Discrete Mathematics for Computer Science

Lecture 17: Relation

Dr. Ming Tang

Department of Computer Science and Engineering
Southern University of Science and Technology (SUSTech)
Email: tangm3@sustech.edu.cn

Cartesian Product

Let $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$, the Cartesian product $A \times B$ is the set of pairs

$$\{(a_1, b_1), (a_2, b_2), ..., (a_1, b_n), ..., (a_m, b_n)\}.$$

Cartesian product defines a set of all ordered arrangements of elements in the two sets.

A subset R of the Cartesian product $A \times B$ is called a relation from the set A to the set B.

Definition: Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product $A \times B$.

Summary on Properties of Relations

- Reflexive Relation: A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.
- Irreflexive Relation: A relation R on a set A is called irreflexive if $(a, a) \notin R$ for every element $a \in A$.
- Symmetric Relation: A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- Antisymmetric Relation: A relation R on a set A is called antisymmetric if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.
- Transitive Relation: A relation R on a set A is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

SUSTech Southern University of Science and Technology

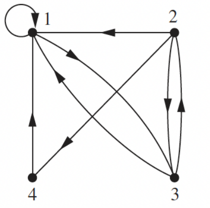
Directed Graph

A directed graph, or digraph, consists of a set V of vertices together with a set E of ordered pairs of elements of V called edges.

The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

Example: Relation R is defined on $\{1, 2, 3, 4\}$:

$$R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}$$



Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called the closure of R with respect to P if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

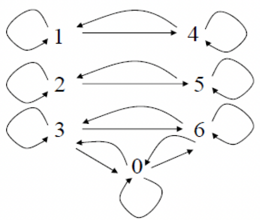
- Transitive Closure
- Path Length and Connectivity Relation
- Transitive Closure and Connectivity Relation

Equivalence Relation

Definition: A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Relation R on $A = \{0, 1, 2, 3, 4, 5, 6\}$ has the pairs:

- (0, 0), (0, 3), (3, 0), (0, 6), (6, 0), (3, 3), (3, 6), (6, 3), (6, 6)
- (1, 1), (1, 4), (4, 1), (4, 4)
- (2, 2), (2, 5), (5, 2), (5, 5)



Equivalence Class and Partition

Equivalence Class

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a, denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

$$[a]_R = \{b : (a, b) \in R\}$$

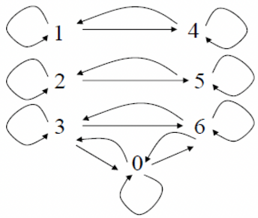
Example: $A = \{0, 1, 2, 3, 4, 5, 6\}$
$R = \{(a, b) : a \equiv b \mod 3\}$

$[0] = [3] = [6] = \{0, 3, 6\}$
$[1] = [4] = \{1, 4\}$
$[2] = [5] = \{2, 5\}$

Equivalence Class

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

(i)aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$



Equivalence Class

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

(i)aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Proof:

- (i) $\rightarrow$ (ii): prove $[a] \subseteq [b]$ and $[b] \subseteq [a]$
  Suppose $c \in [a]$. Then, aRc.
  Because aRb and R is symmetric, we know that bRa.
  Since R is transitive and bRa and aRc, it follows that bRc.
  Hence, $c \in [b]$. This shows that $[a] \subseteq [b]$.

Equivalence Class

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

(i)aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Proof:

- (i) $\rightarrow$ (ii): prove $[a] \subseteq [b]$ and $[b] \subseteq [a]$
- (ii) $\rightarrow$ (iii): Assume that $[a] = [b]$. It follows that $[a] \cap [b] \neq \emptyset$ because $[a]$ is nonempty (because $a \in [a]$ as R is reflexive).
- (iii) $\rightarrow$ (i): Suppose that $[a] \cap [b] \neq \emptyset$. There exists a c such that $c \in [a]$ and $c \in [b]$, i.e., aRc and bRc.
  By the symmetric property, cRb.
  Then by transitivity, because aRc and cRb, we have aRb.

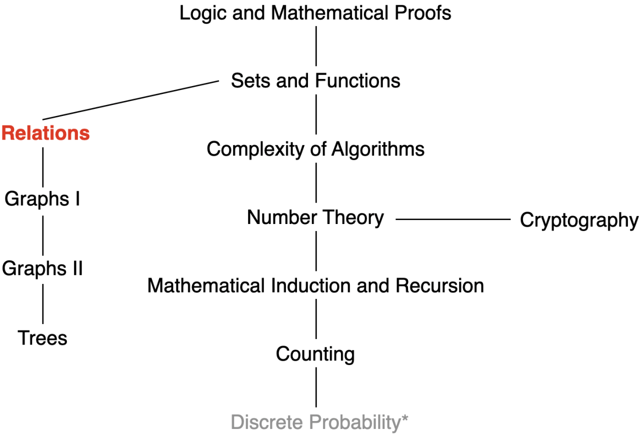
SUSTech Southern University of Science and Technology

Equivalence Classes and Partitions

Theorem: The equivalence classes form a partition of A.

Theorem: Let $\{S_1, S_2, ..., S_i, ...\}$ be a partition of A. Then, there is an equivalence relation R on A, that has the sets S_i as its equivalence classes.

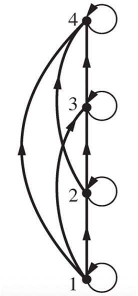
This Lecture

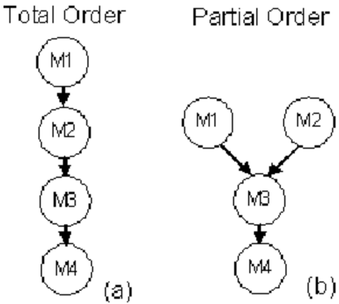


Relation, n-ary Relations, Representing Relations, Closures of Relations, Relation Equivalence, Partial Ordering, ...

Partial Ordering

Definition: A relation R on a set S is called a partial ordering, or partial order, if it is reflexive, antisymmetric, and transitive.





Both (a) and (b) are partial ordering. (a) is total ordering.

Partial Ordering

Definition: A relation R on a set S is called a partial ordering, or partial order, if it is reflexive, antisymmetric, and transitive. A set S together with

a partial ordering R is called a partially ordered set, or poset, denoted by (S, R).

Example: $S = \{1, 2, 3, 4, 5\}$, R denotes the "$\geq$" relation:

- Is R reflexive? Yes
- Is R antisymmetric? Yes
- Is R transitive? Yes

R is a partial ordering; Poset: (S, R) or $(S, \geq)$

Partial Ordering: Example

$S = \{1, 2, 3, 4, 5, 6\}$, R denotes the "$|$" relation

- Is R reflexive? Yes
- Is R antisymmetric? Yes
- Is R transitive? Yes

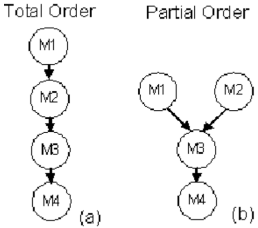
R is a partial ordering; Poset: (S, R) or $(S, |)$

Comparability

The notation $a \preccurlyeq b$ is used to denote that $(a, b) \in R$ in an arbitrary poset (S, R).

The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$.

Definition: The elements a and b of a poset $(S, \preccurlyeq)$ are comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise, a and b are called incomparable.



Example: $S = \{1, 2, 3, 4, 5, 6\}$, R denotes the "$|$" relation.
$2, 4$ are comparable, $3, 5$ are incomparable.

Total Ordering

Definition: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preccurlyeq$ is called a total order or a linear order. A totally ordered set is also called a chain.

Example: $S = \{1, 2, 3, 4, 5, 6\}$, R denotes the "$\geq$" relation S is a chain.

Well-Ordered Set

$(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Example: The set of ordered pairs of positive integers, $\mathbf{Z}^+ \times \mathbf{Z}^+$, with $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 \leq b_2$ (the lexicographic ordering), is a well-ordered set.

The set Z, with the usual $\leq$ ordering, is not well-ordered because the set of negative integers, which is a subset of $\mathbf{Z}$, has no least element.

The Principle of Well-Ordered Induction

The Principle of Well-Ordered Induction: Suppose that $(S, \preccurlyeq)$ is a well-ordered set. Suppose x_0 is the least element of a well ordered set. Then $P(x)$ is true for all $x \in S$, if

Basic Step: $P(x_0)$ is true.

