = y$.

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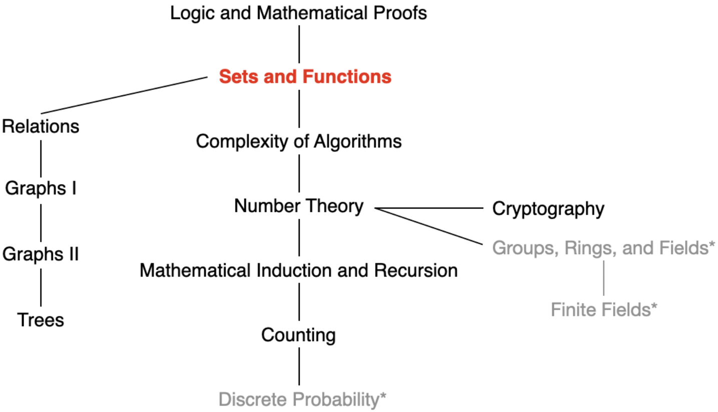
Summary of Function

- Function $f : A \to B$: an assignment of exactly one element of B to each element of A
- Domain, codedomain, image, preimage, range
- One-to-one function
  - also called an injunction or injective function
- Onto function
  - also called a surjection or surjective function
- One-to-one correspondence
  - one-to-one and onto
  - also called a bijection or bijective function
- Inverse function
  - One-to-one correspondence

Next Lecture



Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets