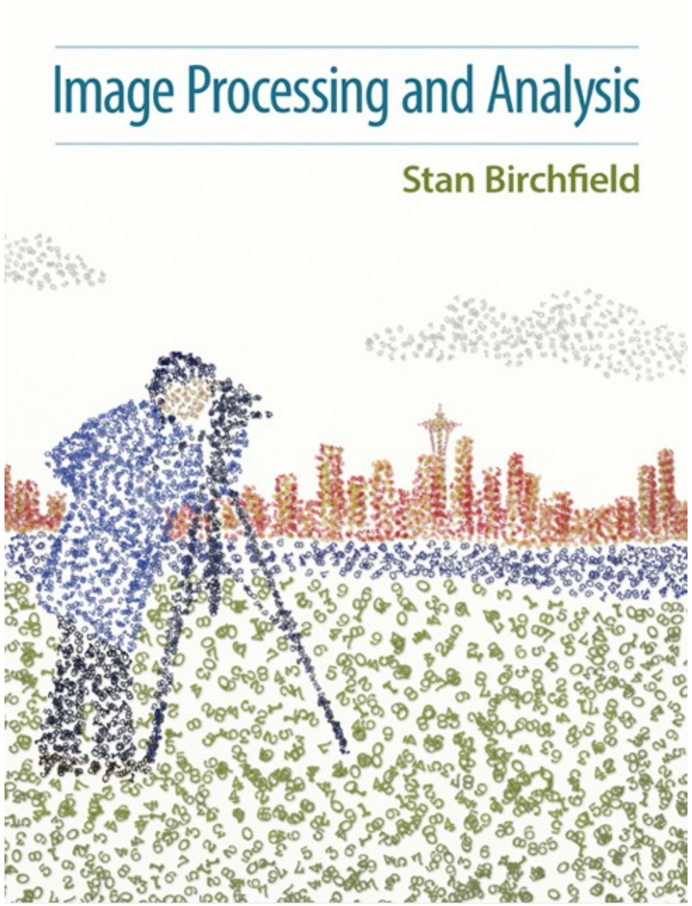
Prof. Kjersti Engan

# ELE510 Image processing and computer vision

Geometry of multiple views, ( chap 13.6 Birchfield) 2023

University of Stavanger

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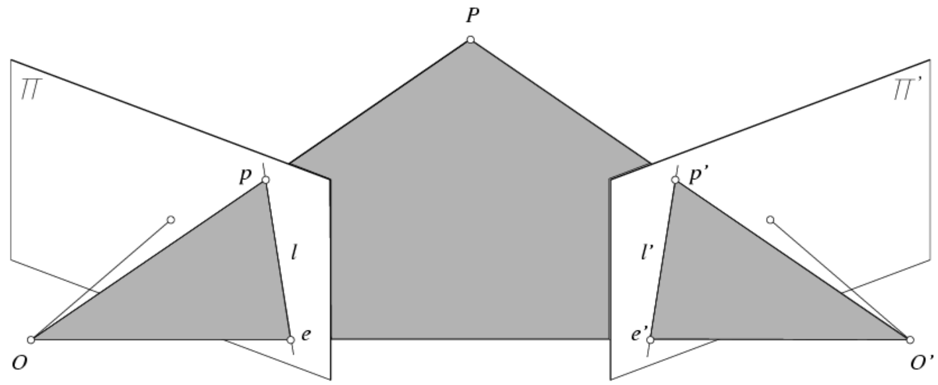
# Geometry of multiple views – 3D reconstruction

Three points from the topic:

1. Epipolar geometry help finding corresponding points
2. Fundamental and essential matrices
3. 3D Reconstruction from camera projection matrices