Inductive Step: For every $y \in S \setminus \{x_0\}$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Or equivalently,
Inductive Step: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

The Principle of Well-Ordered Induction

The Principle of Well-Ordered Induction: Suppose that $(S, \preccurlyeq)$ is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

Inductive Step: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Proof: Suppose it is not the case that $P(x)$ is true for all $x \in S$. Then there is an element $y \in S$ such that $P(y)$ is false.

Consequently, the set $A = \{x \in S | P(x) \text{is false}\}$ is nonempty. Because S is well ordered, A has a least element a.

By the choice of a as a least element of A, we know that $P(x)$ is true for all $x \in S$ with $x \prec a$. By the inductive step, $P(a)$ is true.

This contradiction shows that $P(x)$ must be true for all $x \in S$.

Questions from Section 5 (Induction)

The Well-Ordering Property: Every nonempty set of nonnegative integers has a least element.

The principle of mathematical induction follows from the well-ordering property.

Question from students: Consider the set of negative integers. Although it does not has a least element, it has a greatest element. Can we solve it using mathematical induction?

Yes. We can solve it using the principle of well-ordered induction if we can find a relation $\preccurlyeq$ such that $(S, \preccurlyeq)$ is a well-ordered set.

Questions from Section 5 (Induction)

(i) The principle of mathematical induction, (ii) strong induction, and (iii) well-ordering property are all equivalent principles.

That is, the validity of each can be proved from either of the other two. (See Section 5.2 Exercise 41, 42, 43)

- (i) $\rightarrow$ (ii): The inductive hypothesis of a proof by mathematical induction is part of the inductive hypothesis in a proof by strong induction.
- (ii) $\rightarrow$ (iii) Use strong induction to show that the set of nonnegative integers has a least element.
- (iii) $\rightarrow$ (i) The principle of mathematical induction follows from the well-ordering property.

Questions from Section 5 (Induction)

(i) The principle of mathematical induction, (ii) strong induction, and (iii) well-ordering property are all equivalent principles.

(ii) $\rightarrow$ (iii) Use strong induction to show that the set of nonnegative integers has a least element.

- Suppose the well-ordering property were false; Let S be a nonempty set of nonnegative integers that has no least element
- Let $P(n)$ be the statement "$i \notin S$ for $i = 0, 1, ..., n$".
- Basic Step: $P(0)$ is true, because if $0 \in S$, then S has a least element
- Inductive Step: Suppose $P(n)$ is true. Then, $0 \notin S$, ..., $n \notin S$. Clearly, $n + 1$ cannot be in S, for if it were, it would be the least element. Thus, $P(n + 1)$ is true.
- Thus, by induction, $n \notin S$ for all nonnegative integers n. Thus, $S = \emptyset$.

SUSTech Southern University of Science and Technology

Lexicographic Ordering

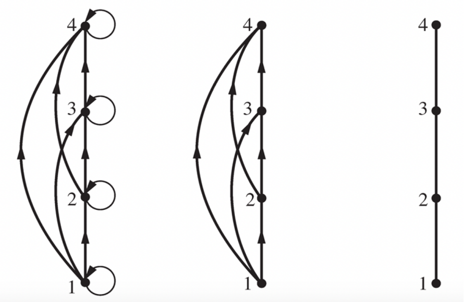
Definition: Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the lexicographic ordering on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), i.e., $(a_1, a_2) \preccurlyeq (b_1, b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ then $a_2 \preccurlyeq_2 b_2$.

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet.This is the same ordering as that used in dictionaries.

- discreet $\prec$ discrete
- discreet $\prec$ discreetness

Hasse Diagram

A Hasse diagram is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

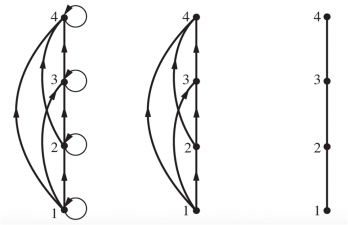


- A partial ordering. The loops are due to the reflexive property.
- The edges that must be present due to the transitive property are deleted.
- The Hasse diagram for the partial ordering (a).

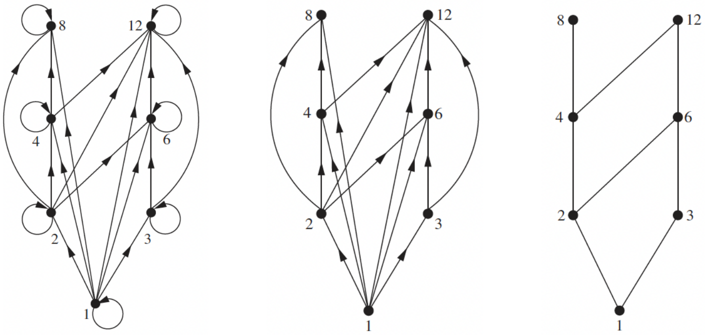
Procedure for Constructing Hasse Diagram

Start with the directed graph of the relation:

- Remove the loops (a, a) present at every vertex due to the reflexive property.
- Remove all edges (x, y) for which there is an element $z \in S$ s.t. $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
- Arrange each edge so that its initial vertex is <u>below</u> the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.



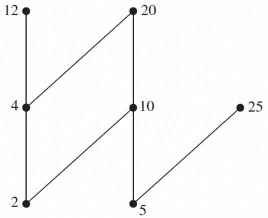
Hasse Diagram Example



Maximal and Minimal Elements

Definition: a is a maximal (resp. minimal) element in poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$ (resp. $b \prec a$).

Example: Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and which are minimal?



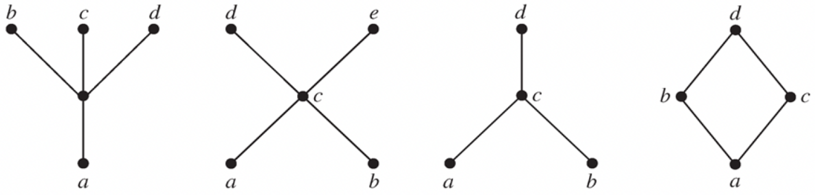
The maximal elements are 12, 20, and 25.

The minimal elements are 2 and 5.

A poset can have more than one maximal element and more than one minimal element.

Greatest and Least Elements

Definition: a is the greatest (resp. least) element of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ (resp. $a \preccurlyeq b$) for all $b \in S$.



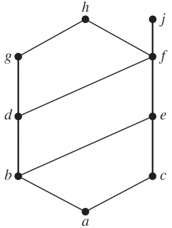
- (a): a least element a, no greatest element
- (b): neither a least nor a greatest element
- (c): no least element., a greatest element d
- (d): a least element a, a greatest element d

Upper and Lower Bound

Definition: Let A be a subset of a poset $(S, \preccurlyeq)$.

- $u \in S$ is called an upper bound (resp. lower bound) of A if $a \preccurlyeq u$ (resp. $u \preccurlyeq a$) for all $a \in A$.
- $x \in S$ is called the least upper bound (resp. greatest lower bound) of A if x is an upper bound (resp. lower bound) that is less than any other upper bounds (resp. lower bounds) of A.

Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist.



g is the least upper bound, b is the greatest lower bound.

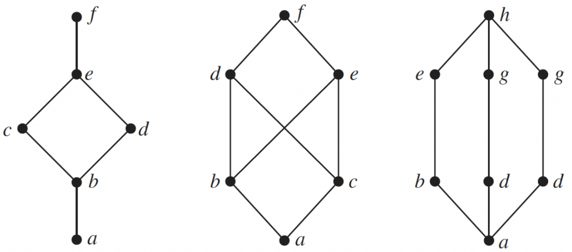
Upper and Lower Bound

Example: Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

- Lower bound of $\{3, 9, 12\}$: 1 and 3; the greatest lower bound: 3.
- Lower bound of $\{1, 2, 4, 5, 10\}$: 1; the greatest lower bound: 1.
- Upper bound of $\{3, 9, 12\}$: multiple of 36; the least upper bound: 36.
- Upper bound of $\{1, 2, 4, 5, 10\}$: multiple of 20; the least upper bound: 20.

Lattices

Definition: A partial ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice.



- (a) and (c): lattices
- (b): not a lattice, because the elements b and c have no least upper bound.

Lattices: Example

Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Solution: Because 2 and 3 have no upper bounds, they certainly do not have a least upper bound. Hence, the first poset is not a lattice.

Every two elements of the second poset have both a least upper bound and a greatest lower bound.

