Beyond Analogy: Qualitative Dimensions of Comparing in Math Class

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Abstract: Comparisons have been shown to be effective for improving students' understanding of mathematical concepts and procedures. The evidence for the effectiveness of comparisons has primarily come from laboratory experiments on analogy and analogy-based interventions in math classrooms rather than investigations of naturalistic math classrooms. As a result, the qualitative aspects of comparisons that may be important for the effectiveness of comparisons have been understudied. This paper describes preliminary research on qualitative dimensions of comparisons. Analysis of video-recordings of a math classroom resulted in identification of 244 naturally occurring comparisons that were coded along five dimensions. Two dimensions related to the ambiguity of comparisons and a system for representing such ambiguity are extensively discussed herein. Potential effects of ambiguity of comparisons on learning are outlined.

Keywords: comparison, analogy, mathematics, qualitative methods, representation

Introduction

Comparing – the act of identifying similarities and differences – underlies many of the things we do every day. Research in cognitive psychology, such as analogical reasoning (e.g., Gentner & Markman, 1997), has been primarily focused on comparisons in laboratory settings with limited applications to classroom contexts. Some research on comparisons in classroom contexts has been situated in math because of its highly pattern-dependent nature. Overall, the consensus across theoretical and applied studies is that making comparisons leads to improvements in procedural and conceptual knowledge in mathematics (e.g., Alfieri, Nokes-Malach & Schunn, 2013), but more work needs to be done to document why and how such improvements might arise.

Many interventions in math classrooms have demonstrated the effectiveness of comparison-focused activities versus "business-as-usual" math teaching (e.g., Rittle-Johnson & Star, 2011; Schwartz & Martin, 2004) but they have not qualitatively examined the mechanisms by which such activities led to learning gains relative to control groups. Providing only quantitative differences between treatment and control groups' changes in test scores and limited description of both the interventions and of the control group conditions makes it more difficult to replicate the findings in additional studies that evaluate comparison as a mechanism. One study has suggested that the frequency with which students alternate discussing examples in a comparison-focused activity might be important for the development of students' conceptual understanding of mathematical content (Schwartz, Chase, Oppezzo, & Chin, 2011). More qualitative data on how students' thoughts or behaviors during the learning experience varied across conditions would provide insight as to what kinds of student behaviors teachers should encourage and inform how student-to-student interactions might productively interact with teacher-to-student interactions before, during, and after a comparison-focused activity (Kapur, 2015).

One study has helped make comparison research more readily applicable by describing qualitative dimensions of comparisons made in naturalistic, non-experimental math classrooms (Richland, Holyoak & Stigler, 2004). These researchers used a very strict definition of analogy, a specific subtype of comparison, to analyze only utterances in which a relationship between two entities was explicitly identified. Although these explicit analogies may be more straightforward to analyze because all of the elements of the mapping are clearly specified, the authors themselves acknowledged that other "hints at mapping" occurred that were not captured by their analysis. Gaining a better understanding of the prevalence and additional qualities of less-clearly defined comparisons as they occur in non-experimental classrooms would further inform teachers as to what teaching with comparisons might look like and help elucidate how students might learn from comparisons other than analogies.

Method

The data analyzed for this study were primarily video recordings of a sixth-grade honors math class during the 2017 spring semester. This class was situated in a well-resourced middle school in a suburb of a large Midwestern city. This class was selected because the teacher's emphasis on inquiry-based learning and

Common Core State Standards mathematical practices – such as finding patterns, considering multiple solution strategies, and reasoning like a mathematician, all of which involve comparisons – made the class a good candidate for observing comparisons.

To identify comparisons systematically, comparisons were defined as any utterance that communicated a relationship between two or more entities. This definition extended the criteria used by Richland et al. (2004) which only included analogies (typically a relationship of similarity) between two specified entities and allowed for the examination of previously undocumented “hints at mapping.” This definition included relationships other than similarity, such as difference (e.g., two different strategies) and relative quality (e.g., a more efficient strategy). Comparisons of more than two entities were also included. Using the operational definition of comparison described above, two researchers collaboratively identified any utterances that included comparison in a subset of the video data selected from two days of class during a statistics unit focused on variability and mean absolute deviation. Any disagreements about which utterances included comparisons were resolved through discussion.

Qualitative dimensions of comparison

After identifying instances of comparison, all instances were iteratively coded along five dimensions to capture the variety of the comparisons. The five dimensions were: 1) type of entities compared, 2) type of relationship, 3) how explicit or implicit the entities were, 4) how explicit or implicit the relations were, and 5) in what context the comparison was made. Two dimensions – the types of entities that were compared and the context in which the comparisons were produced – were also analyzed by Richland et al. (2004) in their analysis of analogies. The other three dimensions emerged from the data as important variables that captured a variety that has not previously been examined in literature on comparisons. For this paper, two of the latter dimensions – how explicit or implicit the entities and relations of the comparisons were – will be presented because these dimensions have not previously been analyzed in the literature and have some potential implications for teaching and learning through comparison.

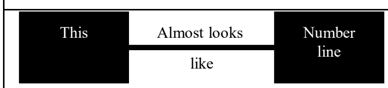
Representational system

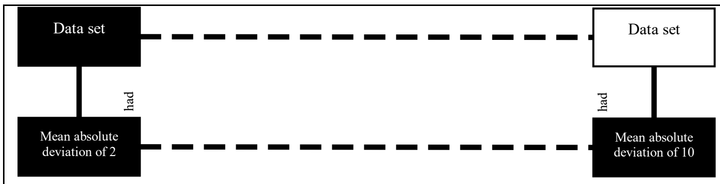
To analyze the comparisons, a representational system was created that is similar to diagrams used in models of structure mapping theory of analogy (Gentner & Markman, 1997). In this representational system, squares represent entities (objects or relationships as entities). If the entity was explicitly mentioned (including by usage of a pronoun), the square was black. If an entity was only implied, it was represented as a white square. Lines represented relationships. If a relationship among entities was explicit, it was shown as a solid line connecting two or more squares. If a relationship among entities was only implied, this relationship was represented with a dotted line connecting two or more squares. Representations that exhibit a range of comparisons, as well as how they were coded on the two dimensions of comparison ambiguity (discussed below) are included in Table 1.

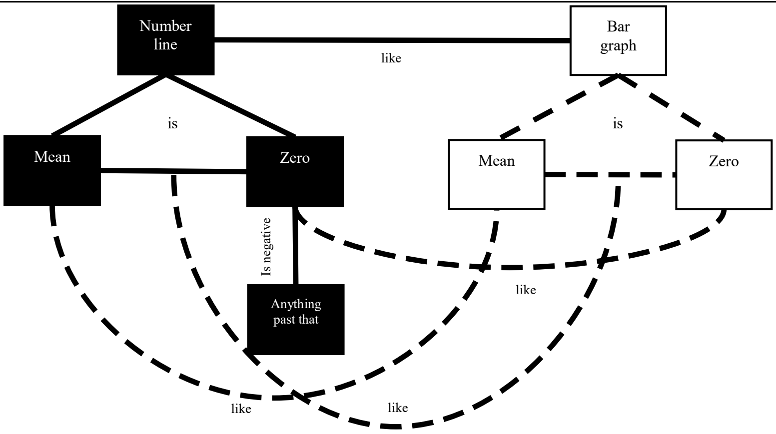
Comparison ambiguity: Explicitness of entities and relations

These representations allowed for categorizing of comparisons according to two dimensions: entity explicitness and relation explicitness. Entity explicitness refers to how many entities, explicit and implicit, were compared in a comparison. If additional entities were mentioned besides the ones that were the main focus of the comparison, the comparison was labeled as including entities at “multiple levels.” Relation explicitness refers to how implicit or explicit the relation was in a comparison. The difference between the two categories of relation explicitness - Alignment and Relation specified - is that comparisons categorized as alignments did not have any obvious relational language such as “similar” or “different”. Instead, these alignment comparisons usually had parallel structure that implied an interpretable relation among entities but did not have connecting words that communicated the relation explicitly. Combining entity explicitness and relation explicitness gave an overall dimension of a comparison’s lack of explicitness, or ambiguity. See Table 2 for counts of comparisons of each type of ambiguity.

Table 1: Representations of Comparisons

Representation and Utterance	Entity explicitness	Relation explicitness
 “This almost looks like a number line here”	Two named	Relation specified

<table>
<tr><td>

"Like, if I were to tell you there was a mean absolute deviation of ten, and then a data set that had a mean absolute deviation of two"</td><td>Two named and multiple levels</td><td>Alignment</td></tr>
<tr><td>

"Like, on a number line, how like, the mean is zero, anything past that is negative."</td><td>One named and multiple levels</td><td>Relation specified</td></tr>
</table>

Table 2: Counts of Comparisons by Category of Ambiguity

	Entity explicitness						
Relation explicitness	One named; other(s) implied	Two named	More than two entities named	One named, multiple levels	Two named, multiple levels	More than two, multiple levels	Sub-totals
Alignment	23	50	18		14		105
Relation specified	37	40	42	3	15	2	139
Sub-totals	60	90	60	3	14	2	244

Results and discussion

Comparisons were frequently made during the selected two days in this mathematics classroom. Casting a much wider and different net than Richland et al. (2004), 244 comparisons were identified in two days of class. Relatively few comparisons – only 16.4% of all comparisons identified – fit the definition of analogy used in Richland et al. (2004) – comparisons with two entities explicitly named and an explicit relationship. Although more comparisons had an explicit relationship (139 vs 105), several of the comparisons with explicit relationships also had ambiguous entities. Even though the most common type of entity explicitness was two explicit entities (90), more comparisons of this type had implicit (50) than explicit relationships (40).

The patterns of ambiguity of relations and entities outlined above reveal that comparisons in forms other than analogies were quite prevalent. Existing models of analogical reasoning already have ways of explaining how we map the entities and relations of a comparison with an explicit relation and explicit entities (e.g., Falkenheiner, Forbus & Gentner, 1989). However, such models do not currently address how we are able to process and accurately interpret comparative statements in which the relation and the entities are more ambiguous.

The implications of entity explicitness, relation explicitness and overall comparison ambiguity for how comparison supports learning are as of yet unclear, but there are a few possibilities. Like in everyday conversation and interactions, the more ambiguity in an action, the more likely it is to be misinterpreted. The same may be true for these ambiguous comparisons. It is likely true that these ambiguities require more in terms of mental resources to identify entities and test out plausible interpretations. One way that these alternative interpretations may affect learning outcomes is that students would not demonstrate as much improvement in their understanding, however it is measured, because processing these comparisons could require too much cognitive load (e.g., Richland, Begolli, Simms, Frausel & Lyons, 2016). With more mental resources devoted to interpreting the comparison, fewer resources may be available for integrating the new information from the comparison into existing knowledge structures. Alternatively, if students are still able to integrate the comparison into their existing knowledge structures, they may incorporate some misalignments into their knowledge structures due to a misinterpretation.

However, another possibility is that ambiguity could lead to greater improvements in students' understanding of whatever is compared. Students may learn more from processing ambiguous comparisons than from processing explicit comparisons for the very reason that students must do a lot more cognitive work trying to interpret ambiguity than they would for a more explicit comparison. This possibility relies on the tenets of constructivism (e.g., Piaget, 1964) that posit that no changes in knowledge occur unless a student does the work of integrating new knowledge with their prior knowledge. Greater ambiguity would likely require more effortful integration than explicit comparisons, so an ambiguous comparison would require at least some connection with existing structures just to interpret it at all as opposed to a more explicit comparison that may integrate less extensively with existing knowledge structures, if at all.

A third possible mechanism for how greater ambiguity could result in improved learning outcomes involves comparisons at multiple levels. It is possible that more ambiguous comparisons are initially more confusing and more likely to be misinterpreted than more explicit comparisons, but having to evaluate several plausible interpretations could reap the previously shown benefits of comparison twice over: testing how each plausible interpretation fits with existing knowledge would involve both comparing the fit across interpretations as well as comparing the entities within each interpretation of the comparison. This process seems to align with how the fit of an interpretation of an analogy is performed according to structure mapping models (Falkenheiner, Forbus, & Gentner, 1989). Future research that examines processing time as well as learning gains resulting from systematic variations in entity and relation ambiguity will help to elucidate which of these possibilities for the mechanism of how ambiguity of comparisons affects learning is most tenable.

Although these results are at present derived from only one classroom, they still have implications for teaching in math classrooms more broadly. Having a framework for the ambiguities of comparisons can aid teachers in identifying aspects of their own and students' comparative statements that may be potentially confusing. Recognizing ambiguous comparisons and thinking through the possible ways that students might misinterpret them can help teachers deduce why a student might have a certain misconception. If less ambiguous comparisons are more beneficial for learning, then it becomes important for teachers to be more careful with their comparative language as well as encourage more explicit and precise communication of comparative statements by students.

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