State-conserving one-dimensional cellular automata with radius one

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Abstract

This paper presents a new way of looking at state-conserving one-dimensional cellular automata. Such cellular automata preserve the distribution of states, *i.e.*, the number of cells in each state, throughout the entire evolution of the system. The tools introduced make it possible to fully characterize and enumerate all such cellular automata with radius one, regardless of the number of states. Surprisingly, it turns out that the number of state-conserving one-dimensional cellular automata with radius one and k states is very closely related to the number of labeled directed graphs with k vertices containing no directed path of length 2.

1 Introduction

Cellular automata (CAs) are discrete dynamical systems that provide a mathematical framework for modeling and predicting various kinds of phenomena, especially physical ones, but not only. In particular, CAs are widely used as models for a system of interactions of particles moving in a grid [1, 2]. In this case, it is most often assumed that the particles are indistinguishable, and the state of a cell corresponds to the number of particles in it at a given time step. In addition, one usually demands that particles can neither disappear nor appear, but only move within the grid. These conditions led to the study of number-conserving CAs, for which the sum of the states of all cells does not change at any time step (see, e.g., [3, 4, 5, 6]).

However, very often the assumption that all particles are identical severely restricts the modeling capabilities and it thus becomes necessary to allow for the existence of different types of particles. In this setting, it is assumed that there can be at most one particle in each cell at any given time step, and that different cell states represent different types of particles. Here, too, it is assumed that particles neither appear nor disappear, but only move within the grid. These assumptions result in the class of so-called *state-conserving* CAs, in which the distribution of states is preserved at each time step, *i.e.*, the number of particles of each type remains constant throughout the evolution of the CA (see, e.g., [7, 8, 9, 10]).

Both the property of being number-conserving and the property of being state-conserving have very simple necessary and sufficient conditions formulated in terms of a local rule, resulting from the general conditions for additive invariants given in [11]. Thanks to these conditions it is possible (at least theoretically) to verify whether a given CA is number-conserving or not, or it is feasible to create a number-conserving CA, but usually the enumeration of all number-conserving CAs with a given neighborhood size and state set is beyond the computational capabilities of computers. Denoting the number of number-conserving one-dimensional local rules with neighborhood size m and with state set $Q = \{0, 1, \ldots, k-1\}$ by $\Lambda_{m,k}$, no general formula for $\Lambda_{m,k}$ is known. Moreover, only a few particular values of $\Lambda_{m,k}$ have been obtained through a computer-assisted search. The exception here is the very nice result by Fukś and Sullivan [12], giving a closed-form expression for the sequence $\Lambda_{2,k}$, *i.e*, when the size of the neighborhood is two.

The situation in the case of state-conserving CAs is no better, despite the fact that the state-conserving condition appears to be easier to study, as it is insensitive to relabeling the elements of the considered state set Q (unlike the number-conserving condition, as shown in [13]). Denoting the number of state-conserving one-dimensional local rules with neighborhood size m and with k states by $\Upsilon_{m,k}$, no general formula for $\Upsilon_{m,k}$ is known either. So far, the only results in this direction have been obtained thanks to computing power and do not go beyond the range of eight states. As in the case of number conservation, we can give a formula for $\Upsilon_{2,k}$, but it is not a great result, because regardless of k, we simply have $\Upsilon_{2,k} = 2$. Indeed, it is easy to see that when the neighborhood size is two, there are only two state-conserving one-dimensional CAs with a given state set: the identity rule and the shift rule. Thus all of them are trivial. The situation is different when we expand the neighborhood to three cells, *i.e.*, when we consider CAs with radius one. So far, the first seven terms of the sequence $(\Upsilon_{3,k})_{k=2}^{\infty}$ are known: $5, 15, 89, 843, 11\,645, 227\,895, 6\,285\,809$ (see [9]), which suggests that as k increases, the value $\Upsilon_{3,k}$ increases significantly. It was also possible to test the rules found from the point of view of reversibility. Unfortunately, it appeared that there are no non-trivial reversible ones (only the identity rule and the shift rules).

The goal of this paper was to find a general formula for $(\Upsilon_{3,k})_{k=2}^{\infty}$ and to provide a characterization of state-conserving one-dimensional CAs with radius one that would, for example, allow us to enumerate all such CAs and to pinpoint which of them are reversible. Surprisingly, both problems turned out to be very simple, thanks to the identification of appropriate tools. In particular, it appeared that $\Upsilon_{3,k} = 2 + s_k$, where s_k is the number of labeled directed graphs with k vertices containing no directed path of length 2 (s_k is also the number of graded posets on a k-set with rank less than 2, thus, the formula for s_k has been known for over 50 years – see [14], for example). Moreover, it turned out that radius one is too small to enable the existence of non-trivial reversible state-conserving one-dimensional CAs, regardless of k. Unlike the study of some other properties of CAs, where the authors focus on finding the least complicated algorithms possible to determine whether a given CA has the examined property or not (see, for example, [15, 16]), in our case we know exactly which of the considered CAs are reversible.

2 Preliminaries

In this paper, we focus on one-dimensional CAs with radius one only. However, in order to be able to discuss certain aspects of the tools introduced here, we need slightly more general definitions, allowing any radius.

For a positive integer n, the n-cell grid $\mathcal{G}_n$ refers to the grid of cells numbered by $0, 1, \ldots, n-1$, *i.e.*,

$$\mathcal{G}_n = \{0, 1, \ldots, n-1\},$$

and it is assumed that the last cell $n-1$ is adjacent to the first cell 0 (*i.e.*, we consider periodic boundary conditions).

Let Q be any finite set of states. A configuration of length n is any mapping from the grid $\mathcal{G}_n$ to Q. The set of all possible configurations of length n is denoted by X_n and is identified with Q^n. The state of cell $i \in \mathcal{G}_n$ in a configuration $\mathbf{x} \in X_n$ is denoted by x_i. The set of all finite configurations is denoted by X^*, *i.e.*,

$$X^* = \bigcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} Q^n \,.$$

For a given state $q \in Q$, let $\delta_q : X^* \to \mathbb{N}$, with $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the function that counts the number of appearances of state q in $\mathbf{x}$, *i.e.*, for any $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \in X_n$:

$$\delta_q(\mathbf{x}) = \delta_{q,x_0} + \delta_{q,x_1} + \ldots + \delta_{q,x_{n-1}} \,,$$

where $\delta_{q,x}$ denotes the Kronecker delta, *i.e.*, $\delta_{q,x} = 1$ if $x = q$ and 0 otherwise.

In each subsequent time step, each cell $i \in \mathcal{G}_n$ updates its state based on its current state and the states of its l left and r right neighbors, where $l, r \in \mathbb{N}$. Formally speaking, there is a local rule $f : \{0,1\}^{l+1+r} \to \{0,1\}$, which generates the global rule $F : X^* \to X^*$ acting on each X_n in the following way: if $\mathbf{x} \in X_n$, then $F(\mathbf{x}) \in X_n$ and

$$F(\mathbf{x})_i = f(x_{i-l}, x_{i-l+1}, \ldots, x_{i+r-1}, x_{i+r}) \,.$$

The number $m = l + 1 + r$ is called neighborhood size and the function f is called an m-input local rule. Due to the nature of CAs with periodic boundary conditions, w.l.o.g., we can set the values of l and r arbitrarily (yet keeping the value $m = l + 1 + r$ unchanged). Usually, it is assumed that l and r are as close as possible, thus, depending on the parity of m, it holds that $l = r$ (then it is said that f has radius r) or $l = r - 1$ (then it is said that f has radius $r - \frac{1}{2}$).

Roughly speaking, being state-conserving simply means that for each initial configuration the number of cells in each state is constant throughout the entire evolution of the system.

Definition 1. *A global rule $F : X^* \to X^*$ is state-conserving if for every state $q \in Q$ it holds that $\delta_q(F(\mathbf{x})) = \delta_q(\mathbf{x})$ for each configuration $\mathbf{x} \in X^*$.*

We will also say that a local rule is state-conserving, having in mind that its corresponding global rule is state-conserving.

