Discrete Mathematics for Computer Science

Lecture 19: Graph

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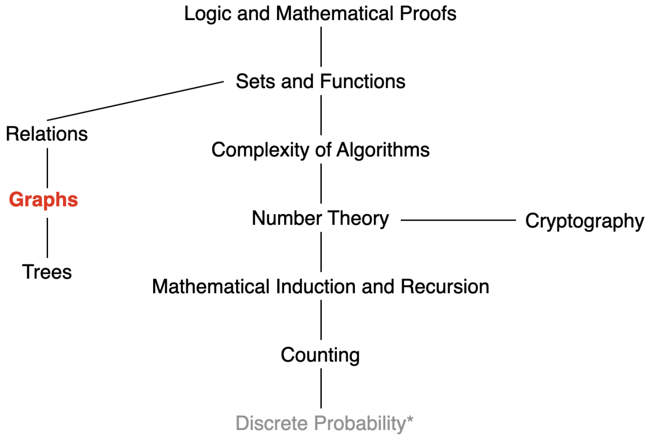


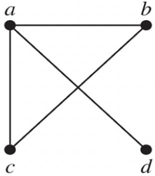
Graph and terminologies, representing graphs and graph isomorphism, connectivity, Euler and Hamiliton path, ...

Adjacency Matrices

Definition: Suppose that $G = (V, E)$ is a simple graph with $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$. The adjacency matrix $\mathbf{A}_G$ of G, is the $n \times n$ zero-one matrix with 1 as its (i, j)-th entry when v_i and v_j are adjacent, and 0 as its (i, j)-th entry when they are not adjacent.

$$\mathbf{A}_G = [a_{ij}]_{n\times n}, \text{ where}$$

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Directed graph?

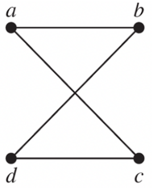
Counting Paths between Vertices

Theorem: Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of vertices. The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)-th entry of $\mathbf{A}^r$.

Note: with directed or undirected edges, multiple edges and loops allowed

Counting Paths between Vertices:

How many paths of length 4 are there from a to d in the graph G?



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d;
a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; a, c, d, c, d

Counting Paths between Vertices

Theorem: The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)-th entry of $\mathbf{A}^r$.

Proof (by induction):

- Basic Step: The number of paths from v_i to v_j of length 1 is the (i, j)-th entry of $\mathbf{A}$.
- Inductive hypothesis: Assume that the (i, j)-th entry of $\mathbf{A}^r$ is the number of different paths of length r from v_i to v_j.
- Inductive Step: $\mathbf{A}^{r+1} = \mathbf{A}^r\mathbf{A}$. The (i, j)-th entry of $\mathbf{A}^{r+1}$ equals

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{ik}a_{kj} + \cdots b_{in}a_{nj}.$$

  - b_{ik}: the (i, k)-th entry of $\mathbf{A}^r$. By the inductive hypothesis, b_{ik} is the number of paths of length r from v_i to v_k;
  - a_{kj}: the (k, j)-th entry of $\mathbf{A}$; the number of path from k to j with length 1;
  - $b_{ik}a_{kj}$: the number of paths from i to j with k as the interior point of length $r + 1$.