- The least upper bound of two elements in this poset is the larger of the elements
- The greatest lower bound of two elements is the smaller of the elements

Hence, this second poset is a lattice.

Topological Sorting

Motivation: A project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks?

Topological sorting: Given a partial ordering R, find a total ordering $\preccurlyeq$ such that $a \preccurlyeq b$ whenever aRb. $\preccurlyeq$ is said compatible with R.

Topological Sorting for Finite Posets

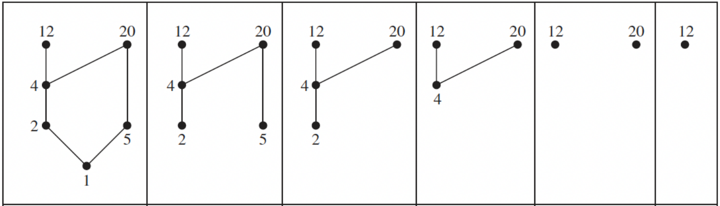
Lemma: Every finite nonempty poset $(S, \preccurlyeq)$ has at least one minimal element.

ALGORITHM 1 Topological Sorting.

procedure *topological sort* ($(S, \preccurlyeq)$: finite poset)
$k := 1$
while $S \neq \emptyset$
    $a_k :=$ a minimal element of S {such an element exists by Lemma 1}
    $S := S - \{a_k\}$
    $k := k + 1$
return $a_1, a_2, \ldots, a_n$ {$a_1, a_2, \ldots, a_n$ is a compatible total ordering of S}

Topological Sorting for Finite Posets

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.



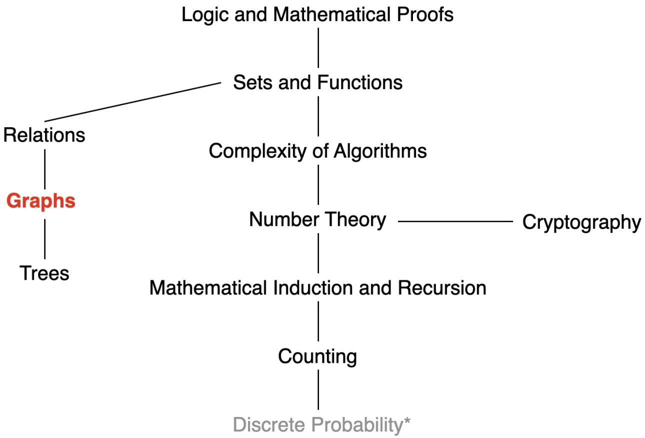
Minimal element chosen 1	5	2	4	20	12

This produces the total ordering

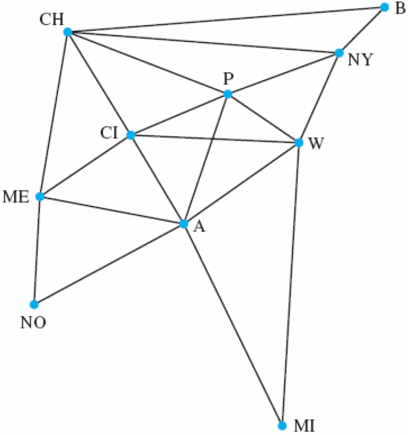
$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$$

Recall the Motivation: A project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks?

This Lecture



Example



- What is the minimum number of links to send a message from *B* to *NO*?

  3: B - CH - ME - NO

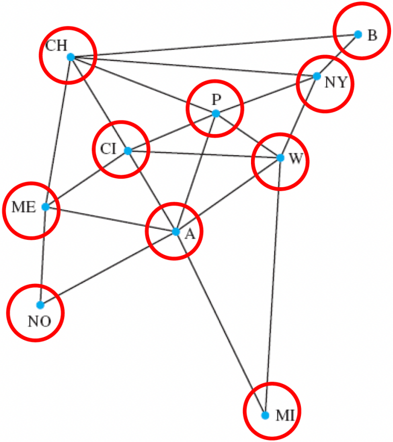
- Which city/cities has/have the most communication links emanating from it/them?

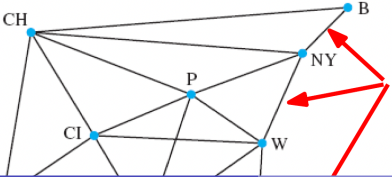
  A: 6 links

- What is the total number of communication links?