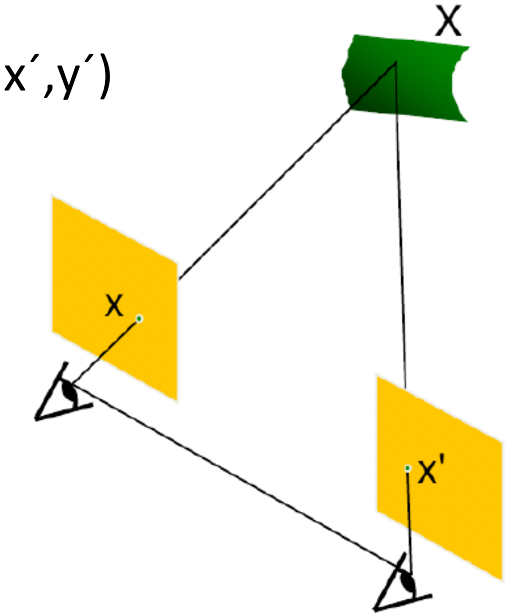


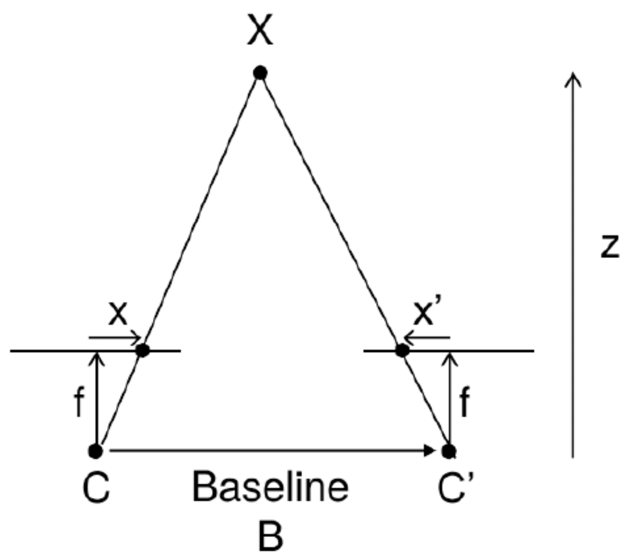
# Depth from Stereo

Goal: recover depth, Z

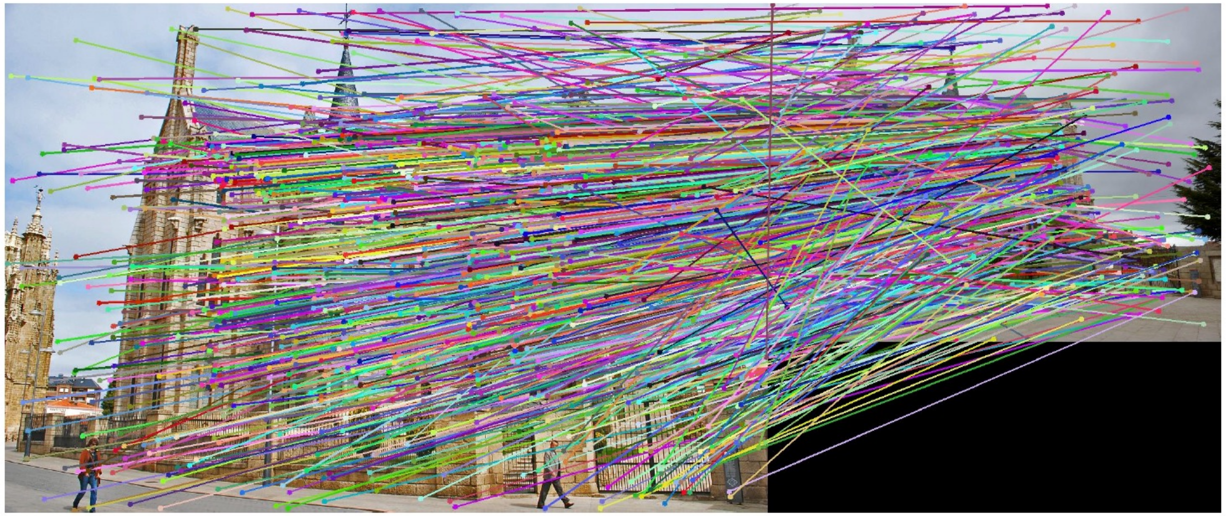
Subproblems:

- Calibration of cameras
- Correspondence - (x,y)<->(x´,y´)
- 3D Reconstruction

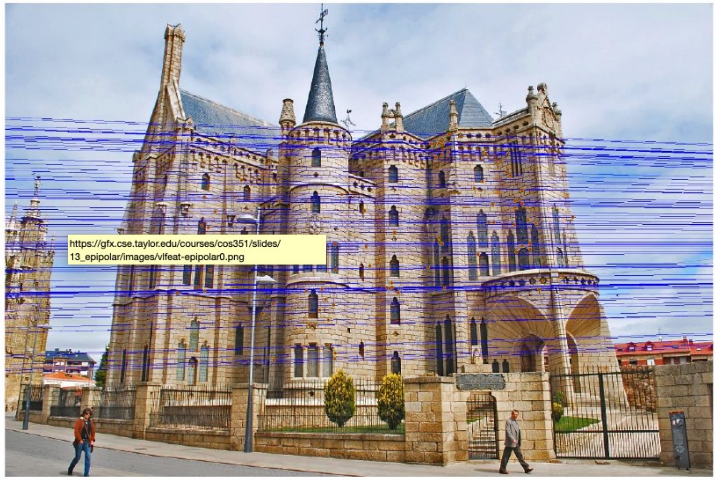


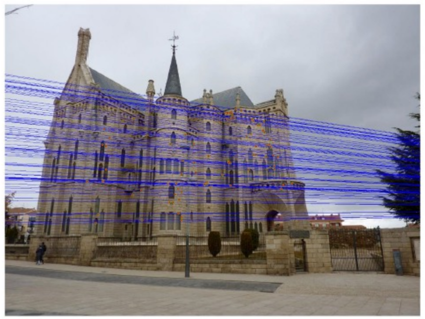



Recap correspondence -- Where are possible corresponding points – lies on the epipolar lines



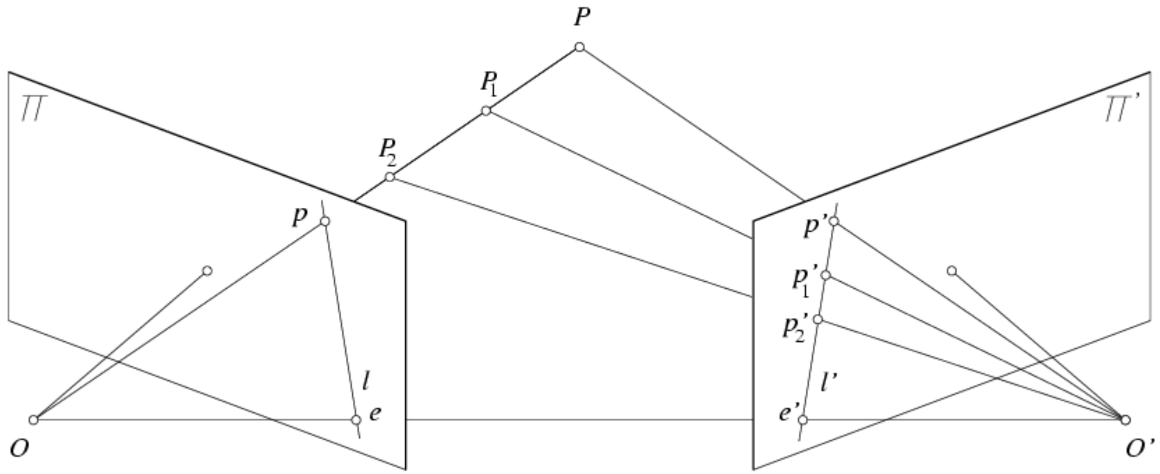
Remember rectified cameras: corresponding points are on the scan lines (y = y′).



# Epipolar Geometry

The *left camera coordinates are used as reference/ world coordinate system*




The *Epipolar Plane*: OO'P
*Epipoles*: e, e'
*Epipolar lines*: l, l'
Image points: p, p'

**Note!** There is a separate *epipolar plane* for each point in the scene.

Potential matches for ***p/ p'*** have to lie on the corresponding epipolar line ***l'/ l***

The optical center (=focal point for pinhole camera) is a distinct point on the other camera's image plane.

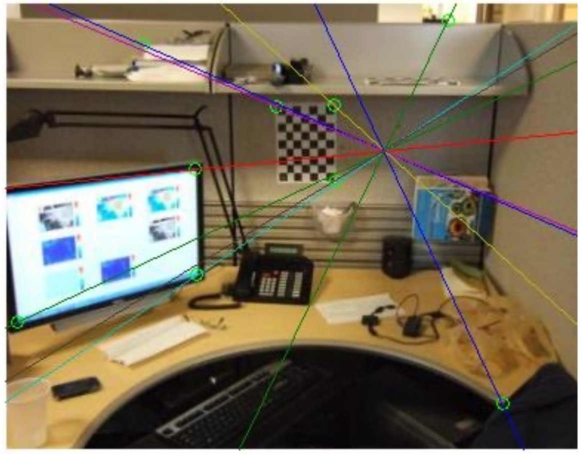
Denoted by **e** and **e´**, called *epipoles or epipolar points*.

Both epipoles **e** and **e´** in their respective image planes and both optical centers **O** and **O´** lie on a single 3D line.

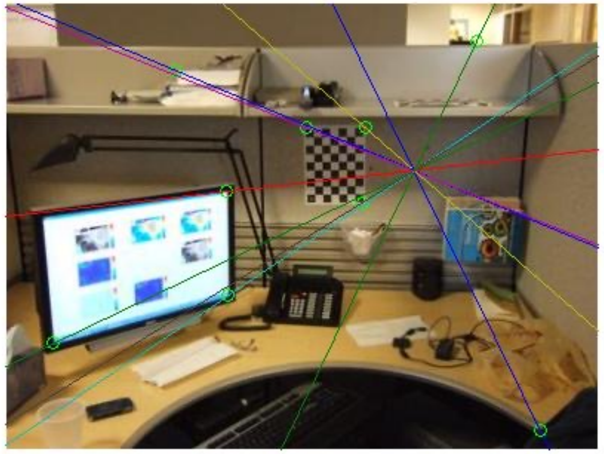
If this line does not cross the image plane, the epipoles are outside the image ( here: inside/visible in the image).

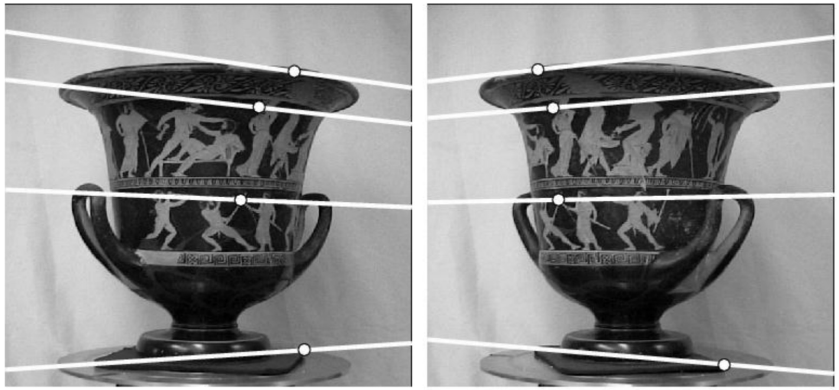
# Epipolar Lines

Inliers and Epipolar Lines in First Image



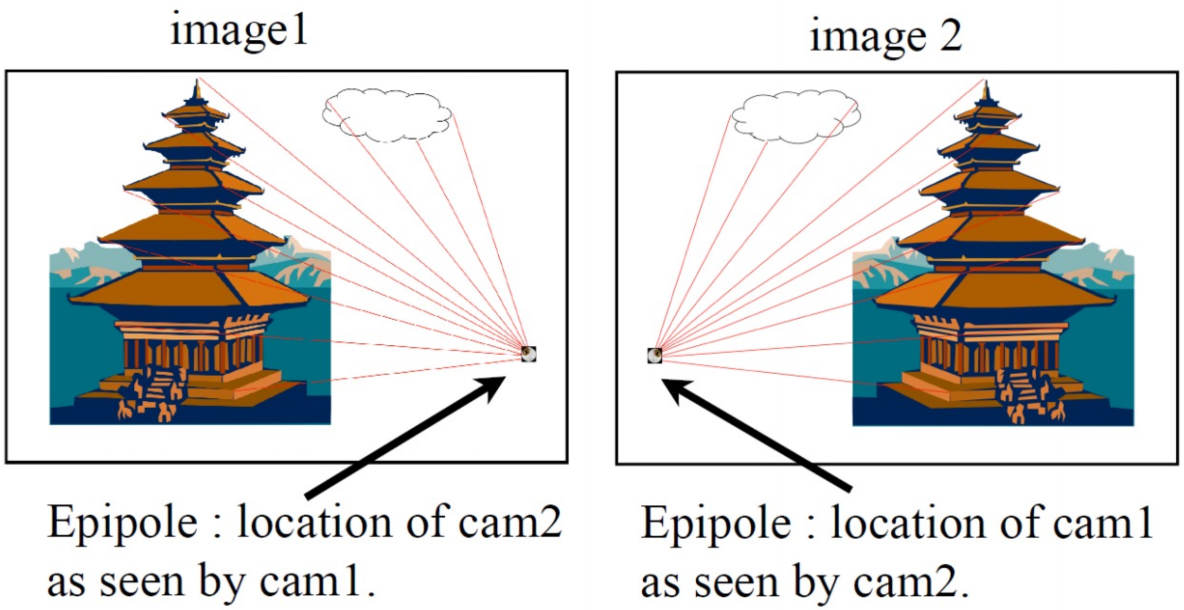
Inliers and Epipolar Lines in Second Image