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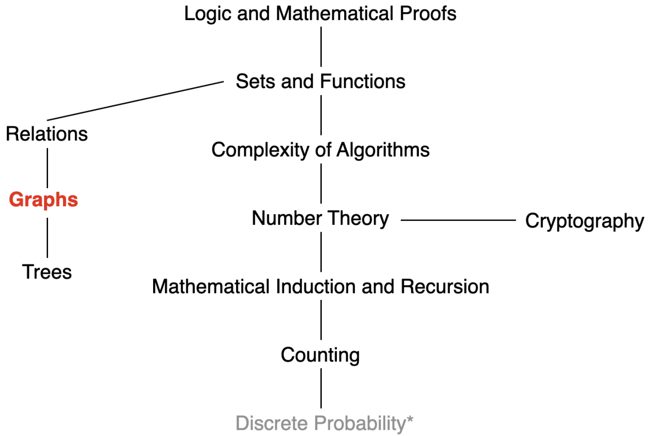
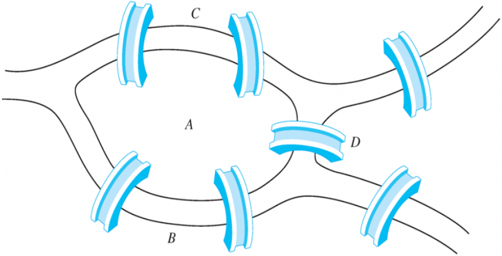
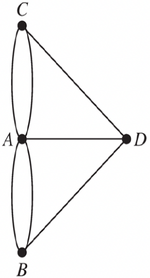


Graph and terminologies, representing graphs and graph isomorphism, connectivity, Euler and Hamiliton path, ...

Euler Paths

Königsberg seven-bridge problem: People wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.



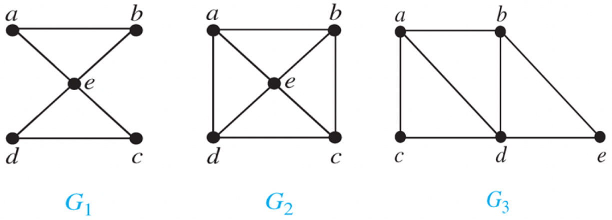


Euler Paths and Circuits

Definition: An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

Recall that a path or circuit is simple if it does not contain the same edge more than once.

Example: Which of the undirected graphs have an Euler circuit? Of those that do not, which have an Euler path?

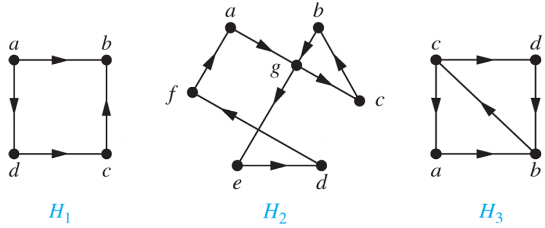


G_1: an Euler circuit, e.g., a, e, c, d, e, b, a;
G_2: neither; G_3: an Euler path, e.g., a, c, d, e, b, d, a, b

Euler Paths and Circuits

Definition: An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

Example: Which of the directed graphs have an Euler circuit? Of those that do not, which have an Euler path?



H_1: neither; H_2: an Euler circuit, e.g., a, g, c, b, g, e, d, f, a; H_3: an Euler path, e.g., c, a, b, c, d, b

Necessary Conditions for Euler Circuits and Paths

Consider undirected graph:

Euler Circuit $\Rightarrow$ The degree of every vertex must be even

- Each time the circuit passes through a vertex, it contributes two to the vertex's degree.
- The circuit starts with a vertex a and ends at a, then contributes two to $deg(a)$.

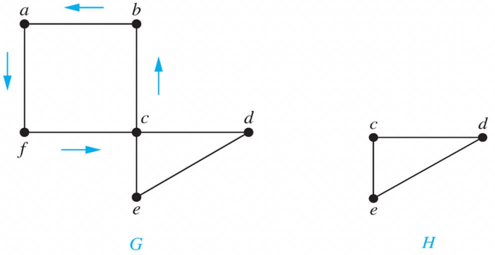
Euler Path $\Rightarrow$ The graph has exactly two vertices of odd degree

- The initial vertex and the final vertex of an Euler path have odd degree.

Are these conditions also sufficient?

Sufficient Conditions for Euler Circuits and Paths

G is a connected multigraph with ≥ 2 vertices, all of even degree.



We will form a simple circuit that begins at an arbitrary vertex a of G, building it edge by edge.

The path begins at a, and it must terminate at a. This is because every time we enter a vertex other than a, we can leave it.

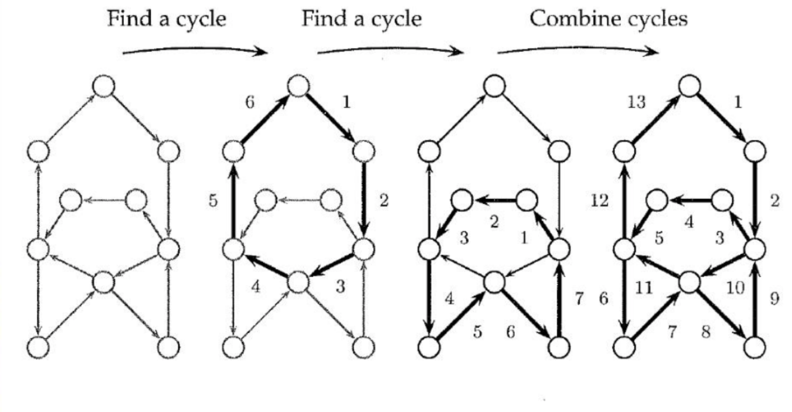
An Euler circuit has been constructed if all the edges have been used. Otherwise, consider the subgraph H obtained from G by deleting the edges already used. Every vertex in H has even degree ...

Algorithm for Constructing an Euler Circuit

ALGORITHM 1 Constructing Euler Circuits.

procedure *Euler*(G: connected multigraph with all vertices of even degree)
circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex
H := G with the edges of this circuit removed
while H has edges
    subcircuit := a circuit in H beginning at a vertex in H that also is an endpoint of an edge of *circuit*
    H := H with edges of *subcircuit* and all isolated vertices removed
    circuit := *circuit* with *subcircuit* inserted at the appropriate vertex
return *circuit* {*circuit* is an Euler circuit}

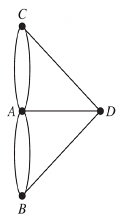
Algorithm for Constructing an Euler Circuit



Euler Circuits and Paths

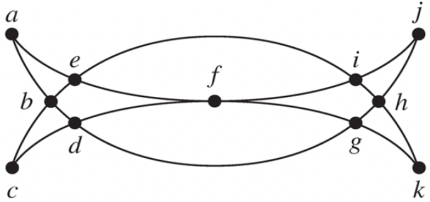
Theorem: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem: A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.



No Euler circuit, no Euler path

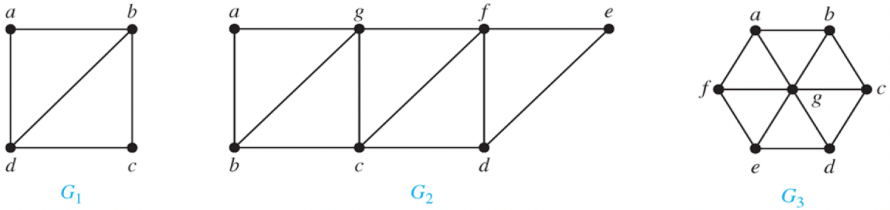
Euler Circuits and Paths: Example



It has such a circuit because all its vertices have even degree.
We will use the algorithm to construct an Euler circuit:

- Form the circuit a, b, d, c, b, e, i, f, e, a;
- Obtain the subgraph H by deleting the edges in this circuit and all vertices that become isolated;
- Form the circuit d, g, h, j, i, h, k, g, f, d in H;
- Splice this new circuit into the first circuit at the appropriate place produces the Euler circuit
  a, b, d, g, h, j, i, h, k, g, f, d, c, b, e, i, f, e, a.

Euler Circuits and Paths: Example



- G_1 contains exactly two vertices of odd degree, namely, b and d. Hence, it has an Euler path that must have b and d as its endpoints.
- G_2 has exactly two vertices of odd degree, namely, b and d. So it has an Euler path that must have b and d as endpoints.
- G_3 has no Euler path because it has six vertices of odd degree.