  20 links

Graph G

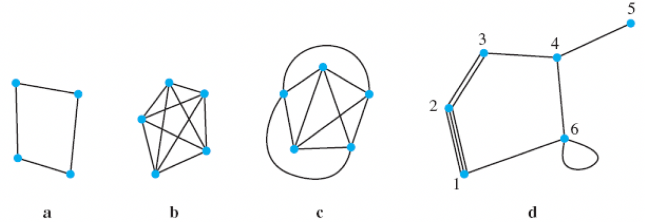




- Consists of a set of vertices V, $|V| = n$
- and a set of edges E, $|E| = m$
- Each edge has two endpoints

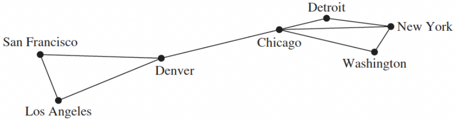
Definition of a Graph

Definition: A graph $G = (V, E)$ consists of a nonempty set V of vertices (or nodes) and a set E of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to be incident to (or connect) its endpoints.

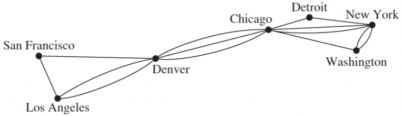


Simple Graph, Multigraph, Pseudograph

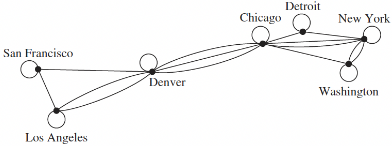
- simple graph: A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices.



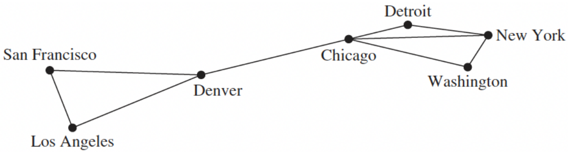
- Multigraph: Graphs that may have multiple edges connecting the same vertices.



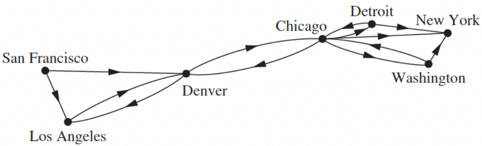
- Pseudograph: Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself.



Directed and Undirected Graph



A directed graph (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

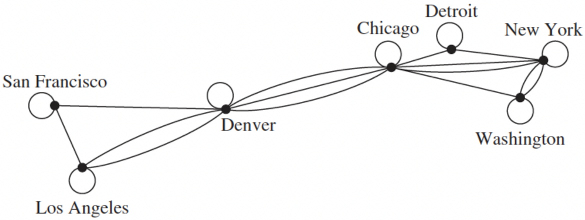


Graph: Example

- Computer networks
- Social networks
- Communication networks
- Information networks
- Software design
- Transportation networks
- Biological networks

Computer Networks

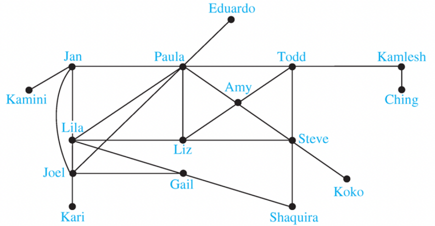
- Vertices: computers
- Edges: connections



Social Networks

- Vertices: individuals
- Edges: relationships

Friendship graphs: undirected graphs where two people are connected if they are friends (in the real world, wechat, or Facebook, etc.)



Social Networks

Influence graphs: directed graphs where there is an edge from one person to another if the first person can influence the second one.

Collaboration graphs: undirected graphs where two people are connected if they collaborate in some way.

- Hollywood graph
- Academic collaboration graph

Undirected Graphs

Definition: Two vertices u, v in an undirected graph G are called adjacent (or neighbors) in G if there is an edge e between u and v. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

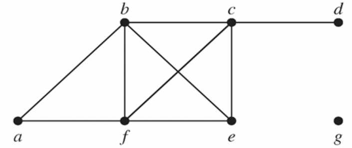
Definition: The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the neighborhood of v.

If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A.

