A Computational Analysis of the Riemann Zeta Function

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Abstract

The Riemann Zeta Function was first introduced by Leonhard Euler in the eighteenth century. In 1736, Euler was able to solve this equation in the real plane when evaluated at even natural numbers. In 1859, Bernhard Riemann extended analysis of the zeta function into the complex plane. Though much work has been done on the zeta function, a closed form solution for zeta evaluated at odd natural numbers does not exist. In our research we analyze the zeta function and search for methods of computing it. Our work uses different methods of computation developed by various mathematicians to compute these unknown values of the zeta function, including the Prime Product of Euler, the Ramanujan Method, and the Modified Euclidean Algorithm Method by Casey. Programming is done in the Python and Julia languages, where we take advantage of the computational features of both. Our goal is compute these numbers past their current accuracy.

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1 Origin of the ζ Function

1.1 The Bernoulli Brothers

Jakob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748), were brothers born in Basel Switzerland. They both received doctorates from the University of Basel, and later became professors there. In fact, Jakob, was Johann's doctoral advisor [14].

In the 1689 Jakob Bernoulli stated the following problem, known as the Basel Problem,

The Basel Problem. *Find a closed form solution to the series:*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots .$$

The Bernoulli Brothers had just proved that the harmonic series diverges, and that this series converges. So the natural next step is to ask, what does it converge to [5]?

Though Jakob was unable to find a closed form solution for $\sum_{n=1}^{\infty} \frac{1}{n^2}$, he was able to bound it. For this he used the series,

$$1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \dots ,$$

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where the kth denominator is the kth triangular number,

$$\frac{k(k+1)}{2}.$$

Then, manipulating this series,

$$
\begin{aligned}
& 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \dots \\
= \ & 2 \left(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots \right), \\
= \ & 2 \left(\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots \right), \\
= \ & 2.
\end{aligned}
$$

Jakob achieved this result using telescoping series [6]. He also knew that,

$$\frac{1}{k^2} > \frac{1}{\frac{k(k+1)}{2}}.$$

Then, he reasoned that,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{\frac{k(k+1)}{2}} &= 2, \\
\sum_{n=1}^{\infty} \frac{1}{n^2} &< 2.
\end{aligned}
$$

But that is a precise as he got, until one of his students, Euler, solved the problem. In fact, he admitted defeat against this problem and wrote a plea for help saying, "If anyone finds and communicates to us that which thus far

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has eluded our efforts, great will be our gratitude." [6]

1.2 Euler

Leonhard Euler (1707-1783), was born in Basel, Switzerland. He studied at
the University of Basel as a student of Johann Bernoulli [14]. He worked
extensively on much of modern Analysis, and Complex Analysis and he gave
us the Euler number, e, and the relationship

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

1.2.1 Initial Attempt

In 1731, Euler was 24 years old, and hard at work to solve the Basel Problem.
His first attempt was a brute force attempt to sum the first hundred terms
of the series [6]. Unfortunately, since the series converges so slowly, this did
provide any real insights for Euler. His next attemp proved slightly more
fruitful. He compared the series to the integral,

$$I = \int_0^{\frac{1}{2}} -\frac{\log(1-t)}{t}dt\,.$$

Letting $z = 1 - t$, he showed that,

$$I \;=\; \int_1^{\frac{1}{2}} -\frac{\log z}{1-z}\,.$$

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Making use of Geometric Series,

$$= \int_1^{\frac{1}{2}} (1 + z + z^2 + z^3 + \ldots) \log z \, dz \,.$$

After some clever serires manipulations, he arrived at the result,

$$I = \log z \left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots \right) - \left(z + \frac{z^2}{4} + \frac{z^3}{9} + \frac{z^4}{16} + \ldots \right) \Big|_1^{\frac{1}{2}} \,.$$

From this Euler produced a formula that converged more quickly and was thus much more computable. That is,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2 2^{n-1}} + (\log 2)^2 \,.$$

1.2.2 Euler's Solution

In 1735, Euler had the breakthrough that,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \,.$$

Euler's Proof. Euler starts with a polynomial $p(x)$ such that,

$$p(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \ldots \,,$$

Note that $p(0) = 1$. Thus to find the roots of $p(x)$, $(x \neq 0)$,

$$
\begin{aligned}
p(x) &= x \left(\frac{1 - x^2/3! + x^4/5! - \cdots}{x} \right) \\
&= \frac{x - x^3/3! + x^5/5! - x^7/7! + x^9/9! - \cdots}{x} \\
&= \frac{\sin x}{x}.
\end{aligned}
$$

Thus $p(x) = 0$, when $\sin x = 0$. This happens when $x = \pm k\pi$ for $k \in \mathbb{N}$.

Then, factoring the polynomial, Euler produced the product,

$$
\begin{aligned}
&p(x) \\
&= \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{-\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{-2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 - \frac{x}{-3\pi}\right)\cdots, \\
&= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\cdots.
\end{aligned}
$$

Then Euler had that,

$$
1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right)x^2 + \cdots.
$$

Equating the coefficients of x^2,

$$
\begin{aligned}
-\frac{1}{3!} &= -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right), \\
&= \frac{1}{-\pi}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right).
\end{aligned}
$$

And thus, Euler achieved the result [6] [13],

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$\square$

1.2.3 The General Solution for Even Powers

In 1748, Euler published, *Introductio in Analysin Infinitorum*. Here he moved past the Basel Problem and demonstrated results for even exponent values up to 10 [8].

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6}, \\
\sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90}, \\
\sum_{n=1}^{\infty} \frac{1}{n^6} &= \frac{\pi^6}{945}, \\
\sum_{n=1}^{\infty} \frac{1}{n^8} &= \frac{\pi^8}{9450}, \\
\sum_{n=1}^{\infty} \frac{1}{n^{10}} &= \frac{\pi^{10}}{93555}.
\end{aligned}
$$

He also found the general solution for all even integer exponents:

$$\sum_{n \geq 1} n^{-2k} = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k},$$

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where B_{2k} is the $2k^{th}$ Bernoulli number. However, all $2k + 1^{th}$ Bernoulli numbers are 0. Thus, for odd integers, this formula does not hold [7].

Though he was unable to solve for ζ evaluated at odd integers, Euler did give us this "faster" way of computing ζ,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p_j \in \mathbb{P}} \frac{1}{1 - (p_j)^{-s}} \, ,$$

where $\mathbb{P}$ is the set of all prime numbers [7].

1.3 Riemann

Bernhard Riemann (1826-1866), was born in Breselenz, in the kingdom of Hanover. He studied in Berlin and at the University of Göttingen. He became a professor at Göttingen under K.F. Gauss. In 1859 he took on the zeta function to impress Gauss, and failed to find a closed form solution for odd powers. He did however succeed in expanding ζ into the complex plane. In 1866 he died at the age of 39 in Italy, of tuberculosis [11].

In 1859 Bernhard Riemann took on this problem and gave it its name. He begins his paper referencing Euler's work,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p_j \in \mathbb{P}} \frac{1}{1 - (p_j)^{-s}} \, ,$$

"The function of the complex variable s which is represented by these two expressions, wherever they converge, I denote by $\zeta(s)$." [12]

1.3.1 Riemann's Functional Equation

Riemann also proved that, for all complex s,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{(1-s)}{2}\right)\zeta(1-s).$$

This shows that $\zeta(-2n) = 0$, for $n \in \mathbb{N}$. These are called the trivial zeros of the zeta function [9].

1.3.2 The Riemann Hypothesis

Perhaps the greatest unsolved problem in mathematics came from Riemann's 1859 paper, which we state here.

The Riemann Hypothesis. *The nontrivial zeros of the Riemann Zeta Function lie on the line $\Re(s) = \frac{1}{2}$ [2].*
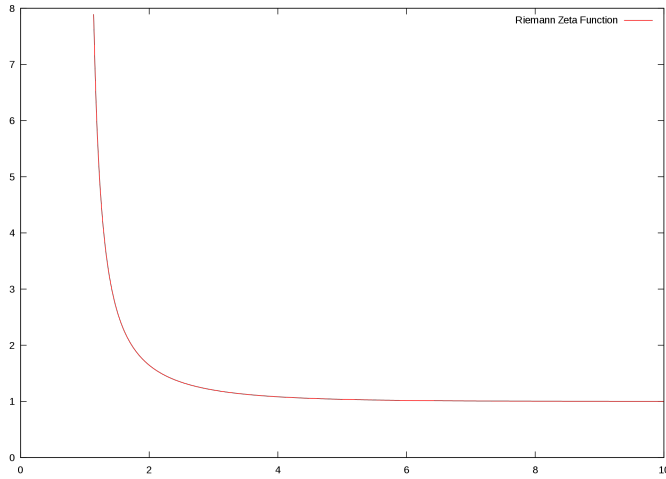

Figure 1: Graph of the Zeta Function in the Real Plane
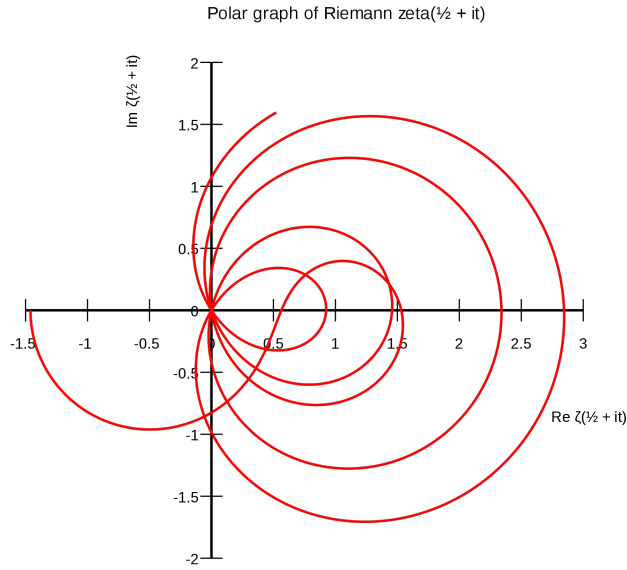
Figure 2: Graph of Zeta with $\Re(\zeta) = \frac{1}{2}$

1.4 Ramanujan

Srinivasa Ramanujan (1887-1920) was born in Erode, Madras Presidency, in India. He was forced to leave Government College in Kumbakonam, after they recinded his scholarship for failing all his non-math courses. In 1913, he send a letter to Hardy, in which he showed that,

$$
\begin{aligned}
\zeta(-1) &= \quad \sum_{n=1}^{\infty} n \quad &= -\frac{1}{12}\,, \\
\zeta(-2) &= \quad \sum_{n=1}^{\infty} n^2 \quad &= 0\,.
\end{aligned}
$$

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In 1914 Hardy arranged a scholarship for Ramanujan at Trinity College in Cambridge [14].

Thanks to Ramanujan we have this method to compute $\zeta(3)$ relatively quickly.

$$\zeta(3) = \frac{8}{7} \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \frac{1}{k^3} - \frac{4}{21} \log^3 2 + \frac{2}{21} \pi^2 \log 2 \,.$$

It is easy to see how this computation is faster than a straightforward computation of $\zeta(3)$, in that we have a $\left(\frac{1}{2}\right)^k$ term in front of $\frac{1}{k^3}$ [10].

2 Basic Properties and Definitions of ζ

Here we state some useful properties of ζ and formulas for its computation that we make use of later.

Like Riemann [12], we definte $\zeta(s)$ as follows,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \,.$$

We will make use of the following theorem [1],

Theorem 1 (The Fundamental Theorem of Arithmetic). *Every integer $n > 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors.*

2.1 Euler Prime Product

By Euler [5], we have that,

Theorem 2.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p_j \in \mathbb{P}} \frac{1}{1 - (p_j)^{-s}} \; .$$

Proof of Theorem 2. First we note that,

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots \; . \tag{1}$$

We also note that by Theorem 1, every natural number can be uniquely represented as a product of prime factors. Then, applying this to the denominators of ζ,

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \dots \; . \tag{2}$$

Then subtracting 1 from 2, we have,

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \dots \; . \tag{3}$$

Repeating this process for all the prime numbers we have,

$$\dots \left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 \; . \tag{4}$$

Then, dividing, we have,

$$\zeta(s) = \prod_{p_j \in \mathbb{P}} \frac{1}{1 - (p_j)^{-s}} \; . \tag{5}$$

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$\square$

2.2 General Solution for Even Powers

Theorem 3. *For $k \in \mathbb{N}$*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k} \,,$$

where $B_{2k} = 2k^{th}$ Bernoulli Number [3], defined by the generating series,

$$\frac{z}{e^z - 1} + \frac{1}{2} z = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n} \,, \ |z| < 2\pi \,.$$

2.3 Monotinicity

Claim 1. *For $s \geq 2$, $\zeta(s+2) < \zeta(s+1) < \zeta(s)$.*

Proof. By using the comparison test for each term, $n > 1$, and noting that,

$$(n+2)^{-s} < (n+1)^{-s} < n^{-s} \,,$$

then,

$$\zeta(s+2) < \zeta(s+1) < \zeta(s) \,.$$

Therefore, we can at least bound all odd values of the zeta function between the surrounding even values. $\square$

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2.4 Ramanujan's Formula

By Ramanujan we have [10],

Theorem 4.

$$\zeta(3) = \frac{8}{7}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \frac{1}{k^3} - \frac{4}{21}\log^3 2 + \frac{2}{21}\pi^2 \log 2 \,.$$

3 Computations

We will move past bounding ζ between known values. Using the following methods, we will bound ζ to arbitrary precision.

3.1 Brute Force Computation

We first took the formula for ζ,

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}\,,$$

and ran computations on this. This is the first method that Euler attempted to solve the Basel Problem. The problem with this method is that $\zeta(s)$ converges very slowly. It takes 1,000 iterations of $\zeta(2)$ to get to 2 decimal places of precision. The time complexity of this method is $O(n)$ since computation requires n iterations.

3.1.1 Julia Code

```
function zeta(s,n)
    sigma=0
    for i=1:n
        sigma+=BigFloat(1)/(BigFloat(i)^s)
    end
    return sigma
end
```

3.2 Euler Prime Product

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p_j \in \mathbb{P}} \frac{1}{1-(p_j)^{-s}} \ .$$

Thanks again to Euler, we have this product of prime numbers that we use to calculate $\zeta(s)$. The beauty of this method is that is converges much more quickly to the solution than the brute force method. But we present ourselves with a new problem, in order to do this computation we need to first generate an arbitrarily large list of prime numbers. Luckily for us, the Julia language has a built in prime number generation method. However it is important for us to briefly dicuss what this method is doing.

In order to generate prime numbers n, we begin at 2, the first prime and loop over a range of length n and check the divisibility of each integer between 2 and $\sqrt{k}$ for each integer $k < n$. This gives our computation time complexity of $O(\sqrt{n})$.

