Recent insights into number-conserving cellular automata

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Abstract This chapter provides a summary of recent results on number-conserving cellular automata, along with a general introduction to the field. We cover results in arbitrary dimensions, with particular attention to the von Neumann neighborhood, while also addressing specific classes of cellular automata, such as Wolfram's well-known elementary cellular automata and non-uniform variants thereof, affine continuous cellular automata and cellular automata on triangular grids.

Key words: Affine continuous cellular automata, cellular automata, non-uniform cellular automata, number conservation, triangular cellular automata

1 Introduction

Number-Conserving Cellular Automata (NCCAs), also called *conservative cellular automata*, constitute a particularly interesting class of Cellular Automata (CAs), in which the states are numbers, and the global rule has the

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special feature of preserving the sum of the states upon every update of the states (and this for all configurations, which are supposed to be either periodic or finite). In other words, we focus on CAs that have the important property — from the point of view of applications — that the sum of the states in any finite configuration remains constant throughout the evolution of the automaton. This property can be related to the natural behavior of many physical systems that are governed by traditional conservation laws (*e.g.*, conservation of mass or energy). The number conservation property can be naturally extended to non-classical CAs (*e.g.*, non-uniform CAs and Continuous CAs (CCAs)) in which the concept "sum of states" can be easily defined.

Historically, when CAs were first introduced in the early works of Ulam and von Neumann [52, 53], number conservation was not considered. The initial focus was mainly on reversibility and related properties. Yet, in the 80s and 90s of the last century, pioneering works on *conservation laws* in CAs were published by multiple authors, most notably including Fredkin and Toffoli [18], Nagel and Schreckenberger [40], Tamotsu Kohyama [33], and Kotze and Steeb [34]. The most important contribution among these early studies on NCCAs has been authored by Hattori and Takesue [24]. Later works by Takesue [50], Pivato [43], Taati [49] and the lesser known, yet important paper by Baird and Fagin [1], continued the exploration and study of conservation laws in CAs, clearly showing their importance and rich, non-trivial properties. Over the years, many different views on and definitions of conservation laws for CAs (in some sense different or broader than the one given for NCCAs) have been proposed. Yet, the number conservation property considered here is the one still gaining the most attention and thus we limit our attention to it. The motivation for it is purely practical, as number conservation is implicitly assumed in a large number of modeling tasks.

In this paper, we focus our attention on some of the most recent and important theoretical results on NCCAs. For the sake of readability, we start by formally defining NCCAs, formulating their main properties and briefly outlining the classical results. The key part of the paper summarizes the results obtained in recent years. Yet, we do not aim to cover the entire domain exhaustively and thus the choice of the problems covered here is determined by our personal area of expertise. Note that there is also a number of important topics in the study of NCCAs that are absent in our review, *e.g.*, computational complexity, universality or relations to specific laws from theoretical physics or results from theoretical computer science. Importantly, aside from summarizing the main findings, we also formulate and briefly discuss some of the key open questions in the field. We hope that this survey will motivate further research in this direction and will help to uncover new properties of NCCAs.

2 Basic definitions of NCCAs

Roughly speaking, a CA is a discrete dynamical system that consists of a regular grid of cells that in consecutive time steps update their states depending on the states of their neighbors, according to some local rule.

More formally, a d-dimensional CA is a quadruple $(\mathcal{C}, Q, V, f)$, where

- $\mathcal{C} \subseteq \mathbb{Z}^d$ is the cellular space;
- Q is the set of states and for our propose we will assume that Q is represented by numbers (*i.e.*, $Q \subset \mathbb{R}$) and that $0 \in Q$;
- $V = (\overrightarrow{\mathbf{v}_1}, \overrightarrow{\mathbf{v}_2}, \ldots, \overrightarrow{\mathbf{v}_m})$ is the neighborhood vector, which means that, for any cell $\mathbf{i} \in \mathcal{C}$, the cells $\mathbf{i} + \overrightarrow{\mathbf{v}_1}, \mathbf{i} + \overrightarrow{\mathbf{v}_2}, \ldots, \mathbf{i} + \overrightarrow{\mathbf{v}_m}$ are its neighbors (it is usually assumed that the zero vector belongs to V, *i.e.*, that the cell itself belongs to its neighborhood);
- $f : Q^m \to Q$ is the local rule: the state of a cell $\mathbf{i}$ in the next time step is given by $f(x_1, x_2, \ldots, x_m)$, where $x_1, x_2, \ldots, x_m$ are the current states of the cells $\mathbf{i} + \overrightarrow{\mathbf{v}_1}, \mathbf{i} + \overrightarrow{\mathbf{v}_2}, \ldots, \mathbf{i} + \overrightarrow{\mathbf{v}_m}$, respectively. Such a local rule is referred to as an "m-input local rule" when emphasizing that the neighborhood size is important.

Usually, the cellular space is infinite and corresponds to the entire space $\mathbb{Z}^d$ or is finite and corresponds to some d-dimensional cuboid of size $\mathbf{N} = (n_1, n_2, \ldots, n_d)$ with periodic boundary conditions, *i.e.*,

$$\begin{aligned} \mathcal{C} &= \mathcal{C}_{\mathbf{N}} = (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_d\mathbb{Z}) \\ &= \{0, 1, \ldots, n_1 - 1\} \times \{0, 1, \ldots, n_2 - 1\} \times \cdots \times \{0, 1, \ldots, n_d - 1\}. \end{aligned}$$

The most commonly used set of states is $\{0, 1, \ldots, k-1\}$, for some integer $k \geqslant 2$, and in this case the CA is called "k-ary" (binary, ternary, quaternary, quinary, and so on).

As for the neighborhood, among the many different ones that could be adopted (see, *e.g.*, [70]), the two most popular ones are the Moore neighborhood and the von Neumann neighborhood. They differ in terms of the metric considered in $\mathbb{Z}^d$: the Moore neighborhood is defined by the Chebyshev distance

$$\mathrm{dist}_{\mathrm{Ch}}(\mathbf{i}, \mathbf{j}) = \max_{1 \leqslant k \leqslant d} |i_k - j_k| , \tag{1}$$

while the von Neumann neighborhood is defined by the Manhattan distance:

$$\mathrm{dist}_{\mathrm{M}}(\mathbf{i}, \mathbf{j}) = \sum_{k=1}^{d} |i_k - j_k| , \tag{2}$$

where $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in \mathcal{C}$ and $\mathbf{j} = (j_1, j_2, \ldots, j_d) \in \mathcal{C}$. From this point of view, these neighborhoods of a cell $\mathbf{i}$ can be seen as the closed unit balls centered at $\mathbf{i}$, *i.e.*, consisting of those cells whose distance (Chebyshev or Manhattan) to $\mathbf{i}$ does not exceed the radius r ($r = 1$ being the most popular

choice). While the Moore neighborhood has a very simple structure (each ball is a d-dimensional cube), the von Neumann neighborhood is a much more interesting and difficult object to study. In the rest of the paper, in the case of the von Neumann neighborhood, we will only consider radius $r = 1$, even if this is not explicitly stated.

A *configuration* is any mapping from the grid $\mathcal{C}$ to Q; the set of all configurations is denoted by $X = Q^{\mathcal{C}}$. When considering finite grids, often not just one specific grid is considered, but all finite grids together and then the set of configurations has a slightly more complicated structure than $X_{\mathbf{N}} = Q^{\mathcal{C}_{\mathbf{N}}}$, namely:

$$X^* = \bigcup_{\mathbf{N}\in\mathbb{N}^d} X_{\mathbf{N}}\,.$$

The latter case is equivalent to considering all spatially periodic configurations on the entire grid $\mathbb{Z}^d$ (see, *i.e.*, [31] for more information). Unless this leads to confusion, we will simply write X to denote the set of configurations under consideration. The state of a cell $\mathbf{i} \in \mathcal{C}$ in a configuration $\mathbf{x} \in X$ is denoted by $x_{\mathbf{i}}$. A configuration $\mathbf{x} \in X$ is called *finite* if at most finitely many cells are assigned a non-zero state, *i.e.*, the set $\{\mathbf{i} \in \mathcal{C} \mid x_{\mathbf{i}} \neq 0\}$ is finite.

A given local rule f induces a *global rule* $F : X \to X$ defined for any $\mathbf{x} \in X$ and $\mathbf{i} \in \mathcal{C}$ as follows

$$F(\mathbf{x})_{\mathbf{i}} = f(x_{\mathbf{i}+\overrightarrow{\mathbf{v}_1}}, x_{\mathbf{i}+\overrightarrow{\mathbf{v}_2}}, \ldots, x_{\mathbf{i}+\overrightarrow{\mathbf{v}_m}})\,.$$

Usually, the global rule F is identified with the cellular automaton.

Generally, a number-conserving CA preserves the sum of all states in any finite configuration throughout the evolution of the automaton, a property that is usually defined in one of the following ways.

Definition 1 (**Finite NC**) A local rule f is *finite number-conserving* if for every finite configuration $\mathbf{x} \in X$ it holds that $\sum_{\mathbf{i}\in\mathcal{C}} F(\mathbf{x})_{\mathbf{i}} = \sum_{\mathbf{i}\in\mathcal{C}} \mathbf{x}_{\mathbf{i}}$.

Definition 2 (**Periodic NC**) A local rule f is *periodic number-conserving* if for every configuration $\mathbf{x} \in X^*$ it holds that $\sum_{\mathbf{i}\in\mathcal{C}} F(\mathbf{x})_{\mathbf{i}} = \sum_{\mathbf{i}\in\mathcal{C}} \mathbf{x}_{\mathbf{i}}$.

Definition 3 (**NC on a given finite grid**) A local rule f is *number-conserving* on a given grid $\mathcal{C}_{\mathbf{N}}$ if for every configuration $\mathbf{x} \in X_{\mathbf{N}}$ it holds that $\sum_{\mathbf{i}\in\mathcal{C}} F(\mathbf{x})_{\mathbf{i}} = \sum_{\mathbf{i}\in\mathcal{C}} \mathbf{x}_{\mathbf{i}}$.

One can also consider all configurations, not necessarily finite ones only, and extend the definition of number conservation. Durand *et al.* [11] discuss different ways of obtaining such extension. However, as it turns out, all these definitions have been shown to be equivalent, with the proviso that, as for Definition 3, $\mathbf{N}$ must be sufficiently large compared to the size of the neighborhood of the local rule f (see, *e.g.*, [11, 4]). If the state set contains not only natural numbers, then the term *density conservation* is preferred. This is the case for CCAs, for example.

The simplest but non-trivial CAs are Elementary CAs (ECAs) – one-dimensional binary 3-input CAs, widely disseminated by Wolfram (see, *e.g.*, [54, 55, 56, 57]). The local rule of an ECA, also called a Wolfram rule, is a function from $\{0,1\}^3$ to $\{0,1\}$ and is usually given by a string $l_7l_6l_5l_4l_3l_2l_1l_0$:

$$f(1,1,1)f(1,1,0)f(1,0,1)f(1,0,0)f(0,1,1)f(0,1,0)f(0,0,1)f(0,0,0)\,,$$

or by a lookup table (LUT) with entries $l_7, l_6, l_5, l_4, l_3, l_2, l_1, l_0$ (see, *e.g.*, Table 1) or by the abbreviation ECA followed by the unique number from 0 up to 255 in the Wolfram code obtained by converting the above binary string to decimal notation [23]. Among all 256 ECAs there are only five number-conserving ones: the identity rule, two shift rules and two traffic rules (see Table 1). The dynamics of these rules is well understood (see, *e.g.*, [21]).

f	lookup table of f								rule name
	111	110	101	100	011	010	001	000	
ECA170	1	0	1	0	1	0	1	0	shift-left rule
ECA184	1	0	1	1	1	0	0	0	traffic-right rule
ECA204	1	1	0	0	1	1	0	0	identity rule
ECA226	1	1	1	0	0	0	1	0	traffic-left rule
ECA240	1	1	1	1	0	0	0	0	shift-right rule

Table 1 The list of number-conserving ECAs with their lookup tables.

3 Characterization of NCCAs

Since their inception, some elegant characterization of NCCAs has been sought after, and indeed many insightful results have been obtained in this regard. Perhaps the most important results are necessary and sufficient conditions expressed in terms of the local rule, where the formulas are inspired by the results of Hattori and Takesue [24] or Takesue [50] concerning general additive conserved quantities.

In one dimension, necessary and sufficient conditions for a CA to be number-conserving are given by Boccara and Fukś in [3] for binary CAs and more generally in [4] for k-ary CAs. Next, Durand *et al.* [11] state necessary and sufficient conditions for a k-ary CA to be number-conserving in two or more dimensions. Furthermore, Moreira [38] extends the work of Durand *et al.* to any finite subset of $\mathbb{Z}$ as set of states. The conditions in [11] and [38] are explicitly addressing the Moore neighborhood (of any radius). Although these conditions can be used for any other neighborhood (as every neighborhood is a subset of a Moore neighborhood with a large enough radius), in some cases

it just is not worth it. In particular, for $d > 1$, it is not advisable to consider the von Neumann neighborhood as a subset of the Moore neighborhood (with radius one) as the first one has only $2d + 1$ cells, while the second one has as many as 3^d cells. For this reason, it makes sense to look for necessary and sufficient conditions for d-dimensional CAs to be number-conserving, dedicated specifically to the von Neumann neighborhood.

Such conditions were stated by Wolnik *et al.* [61] and they hold for any state set $Q \subset \mathbb{R}$, irrespective of whether it is finite or not. The main idea of that paper was to reduce the expression of the condition to the simplest possible neighborhood configurations: *monomers* (there is at most one non-zero state) and *dimers* (there are at most two non-zero states)[1]. This was possible since the intersection of the von Neumann neighborhoods of two different cells consists of at most two cells. In fact, the mentioned work does not present just one, but an entire family of necessary and sufficient conditions for d-dimensional CAs with the von Neumann neighborhood to be number-conserving, since for a given d there are exactly $(2d + 1) \cdot 2^{d^2}$ possible formulations of this condition. For example, for $d = 2$ the presented formula can be written in 80 different ways, which may differ in the number of components. The same holds for $d = 3$, resulting in as many as $7 \cdot 2^9$ possible formulations. In each particular case we can choose the version that suits us best. This approach is especially useful to describe number-conserving rules satisfying some additional conditions. In particular, this comes in handy in cases where these additional conditions result in dependencies between monomers or dimers, which happens, for example, in the case of local rules with some kind of symmetry, the most natural one being rotation symmetry (see, *e.g.*, [15, 28, 66, 69]).

Admittedly, the result in [61] is no revolutionary concept. In fact, it is obtained thanks to the great determination in trying to translate the one-dimensional result [4] into more dimensions and to overcome the associated technical problems. Moreover, for $d = 2$ such conditions were known and not only for the von Neumann neighborhood, but also for all other neighborhoods containing up to five cells [37]. However, the theory presented in [61] allows us to look at the issue in a broader way. For example, it can also be applied to the case $d = 1$ and then as many as six equivalent formulations of the necessary and sufficient conditions for a one-dimensional 3-input CA to be number-conserving are obtained, one of which is shown in [4].

Another, no less popular way of characterizing NCCAs is the so-called *motion representation* — a representation of an NCCA based on particle movements between cells. More precisely, the state of a cell is interpreted as the number of indistinguishable particles in it. The particles move between the cells and each particle is neither divided nor does it vanish. The global rule is interpreted as a regulation that governs the interaction of these movements. Of course, this interpretation imposes the following constraint: Q must be a

[1] We owe these terms to Nazim Fatès.

subset of natural numbers. Therefore such motion representation is considered only in the case of the k-ary NCCAs.

The motion representation of one-dimensional NCCAs has been extensively investigated, including some papers that definitely deserve to be mentioned:

- Boccara and Fukś [3] – The authors introduce the term "motion representation" for one-dimensional binary m-input NCCAs. Moreover, they find all such NCCAs for $m \leqslant 5$ and present the motion representation for some of them.
- Boccara and Fukś [4] – The authors find all one-dimensional ternary 2-input and 3-input NCCAs and give their motion representations.
- Pivato [43] – The author shows that every one-dimensional NCCA can be interpreted using motion representations. The same result was independently obtained by Fukś [19].
- Moreira, Boccara and Goles [39] – The authors formalize motion representations as *particle automata* and show that any one-dimensional NCCA can be uniquely characterized by some kind of canonical form of motion representation.
- Boccara and Fukś [5] – The authors generalize the motion representation in a way that is valid for non-conservative CAs as well.

A very interesting characterization of one-dimensional NCCAs was recently suggested by Redeker [45], who proposed *flow functions*, which describe how many particles cross the boundary between two cells, depending on the neighborhood of this boundary. Based on this tool, he introduced two related universal constructions of one-dimensional NCCAs. Although some kinds of flow functions in the case of one-dimensional NCCAs did already appear in the literature (see, *e.g.*, [24, 27, 43]), they had not been used for an exhaustive construction as in [45].

In the case of two dimensions, the topic of motion representation is not that well explored. Kari *et al.* [32] showed that the dynamics of any two-dimensional NCCA can be expressed in terms of particle displacements. However, as the authors point out, there is an infinite number of motion representations for such an NCCA and it is hard to define a "canonical" one. Note that their results concern any neighborhood type. A characterization of two-dimensional NCCAs with the von Neumann neighborhood was provided by Tanimoto and Imai [51]. Their result is stated in terms of some flow functions (in the vertical, horizontal and diagonal direction). Unfortunately, the flow functions are not uniquely defined, therefore for a given two-dimensional NCCA with the von Neumann neighborhood there are also many different descriptions rather than a "canonical" one. When it comes to more dimensions, there are practically no results on motion representation of NCCAs. Although it is rather clear that the results from [32] can be generalized to three or more dimensions, research in this direction has not been continued.

Recently, a very useful mathematical tool allowing for the characterization of d-dimensional NCCAs with the von Neumann neighborhood has been

put forward. As mentioned earlier, the necessary and sufficient conditions presented in [61] can be formulated in $(2d+1)2^{d^2}$ different ways and although they all are equivalent, the obtained formulas can differ in the number of terms. A better understanding of this fact has led to the new idea described by Wolnik *et al.* [67]. It has been proven that the local rule of any d-dimensional NCCA with the von Neumann neighborhood can be decomposed into two parts: *a split function* and *a perturbation*. The definition of a split function reflects the obvious fact that if an initial configuration has only one cell in a non-zero state, then in the next time step an NCCA must redistribute this state to the cells located in the neighborhood in such a way that the redistributed parts belong to the state set Q. In other words, this state splits, which happens according to some recipe depending on the state. Thus, split functions act as follows: each state splits according to its recipe irrespective of the states of its neighbors. Of course, split functions do not have to be local rules. Indeed, as each state splits independently, it may happen that the sum of the splitted constituents ending up in one cell from different neighbors does not belong to the state set Q. A perturbation is, roughly speaking, the only possible correction of a split function bringing the values back to Q, thus allowing to express the local rule of an NCCA: it takes the value zero at all monomers and transforms each configuration into one whose sum of states is zero. The main theorem of [67] reads as follows.

Theorem 1 *Let f be the local rule of a d-dimensional CA with the von Neumann neighborhood. This CA is number-conserving if and only if there exist a split function h and a perturbation g such that $f = h + g$. Moreover, for a given local rule f, the functions h and g are uniquely determined.*

One of the advantages of the above theorem is the following. For the enumeration of all d-dimensional NCCAs, we can consider each split function h separately and find all perturbations g that are *compatible with* h in the sense that $h + g$ takes values in Q. This task is ideally suited for a computer. First of all, the set $\mathcal{S}$ of all split functions has a very simple structure. Indeed, it can be identified with the Cartesian product $\bigtimes_{q\in Q_+} S_q$ where

$$S_q = \left\{(x_1, x_2, \ldots, x_{2d+1}) \mid x_1 + x_2 + \ldots + x_{2d+1} = q,\ x_1, x_2, \ldots, x_{2d+1} \in Q\right\}.$$

In particular, if Q is finite, then the cardinality of the set $\mathcal{S}$ equals $\prod_{q\in Q_+} |S_q|$. For example, in the case of k-ary CAs, it holds for each $q \in \{1, \ldots, k-1\}$ that $|S_q| = \binom{2d+q}{q}$ (see, *e.g.*, [46]), so, we have

$$|\mathcal{S}| = \binom{2d+1}{1}\binom{2d+2}{2}\cdots\binom{2d+k-1}{k-1}. \tag{3}$$

Table 2 lists $|\mathcal{S}|$ for certain values of d and state sets Q, with the number of all local rules in parentheses. One can see that the number of split functions

is definitely smaller than the number of all local rules. For example, in the two-dimensional case and $Q = \{0, 1, 2\}$, there are only 75 split functions, while there are 3^{3^5} local rules.

	$d = 1$	$d = 2$	$d = 3$	$d = 4$
$Q = \{0, 1\}$	3 (256)	5 (2^{32})	7 (2^{128})	9 (2^{512})
$Q = \{0, 1, 2\}$	18 (3^{27})	75 (3^{243})	196 (3^{2187})	405 (3^{19683})
$Q = \{0, 1, 2, 3\}$	180 (4^{64})	2625 (4^{1024})	16464 (4^{16384})	66825 (4^{262144})

Table 2 The number of all split functions versus the number of all local rules for dimension d and state set Q.

On the other hand, the set $\mathcal{P}$ of all perturbations also has a very nice structure – it is a linear space – and therefore it can be very easily described in terms of the elements of a basis of this space and the cardinality of the basis is equal to $d^2|Q_+|^2$. In Table 3 the dimension of $\mathcal{P}$ in the case of k-ary CAs is given for $d \leqslant 4$ and $k \leqslant 4$.

	$d = 1$	$d = 2$	$d = 3$	$d = 4$
$Q = \{0, 1\}$	1	4	9	16
$Q = \{0, 1, 2\}$	4	16	36	64
$Q = \{0, 1, 2, 3\}$	9	36	81	144

Table 3 The dimension of the linear space $\mathcal{P}$ for dimension d and state set Q.

To what extent the decomposition theorem allows to reduce the computational complexity was illustrated in [67] on the example of finding the complete list of three-dimensional ternary NCCAs. Before the introduction of this method, this case had been beyond the capabilities of presently available computers (for example, using the theory of Durand *et al.* [11] would require checking $3^{3^7} \approx 2.9 \cdot 10^{1043}$ local rules).

Moreover, this new approach to the study of d-dimensional NCCAs with the von Neumann neighborhood allowed to answer some theoretical questions in the field. In [67], the hypothesis was made that in the binary case, increasing the dimension of the space does not result in the appearance of a new kind of number-conserving CAs. The main result of [58] confirmed this hypothesis.

Theorem 2 *Let $d \geqslant 1$. There are exactly $4d+1$ d-dimensional binary NCCAs with the von Neumann neighborhood: the identity rule, the shift rules and the traffic rules in each of the $2d$ directions.*

In this way, the structure of the d-dimensional binary NCCAs with the von Neumann neighborhood became fully unveiled and it appeared that regardless

of the dimension d, all of these cellular automata are trivial, as they are intrinsically one-dimensional. Moreover, it became possible to describe the dynamics of such automata in detail [58].

In [59], the decomposition theorem allowed to characterize all reversible d-dimensional ternary NCCAs with the von Neumann neighborhood.

Theorem 3 *Let $d \geqslant 1$. Any reversible d-dimensional ternary NCCA with the von Neumann neighborhood is a shift rule.*

Thus it turned out that all reversible d-dimensional ternary NCCAs with the von Neumann neighborhood are trivial: each such CA acts independently in every dimension of the grid and we observe a synchronous action as aggregated result. It confirmed the hypothesis that three states are too few to enable the existence of nontrivial reversible NCCAs with the von Neumann neighborhood in any dimension.

As shown, the decomposition theorem is a strong tool that helps to solve many specific problems in the analysis of NCCAs. Sadly, so far it is only applicable for the von Neumann neighborhood. Thus, the following question is crucial for future studies in the field.

Question 1 The decomposition theorem applies only to the von Neumann neighborhood. A natural question arises whether similar characterizations can be obtained for neighborhoods of a different kind, such as those defined in [70].

4 Enumerations of NCCAs

One of the important research questions relates to the enumeration of all NCCAs for a given dimension, state set and neighborhood type. In Table 4 the numbers of one-dimensional k-ary NCCAs (*i.e.*, with state set $Q = \{0, 1, \ldots, k-1\}$) are given for the known cases. Up to our knowledge, this table represents the current state-of-the-art.

k \ m	2	3	4	5	6	7
2	**2**	**5**	**22**	**428**	**133 184**	**1 571 814 309**
3	**4**	**144**	**5 448 642**	?	?	?
4	**10**	**89 585**	?	?	?	?
5	**30**	**1 876 088 314**	?	?	?	?
6	**106**	?	?	?	?	?

Table 4 Number of one-dimensional m-input NCCAs with state set $Q = \{0, 1, \ldots, k-1\}$.

Below we summarize the sources of the numbers listed in Table 4:

- $k = 2$, $m \leqslant 5$ – the numbers can be derived from Boccara and Fukś [3];
- $k = 2$, $m = 7$ can be found in an internship report of Miquey [35] on so-called State-Conserving CAs which for $k = 2$ are equivalent to NCCAs;
- $k = 3$, $m \leqslant 3$ – the work of Boccara and Fukś [4];
- $k = 2$, $m \leqslant 6$ and $k = 3$, $m \leqslant 4$ and $k = 4$, $m \leqslant 3$ were also given (along with the full list of rules) on the web page of Moreira [36];
- $k = 5$, $m = 3$ – this number has been obtained by Dziemiańczuk using the split and perturb decomposition, and this result is still unpublished;
- $m = 2$ and arbitrary k – this is based on Fukś and Sullivan [20]. Note that in this case the authors managed to find and prove a formula that holds for all k.

Filling in missing entries in Table 4 is possible, but seems relatively uninteresting as the numbers grow very quickly with the growth of the state set and neighborhood. Hence, storing the complete lists of rules is problematic. Therefore further studies concentrated on finding NCCAs fulfilling some additional conditions.

A very interesting line of research corresponds to the study of reversible one-dimensional NCCAs with state set $Q = \{0, 1, \ldots, k-1\}$. Due to work of García–Ramos [22], we know such 2-input CAs are relatively simple: each of them is a product of shift rules and the identity rule. The 3-input case is richer and Table 5 summarizes the current knowledge on it.

k	2	3	4	5	6	7
	3	**3**	**21**	**21**	**471**	**1 669**

Table 5 Number of reversible one-dimensional 3-input NCCAs with state set $Q = \{0, 1, \ldots, k-1\}$.

The findings listed in Table 5 are due to:

- $k \leqslant 4$ – the paper of Imai *et al.* [29];
- $k = 5, 6, 7$ – the paper of Wolnik *et al.* [63]; the complete list of rules is available online [16].

The ongoing interest in reversible one-dimensional 3-input NCCAs yields interesting questions.

Question 2 Is it true for prime k that every reversible one-dimensional 3-input k-ary NCCA (except of shifts) has finite order?

It is relatively easy to check that for a composite number k, one can construct a reversible NCCA with an infinite order. But what happens when k is prime? Simulations only allow to examine $k = 2, 3, 5, 7$ as for higher values of k the problem becomes too computationally intensive. See Wolnik *et al.* [62] for more details and some initial intuitions.

So far only one-dimensional NCCAs have been considered in this section. For two dimensions little is known, mainly limited to the von Neumann neighborhood. It has been shown that there are only **9** two-dimensional binary NCCAs with the von Neumann neighborhood: the identity rule and shift rules (see, *e.g.*, the unpublished report by Moreira [37]). As already discussed, it has been shown recently that the same is true for arbitrary dimension d (see Theorem 2).

An enumeration is also known in the case of a ternary state set ($k = 3$), a result due to Dzedzej *et al.* [14]. The complete list of **1327** two-dimensional ternary NCCAs with the von Neumann neighborhood is available online [12].

Similarly to the one-dimensional case, it is expected that a further increase of k will lead to infeasibly high numbers, making further exploration not that interesting. Therefore, additional assumptions are again considered to limit the rule space.

Firstly, the case of reversible two-dimensional NCCAs with the von Neumann neighborhood has been considered. In the ternary case ($k = 3$), it has been shown that only the identity and shift rules are reversible. Moreover, the same holds for higher d as already mentioned in Theorem 3. The case $k = 4$ has been solved by Dzedzej *et al.* [13] – there are **65** two-dimensional quaternary NCCAs with the von Neumann neighborhood.

Secondly, rotation-symmetric two-dimensional NCCAs with the von Neumann neighborhood have been studied. The known numbers are listed in Table 6. The results for $k \leqslant 5$ are due to Imai *et al.* [28], $k = 6$ has been solved by Dzedzej *et al.* [15], while the results for $k = 7$ have been made available online by Nenca *et al.* [41].

k	2	3	4	5	6	7
	1	**1**	**1**	**4**	**116**	**30 144**

Table 6 Number of rotation-symmetric two-dimensional NCCAs with the von Neumann neighborhood and k states (assuming state set $Q = \{0, 1, \ldots, k-1\}$).

So far, we have considered cases where the state set is of the particular form $Q = \{0, 1, \ldots, k-1\}$. The papers [15, 28] show that for $k \leqslant 6$, one can use a different state set $\widetilde{Q}$ with k elements while still being able to enumerate all the NCCAs. The exact number depends on the properties of $\widetilde{Q}$, namely its "arithmeticity". Yet, all of the obtained NCCAs are equivalent to the rules found for Q, irrespective of the choice of $\widetilde{Q}$. Surprisingly, it was shown by Wolnik *et al.* [69] that the situation is different for $k = 7$. By a careful selection of $\widetilde{Q}$, one can obtain new NCCAs that are not equivalent to any of the rules found for Q.

Finally, one can also consider three-dimensional NCCAs with the von Neumann neighborhood. The case $k = 2$ has already been commented, *i.e.*, there are **13** such NCCAs characterized by Theorem 2. For $k = 3$ we know

that there are **5 302** such NCCAs [67]. As in the one- and two-dimensional cases, we believe that for higher k, the numbers are so large that enumeration would become unfeasible.

Rotation-symmetric three-dimensional NCCAs with the von Neumann neighborhood have been studied by Wolnik *et al.* [66] and due to this work we know that for $k \leqslant 6$ there is only one NCCA – the identity rule. For $k = 7$ an additional, non-trivial NCCA emerges and its dynamics has been discussed in [66]. Thus for $k = 7$, there are **2** rotation-symmetric three-dimensional NCCAs with the von Neumann neighborhood. Higher k have not been considered so far and remain an open problem.

All the results discussed so far were limited to the von Neumann neighborhood in different dimensions. Very little is known about NCCAs defined on different neighborhoods. For instance, Moreira [37] studied the problem for arbitrary two-dimensional neighborhoods on rectangular grids and derived necessary and sufficient conditions for NCCAs for all two-dimensional neighborhoods containing at most five cells.

When it comes to the Moore neighborhood, the only results known to the authors can be found in Ishizaka *et al.* [30], where 2×2 and 2×3 two-dimensional neighborhoods were considered. In addition, they proposed an algorithm that allows to generate all two-dimensional binary "motion-representable" NCCAs defined on $n \times m$ neighborhoods. Yet, as far as we know, this line of research has not been explored further.

Although complete lists of NCCAs have been found in many different settings, finding and proving formulas that hold for any number of states still poses a challenge.

Question 3 The sequence of numbers in the second column (for $m = 2$) of Table 4 has a known general formula (see [20]). Is it possible to provide general formulas for other columns of this table? And for the numbers shown in Tables 5 and 6?

5 Characterization of density-conserving ACCAs

In the previous sections we considered NCCAs with a finite state set. Yet, as mentioned in the introduction, one can extend the definition of number conservation to CAs with an infinite state space. Here, we consider one-dimensional 3-input Affine Continuous Cellular Automata (ACCAs), a simple subclass of CCAs for which the preservation of the sum of states has been quite well researched. Roughly speaking, ACCAs are a generalization of Fuzzy CAs, which are obtained from binary CAs by "fuzzification" of their disjunctive normal form [17]. Starting from the ECAs, the fuzzification process produces only 256 Fuzzy CAs (each ECA corresponds to a unique Fuzzy CA),

while further generalization produces infinitely many one-dimensional 3-input ACCAs.[2]

The local rule of an ACCA is affine in each variable and is defined by the following formula:

$$\begin{aligned} f(x,y,z) = {} & l_0(1-x)(1-y)(1-z) + l_1(1-x)(1-y)z + l_2(1-x)y(1-z) \\ & + l_3(1-x)yz + l_4x(1-y)(1-z) + l_5x(1-y)z \\ & + l_6xy(1-z) + l_7xyz \,, \end{aligned} \tag{4}$$

where $l_0, \ldots, l_7$ are the LUT entries.

To understand the difference between ECAs, Fuzzy CAs and ACCAs, it is worth emphasizing that the local rules of all these CA families are defined by Eq. (4). However, in the case of ECAs both LUT entries and inputs are binary, in the case of Fuzzy CAs the LUT entries are still binary, while the inputs belong to the unit interval $[0,1]$, whereas in the case of ACCAs both LUT entries and inputs belong to $[0,1]$.

It is not hard to prove that the local rule f of a density-conserving ACCA can be written as:

$$f(x,y,z) = (a+b+c-1)(x-z)y + (1-a-c)x + cy + az \,, \tag{5}$$

where $a, b, c \in [0,1]$ and $a, b \leqslant 1-c$. The local rules thus depend on three parameters and can be represented graphically in the Cartesian coordinate system as a pyramid P (Fig. 1).

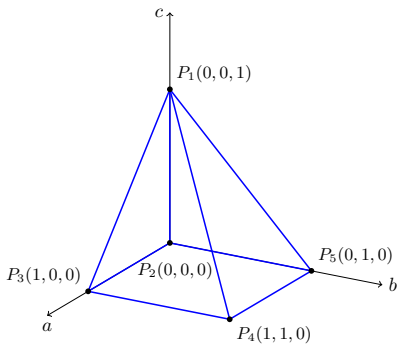

Fig. 1 Parameterization of the set of density-conserving one-dimensional 3-input ACCAs.

The apex $P_1(0,0,1)$ of the pyramid corresponds to the Fuzzy CA obtained by fuzzification of the identity rule (ECA 204), while the vertices $P_2(0,0,0)$,

[2] Until the end of this section, unless otherwise stated, when we write ACCAs we mean one-dimensional 3-input ACCAs.

$P_3(1,0,0)$, $P_4(1,1,0)$, $P_5(0,1,0)$ of its base correspond to fuzzy versions of the following ECAs: the traffic-right rule (ECA 184), the shift-left rule (ECA 170), the traffic-left rule (ECA 226) and the shift-right rule (ECA 240), respectively.

Wolnik *et al.* [60] described the dynamical properties of density-conserving ACCAs on finite grids $\mathcal{C}_N$, where $N \in \mathbb{N}$. We summarize the main findings.

The case $c = 1$. In this case, there is only one density-conserving rule (found at the top vertex of the pyramid P): the identity rule. So, its dynamics is very poor, but, as we show below, for c very close to one, the behavior of the corresponding ACCAs is strictly different from that of the identity rule.

The case $0 < c < 1$. Local rules in this case can be found anywhere in P, except for its bottom plane and apex. It appears that each initial configuration evolves to a homogeneous one, as stated in the following theorem.

Theorem 4 *Let f be the local rule of a density-conserving ACCA parameterized by $(a,b,c) \in P$ and let F be the corresponding global rule. If $0 < c < 1$, then for any $\mathbf{x} \in X_{\mathbf{N}}$, it holds that:*

$$\lim_{t\to\infty} F^t(\mathbf{x}) = (\rho(\mathbf{x}), \rho(\mathbf{x}), \ldots, \rho(\mathbf{x}))\,, \tag{6}$$

where $\rho(\mathbf{x})$ is the density of $\mathbf{x}$, i.e., $\rho(\mathbf{x}) = \frac{1}{N}(x_0 + x_1 + \ldots + x_{N-1})$.

The case $c = 0$. In this case, the dynamics has a much more complex behavior than in the case $0 < c < 1$. Without going into detail, the most important facts are the following. For odd N and $0 < a, b < 1$ the convergence in (6) still holds. However, for even N, the system has periodic points with period 2: $(\alpha, \beta, \alpha, \beta, \ldots, \alpha, \beta)$, where $\alpha, \beta \in [0,1]$. Moreover, if $a \in \{0,1\}$ or $b \in \{0,1\}$, then there exist points with period N, irrespective of N being odd or even.

Question 4 The above detailed characterization only holds in the one-dimensional case and the question is whether it can be realized for $d = 2$.

At the moment, only results of computer simulations are available (see [8]).

6 Triangular NCCAs

Wolnik *et al.* [68] adapted the approach presented in [67] for NCCAs defined on a regular square grid, *i.e.*, the split-and-perturb decomposition of a number-conserving local rule, to the study of number conservation of two-dimensional CAs defined on a regular triangular grid.

In order to understand the discussion, it is worth noting that a triangular tessellation of a plane into equilateral triangles produces two types of cells depending on the orientation of the triangle, namely $\triangle$- and $\triangledown$-cells. This raises the problem of defining local rules, which, looking at the fundamental

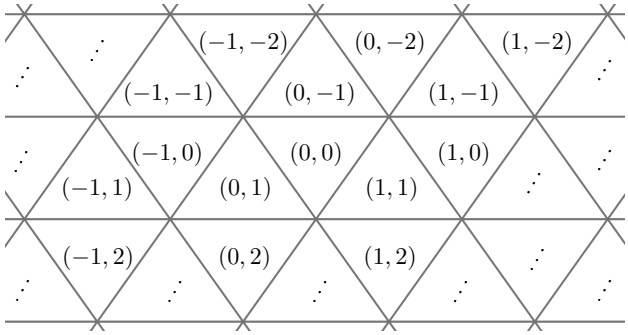

Fig. 2 Fragment of a triangular tessellation of a plane into equilateral triangles. The cell $(0, 0)$ is an example of a $\bigtriangledown$-cell, while the cell $(0, 1)$ is an example of a $\bigtriangleup$-cell.

assumptions about CAs, should be the same for all cells. This problem is handled in two ways. Firstly, one can define two independent local rules: the first one for $\bigtriangleup$-cells and the second one for $\bigtriangledown$-cells. Secondly, one can also consider local rules that are rotation-symmetric, because then the orientation of the triangles does not matter. Wolnik *et al.* [68] used the latter solution and focused on the simplest triangular CAs: automata that update the states of their cells on the basis of the states of the adjacent cells only (which is a natural adaptation of the von Neumann neighborhood to the triangular case).

Formally speaking, the global rule $F : X \to X$ of such a triangular CA is defined by a local rule $f : Q^4 \to Q$ in the following way. For any $\mathbf{x} \in X$ and $\mathbf{i} \in \mathcal{G}$, we put

$$F(\mathbf{x})_{\mathbf{i}} = f(x_{\mathbf{i}}, x_{\mathbf{j}_1(\mathbf{i})}, x_{\mathbf{j}_2(\mathbf{i})}, x_{\mathbf{j}_3(\mathbf{i})}) \,, \tag{7}$$

where $\mathbf{j}_1(\mathbf{i})$, $\mathbf{j}_2(\mathbf{i})$, $\mathbf{j}_3(\mathbf{i})$ are the cells adjacent to cell $\mathbf{i}$, numbered clockwisely. To ensure the correctness of this definition, it is assumed that the local rule f is rotation-symmetric, *i.e.*, for any $q, p_1, p_2, p_3 \in Q$, it holds that $f(q, p_1, p_2, p_3) = f(q, p_2, p_3, p_1)$.

It turned out that such a choice of neighborhood allows to prove a theorem analogous to Theorem 1, and in addition it was possible to obtain a very simple description of the perturbation space. Moreover, it appeared that each number-conserving triangular local rule (for which we assumed only rotation symmetry) is also permutation-symmetric, *i.e.*, for any permutation $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$ and for any $q, p_1, p_2, p_3 \in Q$, it holds that

$$f(q, p_{\pi(1)}, p_{\pi(2)}, p_{\pi(3)}) = f(q, p_1, p_2, p_3) \,.$$

In contrast to the case of a square grid, in the case of a triangular grid, the split-and-perturb decomposition approach allows to enumerate all k-ary NCCAs, regardless of the value of k. This became possible because, in the triangular case and for the state set $Q = \{0, 1, \ldots, k-1\}$, the number of perturbations that are compatible with a given split function h appeared

to be independent of h. The main result of [68] regarding triangular k-ary NCCAs is the following.

Theorem 5 *Let $k > 1$ be an integer and denote $K = \left\lfloor \frac{k-1}{3} \right\rfloor$. The number of all k-ary triangular NCCAs equals $\mathrm{N}(k)$, where*

$$\mathrm{N}(2) = 1,\ \mathrm{N}(3) = 1,\ \mathrm{N}(4) = 2,\ \mathrm{N}(5) = 16,\ \mathrm{N}(6) = 512,$$

while for $k \geqslant 7$

$$\mathrm{N}(k) = 1^{E_1} \cdot 2^{E_2} \cdot \ldots (K-1)^{E_{K-1}} \cdot K^{E_K} \cdot (K+1)^{E_{K+1}} \ , \tag{8}$$

where the exponents E_i are defined as follows:

$$E_1 = 7 \text{ and for } i = 2, 3, \ldots, K-1,\ E_i = \begin{cases} 9i, & \text{if } i \text{ is even} \\ 9i-3, & \text{if } i \text{ is odd} \end{cases}$$

$$E_K = \begin{cases} 14n, & \text{if } k-1 = 6n \\ 17n, & \text{if } k-1 = 6n+1 \\ 18n, & \text{if } k-1 = 6n+2 \\ 14n+5, & \text{if } k-1 = 6n+3 \\ 17n+6, & \text{if } k-1 = 6n+4 \\ 18n+6, & \text{if } k-1 = 6n+5 \end{cases}, \quad E_{K+1} = \begin{cases} n, & \text{if } k-1 = 6n \\ 4n+1, & \text{if } k-1 = 6n+1 \\ 9n+3, & \text{if } k-1 = 6n+2 \\ n+1, & \text{if } k-1 = 6n+3 \\ 4n+4, & \text{if } k-1 = 6n+4 \\ 9n+9, & \text{if } k-1 = 6n+5 \end{cases}$$

The split-and-perturb decomposition also allows for a given $k > 1$ to generate the complete list of all k-ary triangular NCCAs (see Algorithms 1 and 2 in [68]). However, as can be seen in Table 7, already for $k = 9$ the size of this set becomes too large for such a procedure to make practical sense.

k	$\mathrm{N}(k)$	k	$\mathrm{N}(k)$
2	$1^2 = 1$	9	$1^7 \cdot 2^{18} \cdot 3^{12} = 139\,314\,069\,504$
3	$1^4 = 1$	10	$1^7 \cdot 2^{18} \cdot 3^{19} \cdot 4^2 \approx 4.9 \cdot 10^{15}$
4	$1^6 \cdot 2^1 = 2$	11	$1^7 \cdot 2^{18} \cdot 3^{23} \cdot 4^8 \approx 1.6 \cdot 10^{21}$
5	$1^7 \cdot 2^4 = 16$	12	$1^7 \cdot 2^{18} \cdot 3^{24} \cdot 4^{18} \approx 5.1 \cdot 10^{27}$
6	$1^7 \cdot 2^9 = 512$	13	$1^7 \cdot 2^{18} \cdot 3^{24} \cdot 4^{28} \cdot 5^2 \approx 1.3 \cdot 10^{35}$
7	$1^7 \cdot 2^{14} \cdot 3^1 = 49\,152$	14	$1^7 \cdot 2^{18} \cdot 3^{24} \cdot 4^{34} \cdot 5^9 \approx 4.3 \cdot 10^{43}$
8	$1^7 \cdot 2^{17} \cdot 3^5 = 31\,850\,496$	15	$1^7 \cdot 2^{18} \cdot 3^{24} \cdot 4^{36} \cdot 5^{21} \approx 1.7 \cdot 10^{53}$

Table 7 The numbers $\mathrm{N}(k)$ for $k \leqslant 15$, specifying the numbers of all k-ary triangular NCCAs.

Regarding this topic, two following questions arise immediately.

Question 5 Can we enumerate all k-ary triangular NCCAs when two local rules are allowed: one for $\triangle$-cells and the second for $\triangledown$-cells?

Question 6 The neighborhood considered above is equivalent to the von Neumann neighborhood. What does the characterization of triangular NCCAs with a neighborhood equivalent to the Moore neighborhood look like?

7 Non-uniform NCCAs

In recent years, one-dimensional CAs that are not required to be spatially homogeneous are gaining more and more interest. The reason is that the assumption that all cells in a given CA use the same local rule to generate their next state is in many cases a great inconvenience (for several examples, see Dennunzio *et al.* [9]). Abandoning this assumption leads to very interesting dynamical systems called non-uniform or hybrid CAs (ν-CAs), in which each single cell is allowed to have its own local rule. Such systems look spectacular in computer simulations, but are very difficult to analyze theoretically. Although research into ν-CAs began in the late 1980s (see Pries *et al.* [44]), many fundamental questions still remain unanswered.[3] In fact, there are only a few papers about general ν-CAs (see, *e.g.*, [9, 6, 10, 47]), while the vast majority deals with the so-called non-uniform elementary CAs (ν-ECAs), in which each cell is allowed to have its own local rule belonging to Wolfram's set $\mathcal{W}$ of 256 ECAs (see, *e.g.*, [7, 26, 48]).

More formally, a ν-ECA H is given by a finite rule sequence $[f_0, f_1, \ldots, f_{n-1}]$ or by an infinite rule sequence $[f_i]_{i\in\mathbb{Z}}$, where each $f_i \in \mathcal{W}$. It acts on a given binary configuration $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \in \{0,1\}^n$ (resp. $\mathbf{x} \in \{0,1\}^{\mathbb{Z}}$) in the usual way:

$$H(\mathbf{x})_i = f_i(x_{i-1}, x_i, x_{i+1})\,. \tag{9}$$

If the considered grid is finite, then we have to choose a type of boundary condition and the two most commonly used ones are:

(P) *periodic boundary condition*: it is assumed that $x_{-1} = x_{n-1}$ and $x_n = x_0$, *i.e.*, all operations on indices are performed modulo n (we denote this case by $[f_0, f_1, \ldots, f_{n-1}]_{\mathbf{P}}$);
(N) *null boundary condition*: it is assumed that $x_{-1} = x_n = 0$ (we denote this case by $[f_0, f_1, \ldots, f_{n-1}]_{\mathbf{N}}$).

The definition of number conservation remains generally unchanged.

Definition 4 A ν-ECA H on $\mathcal{C} \subseteq \mathbb{Z}$ is called number-conserving if $\sum_{i\in\mathcal{C}} H(\mathbf{x})_i = \sum_{i\in\mathcal{C}} \mathbf{x}_i$ for each finite configuration $\mathbf{x} \in X$.

However, it should be noted that the above definition is one of two really different definitions being considered for ν-CAs on the infinite grid. In the case of ν-ECAs, notwithstanding, they are equivalent (see Proposition 1 in [10]).

[3] More information on the historical details can be found, *e.g.*, in [2].

The study of number conservation of ν-CAs started with the very general results in [10] and as far as the entire class of all ν-CAs is concerned there is nothing more. However, when narrowing the class of ν-CAs to ν-ECAs, the situation looks much better. Some preliminary results on number conservation of ν-ECAs appeared in [25]. Then in [42] many interesting facts and hypotheses based on computer simulations were formulated. All these simulations concerned ν-ECAs on finite grids with periodic boundary conditions only. Currently, research on number conservation of ν-ECAs can be considered completed. Both the characterization of all number-conserving ν-ECAs on finite grids as well as on the infinite grid have been established. In both cases, the results turned out to be surprising to say the least.

The case of finite grids is solved to the last detail in [64]. This was made possible by observing that, contrary to expectations, ν-ECAs with periodic boundary conditions and ν-ECAs with null boundary conditions are very closely related to each other. It turned out that each number-conserving ν-ECA on a finite grid with null boundary conditions is very simple, since it can be at most a concatenation of the following four types of ν-ECAs (to enlighten the description, we appeal to the particle interpretation of binary CAs):

- $\boxed{\text{Id}} = [\text{ECA}204]_{\mathbf{N}}$ on a one-cell grid – This ν-ECA is stationary, which means that a particle (if there is any) stays in the cell at all times.
- $\boxed{\text{S}} = [\text{ECA}170, \text{ECA}240]_{\mathbf{N}}$ on a two-cell grid – The dynamics of this ν-ECA can be described as follows: in each consecutive time step a particle from the left cell (if there is any) moves to the right cell and a particle from the right cell (if there is any) moves to the left cell. Thus, each configuration is periodic with period two.
- $\boxed{\text{A}k} = [\text{ECA}136, (\text{ECA}184)^k, \text{ECA}252]_{\mathbf{N}}$, with $k \geqslant 0$, on a $(k+2)$-cell grid – The first $k+1$ cells act according to the traffic-right rule, while the last one stops this traffic. As a result, all particles are lining up one after the other at the end of the grid and then the particles stop moving. Thus each configuration is eventually fixed.
- $\boxed{\text{B}k} = [\text{ECA}238, (\text{ECA}226)^k, \text{ECA}192]_{\mathbf{N}}$, with $k \geqslant 0$, on a $(k+2)$-cell grid – This ν-ECA behaves as follows: all particles move according to the traffic-left rule, but the first cell stops this traffic and as a consequence all particles line up one after the other at the beginning of the grid. Here also each configuration is eventually fixed.

Note that k may be zero and then $\boxed{\text{A}0}$ and $\boxed{\text{B}0}$ reduce to $[\text{ECA}136, \text{ECA}252]_{\mathbf{N}}$ and $[\text{ECA}238, \text{ECA}192]_{\mathbf{N}}$, respectively. Moreover, $\boxed{\text{B}k}$ is the reflection of $\boxed{\text{A}k}$.

For example, in the fairly large set of all 256^5 ν-ECAs on a five-cell grid with null boundary conditions there are only 64 ones that are number-conserving. Here are some of them:

$\boxed{\text{Id}}\boxed{\text{Id}}\boxed{\text{Id}}\boxed{\text{B}0}$ $\boxed{\text{Id}}\boxed{\text{A}0}\boxed{\text{Id}}\boxed{\text{Id}}$ $\boxed{\text{A}0}\boxed{\text{S}}\boxed{\text{Id}}$ $\boxed{\text{B}0}\boxed{\text{S}}$ $\boxed{\text{Id}}\boxed{\text{A}2}$

The main results concerning ν-ECAs on finite grids with null boundary conditions are summarized in the following theorem.

Theorem 6 *A ν-ECA $H = [f_0, f_1, \ldots, f_{n-1}]_{\mathbf{N}}$ is number-conserving if and only if $H \in \mathrm{Con}$, where* Con *is the set containing* $\boxed{\mathrm{Id}}$, $\boxed{\mathrm{S}}$, $\boxed{\mathrm{A}k}$, $\boxed{\mathrm{B}k}$, *for all $k \geqslant 0$, and all their concatenations.*

The number of number-conserving ν-ECAs with null boundary conditions on an n-cell grid is equal to F_n^2, where $(F_n)_{n=0}^{\infty}$ is the Fibonacci sequence, defined by $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$.

Moreover, a ν-ECA $H = [f_0, f_1, \ldots, f_{n-1}]_{\mathbf{N}}$ is number-conserving and reversible if and only if $H \in \mathrm{RCon}$, where RCon *is the set containing* $\boxed{\mathrm{Id}}$, $\boxed{\mathrm{S}}$ *and all their concatenations. The number of all reversible number-conserving ν-ECAs on an n-cell grid with null boundary conditions is equal to F_n.*

So it turned out that allowing cells to have their own elementary local rule does not result in any new type of dynamics. Indeed, even if a given ν-ECA is a concatenation of some blocks $\boxed{\mathrm{Id}}$, $\boxed{\mathrm{S}}$, $\boxed{\mathrm{A}k}$ and $\boxed{\mathrm{B}k}$, each component "lives its own life", as if there were impermeable barriers between adjacent components. Consequently, for all such ν-ECAs each configuration is eventually fixed or periodic with period two (depending on whether the CA contains the $\boxed{\mathrm{S}}$ component or not).

And what about periodic boundary conditions? Obviously, one can transform each number-conserving ν-ECA with null boundary conditions into a number-conserving ν-ECA with periodic boundary conditions by "gluing" together the ends of its grid. As the theorem below shows, there are no other number-conserving ν-ECAs with periodic boundary conditions (except, of course, the classical ECAs such as shift rules and traffic rules).

Theorem 7 *Let $n \geqslant 5$. A ν-ECA $H = [f_0, f_1, \ldots, f_{n-1}]_{\mathbf{P}}$ is number-conserving if and only if*

$$H \in \{[(\mathrm{ECA170})^n]_{\mathbf{P}}, [(\mathrm{ECA184})^n]_{\mathbf{P}}, [(\mathrm{ECA226})^n]_{\mathbf{P}}, [(\mathrm{ECA240})^n]_{\mathbf{P}}\},$$

or there exists $i \in \{0, 1, \ldots, n-1\}$ such that

$$[f_i, f_{i+1}, \ldots, f_{i-1}]_{\mathbf{P}} = [f_i, f_{i+1}, \ldots, f_{i-1}]_{\mathbf{N}} \in \mathrm{Con}.$$

In the above theorem there is the assumption $n \geqslant 5$, as indeed grids of length three or four allow for other types of number-conserving ν-ECA with periodic boundary conditions. Their dynamics is a bit more complicated (if we can talk about the complexity of the dynamics of such simple CAs at all, in which each configuration must be repeated no later than every 16th time step). A detailed description of these cases can be found in [64].

Having full information about what all number-conserving ν-ECAs on finite grids look like, in particular, that regardless of the length of the grid, they are built from the same four types of blocks, it was expected that the case of the infinite grid $\mathbb{Z}$ can be treated as a limit case of finite grids. Because it

is clear that one can use the blocks $\boxed{\text{Id}}$, $\boxed{\text{S}}$, $\boxed{\text{A}k}$ and $\boxed{\text{B}k}$ to get some examples of number-conserving ν-ECAs on $\mathbb{Z}$:

$$\cdots\boxed{\text{Id}}\boxed{\text{B12}}\boxed{\text{S}}\boxed{\text{A9}}\boxed{\text{S}}\boxed{\text{Id}}\boxed{\text{A5}}\boxed{\text{Id}}\boxed{\text{Id}}\boxed{\text{A3}}\boxed{\text{A7}}\cdots$$

Additionally, we can modify the block $\boxed{\text{A}k}$ (as well as $\boxed{\text{B}k}$) to get two new blocks by increasing k to infinity:

$$\boxed{\text{A}\infty} = [\text{ECA136}, \text{ECA184}, \text{ECA184}, \text{ECA184}, \ldots]_{\mathbf{N}}\,,$$
$$\boxed{\infty\text{A}} = [\ldots, \text{ECA184}, \text{ECA184}, \text{ECA184}, \text{ECA252}]_{\mathbf{N}}\,.$$

And then we can use these new blocks to get new examples:

$$\boxed{\infty\text{B}}\boxed{\text{Id}}\boxed{\text{S}}\boxed{\text{S}}\boxed{\text{A15}}\boxed{\text{Id}}\boxed{\text{A0}}\boxed{\text{A6}}\boxed{\text{B5}}\boxed{\text{Id}}\boxed{\text{S}}\boxed{\text{Id}}\boxed{\text{B8}}\boxed{\text{A}\infty}$$

It might have been expected that the case of $\mathbb{Z}$ is not qualitatively different from the case of finite grids, which is precisely the case with uniform ECAs, where the same five rules are number-conserving on each finite grid as well as on $\mathbb{Z}$. However, it turned out that the world of number-conserving ν-ECAs on $\mathbb{Z}$ is much more surprising. It suffices to look at the following example:

$$[\ldots, \text{ECA204}, \text{ECA204}, \text{ECA204}, \text{ECA0}, \text{ECA240}, \text{ECA240}, \text{ECA240}, \ldots]$$

The left half of this ν-ECA is stationary (it contains copies of the identity rule ECA204 only), while the right half starts with ECA0 and then there are copies of the shift-right rule ECA240, so the particles from this part simple go simultaneously to the right. It is therefore obvious that this ν-ECA is number-conserving, despite the fact that ECA0 is not double-legal[4] Moreover, the conjugation of this ν-ECA is not number-conserving at all! Both these facts do not appear in the case of classical (uniform) number-conserving CAs (regardless of the type of grid) and even in the case of ν-CAs on finite grids.

Finally, an exhaustive characterization of all number-conserving ν-ECAs on $\mathbb{Z}$ was found, from which it follows that the case of an infinite grid can in no way be treated as a limit case of finite grids, as there are many number-conserving ν-ECAs on $\mathbb{Z}$ that have no analogous counterpart on finite grids. Although computer experiments are completely useless to assist in finding these ν-ECAs (one can use simulations on finite grids only), Wolnik *et al.* [65] found a suitable mathematical tool that allowed to solve this problem completely. For more details and a description of all number-conserving ν-ECAs on $\mathbb{Z}$, please see [65].

During the research on number conservation of ν-ECAs, it became clear that some of the observed facts are closely related to the simplicity of elementary

[4] Double-legal local rules are those satisfying $f(0,0,0) = 0$ and $f(1,1,1) = 1$. For uniform ECAs, being double-legal is a natural necessary condition for number conservation. Surprisingly, for non-uniform CAs on infinite grids this is no longer the case, as a number-conserving ν-ECA can have components that are not double-legal.

CAs: in ν-ECAs all rules are binary and have three inputs. Therefore, further investigation should try to answer the following main questions.

Question 7 How does an enlargement of the set of states or an increase of the radius affect one-dimensional number-conserving ν-CAs?

Question 8 Can an equally simple characterization be obtained for two-dimensional binary ν-CAs with relatively small neighborhoods?

8 Conclusions

In this survey paper, we tried to summarize the state-of-the-art on the theoretical knowledge about NCCAs. The road leading to the results outlined had two lanes. Initially, a lot of work went into computer simulations and attempts to exhaustively scan through large rule spaces in search for answers. This led to some interesting findings and, most importantly, allowed to formulate some of the early hypotheses. Yet, obvious computational limits were reached. Only the application of mathematical reasoning allows to break through these computational barriers. We believe that this approach of intertwining computational and analytical methods should be continued. Hopefully, it will result in answering some of the open questions formulated in this paper.

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