Gate-level Minimization

CS207 Chapter 3

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Gate-level minimization

task to find optimal

- The complexity of digital logic gates to implement a Boolean function is directly related to the complexity of algebraic expression.
- Gate-level minimization is the design task of finding an optimal gate-level implementation of Boolean functions describing a digital circuit.
  - Difficult by hand for more than few inputs.
  - Typically by computer, need to understand the underlying principle.

Can only solve up to 4 variable

other by EDA tool.

Electronic design automation

But the principle is same

The map method

also called gate-level minimization

put forward (an idea or plan) for consideration or discussion by others

- The map method, first proposed by Veitch and slightly improvised by Karnaugh, provides a simple, straightforward procedure for the simplification of Boolean functions.
  - Called *Karnaugh map*. k map
- The map is a diagram consisting of *squares*. For n variables on a Karnaugh map there are 2^n numbers of squares.
  - Each square or cell represents one of the minterms.
  - Since any Boolean function can be expressed as a sum of minterms, it is possible to recognize a Boolean function graphically in the map from the area enclosed by those squares whose minterms appear in the function.

based on minterm & maxterm

1. Select K-map according to the number of variables.
2. Identify minterms or maxterms as given in problem.
3. For SOP put 1's in blocks of K-map respective to the minterms (0's elsewhere).
4. For POS put 0's in blocks of K-map respective to the maxterms(1's elsewhere).
5. Make rectangular groups containing total terms in power of two like 2,4,8 ..(except 1) and try to cover as many elements as you can in one group.
6. From the groups made in step 5 find the product terms and sum them up for SOP form

Two-variable K-map

one and only one input to produce 1
refer to a term that has

A & B 2^2

- A two-variable system can form four minterms

⇓ small squares (cells)

注意 K-map 画法.

$A \backslash B$	0	1
0	m_0	m_1
1	m_2	m_3

$A \backslash B$	0	1
0	$A'B'$	$A'B$
1	AB'	AB

$A \backslash B$	0	1
0	0	1
1	1	1

unique K map

- The two-variable Karnaugh map is a useful way to represent any of the 16 Boolean functions.

unique truth table

  - Example:

$$A + B = A(\underbrace{B + B'}_{=1}) + B(\underbrace{A + A'}_{=1})$$

add $(x+x')$ for missing term x

$$= AB + AB' + AB + A'B = AB + AB' + A'B$$

label minterm appear at this place 1
other 0

  - So the squares corresponding to AB, AB', and $A'B$ are marked with 1.

Three-variable K-map

- Since there are eight minterms for three variables, the map consists of eight cells or squares.
  - Minterms are arranged, not according to the binary sequence, but according to the sequence similar to the gray code.
  - **Between two consecutive rows or columns, only one single variable changes its logic value from 0 to 1 or from 1 to 0.**

别写反

$A \backslash BC$	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

the first & last column is adjecent too

adjacent column

Only 1 bit is changed

||

principle of K map

unique K map of 3 variables

$(5)_{10} = (101)_2$

A B C

in 2 variable case

there's no need to

consider adjecent

Three-variable K-map

- To understand the usefulness of the map for simplifying the Boolean functions, we must observe the basic properties of the adjacent squares.
  - Any two adjacent squares in the Karnaugh map differ by only one variable, which is complemented in one square and uncomplemented in one of the adjacent squares.
  - The sum of two minterms can be simplified to a single AND term consisting of less number of literals.
  - $m_1 + m_5 = A'B'C + AB'C = (A' + A)B'C = B'C$

$(101)_2 = (5)_{10}$

||

row & column

BOTH adjecent

A \ BC	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

adjecent in K map

$\Longrightarrow$ CAN simplification

Three-variable K-map

- Example: Simplify the Boolean function $F = \underline{A'BC} + A'BC' + AB'C' + AB'C$.

adjecent 1s

minterm
$\Downarrow$
can be represented as a cell

$A \backslash BC$	00	01	11	10
0	0	0	1	1
1	1	1	0	0

adjecent 1s
$\Downarrow$
bigger cell
$\Downarrow$
a term (simper form)

- The first row: $A'BC + A'BC' = A'B$.
- The second row: $AB'C' + AB'C = AB'$.
- $F = A'B + AB'$.

Three-variable K-map

- Example: Simplify the Boolean function $F = \underbrace{\overset{3}{A'BC} + \overset{4}{AB'C'} + \overset{7}{ABC} + \overset{6}{ABC'}}_{= 1}$.

$A \backslash BC$	00	01	11	10
0	0	0	3 1	0
1	4 1	0	7 1	6 1

* how to create adjecent

try to group every term

- The third column: $A'BC + ABC = BC$.
- The second row: $AB'C' + ABC' = AC'$.
- $F = BC + AC'$.

fixed value ($1 \Rightarrow x$ $0 \Rightarrow x'$) in group!

Three-variable K-map

- Example: Simplify the Boolean function $F = \sum(1, 2, 3, 5, 7)$.

$A \backslash BC$	00	01	11	10
0	0	1	1	1
1	0	1	1	0

1 can be reused

Don't forget th SKIP!