3.2.1 Julia Code

```
function euler(s,data)
    primes = readcsv(data)
    prod=1
    for p in primes
        prod*=(1)/(1-(1/BigFloat(p^s)))
    end
    return prod
end
```

After 1,000,000 iterations:
1.202056903159594

3.2.2 Python Code

```
from mpmath import *

def eu_iden(s, data,digits=100):
    mp.dps=digits
    primes = []
    with open(data) as inputfile:
        for line in inputfile:
            try:
                primes.append(int(line))
            except ValueError:
                pass
    prod=1
    for p in primes:
        prod*=(1-(1/mpf(p**s)))**(-1)
    inputfile.close()
    return prod
```

After 1,000,000 iterations:

1.2020569031595942

3.2.3 Comparison

Both Python and Julia have computational advantages and disadvantages. Julia, is a relatively new language, and runs primarily on compiled C code, which gives it its characteristic speed. The issue with Julia, is floating point precision. Hopefully this will not be an issue in the near future. However, due how new and still unestablished the language is, there does not yet exist a precision library in Julia that allows for more than 256 bits of floating point precision. While this suffices for relatively small computations, this limits our ability to tighten the boundaries on unknown values of ζ.

Python, in contrast, is an older language that is established as a scientific computing language. As such there exist several libraries to handle floating point precision, which means that precision is entirely dependent upon the size of the machine the computation is run on. However, Python's greatest shortcoming is its speed, or rather, its lack thereof. Computations that take minutes using Julia or C, could take hours in Python. Thus the optimal strategy for using Euler's prime product is to produce a list of prime numbers quickly using Julia, and then run a large, but precise computation using Python.

3.3 Ramanujan Method

To compute $\zeta(3)$ we have a relatively fast formula thanks to Ramanujan. The Ramanujan formula,

$$\zeta(3) = \frac{8}{7}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k}\frac{1}{k^{3}} - \frac{4}{21}\log^{3}2 + \frac{2}{21}\pi^{2}\log 2\,.$$

It is easy to see how this computation is faster than a straightforward computation of $\zeta(3)$, in that we have a $\left(\frac{1}{2}\right)^{k}$ term in front of $\frac{1}{k^{3}}$ [10]. Using the Ramanujan Formula we were able to produce the following code and results.

3.3.1 Julia Code

```
function ramanujan(k)
    sigma = zeros(BigFloat,k)

    for i=1:k
                sigma[i]=BigFloat(((1/2)^i)*(1/i^3))
        end
        diff1=BigFloat((4/21)*log(2)^3)
        diff2=BigFloat(2/21)*BigFloat(pi^2)*BigFloat(log(2))
        result=BigFloat(8/7)*BigFloat(sum(sigma)) - diff1 + diff2
        return result
    end
```

3.3.2 Python Code

```python
import numpy as np
from math import log
from mpmath import *
mp.prec+=203

def ramanujan(k):
    k=mpf(k)
    array=np.zeros_like(np.arange(k))
    for i in np.arange(1,k+1):
        array[i-1]=(mpf(0.5)**mpf(i))*(mpf(1)/mpf(i)**mpf(3))
    diff1=(mpf(4)/mpf(21))*mpf(np.log(2))**(mpf(3))
    diff2=(mpf(2)/mpf(21))*mpf(np.pi)**mpf(2)*mpf(np.log(2))
    sigma=mpf(8)/mpf(7)*np.sum(array) -diff1 + diff2
    return sigma
```

3.3.3 Notes on Computation

Unlike the Euler product, the Ramanujan formula does not require us to generate a list of prime numbers to run the computation which greatly reduces our time of computation. Furthermore it converges relatively quickly thanks to the $(\frac{1}{2})^k$ in the sum. In terms of our computation, not only is this method highly computatable, but it is also easily parallelizable. Meaning the computation can be split between processors and even between different machines on a distributed computing network. The only limitation with this method is that Ramanujan's formula only works for $\zeta(3)$. We are thus left to other methods to compute other odd values of ζ.

3.4 MEA Method

The MEA Method by Casey uses probability to compute values of ζ. First, we make use of the following claim and theorem [3].

Claim 2. *Given n ($n \geq 2$) randomly chosen positive integers $\{k_1, \ldots, k_n\}$,*

$$P\{\gcd(k_1, \ldots, k_n) = 1\} = [\zeta(n)]^{-1} .$$

Theorem 5. *Let $N_n(\ell) = card\{(k_1, \ldots, k_n) \in \{1, \ldots, \ell\}^n : \gcd(k_1, \ldots, k_n) = 1\}$. Then for $n \geq 2$ we have that*

$$\lim_{\ell \to \infty} \frac{N_n(\ell)}{\ell^n} = [\zeta(n)]^{-1} .$$

Let $\lfloor x \rfloor$ denote the floor function of x. Then using the previous claim and theorem, we produce a formula for computation in the following lemma [4].

Lemma. *Let $N_n(\ell) = card\{(k_1, \ldots, k_n) \in \{1, \ldots, l\}^n : \gcd(k_1, \ldots, k_n) = 1\}$ is the number of relatively prime elements in $\{1, \ldots, l\}^n$. Then,*

$$N_n(\ell) = \ell^n - \sum_{p_i} \left(\lfloor \frac{\ell}{p_i} \rfloor \right)^n + \sum_{p_i < p_j} \left(\lfloor \frac{\ell}{p_i \cdot p_j} \rfloor \right)^n - \sum_{p_i < p_j < p_k} \left(\lfloor \frac{\ell}{p_i \cdot p_j \cdot p_k} \rfloor \right)^n + \ldots .$$

3.4.1 Discussion of Computation

Computing using this method requires a recursive algorithm. We first generate a list of prime numbers as we did in the Euler Product Method. Then we must recurse over this list of primes and calculate an infinite sum of infi-

nite sums. Each kth sum within the outer sum requires us to compute every permutation of k primes. Thus for n primes we are running more than n^n computations. While is highly inefficient, our use of the floor function gives this method its merit. Because of the floor function we are not only dealing with integers, instead of floating point numbers, but for any given ℓ value eventually the product of primes will bring the quotient to 0. So each sum does not need to be calculated infinitely. Instead each kth sum only needs to be calculated until its quotient goes to 0. Further, we only need to compute sums until the kth prime is greater than our ℓ value.

4   Results

Using the methods from Section 3, we have achieved the following results.

$n = 1,000,000$	Julia	Python
Brute Force	1.202056903159	1.202056903159
Ramanujan Method	1.202056903159594	1.2020569031595942

Euler Product	Julia	Python
$\zeta(3)$	1.202056903159594	1.202056903159594
$\zeta(5)$	1.036927755143369	1.036927755143369

If, for odd s,

$$\zeta(s) = \frac{\pi^s}{??},$$

then, $?? \approx \ldots$

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MEA Method	$\ell = 10^6$	$\ell = 10^8$
$\zeta(3)^{-1} \cdot \pi^3$	25.79435196830	25.7943501926105
$\zeta(5)^{-1} \cdot \pi^5$	295.121515513789	295.121509986379

4.1   Bounds of ζ

By Claim 1,

$$\zeta(2) < \zeta(3) < \zeta(4) < \zeta(5) < \zeta(6) \,.$$

So we have that,

$$\frac{\pi^2}{6} < \zeta(3) < \frac{\pi^4}{90} < \zeta(5) < \frac{\pi^6}{945} \,.$$

But by our analysis, we are able to bound further. Specifically we can say that,

$$\frac{10\pi^3}{258} < \ \zeta(3) \ < \frac{4\pi^3}{103},$$
$$\frac{20\pi^5}{5903} < \ \zeta(5) \ < \frac{25\pi^5}{7378} \,.$$

Our results, though not exact, present bounds on $\zeta(3)$ and $\zeta(5)$ that are stricter than Euler and Riemann were ever able to produce. As modern computing methods become faster and allow for more precision, these bounds will become tighter and tighter and eventually converge to our result.

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