Note that the property of being state-conserving is robust to a relabeling of the elements of the considered state set Q. Strictly speaking, suppose that two finite equinumerous sets Q and $\widetilde{Q}$ are given and let $\phi : Q \to \widetilde{Q}$ be a bijection. Then for each one-dimensional CA with state set Q and m-input local rule, one can define a CA with state set $\widetilde{Q}$ by using the following m-input local rule:

$$\widetilde{f}(\phi(q_1), \phi(q_2), \ldots, \phi(q_m)) = \phi(f(q_1, q_2, \ldots, q_m)) \,.$$

These CAs are indistinguishable in the following sense: if we choose some collection of colors to represent the states of $\widetilde{Q}$ and then for every state $q \in Q$ we use the same color as for the state $\phi(q)$, then the corresponding space-time diagrams for the CAs will be exactly the same. This implies that f is state-conserving if and only if $\widetilde{f}$ is state-conserving.

Remark 1. *W.l.o.g., we may always assume that the state set Q equals $\{0, 1, \ldots, k-1\}$ for some integer $k \geq 2$.*

We start with the following very useful fact (see, e.g., [9]).

Lemma 1. *Let f be an m-input local rule. If f is state-conserving, then it is internal,* i.e., *$f(q_1, q_2, \ldots, q_m) \in \{q_1, q_2, \ldots, q_m\}$, for each $q_1, q_2, \ldots, q_m \in Q$. In particular, each state is quiescent,* i.e., *for each $q \in Q$ it holds that $f(q, q, \ldots, q) = q$.*

This obvious lemma leads us to argue that a neighborhood size of 2 does not suffice to define a non-trivial state-conserving CA.

Theorem 1. *For any state set Q there are only two 2-input local rules that are state-conserving: the identity rule and the shift (to the left) rule.*

Proof. Let x and y be any two different states in Q. For $\mathbf{x} = (x, y) \in X_2$, we have $F(\mathbf{x}) = (f(x, y), f(y, x))$, so, if F is state-conserving, then $f(x, y) = x$ and $f(y, x) = y$, or $f(x, y) = y$ and $f(y, x) = x$. We now claim that either for all $x, y \in Q$ it holds that $f(x, y) = x$ or for all $x, y \in Q$ it holds that $f(x, y) = y$. To this end, it suffices to show that if for two different states x, y it holds that $f(x, y) = x$, then already for any (x', y') we have $f(x', y') = x'$.

Thus, suppose that for two different states $x, y \in Q$ we have $f(x, y) = x$ (which implies $f(y, x) = y$). Let $y' \in Q \backslash \{x\}$. Since $F(x, y', y) = (f(x, y'), f(y', y), y)$, it must hold that $f(x, y') = x$, otherwise the state x will disappear. Now, let $x' \in Q$. Since $F(x, x', y') = (f(x, x'), f(x', y'), f(y', x)) = (x, f(x', y'), y')$, it must hold that $f(x', y') = x'$. □

The above result shows that in order to obtain non-trivial state-conserving CAs, one must consider local rules with a neighborhood of at least three cells. Therefore, the purpose of the remainder of this paper is to fully characterize all state-conserving one-dimensional CAs whose local rule has radius one.

3 A characterization of state-conserving one-dimensional CAs with radius one

In this section, we will consider only one-dimensional CAs with radius one, *i.e.*, with a neighborhood of size three, even if it is not explicitly written.

In the binary case, *i.e.*, when $Q = \{0, 1\}$, we are dealing with the so-called Elementary CAs (ECAs), widely disseminated by Wolfram (see, e.g., [17, 18, 19]). As is well known, there are only five state-conserving ECAs (in the binary case being state-conserving is the same as being number-conserving). We present these rules by means of their Wolfram code and their lookup table (LUT) in Table 1.

		111	110	101	100	011	010	001	000
ECA204	identity rule	1	1	0	0	1	1	0	0
ECA170	shift-left rule	1	1	1	1	0	0	0	0
ECA240	shift-right rule	1	0	1	0	1	0	1	0
ECA184	traffic-right rule	1	0	1	1	1	0	0	0
ECA226	traffic-left rule	1	1	1	0	0	0	1	0

Table 1: The only ECAs that are state-conserving.

The CA using the identity rule keeps any initial configuration unchanged for all time steps. Also the shift rules do not change the pattern of a configuration, but move it to the left or to the right. The traffic rules, which are often used as the simplest models of road traffic flow, are a bit more interesting. Interpreting the states 1 and 0 as 'car' and 'empty space', respectively, we can describe their dynamics as follows: cars are moving only if there is an unoccupied space in front of them. The mechanics of these rules are very well explained, for example, in [20] (in particular, it is known that the traffic rules are not reversible). However, for our purposes, a completely different characterization of the traffic rules will be turn out to be far more useful, namely a description in the language of swaps.

By a swap $\langle p, q\rangle$ (with $p, q \in Q$ and $p \neq q$) we refer to the action that each occurrence of the pattern (p, q) in the configuration is converted by the CA into the pattern (q, p). More precisely, if $\mathbf{x} \in X_n$ and, for some $i \in \{0, 1, \ldots, n-1\}$, we have $x_i = p$ and $x_{i+1} = q$, then $F(\mathbf{x})_i = q$ and $F(\mathbf{x})_{i+1} = p$. In other words, if a particle of type p has as its right neighbor a particle of type q, then the two particles simply swap places. It is easy to see that the traffic-right rule is simply the swap $\langle 1, 0\rangle$, while the traffic-left rule is simply the swap $\langle 0, 1\rangle$. Note that the swaps $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$ cannot coexist in a state-conserving ECA, since these patterns are not disjoint and, for example, the configuration 010 should then go to 101.

Now, let $f : Q^3 \to Q$ be the local rule of a state-conserving one-dimensional CA with radius one and let F be the corresponding global rule. For any two different states $p, q \in Q$, we define $F_{p,q}$ as the restriction of F to $X_{p,q}$, where $X_{p,q}$ denotes the subset of X^* consisting only of those configurations that do not contain states other than p and q, *i.e.*, $X_{p,q} = \bigcup_{n=1}^{\infty} \{p, q\}^n$. Note that $F_{p,q}$ is the global rule of the state-conserving one-dimensional CA with radius one induced by the local rule $f_{p,q} : \{p, q\}^3 \to \{p, q\}$ being the restriction of f to $\{p, q\}^3$. From this simple observation we immediately get the following fact.

Theorem 2. *Let F be the global rule of a one-dimensional state-conserving CA with radius one and state set Q. For any two different states $p, q \in Q$ the global rule $F_{p,q}$ is either the shift-left rule, the shift-right rule, the identity rule, the swap $\langle p, q\rangle$ or the swap $\langle q, p\rangle$.*

Moreover, the following is obvious.

Remark 2. *If for some $p, q \in Q$ the restriction $F_{p,q}$ is a swap, then $F_{p,q}$ is not reversible (on $X_{p,q}$), hence also F is not reversible (on X^*).*

Table 2 presents all 15 one-dimensional state-conserving ternary CAs (*i.e.*, $Q = \{0, 1, 2\}$) with radius one, together with the corresponding restrictions $F_{0,1}$, $F_{0,2}$ and $F_{1,2}$.

lookup table	$F_{0,1}$	$F_{0,2}$	$F_{1,2}$
210210210210210210210210210	Sh-L	Sh-L	Sh-L
210222222210111000210111000	Id	$\langle 2, 0\rangle$	$\langle 2, 1\rangle$
212222000212111000212111000	Id	Id	$\langle 2, 1\rangle$
212222010212111010212000010	$\langle 0, 1\rangle$	Id	$\langle 2, 1\rangle$
220110222220110111220110000	$\langle 1, 0\rangle$	$\langle 2, 0\rangle$	Id
220111222220111000220111000	Id	$\langle 2, 0\rangle$	Id
222110000222110111222110000	$\langle 1, 0\rangle$	Id	Id
222111000222111000222111000	Id	Id	Id
222111010222111010222000010	$\langle 0, 1\rangle$	Id	Id
222111200222111200000111200	Id	$\langle 0, 2\rangle$	Id
222111210222111210000000210	$\langle 0, 1\rangle$	$\langle 0, 2\rangle$	Id
222210000111210111222210000	$\langle 1, 0\rangle$	Id	$\langle 1, 2\rangle$
222211000111211000222211000	Id	Id	$\langle 1, 2\rangle$
222211200111211200000211200	Id	$\langle 0, 2\rangle$	$\langle 1, 2\rangle$
222222222111111111000000000	Sh-R	Sh-R	Sh-R

Table 2: The only one-dimensional ternary CAs with radius one that are state-conserving with their corresponding restrictions $F_{0,1}$, $F_{0,2}$ and $F_{1,2}$. Sh-L, Sh-R, Id denote the shift-left, shift-right and identity rule, respectively.

Looking at Table 2, two important observations can be made. First, different CAs in this table have different sequences $(F_{0,1}, F_{0,2}, F_{1,2})$, *i.e.*, the sequence $(F_{0,1}, F_{0,2}, F_{1,2})$ uniquely defines the corresponding rule. Second, if one of $F_{0,1}$, $F_{0,2}$, $F_{1,2}$ is a shift rule, then all of them are the same shift rule. It turns out that both of these regularities hold for any state set Q.

We start with proving that knowledge of all $f_{p,q}$ uniquely determines f.

Theorem 3. *Let $f : Q^3 \to Q$ be the local rule of a state-conserving one-dimensional CA with radius one. If we know all corresponding local rules $f_{p,q}$ (for any two different states $p, q \in Q$), then we know the entire f.*

Proof. Let p, q, s be any three states in Q and let us consider the configuration $\mathbf{x} = (p, p, q, s, s) \in X_5$. In the next time step, we obtain

$$F(\mathbf{x}) = (f(s, p, p), f(p, p, q), f(p, q, s), f(q, s, s), f(s, s, p)) .$$

Since the values $f(s, p, p) = f_{s,p}(s, p, p)$, $f(p, p, q) = f_{p,q}(p, p, q)$, $f(q, s, s) = f_{q,s}(q, s, s)$ and $f(s, s, p) = f_{s,p}(s, s, p)$ are known, we can calculate $f(p, q, s)$ from the fact that f is state-conserving. $\square$

Now we prove that if at least one $F_{p,q}$ is a shift rule, then F is the shift rule in the same direction. Again, it is more convenient to formulate this result in terms of the local rule.

Theorem 4. *Let $f : Q^3 \to Q$ be the local rule of a state-conserving one-dimensional CA with radius one. If there is a pair $p, q \in Q$ such that the local rule $f_{p,q}$ is the shift-left (resp. shift-right) rule, then f is the shift-left (resp. shift-right) rule,* i.e., *for any $x, y, z \in Q$ it holds that $f(x, y, z) = z$ (resp. $f(x, y, z) = x$).*

Proof. Let $f_{p,q}$ be the shift-right rule. It suffices to show that for any $x, y, z \in Q$ it holds that $f(x, y, z) = x$, knowing only eight values of f, namely

$$f(p, p, p) = p, \quad f(p, p, q) = p, \quad f(p, q, p) = p, \quad f(p, q, q) = p,$$

$$f(q, p, p) = q, \quad f(q, p, q) = q, \quad f(q, q, p) = q, \quad f(q, q, q) = q.$$

Let $x \in Q \setminus \{p, q\}$. Since

$$F(p, p, p, x, q, q, q,) = (q, p, f(p, p, x), f(p, x, q), f(x, q, q), q, q) ,$$

the values $f(p, p, x)$, $f(p, x, q)$, $f(x, q, q)$ should correspond to p, p, x, in some order. According to Lemma 1, it holds that $f(x, q, q) = x$, and thus $f(p, p, x) = f(p, x, q) = p$.

So far, we have shown that for any $x \in Q$ it holds that

$$f(p, p, x) = p, \quad f(p, x, q) = p, \quad f(x, q, q) = x. \tag{1}$$

Now, let $x \in Q$ and $y \in Q \setminus \{p, q\}$. According to Eq. (1), it holds that

$$F(p, p, x, y, q, q) = (q, p, f(p, x, y), f(x, y, q), y, q) .$$

Hence, the values $f(p, x, y)$, $f(x, y, q)$ should correspond to p, x. The only possibility is that

$$f(p, x, y) = p, \quad f(x, y, q) = x. \tag{2}$$

Finally, let $x, y \in Q$ and $z \in Q \setminus \{p, q\}$. Then

$$F(p, x, y, z, q, q) = (q, p, f(x, y, z), y, z, q) ,$$

which yields $f(x, y, z) = x$ and concludes the proof in the case of the shift-right rule. The case of the shift-left rule is proven analogously. $\square$

To describe all other state-conserving CAs, *i.e.*, those that are not shift rules, we introduce the following notation. By $\mathcal{S}_Q$ we denote the set of all possible swaps:

$$\mathcal{S}_Q = \{\langle p, q\rangle \mid p, q \in Q \wedge p \neq q\} .$$

A subset S of $\mathcal{S}_Q$ is called *admissible* if it does not contain two swaps $\langle p_1, q_1\rangle$ and $\langle p_2, q_2\rangle$ such that $q_1 = p_2$. This implies that the patterns of the swaps constituting S are pairwise disjoint (do not overlap). Note that the empty subset is admissible. We denote the number of admissible subsets of a state set $\mathcal{S}_Q$ with k states by s_k. The following fact follows immediately from the definition.

Theorem 5. *Let Q be a finite state set. There is one-to-one correspondence between the admissible subsets of $\mathcal{S}_Q$ and the state-conserving one-dimensional CAs with state set Q and with radius one that are not shift rules. As the consequence, the number of state-conserving one-dimensional CAs with k states and radius one equals $2 + s_k$.*

To obtain a formula for s_k, we show that for a given k-element set Q (for convenience, let $Q = \{0, 1, \dots, k-1\}$), there is a one-to-one correspondence between the set of admissible subsets of $\mathcal{S}_Q$ and the set $\mathcal{G}_k$ of labeled directed graphs with vertices $\{0, 1, \dots, k-1\}$ having no directed path of length 2.

An admissible subset $S \subset \mathcal{S}_Q$ can be represented as a directed graph $G_S = (Q, S)$ with the state set Q as the vertices and the swaps in S as the arcs. Note that the definition of an admissible subset implies that the graph G_S does not contain a directed path of length 2, so $G_S \in \mathcal{G}_k$. On the other hand, if G is any directed graph in $\mathcal{G}_k$, then taking its set of arcs we obtain a subset of $\mathcal{S}_Q$ that does not contain two ordered pairs (p_1, q_1) and (p_2, q_2) such that $q_1 = p_2$, given the assumption that G does not contain a path of length 2. Hence, the obtained subset is admissible.

For example, Figure 1 displays the graph $G_S = (Q, S)$ for the state set $Q = \{0, 1, \dots, 6\}$ and the admissible subset $S = \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,4\rangle, \langle 3,1\rangle, \langle 3,4\rangle, \langle 6,4\rangle\}$.

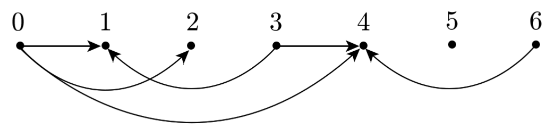

Figure 1: The directed graph $G \in \mathcal{G}_7$ for the set $Q = \{0, 1, 2, \dots, 6\}$ and the admissible subset $S = \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,4\rangle, \langle 3,1\rangle, \langle 3,4\rangle, \langle 6,4\rangle\}$.

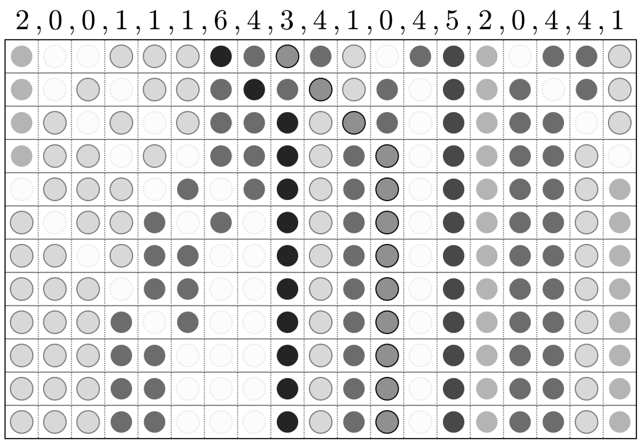

Figure 2: Space time diagram of the CA from Fig. 1. The initial configuration is given above the first row of the diagram.

Thus $s_k = |\mathcal{G}_k|$. The sequence $(|\mathcal{G}_k|)_{k=0}^{\infty}$ is known and it is denoted by A001831 in [21] . Its first few elements are given by:

$$1, 1, 3, 13, 87, 841, 11643, 227893, 6285807, 243593041, \ldots .$$

The following theorem gives the general formula for $|\mathcal{G}_k|$. The proof of this fact is a simple exercise in combinatorics, but we present it for readers' convenience.

Theorem 6. $|\mathcal{G}_0| = 0$ *and for* $k \geq 1$ *it holds that*

$$|\mathcal{G}_k| = \sum_{i=0}^{k} \binom{k}{i} (2^{k-i} - 1)^i.$$

Proof. Let $\mathcal{G}_k^{(i)}$ denote the subset of $\mathcal{G}_k$ containing graphs having exactly i vertices with outgoing arcs, *i.e.*, with a positive outdegree (e.g., the graph shown in Figure 1 belongs to $\mathcal{G}_7^{(3)}$ because it has exactly three vertices with outgoing arcs). Then $\mathcal{G}_k^{(0)} \cup \mathcal{G}_k^{(1)} \cup \ldots \cup \mathcal{G}_k^{(k)}$ establishes a partition of $\mathcal{G}_k$, so $|\mathcal{G}_k| = \sum_{i=0}^{k} |\mathcal{G}_k^{(i)}|$.

Each graph in $\mathcal{G}_k^{(i)}$ has k vertices and exactly i of them have a positive outdegree. These vertices can be chosen from the set $\{0, 1, \ldots, k-1\}$ in $\binom{k}{i}$ ways. Having fixed these i vertices, say $v_1, \ldots, v_i$, we know that each of them has at least one outgoing arc. However, since there cannot be any directed path of length 2, the arcs outgoing from $v_1, \ldots, v_i$ cannot go to $v_1, \ldots, v_i$. Thus the arcs outgoing from v_1 end in some vertices forming a non-empty subset of $\{0, 1, \ldots, k-1\} \setminus \{v_1, \ldots, v_i\}$. The number of such subsets equals $2^{k-i} - 1$. The same holds for v_2 and so on. Therefore, $|\mathcal{G}_k^{(i)}| = \binom{k}{i} \left(2^{k-i} - 1\right)^i$ and finally

$$|\mathcal{G}_k| = \sum_{i=0}^{k} |\mathcal{G}_k^{(i)}| = \sum_{i=0}^{k} \binom{k}{i} \left(2^{k-i} - 1\right)^i,$$

which concludes the proof. □

As a consequence, we get the following corollary.

Corollary 1. *Let* $\Upsilon_{3,k}$ *denote the number of state-conserving one-dimensional local rules with* k *states and with radius one. Then for* $k \geq 2$*, it holds that*

$$\Upsilon_{3,k} = 2 + \sum_{i=0}^{k} \binom{k}{i} \left(2^{k-i} - 1\right)^i.$$

In particular, the first few elements of the sequence $(\Upsilon_{3,k})_{k=2}^{\infty}$ *read as follows*

$$5, 15, 89, 843, 11\,645, 227\,895, 6\,285\,809, 24\,3593\,043, 13\,262\,556\,723, 1\,014\,466\,283\,293, \ldots$$

Corollary 2. *Among all state-conserving one-dimensional CAs with* k *states and with radius one there are only three reversible ones: the shift-left rule, the shift-right rule and the identity rule (for which the corresponding admissible subset is the empty set).*

4 Conclusions

In this paper, we have presented the results of our investigation of state-conserving one-dimensional CAs that can be used to model the movement of distinguishable particles between the cells of some regular grid. Assuming that in subsequent time steps the particles move to an adjacent cell only, and that there can be at most one particle at a time in every cell, we obtain a rich family of such CAs. However, regardless of k, there are no non-trivial reversible ones among them – radius one is too small (similarly, as it turned out that to design a non-trivial reversible number-conserving d-dimensional CAs with the von Neumann neighborhood, regardless of d, three state are too few [22]). This makes it clear that the study of state-conserving CAs should be extended to CAs with larger radii, or even to CAs in higher dimensions. We plan to take up these directions in the near future.

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