① square can be represed by one literal

② group can be different from 2 cells

- $F = C + A'B$.

Why the "1" in cells can be reused ?

- As adjacent cells differs only by one variable, it can be group, and the group can be represented by the literals that do not changed. Because for SOP, $A+A'=1$, the $A'BC+ABC=BC$.
- And for repeat, $A+A=A$.

So $A'BC + AB'C + ABC$

$= A'BC + AB'C + ABC + ABC$

$= BC + AC$

Three-variable K-map

- Example: Simplify the Boolean function $F = \sum(0, 2, 4, 5, 6)$.

$A \backslash BC$	00	01	11	10
0	1	0	0	1
1	1	1	0	1

Don't forget the adjecent betwen the first & last columns.

- $F = C' + AB'$.

Four-variable K-map

There have European K-map as well.

we introduce the US K-map

- Similar to the method used for two-variable and three-variable Karnaugh maps, four-variable Karnaugh maps may be constructed with 16 squares consisting of 16 minterms.

NOT 1 start from 0

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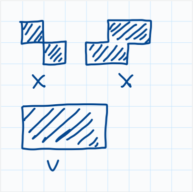
$AB \backslash CD$	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

TWO SKIP

Four-variable K-map

- Two, four, or eight adjacent squares can be combined to reduce the number of literals in a function.
- The squares of the top and bottom rows as well as leftmost and rightmost columns may be combined.
  - When two adjacent squares are combined, it is called a *pair* and represents a term with three literals.
  - Four adjacent squares, when combined, are called a *quad* and its number of literals is two.
  - If eight adjacent squares are combined, it is called an *octet* and represents a term with one literal.
  - If, in the case all sixteen squares can be combined, the function will be reduced to 1.



Four-variable K-map

ALWAYS & minimize } ⟹ use K-map

⟹ 4 variables K-map the max idex is 15

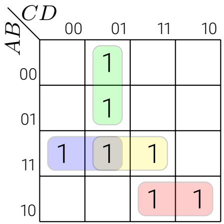
- Example: Simplify the Boolean function $F = m_1 + m_5 + m_{10} + m_{11} + m_{12} + m_{13} + m_{15}$.
  - $A'B'C'D + A'BC'D = A'C'D$,
  - $ABC'D' + ABC'D = ABC'$,
- $F = A'C'D + ABC' + ACD + AB'C$.
- This reduced expression is not a unique one.
  - If pairs are formed in different ways, the simplified expression will be different.

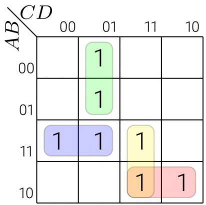
$AB \backslash CD$	00	01	11	10
00		1		
01		1		
11	1	1	1	
10			1	1

⟹ compare with next page's K-map

Four-variable K-map

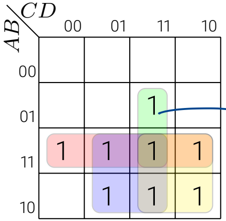
- Example: Simplify the Boolean function $F = m_1 + m_5 + m_{10} + m_{11} + m_{12} + m_{13} + m_{15}$.
- $F = A'C'D + ABC' + ABD + AB'C$.





Four-variable K-map

- Example: Simplify the Boolean function $F = \sum(7, 9, 10, 11, 12, 13, 14, 15)$.
- $F = AB + AC + AD + BCD$.



NOT let it be alone. it can be connected to other

Four-variable K-map

make a schematic or technical drawing of that shows how things work or how they are constructed

- Example: Plot the logical expression $F(A,B,C,D) = ABCD + AB'C'D' + AB'C + AB$ on a four-variable Karnaugh map.

$$
\begin{aligned}
& F(A,B,C,D) \\
&= ABCD + AB'C'D' + AB'C + AB \\
&= ABCD + AB'C'D' + AB'C(D+D') \\
&\quad + AB(C+C')(D+D') \\
&= \dots \\
&= \sum(8,10,11,12,13,14,15) \\
&= AB + AC + AD'
\end{aligned}
$$

$AB \backslash CD$	00	01	11	10
00				
01				
11	1	1	1	1
10	1		1	1

this step can be left out.
we can just use this to fill 1 in K-map

Four-variable K-map

- Simplify the expression $F(A, B, C, D) = \sum(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$.

$AB \backslash CD$	00	01	11	10
00	1	1		1
01	1	1		1
11	1	1		1
10	1	1		

Four-variable K-map

- Simplify the expression $F(A,B,C,D) = A'B'C' + B'CD' + A'BCD' + AB'C'$.

$$
\begin{aligned}
&F(A,B,C,D) \\
&= A'B'C'(D+D') + B'CD'(A+A') \\
&\quad + A'BCD' + AB'C'(D+D') \\
&= A'B'C'D + A'B'C'D' + AB'CD' \\
&\quad + A'B'CD' + A'BCD' + AB'C'D \\
&\quad + AB'C'D' \\
&= \sum(0,1,2,6,8,9,10) \\
&= B'C' + B'D' + A'CD'
\end{aligned}
$$

AB \ CD	00	01	11	10
00	1	1		1
01				1
11				
10	1	1		1

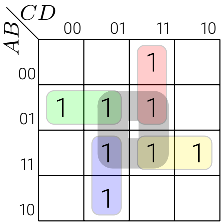
The blue grow covers 4 corner

✻ corners are adjacent

⟹ alway creat corner groups FIRST

Four-variable K-map

- Simplify the expression $F(A,B,C,D) = \sum(3,4,5,7,9,13,14,15)$.
  - It may be noted that one quad can also be formed, but it is redundant as the squares contained by the quad are already covered by the pairs which are essential.
- $F = A'BC' + A'CD + AY'D + ABC.$



Four-variable K-map

- Simplify the expression $F(A, B, C, D) = \prod(0, 1, 4, 5, 6, 8, 9, 12, 13, 14)$.
  - The above expression is given in respect to the maxterms.
  - 0's are to placed instead of 1's at the corresponding maxterm squares.
- $F = C(B' + D)$.

In maxterm. If B=1 ⇒ write B'

connect B&D use "+"

AB \ CD	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	0	1	0
10	0	0	1	1

while 1s represent minterms, 0s represent maxterms. So we can group 0s instead of group 1s.

Simplification of POS

1. $F(A.B.C.D) = \Pi(\underbrace{1.2.5.7\ldots}_{\text{write a "0" in Kmap}})$

2. for the literals unchanged in a group.

$$\begin{cases} 0 \rightarrow A \\ 1 \rightarrow A' \end{cases}$$

3. different literals in one group are connected by addition
4. different groups are connected by multiplication

Four-variable K-map

- Simplify the expression $F(A,B,C,D) = \prod(0,1,4,5,6,8,9,12,13,14)$.
  - The other way to achieve the minimized expression is to consider the 1's of the Karnaugh map.
- $F = CD + B'C = C(B'+D)$.

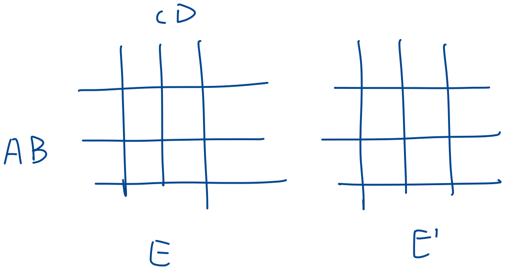
group 0 and group 1
give the same result

$AB \backslash CD$	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	0	1	0
10	0	0	1	1

Five-variable K-map

will NOT be tested

- Karnaugh maps with more than four variables are not simple to use.
  - The number of cells or squares becomes excessively large and combining the adjacent squares becomes complex.
  - A five-variable Karnaugh map contains 2^5 or 32 cells.

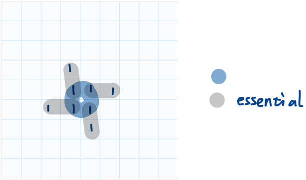


Prime Implicants

- A *prime implicant* is a product term obtained by combining the maximum possible number of adjacent squares in the map.
- The prime implicants of a function can be obtained from the map by combining all possible maximum numbers of squares.
  - If a minterm in a square is covered by only one prime implicant, that prime implicant is said to be *essential*.

can NOT be remove

- Gate-level minimization:
  - Determine all essential prime implicants.
  - Find other prime implicants that cover remaining minterms.
  - Logical sum all prime implicants.



Don’t care conditions

- In practice, Boolean function is not specified for certain combinations of input variables.
  - Input combinations never occur during the process of a normal operation.
  - Those input conditions are guaranteed never to occur.
- Such input combinations are called *don’t-care conditions*.
- These input combinations can be plotted on the Karnaugh map for further simplification.
  - The don’t care conditions are represented by d or X in a K-map.
  - They can be either 1 or 0 upon needed.

and they are indepedent

Don't care conditions

function

- Simplify the expression $F(A,B,C,D) = \sum(1,3,7,11,15)$, $d = \sum(0,2,5)$.
- $F = A'B' + CD$.

don't care

$AB \backslash CD$	00	01	11	10
00	X	1	1	X
01		X	1	
11			1	
10			1	

0/1

Don’t care conditions

- Simplify the expression $F(A, B, C, D) = \sum(1, 3, 7, 11, 15), d = \sum(0, 2, 5)$.
- $F = A'D + CD$.

$AB \backslash CD$	00	01	11	10
00	X	1	1	X
01		X	1	
11			1	
10			1	

$F(A,B,C,D) = \sum(0,1,2,3) + d(4,5)$

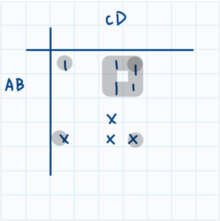
$F(A \cdot B \cdot C \cdot D) = \prod(0\ 1 \cdot 2) + d(4.5)$

are both okay.

More examples

- Using the Karnaugh map method obtain the minimal sum of the products expression for the function $F(A, B, C, D) = \sum(0, 2, 3, 6, 7) + d(8, 10, 11, 15)$.

four variable



$\Rightarrow B'D' + A'C$