Applications of Euler Paths and Circuits

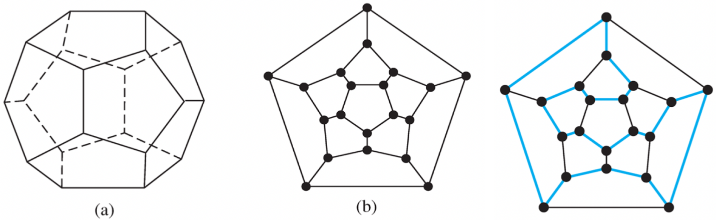
Finding a path or circuit that traverses each

- street in a neighborhood
- road in a transportation network
- link in a communication network
- ...

Hamilton Paths and Circuits

Euler paths and circuits contained every edge only once.

What about containing every vertex exactly once?

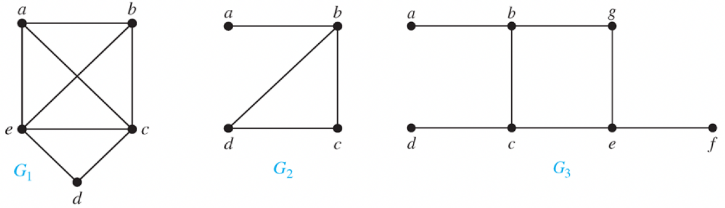


(a) (b)

Hamilton Paths and Circuits

Definition: A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path, and a simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit.

Example: Which of these simple graphs has a Hamilton circuit or, if not, a Hamilton path?



- G_1 has a Hamilton circuit: a, b, c, d, e, a;
- G_2 has no Hamilton circuit (because containing every vertex must contain the edge a, b twice), but it has a Hamilton path;
- G_3 has neither, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once.

Sufficient Conditions for Hamilton Circuits

No known simple necessary and sufficient conditions are known for the existence of a Hamilton circuit.

But, there are some useful sufficient conditions.

Dirac's Theorem: If G is a simple graph with $n \geq 3$ vertices such that the degree of every vertex in G is $\geq n/2$, then G has a Hamilton circuit.

Ore's Theorem: If G is a simple graph with $n \geq 3$ vertices such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices, then G has a Hamilton circuit.

Example: Show that K_n has a Hamilton circuit whenever $n \geq 3$.

Hamilton path problem $\in$ NP-Complete

Applications of Hamilton Paths and Circuits

A path or a circuit that visits each city, or each node in a communication network exactly once, can be solved by finding a Hamilton path.

Traveling Salesperson Problem (TSP) asks for the shortest route a traveling salesperson should take to visit a set of cities.

the decision version of the TSP $\in$ NP-Complete

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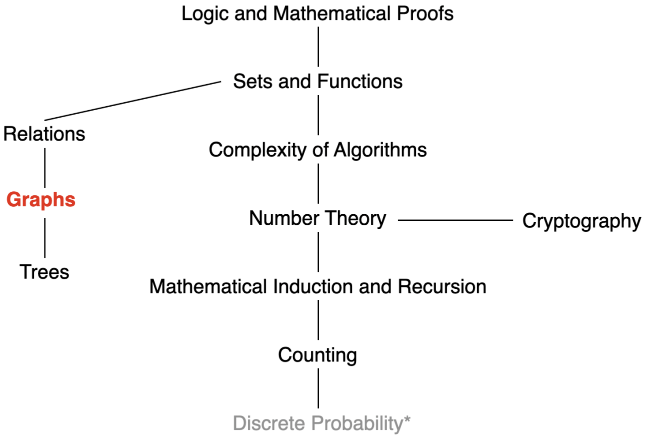
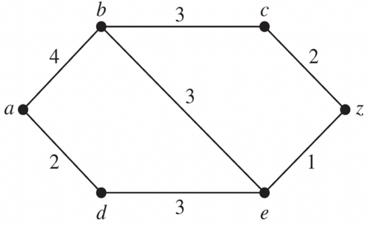


Graph and terminologies, representing graphs and graph isomorphism, connectivity, Euler and Hamiliton path, shortest-path problem ...

Shortest Path Problems

Using graphs with weights assigned to their edges

Such graphs are called weighted graphs and can model lots of questions involving distance, time consuming, fares, etc.



What is the length of a shortest path between a and z?

Dijkstra's Algorithm

S: a distinguished set of vertices;
$L(v)$: the length of a shortest path from a to v that contains only the vertices in S as the interior vertices.

(i) Set $L(a) = 0$ and $L(v) = \infty$ for all v, $S = \emptyset$

(ii) While $z \notin S$

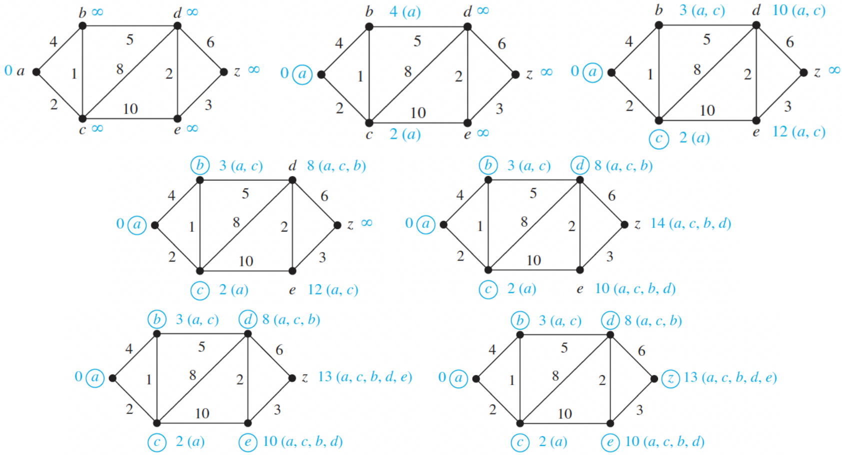
$u :=$ a vertex not in S with $L(u)$ minimal

$S := S \cup \{u\}$

For all vertices v not in S

$$L(v) := \min\{L(u) + w(u, v), L(v)\}$$

Dijkstra's Algorithm



$S = \emptyset$

$L(a) = 0,\ L(b) = \infty,\ L(c) = \infty,\ L(d) = \infty,\ L(e) = \infty,\ L(z) = \infty$

$S = \{a\}$

$L(a) = 0,\ L(b) = 4,\ L(c) = 2,\ L(d) = \infty,\ L(e) = \infty,\ L(z) = \infty$

Dijkstra's Algorithm

Dijkstra's algorithm is a hueristic algorithm, but ...

Theorem: Dijkstra's algorithm finds the length of a shortest path between two vertices in a connnected simple undirected weighted graph.

Proof by induction ... (P713 on textbook)

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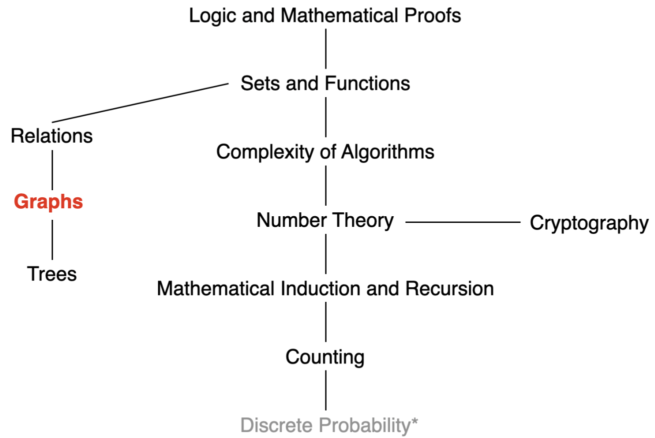
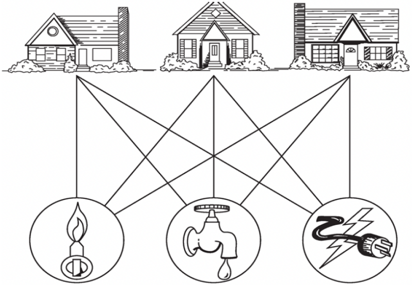


..., Euler and Hamiliton path, shortest-path problem, Planar Graphs, ...

Planar Graphs

Join three houses to each of three separate utilities.

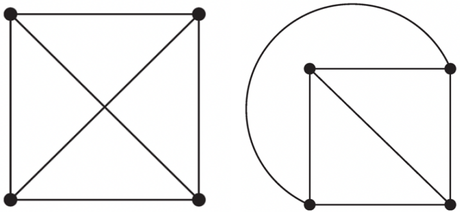


Can this graph be drawn in the plane such that no two of its edges cross?
Complete bipartite graph $K_{3,3}$

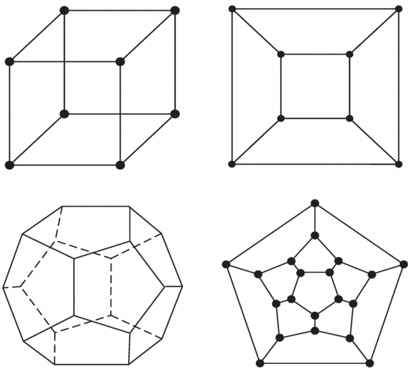
Planar Graphs

Definition: A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.

Example: Is K_4 planar?



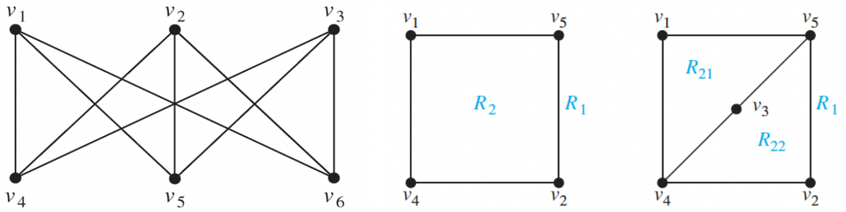
Planar Graphs: Example



- We can show that a graph is planar by displaying a planar representation.
- It is harder to show that a graph is nonplanar.

Planar Graphs: Example

Is $K_{3,3}$ planar?

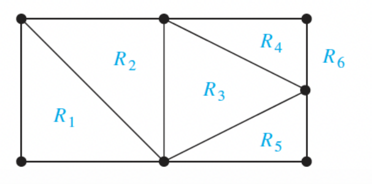


Any attempt to draw $K_{3,3}$ in the plane with no edges crossing is doomed.

- In any planar representation of $K_{3,3}$, the vertices v_1 and v_2 must be connected to both v_4 and v_5.
- These four edges form a closed curve that splits the plane into two regions, R_1 and R_2.
- The vertex v_3 is in either R_1 or R_2. Suppose v_3 is in R_2, there is no way to place the final vertex v6 without forcing a crossing. ....

Euler's Formula

A planar representation of a graph splits the plane into regions, including an unbounded region.



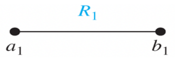
Theorem (Euler's Formula): Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then, $r = e - v + 2$.

Euler's Formula

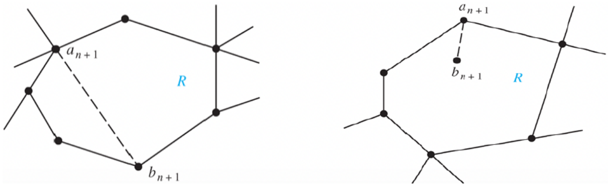
Theorem (Euler's Formula): Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then, $r = e - v + 2$.

Proof (by induction): We will prove the theorem by successively adding an edge at each stage.