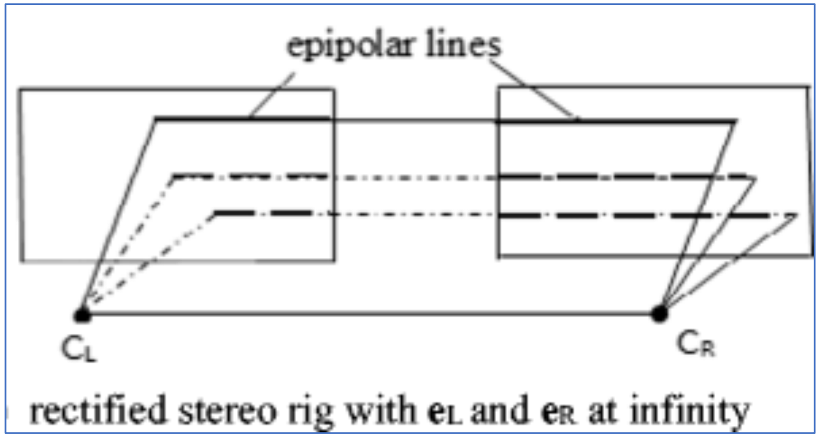


- Potential matches for ***p*** have to lie on the corresponding epipolar line ***l’***.
- Potential matches for ***p’*** have to lie on the corresponding epipolar line ***l***.

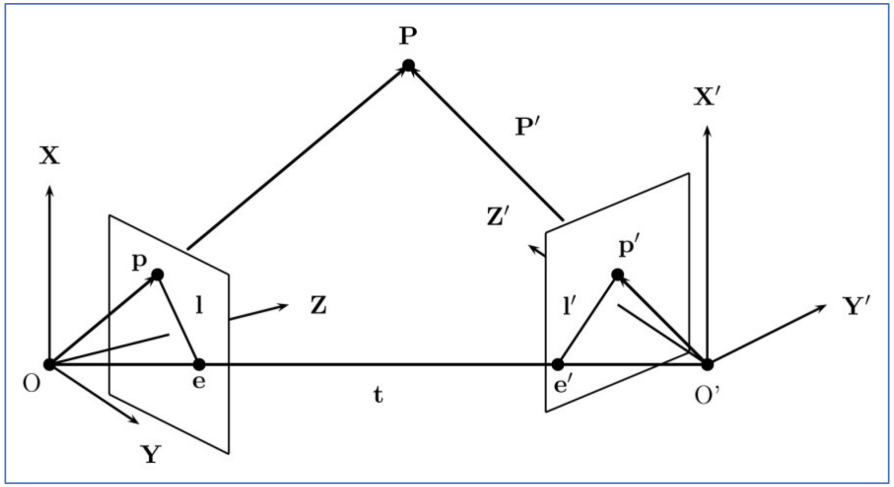
Epipolar plane and line is new for each physical point. Epipoles are constant - it is the focal point/origo of the other camera seen in the image, i.e. only one point. All epipolar lines goes through the epipole – i.e. intersection of epipolar lines revieals the epipoles.




Where is the epipoles for rectified cameras?



# Relationship between camera coordinates for the two cameras




The *left camera coordinates are used as world/reference*.

Right camera coordinates are denoted by primed symbols ‘

There is a rotation, **R**, and a translation, **t**, between the right and left coordinate systems (i.e. cameras). **R** is defined rotation to right camera described with left camera axis. Baseline translation **(t)** is given in left coordinates.
We get:

$$\mathbf{P} = \mathcal{R}\mathbf{P}' + \mathbf{t} \quad \Leftrightarrow \quad \mathbf{P}' = \mathcal{R}^{-1}(\mathbf{P} - \mathbf{t}) = \mathcal{R}^{T}\mathbf{P} - \mathcal{R}^{T}\mathbf{t}$$

Here P and P’ refers to the 3D world point described with left and right camera coordinate systems respectively.

# The Fundamental Matrix

- The fundamental matrix **F** capture the relative geometry between two cameras in pixel values. Let x and x' be corresponding points: $$(x_i, y_i) \Leftrightarrow (x'_i, y'_i)$$

$$\mathbf{x}^T \mathbf{l} = 0$$
$$\mathbf{x}'^T \mathbf{l}' = 0$$

Because the point lie on the epipolar lines

- How du we find the epipolar line *l'* in the second image associated with the point **x** in the first image? Has to be a linear function: $$\boxed{l' = \mathbf{F}\mathbf{x}}$$

We call F the fundamental matrix

# The Fundamental Matrix

$$(x_i, y_i) \Leftrightarrow (x'_i, y'_i)$$

- Given the fundamental matrix **F**, the epipolar line *l'* in the second image associated with the point **x** in the first image is given by:

$$\boxed{l' = \mathbf{F}\mathbf{x}}$$

- Similarly, the epipolar line l in the first image associated with the point **x'** in the second image is given by

$$\boxed{l = \mathbf{F}^T\mathbf{x}'}$$

$$\mathbf{x}^T\mathbf{l} = 0$$
$$\mathbf{x}'^T\mathbf{l}' = 0$$

Because the point lie on the epipolar lines

$$\Rightarrow \quad \mathbf{x}'^T\mathbf{F}\mathbf{x} = 0$$
$$\mathbf{x}^T\mathbf{F}^T\mathbf{x}' = 0$$

# Fundamental matrix – rectified cameras

For rectified cameras:   y - y' = 0

$$\mathbf{x'}^T \mathbf{F} \mathbf{x} = \begin{pmatrix} x' & y' & 1 \end{pmatrix} \mathbf{F} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

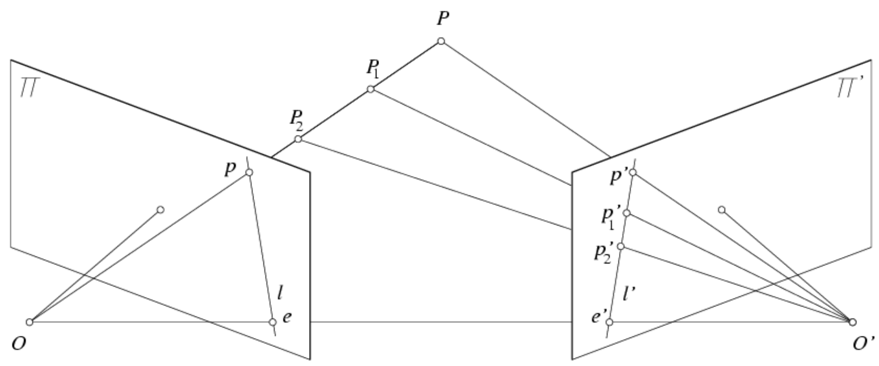
# Fundamental matrix and epipoles

The fundamental matrix also encodes the epipoles **e, e’.** since these are a point on the line

$$\left.\begin{aligned} \mathbf{l}^T\mathbf{e} &= 0 \\ \mathbf{l} &= \mathbf{F}^T\mathbf{x}' \end{aligned}\right\} \quad \mathbf{x}'^T\mathbf{F}\mathbf{e} = 0$$

This must be true for any **x**’. This means:

$$\mathbf{F}\mathbf{e} = 0 \qquad \mathbf{F}^T\mathbf{e}' = 0$$



# The Essential Matrix

- Like the fundamental matrix **F**, the essential matrix, **E**, captures the geometric relationship between the cameras in a compact matrix form.
- The fundamental matrix relates *uncalibrated* cameras (pixel values), while the essential matrix relates *calibrated* cameras (meter).
- If we know **F** or **E** we can find the other with the help of K and K' (the matrices of intrinsic camera parameters for the two cameras)
- Will show that the relationship between the two matrices is:

$$\boxed{\mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}, \text{ or } \mathbf{E} = \mathbf{K}'^{\top}\mathbf{F}\mathbf{K}}$$

- E has 5 parameters: 3 for rotation and 2 for translation in direction between the cameras.

# The Essential Matrix

K and K' are the intrinsic calibration matrices for the two cameras

$\bar{\mathbf{x}} \text{ and } \bar{\mathbf{x}}'$ are metric coordinates $\mathbf{x} \text{ and } \mathbf{x}'$ are pixel coordinates

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

# Reconstruction

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \propto \mathbf{P} \begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix}$$

**Recap – perspective imaging**
**P** is camera projection matrix

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \propto \underbrace{\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}}_{\mathbf{P}_{\{3\times 4\}}} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

$$\mathbf{P}_{\{3\times 4\}} = \underbrace{\begin{bmatrix} \alpha & \gamma & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{K}_{\{3\times 3\}}} [\mathbf{R}_{\{3\times 3\}} \ \mathbf{t}_{\{3\times 1\}}]$$

# Estimating the fundamental matrix

- If we let **x** and **x'** be points in the two images, then we can rewrite the fundamental matrix equation by explicitly listing the individual elements of the matrix and vectors as follows:

$$\boxed{\mathbf{x'}^T \mathbf{F} \mathbf{x} = 0} \qquad \mathbf{x'}^{\mathsf{T}} \mathbf{F} \mathbf{x} = [x' \ \ y' \ \ 1] \underbrace{\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

- Rearranged we can collect the nine unknowns ( $f_{ij}$) in a vector:

$$\mathbf{A}_{\{n \times 9\}} \mathbf{f}_{\{9 \times 1\}} = \mathbf{0}_{\{9 \times 1\}}$$

- Where $n \geq 8$ is the number of corresponding pairs $\boxed{(x_i, y_i) \Leftrightarrow (x'_i, y'_i)}$
- Can solve for F, also need to make sure is a valid fundamental matrix

# Estimate and decomposing the Essential Matrix

- From **F** and **K, K'** we can find **E**: $\boxed{\mathbf{F} = \mathbf{K}'^{-\mathsf{T}}\mathbf{E}\mathbf{K}^{-1}, \text{ or } \mathbf{E} = \mathbf{K}'^{\mathsf{T}}\mathbf{F}\mathbf{K}}$
- Given an essential matrix **E**, we would like to be able to extract the translation vector **t** and rotation matrix **R**. This describes the rotation and translation between the two cameras.

- If we have the intrinsic parameters ( **K**, and **K´**) and **R** and **t**, we can find the camera projection matrices **P**, and **P´** $\boxed{\mathbf{P} = \mathbf{K}\,[\,\mathbf{R} \;\; \mathbf{t}]}$

- Decomposing **E to find R and t** can be done by computing its SVD and applying some properties. We will not dig into that – some details in the book.

# Finding the Camera projection Matrices

- The Essential matrix, E, can be decomposed into $[\mathbf{R}_{\{3\times3\}}\mathbf{t}_{\{3\times1\}}]$

describing rotation and translation between the two cameras.

- When that is done we can estimate the camera matrices. The left camera coordinate system is the world system, so no rotation and translation for P:

$$\mathbf{P} = \mathbf{K}[\mathbf{I}_{\{3\times3\}}\mathbf{0}_{\{3\times1\}}]$$

$$\mathbf{P}' = \mathbf{K}'[\mathbf{R}_{\{3\times3\}}\mathbf{t}_{\{3\times1\}}]$$

**ALGORITHM 13.15** Estimate the projection matrices of a pair of internally calibrated cameras

EstimateCameraProjectionMatrices$(\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}_{i=1}^{n}, \mathbf{K}, \mathbf{K}')$

**Input:** $n$ corresponding pairs of points $(x_i, y_i) \leftrightarrow (x'_i, y'_i)$ between two images intrinsic camera parameters $\mathbf{K}$ and $\mathbf{K}'$

**Output:** the $3 \times 4$ camera projection matrices $\mathbf{P}$ and $\mathbf{P}'$

$\mathbf{F} \leftarrow$ EightPointFundamental$(\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}_{i=1}^{n})$

$\mathbf{E} \leftarrow \mathbf{K}'^{\mathsf{T}}\mathbf{F}\mathbf{K}$

$\mathbf{R}, \mathbf{t} \leftarrow$ DecomposeEssentialMatrix$(\mathbf{E})$

$\mathbf{P} \leftarrow \mathbf{K}\,[\mathbf{I}_{\{3\times3\}} \quad \mathbf{0}_{\{3\times1\}}]$

$\mathbf{P}' \leftarrow \mathbf{K}'\,[\mathbf{R} \quad \mathbf{t}]$

**return** $\mathbf{P}, \mathbf{P}'$

# Calibration stereo - summary

Calibration for stereo setup means finding P and P' camera projection matrices:

1) Use for example Zhangs calibration algorithm to find K and K' (each camera alone). This gives us P. $\boxed{\mathbf{P} = \mathbf{K}[\mathbf{I}_{\{3\times3\}}\mathbf{0}_{\{3\times1\}}]}$

2) Need at least 5, typically 8 correspondence points x,x' to be able to estimate the fundamental matrix F

( need to search a larger space. Dont know the epipolar lines yet)

3) Find the essential matrix E from F, K and K'

4) Decompose E to get R and t – this will give us P´ $\boxed{\mathbf{P}' = \mathbf{K}'[\mathbf{R}_{\{3\times3\}}\mathbf{t}_{\{3\times1\}}]}$

# Doing 3D reconstruction

- When we have estimated F and the camera projection matrices P and P’, i.e. calibrated our stereo vision setup, we can use it to reconstruct 3D world:

1. Find corresponding points (**x, x**’). We know the fundamental matrix **F**, so this is a 1D search on the epipolar lines: $\boxed{l' = \mathbf{F}\mathbf{x}}$

2. Estimate the 3D world point **w** from (**x, x’, P** and **P’). How?**

# Estimating 3D Point Coordinates

- Given the camera projection matrices $\mathbf{P}_{\{3x4\}}$ and $\mathbf{P'}_{\{3x4\}}$, the 3D coordinates of a world point **w** can be estimated from its projections onto the two images.

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} \\ \mathbf{p}_2^\mathsf{T} \\ \mathbf{p}_3^\mathsf{T} \end{bmatrix} \quad \text{and} \quad \mathbf{P}' = \begin{bmatrix} \mathbf{p}_1'^\mathsf{T} \\ \mathbf{p}_2'^\mathsf{T} \\ \mathbf{p}_3'^\mathsf{T} \end{bmatrix}$$

- Let $\mathbf{w}_{\{4\times1\}}$ be the homogeneous coordinates of the world point. coordinates of the image point (in pixels) are then given by:

$$(x, y) = \left( \frac{\mathbf{p}_1^\mathsf{T}\mathbf{w}}{\mathbf{p}_3^\mathsf{T}\mathbf{w}}, \frac{\mathbf{p}_2^\mathsf{T}\mathbf{w}}{\mathbf{p}_3^\mathsf{T}\mathbf{w}} \right) \quad \text{and} \quad (x', y') = \left( \frac{\mathbf{p}_1'^\mathsf{T}\mathbf{w}}{\mathbf{p}_3'^\mathsf{T}\mathbf{w}}, \frac{\mathbf{p}_2'^\mathsf{T}\mathbf{w}}{\mathbf{p}_3'^\mathsf{T}\mathbf{w}} \right)$$

- Solve for the unknown (**w**), as shown on notes.

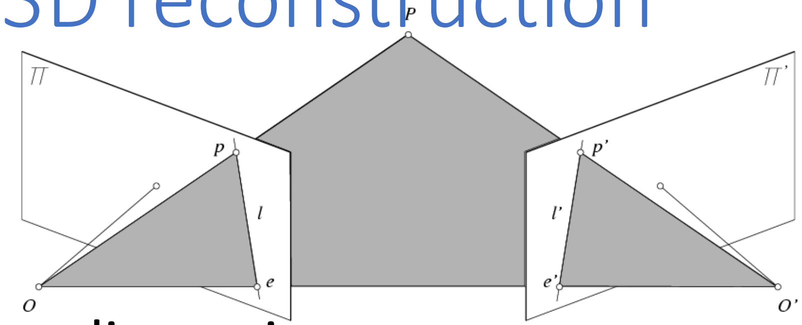
# 3D Reconstruction from Image Pairs - summary

- Find interest points (Harris-Stephens / SIFT )
- Match interest points
- Compute fundamental matrix F
- Calibrate cameras (Zhangs), find K, K´
- Estimate E from F, K and K´, find R, t
- Compute camera matrices P and P’ from K, K´, R and t
- For each matching image points x and x’, compute world point in scene using (x,y), (x´,y´) P and P´ (Use F to help find matching pairs)

Calibrating stereo setup

We can use the fundamental matrix **F** to limit correspondence to 1D search for general stereo camera positions in the same way as is possible for rectified stereo

# Geometry of multiple views – 3D reconstruction




Three points from the topic:

1. Epipolar geometry help finding corresponding points
   - ✓ Epipolar lines limits search space from 2D to 1D
2. Fundamental and essential matrices
   - ✓ Fundamental matrix (F) – relationship between the images ( image coordinates). Essential matrix € – relationship between cameras
3. 3D Reconstruction from camera projection matrices
   - ✓ Can find world point from P and P´ and (x,y)-(x´,y´)