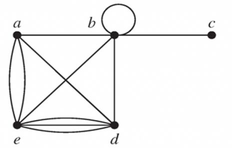
Definition: The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex v is denoted by $deg(v)$.

Undirected Graphs: Example

What are the degrees and neighborhoods of the vertices in the graph G?



$deg(a) = 2$, $deg(b) = deg(c) = deg(f) = 4$, $deg(d) = 1$, $deg(e) = 3$, and $deg(g) = 0$.
$N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \emptyset$.



$deg(a) = 4$, $deg(b) = deg(e) = 6$, $deg(c) = 1$, and $deg(d) = 5$.
$N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$.

Undirected Graphs

Theorem (Handshaking Theorem): If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} deg(v)$$

(Note that this applies even if multiple edges and loops are present.)

Because each edge contributes two degrees.

Directed Graphs

Definition: An directed graph $G = (V, E)$ consists of V, a nonempty set of vertices, and E, a set of directed edges.

Each edge is an ordered pair of vertices. The directed edge (u, v) is said to start at u and end at v.

Definition: Let (u, v) be an edge in G. Then

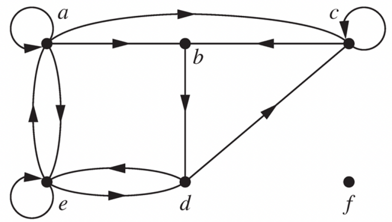
- u is the initial vertex of the edge and is adjacent to v,
- and v is the terminal vertex of this edge and is adjacent from u.

The initial and terminal vertices of a loop are the same.

Directed Graphs

Definition: The in-degree of a vertex v, denoted by $deg^-(v)$, is the number of edges which terminate at v. The out-degree of v, denoted by $deg^+(v)$, is the number of edges with v as their initial vertex.

Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.



The in-degrees are $deg^-(a) = 2$, $deg^-(b) = 2$, $deg^-(c) = 3$, $deg^-(d) = 2$, $deg^-(e) = 3$, and $deg^-(f) = 0$.
The out-degrees are $deg^+(a) = 4$, $deg^+(b) = 1$, $deg^+(c) = 2$, $deg^+(d) = 2$, $deg^+(e) = 3$, and $deg^+(f) = 0$.

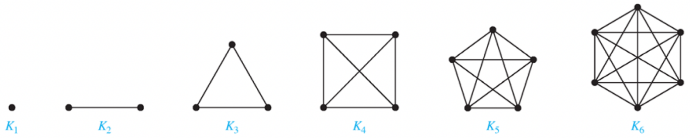
Directed Graphs

Theorem: Let $G = (V, E)$ be a graph with directed edges. Then,

$$|E| = \sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v)$$

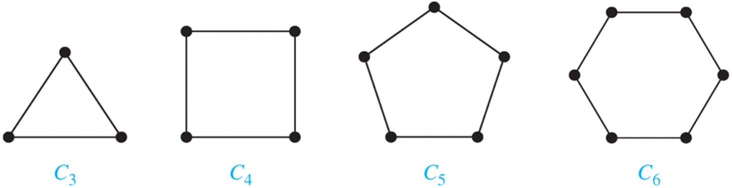
Complete Graphs

A complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



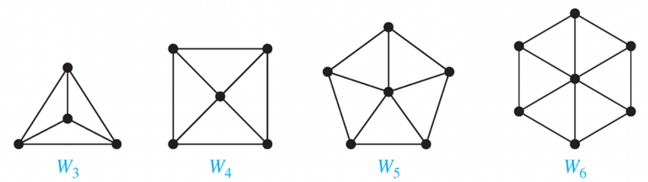
Cycles

A cycle C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \ldots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.



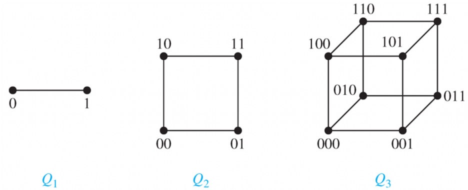
Wheels

A wheel W_n is obtained by adding an additional vertex to a cycle C_n.



N-dimensional Hypercube

An n-dimensional hypercube, or n-cube, Q_n is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices that differ in exactly one bit position.



How many edges? $n2^{n-1}$

Construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit.