Sequencing Arithmetic, Area, and Algebraic Instruction for Teaching the Distributive Principle

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Abstract: The current study examined the effect of different instructional sequences for teaching the distributive principle for multi-digit multiplication. Third graders experienced one of four sequences: Algebra-Arithmetic, Algebra-Area, Arithmetic-Algebra, and Area-Algebra. All sequences produced improvements in solving practiced 2D × 2D problems. Critically, the Area-Algebra and Arithmetic-Algebra sequences had the largest learning effects, with the former better for participants with some prior knowledge of 2D × 2D multiplication and the latter better for participants with no prior knowledge. All sequences also produced improvements on unpracticed transfer problems, with the Area-Algebra sequence producing the largest learning effects. Finally, improvements were most associated with prior knowledge of single-digit addition and subtraction – but not multiplication – facts. The implications of these results are discussed and future directions outlined.

Keywords: multi-digit multiplication, distributive principle, mathematics education

Introduction

The current study implemented three instructional approaches for teaching the distributive principle for solving multi-digit multiplication problems. In particular, it investigated the optimal *combination* and *sequencing* of arithmetic, area (i.e., spatial), and algebra instructional approaches (Schwartz & Bransford, 1998).

The distributive principle for multi-digit multiplications refers to the fact that one of two factors can be split into two or more parts, each part multiplied by the other factor separately, and the partial products added (Anghileri, 1999). This principle is important for learning to solve multi-digit multiplication problems. For example, Liu, Ding, Gao, and Zhang (2015) found that fourth graders solved distributive-format problems (e.g., 25 × (10 + 2)) more quickly and accurately than conventional-format problems (e.g., 25 × 12). This finding suggests that recognizing and applying the distributive property to compute partial products (e.g., 25 × 10 and 25 × 2) improves performance of solving multi-digit multiplication problems.

Previous studies have implemented a variety of instructional approaches to teach the distributive principle (Baroody, 1999; Lee, 2014). The current study focused on three – *arithmetic, area,* and *algebra*. The arithmetic approach takes advantage of concepts such as repeated addition and relational arithmetic fact knowledge (Baroody, 1999). The area approach is often promoted by mathematics educators as being the most comprehensible to students (Lee, 2014). It takes advantage of visualizing partial products and decomposing factors. In particular, it supports the shift from repeated addition reasoning into multiplicative reasoning (i.e., factoring) (Cooney, Swanson, & Ladd, 1988). The algebra approach applies the distributive property to re-express multi-digit multiplication as the sum of a series of simpler multiplications, and can be seen as an elementary form of symbolic algebraic reasoning. The arithmetic and area approaches are more concrete to students in making use of familiar arithmetic expressions and visual diagrams. The algebra approach is more abstract to students in employing unfamiliar arithmetic expressions with missing operands.

The goal of the current study was to determine the optimal combination and sequencing of the different approaches for teaching the distributive principle in multi-digit multiplication. Candidate sequences were evaluated by changes in arithmetic proficiency across a pre-test and two post-tests. There were three research questions: (1) Do different instructional combinations and sequences lead to better performance on *practiced* multi-digit multiplication problems? (2) Do different instructional approaches and sequences lead to better performance on *transfer* multiplication problems? (3) What is the relationship between prior knowledge of single-digit arithmetic facts for the four operations and proficiency in multi-digit multiplication?

Methods

Participants

The original participants were 120 3rd graders at an elementary school in Seoul, South Korea. We only analyzed the data of the 96 participants who (1) completed all pre- and post-tests, (2) completed two weeks of practice, and (3) had scores on the final assessment of practiced 2D × 2D multiplication within ± 2 *SD*s of the overall sample. Among the final 96 participants, 61 students had already learned 2D × 2D multiplication in advance through a private academy or tutor. Thus, we separately analyzed the effect of our instruction on the prior learning group and the no-prior learning group.

Design
We used a pretest-intervention-posttest design. Participants were randomly assigned to four different sequences: 1) *Algebra-Arithmetic,* 2) *Algebra-Area,* 3) *Arithmetic-Algebra,* and 4) *Area-Algebra*. We also considered participants' prior learning of 2D × 2D multiplication, i.e., the prior learning vs. no-prior learning grouping factor. Students practiced one instructional approach in the first week and the other in the second week according to their instruction condition. On the final day of each week, they were administered post-test assessments measuring arithmetic proficiency in multi-digit multiplication.

Materials

Instructions for teaching distributive principle
We developed Arithmetic, Area, and Algebra instructional approaches for teaching the distributive principle for multi-digit multiplication (see Figure 1). In the arithmetic version, students solved multi-digit multiplication problems by retrieving arithmetic facts, recognizing relations between several given multiplication equations. In the area version, they solved problem by computing partial products by counting the number of squares in a grid. In the algebra version, they solved problems by decomposing factors based on their place value.

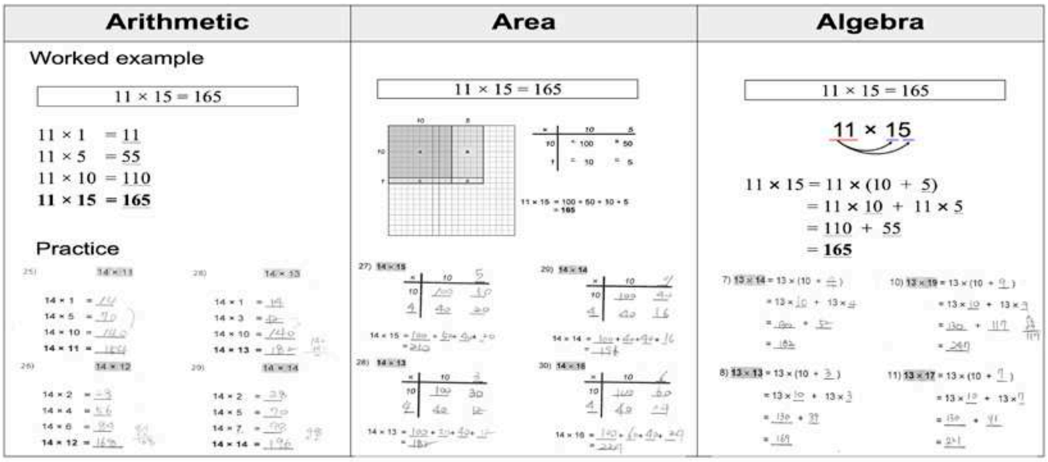

Figure 1. Arithmetic, area, and algebra instructional approaches for teaching the distributive principle.

For each instructional approach, practice items were formed by factors ranging between 11 and 19. Across a week of instruction, the difficulty of the items increased: the "easiness" of the factors decreased (e.g., 11 × 15 → 13 × 14) and the size of the products increased (e.g., 11 × 13 → 18 × 19). The three approaches were combined into four sequences of *a priori* interest: Algebra-Arithmetic, Algebra-Area, Arithmetic-Algebra, and Area-Algebra. Importantly, the algebra approach was included in each sequence.

Assessment
We developed assessments to measure arithmetic factual knowledge and arithmetical proficiency in multi-digit multiplication. Twenty 2D × 2D multiplication problems with factors in the range 11-19 were used at pre-test and at post-tests 1 and 2. The number of correct answers produced in fifteen minutes was the dependent variable. This measure was used to evaluate research question (1) concerning learning. Five 2D × 2D problems which factors greater than 20, five 2D × 2D algebra format problems (e.g., 12 × __ = 156), and five 3D × 1D problems were used at both post-tests to evaluate research question (2) concerning transfer. The number of correct answers produced in ten minutes was the dependent variable. Finally, single-digit addition/subtraction, multiplication, and division (40 items each) were used to measure arithmetical factual knowledge at pre-test. For each of these three measures, the number of correct answers produced in one minute was coded. This measure was used to investigate research question (3) concerning individual differences.

Procedure

Prior to the intervention, participants completed the pre-test measures of 1D arithmetic and 2D x 2D multiplication. They then solved 36 multi-digit multiplication problems every day during the first week using the approach indicated by the workbook of the condition to which they had been assigned. On the final day of the first week, they completed the learning and transfer measures of post-test 1. The second week followed the same structure, but for a different approach, and finished with post-test 2.

Findings

Preliminary analysis showed that there were no significant differences in the pre-test scores across the four sequences in each of the prior learning and the no-prior learning groups. This indicates that our random assignment was successful.

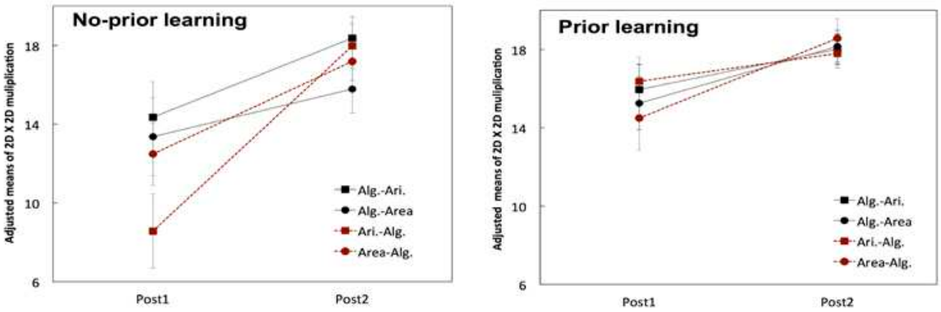

Figure 2. Practiced 2D × 2D multiplication scores (max =20) by time, prior learning, and sequence.

The first research question was whether different instructional combinations and sequences lead to better performance on practiced multi-digit multiplication problems. We addressed this in a three-way repeated measures ANCOVA on post-test scores of practiced 2D × 2D multiplication problems with time, sequence, and prior learning as factors and pre-test scores as a covariate (see Figure 2). There was a main effect of time ($F(1, 87) = 63.577$, $p < .001$, $\eta_p^2 = .422$) and prior learning ($F(1, 87) = 4.846$, $p = .030$, $\eta_p^2 = .053$). Although there was no main effect of sequence ($p = .545$), scores on post-test 2 were significantly improved compared to pre-test scores for all four sequences. Follow-up one-way repeated measures ANOVAs on students' scores on practiced 2D × 2D multiplication problems showed significant improvement for each sequence. The effect sizes were largest in the Arithmetic-Algebra ($\eta_p^2 = .582$) and Area-Algebra ($\eta_p^2 = .568$) sequences, which experienced the algebra approach in the second week after experiencing one of the concrete approaches in the first week. There were also interaction effects of prior learning × time ($F(1, 87) = 4.363$, $p = .040$, $\eta_p^2 = .048$) and prior learning × time × condition ($F(3, 87) = 3.599$, $p = .017$, $\eta_p^2 = .110$). Participants in the no-prior learning group benefitted most from the Arithmetic-Algebra sequence, whereas those in the prior learning group benefitted most from the Area-Algebra sequence.

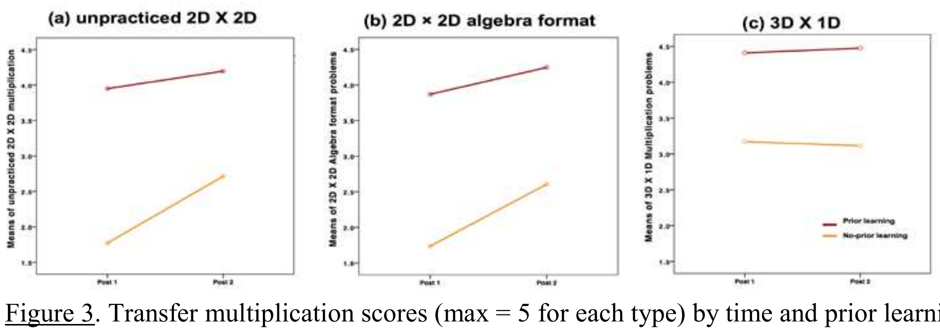

Figure 3. Transfer multiplication scores (max = 5 for each type) by time and prior learning.

The second research question was whether different instructional approaches and sequences lead to better performance on *transfer* multiplication problems. We addressed this in three-way repeated measures ANOVAs with time, sequence, and prior learning as factors and post-test scores on the three transfer measures as dependent variables: (a) unpracticed 2D × 2D problems, (b) 2D × 2D algebra format problems (e.g., 12 × __ = 156), and (c) 3D × 1D problems. There was no main effect of sequence, and therefore Figure 3 collapses across this factor. There was a main effect of prior learning on all transferred multiplication problems ($F_a(1, 88)$

$= 45.405$, $p_a < .001$, $\eta_{a,p}^2 = .340$; $F_b(1, 88) = 11.315$, $p_b = .001$, $\eta_{b,p}^2 = .114$; $F_c(1, 88) = 24.587$, $p_c < .001$, $\eta_{c,p}^2 = .218$). Critically, there was a main effect of time on two types of transfer problems ($F_a(1, 88) = 16.717$, $p_a = .001$, $\eta_{a,p}^2 = .126$; $F_b(1, 88) = 4.269$, $p_b = .042$, $\eta_{b,p}^2 = .046$). For the Area-Algebra sequence, there was the largest improvement from post-test 1 ($M = 2.57$ $SD = 2.01$) to post-test 2 ($M = 3.71$, $SD = 1.73$) on unpracticed 2D × 2D multiplication problems ($F_a(1, 20) = 6.982$, $p_a = .016$, $\eta_{a,p}^2 = .259$).

The third research question concerned the relationship between prior knowledge of single-digit arithmetic facts and arithmetic proficiency in multi-digit multiplication after two weeks of practice learning the distributive principle. We computed bivariate correlations between pre-test scores on single-digit addition/subtraction, multiplication, and division, and post-test 2 scores on practiced 2D × 2D multiplication problems. The single-digit addition/subtraction scores were correlated with practiced 2D × 2D multiplication scores for both the prior learning group ($r = .502$, $p < .001$) and the no-prior learning group ($r = .473$, $p < .001$). We additionally conducted a linear regression analysis to assess the overall relationship between single-digit arithmetic performance and practiced 2D × 2D multiplication at post-test 2 after controlling for prior learning. The results showed that 27.4% of the variance in the dependent measure was accounted for by single-digit arithmetic performance (F (4,91) = 8.580, $p < .001$), with single-digit addition/subtraction knowledge the only significant predictor (*stand.β* = .417, $p = .003$).

Conclusions and implications

The first research question was whether different instructional sequences lead to better performance on practiced multi-digit multiplication problems. In fact, all four sequences resulted in improved performance at post-test 2 on the practiced 2D x 2D problems. However, the Area-Algebra and Arithmetic-Algebra sequences had the largest learning effects. This suggests that there may be some advantage to first practicing with more familiar arithmetic representations or more concrete area representations *before* practicing with more abstract algebra representations (Fyfe, McNeil, & Borjas, 2015). Moreover, the effect of instructional sequence differed as a function of prior learning. For participants with prior learning of 2D x 2D multiplication, the Area-Algebra sequence produced the largest improvements, whereas for those with no-prior learning, the Arithmetic-Algebra sequence produced the largest improvements. Future research should follow up on these findings.

The second research question was whether different instructional sequences lead to better performance on novel transfer problems. All four sequences produced improved performance from post-test 1 to post-test 2 on the unpracticed 2D × 2D problems and the 2D × 2D algebra format problems. Again, there were some differences in efficacy. The Area-Algebra sequence led to the greatest improvement in solving unpracticed 2D × 2D multiplication problems. One possible explanation of this finding is that using an area approach to visualize decomposed factors produced a positive transfer effect on dealing with large unpracticed multiplicands. Future research should evaluate this possibility.

The third research question was whether there is a relationship between knowledge of single-digit arithmetic facts at pre-test and proficiency in multi-digit multiplication at post-test 2, after two weeks of practice learning the distributive principle. A regression analysis revealed that there is a relationship and, surprisingly, it is driven by knowledge of single-digit addition and subtraction facts, not single-digit multiplication or division facts. One interpretation of this finding is that multi-digit multiplication depends less on using multiplication to *compute* partial products and more on using addition and subtraction to *combine* partial products. Future research should evaluate this interpretation.

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