- Basic Step: $r_1 = e_1 - v_1 + 2$

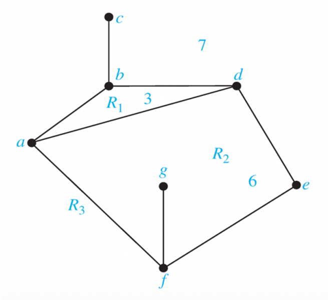


- Inductive Hypothesis: $r_k = e_k - v_k + 2$
- Inductive step: Let $\{a_{k+1}, b_{k+1}\}$ be the edge that is added to G_k to obtain G_{k+1}.



The Degree of Regions

Definition: The degree of a region is defined to be the number of edges on the boundary of this region. When an edge occurs twice on the boundary, it contributes two to the degree.



Corollaries

Corollary 1: If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Proof: The degree of every region is at least 3.

- G is simple
- $v \geq 3$

The sum of the degrees of the regions is exactly twice the number of edges in the graph:

$$2e = \sum_{\text{all regions } R} deg(R) \geq 3r$$

Hence, $(2/3)e \geq r$. By Euler's formula (i.e., $r = e - v + 2$), $e \leq 3v - 6$.

Corollaries

Corollary 2: If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

Proof (by Contradiction):
If G has one or two vertices, the result is true.

If G has at least three vertices, by Corollary 1, $e \leq 3v - 6$, so $2e \leq 6v - 12$.

- If the degree of every vertex were at least six, then we would have $2e = \sum_{v \in V} deg(v) \geq 6v$ (by handshaking theorem).
- This contradicts the inequality $2e \leq 6v - 12$.

It follows that there must be a vertex with degree no greater than five.

Corollary 3: In a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Examples

Show that K_5 is nonplanar.

$v = 5$ and $e = 10$.

Using Corollary 1: If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Show that $K_{3,3}$ is nonplanar.

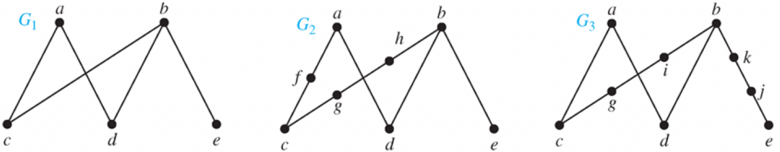
$v = 6$ and $e = 9$.

Using Corollary 3: In a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Kuratowski's Theorem

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an elementary subdivision.

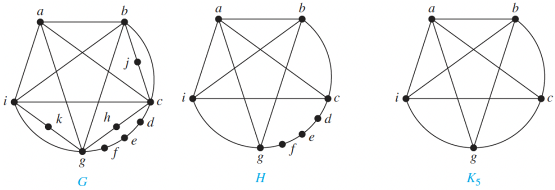
The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.



Kuratowski's Theorem

Theorem: A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Example:

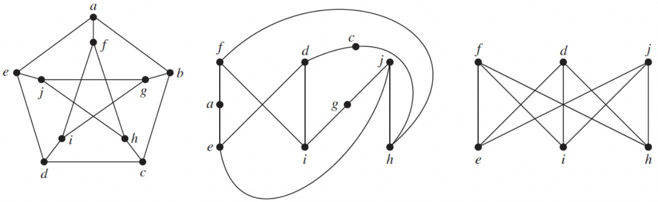


G has a subgraph H homeomorphic to K_5.

- H is obtained by deleting h, j, and k and all edges incident with these vertices.
- H is homeomorphic to K_5 because it can be obtained from K_5 by a sequence of elementary subdivisions.

Hence, G is nonplanar.

Kuratowski's Theorem: Example



(a) (b) H (c) $K_{3,3}$

G has a subgraph H homeomorphic to $K_{3,3}$.

- The subgraph H of the Petersen graph obtained by deleting b and the three edges that have b as an endpoint,
- H is homeomorphic to $K_{3,3}$, with vertex sets $\{f, d, j\}$ and $\{e, i, h\}$, because it can be obtained by a sequence of elementary subdivisions.

Hence, G is nonplanar.