Dragging as a Referential Resource for Mathematical Meaning Making in a Collaborative Dynamic-Geometry Environment

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Abstract: This paper focuses on the referential roles played by dragging moves on dynamic-geometry representations in a collaborative-geometry problem-solving context. Through an interaction analysis of chat excerpts where dragging is used by a team of students to explore the geometric properties of a given polygon, the paper investigates the role of dragging on the facilitation of joint mathematical meaning making online. Our qualitative findings suggest that the indexical properties of the dynamic constructions are specified and recalibrated through the coordination of dragging actions with textual chat, where the two types of actions mutually elaborate each other.

Keywords: dynamic geometry, meaning making, referential practices

Introduction

Dynamic Geometry Systems (DGS) such as GeoGebra, Geometer's Sketchpad and Cabri offer unique affordances for exploring and making sense of geometry (Arzarello et al., 2002; Hölzl, 1996). The visual interface provided by such micro-worlds allows students to construct geometric objects by using elements of Euclidean Geometry such as points, lines and circles through a digital analog of compass and straight-edge constructions. More importantly, the object-oriented design of these micro-worlds allows students to dynamically act on these constructions by dragging their constitutive elements, which helps them to interactively explore the implications of the dependencies within those constructions (Stahl, 2013). By developing increasingly more purposeful dragging strategies, students may notice how a family of Euclidean constructions relate to each other and whether specific invariants are present in that family of figures (Arzarello et al., 2002). Therefore, such dynamic representations can be instrumental in helping students develop a deeper understanding of geometry by making otherwise obscure ideas/theorems in geometry more accessible. The possibility of testing an invariant across a continuum of cases can also help students to develop intuitions for generalizations that go beyond the particular construction view at hand (Leung, 2008).

The nature of the dragging actions through which geometric constructions are manipulated and explored, and their role in facilitating students' understanding of geometry concepts have been investigated by several studies in the math-education literature (Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010; Leung, 2008; Lopez-Real & Leung, 2006; Hölzl, 1996). In particular, Arzarello et al. (2002, p.67) proposed a hierarchy of dragging modalities that distinguish *wandering dragging* (randomly moving basic points to fish for interesting configurations or regularities in the dynamic diagram), *bounded dragging* (moving a restricted point), *guided dragging* (moves aimed to give the dynamic drawing a particular shape), *dummy locus dragging* (moves that reveal that a point is restricted to move on a specific path), *line dragging* (drawing new points along a line in order to keep the regularity of the figure), *linked dragging* (linking a point to an object and moving it onto that object) and *the dragging test* (moves aimed to test if a particular property of the current shape is preserved). These dragging actions are employed by students at different stages of their problem-solving activity, which provide insights into their reasoning with dynamic representations. In particular, wandering and guided dragging are employed during exploration/discovery phases, dummy locus dragging often hints at the construction of a conjecture, and the dragging test is often used to validate/justify conjectures. Therefore, dragging is treated as a key process facilitating the development of cognitive structures that bridge perceptual observations with formal accounts of deductive reasoning in geometry (Arzarello et al., 2002).

The focus of the studies reviewed thus far has been on the individual learner developing a sense of understanding through his/her interaction with dynamic representations. However, acting on these dynamic resources also has a social significance, which changes the problem context not only for the actor himself but also for collaborators witnessing those actions. At the individual unit of analysis, the meaning-making role of the dragging actions may be difficult to investigate from the actions themselves or think-aloud protocols. In a collaborative problem-solving situation, such actions become resources for joint meaning making, which are acted upon, referred to, reasoned with, and questioned in collaborative discourse (Stahl, 2009). Therefore, such collaborative activities present a perspicuous setting for researchers to explore how actions with and around dynamic geometry objects facilitate the development of shared mathematical understanding.

This paper focuses on the referential roles played by dragging moves on dynamic-geometry representations in a collaborative geometry problem-solving context. Our interest is motivated by recent CSCL studies that treat collaborative problem solving as "discovery work," in which collaborators work out the indexical details of their joint situation by calibrating and recalibrating references to relevant constituent elements of their shared task and its evolving solution (Zemel & Koschmann, 2013; Koschmann & Zemel, 2011). Indexical expressions refer to those linguistic resources whose sense depends on the context of the utterance. Through a process of calibrating and recalibrating references to an evolving space of persistently available diagrams, participants increasingly specify what those representations mean for them as part of an evolving solution account. Referring expressions initially function as a place holder for what is currently not known, which gets specified further (i.e., thingified) as subsequent actions and references modify their sense in interaction (Koschmann & Zemel, 2011). We argue that dragging actions have a similar referential role, which facilitates the discovery process with dynamic representations in geometry. Through an interaction analysis (Jordan & Henderson, 1995) of excerpts where dragging is used to explore the geometric properties of a given polygon, we identify the role of dragging on the facilitation of joint mathematical meaning making by studying how the dragging is used to increasingly specify the indexical properties of the dynamic construction.

Methods and data

The excerpts analyzed in this paper are obtained from the Virtual Math Teams (VMT) Spring Fest organized by the Math Forum in 2013. The analyzed chat session is part of a broader curriculum-development activity including the use of dynamic geometry in math classes supported through online collaborative-learning activities in the VMT system. The team consisted of Fruitloops, Cornflakes and Cheerios who are female students about 14 years old, who have not yet studied geometry. This team completed seven hour-long chat sessions of dynamic-geometry tasks in the VMT environment before they met for the session from which the excerpts were obtained. In this session, the team was given a set of 21 different quadrilaterals and was asked to (a) identify their dependencies, and (b) tell how each of them was constructed. The task description suggests participants drag the vertices of each quadrilateral to see what is special about each one. The GeoGebra application also hints at the presence of dependencies/constraints by shading vertices that are dependent on other points. Excerpts from this chat session were subjected to interaction analysis to investigate the referential roles fulfilled by dragging actions.

During the session, participants interacted through the VMT environment (Stahl, 2009), which provides a chat interface with an integrated electronic drawing area with collaborative dynamic-geometry drawing capabilities. The dynamic drawing area is based on GeoGebra, a popular dynamic-geometry application. The VMT environment allows a group of users to co-construct and discuss shared dynamic-geometry objects online. Access to the drawing area is managed through a turn-taking mechanism, which allows only one user at a time to construct or manipulate dynamic objects. The VMT system also supports researchers by providing re-playable logs of these sessions for analysis, allowing step-by-step walkthroughs of drawing and typing actions that took place during the online student sessions. The excerpts discussed in this paper involve manipulation of dynamic objects through dragging moves—which is challenging to present in a text document. For that reason, screenshots that capture intermediary states of the dragging actions are provided to complement the chat logs.

Analysis

Excerpt 1

The excerpt starts when the team decides to move on to polygon #2 (i.e., EFGH). Before exploring polygon #2, the team took turns to explore polygon #1 (i.e., ABCD) by dragging each of its four vertices; they quickly concluded that none of the vertices had any dependencies (i.e., they are free points). In line 1, Cornflakes announces that she will explore polygon EFGH, and then she takes control of the drawing area. Cornflakes first drags vertex F. The series of screenshots displayed in Figure 1 shows how polygon #2 changes while Cornflakes is dragging point F. Cornflakes drags point F up, left, down and then right, tracing almost a complete circle around point E in a counter-clockwise direction. Note that points E and H are unaffected by this drag, but point G is apparently moving as F is dragged.

1. **cornflakes** (3:20:26): ill do polygon efgh
2. ***cornflakes** (3:20:33): takes control of the drawing area*
3. **cheerios** (3:20:37): just say the number its easier

4. ***cornflakes** (3:20:40-3:20:50): drags points F (Figure 1), H, D, E (Figure 2) and G (drags on H, D, and G were not visualized in the figures. D is part of another polygon the team worked on prior to EFGH, which is not displayed).*
5. ***cornflakes** (3:20:52): releases control of the drawing area*
6. **cornflakes** (3:21:17): okasy polygon 2 has all points moving except point g
7. **cornflakes** (3:21:28): and point g is also a different color
8. **cheerios** (3:21:40): do u think it is restricted
9. **cheerios** (3:21:44): or constrained
10. **fruitloops** (3:21:49):i feel like poly 1 and poly 2 are almost exactly the same except that poly 2 had one point that is a lighter shade
11. **fruitloops** (3:22:04): can i try moving it?
12. **cornflakes** (3:22:17): sure
13. **fruitloops** (3:22:25): and @ cheerios , i dont know for sure

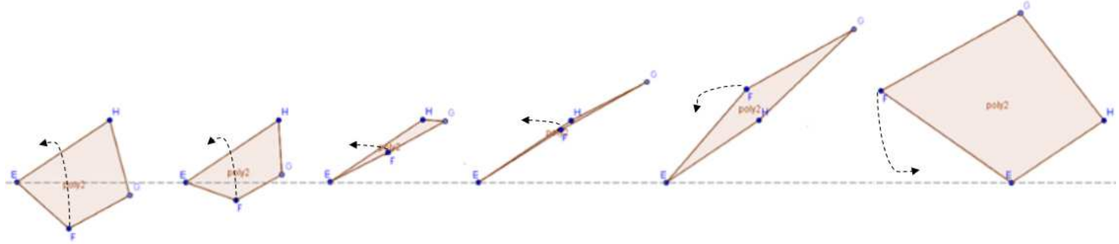

Figure 1. Cornflakes drags vertex F counter-clockwise around point E. Each screen shot corresponds to different stages of the polygon as F is dragged. The dashed lines are provided to aid the interpretation of the dynamic changes enacted by the dragging action. The dashed line over E, which was not affected by the drag, is provided as an anchor to aid the visual comparison between stages. Arrows show the direction of the drag.

Next, Cornflakes drags vertex H up and down. Then she begins to drag point E. The steps of this dragging action are displayed in Figure 2 below. Point E is slightly moved to up-right and then to bottom-left, which did not seem to affect any other vertex, except very minor shifts on G's position. Finally, Cornflakes drags point G. Point G is moved up-left and then down-right slightly as a consequence of this dragging action. None of the other vertices seem to be affected.

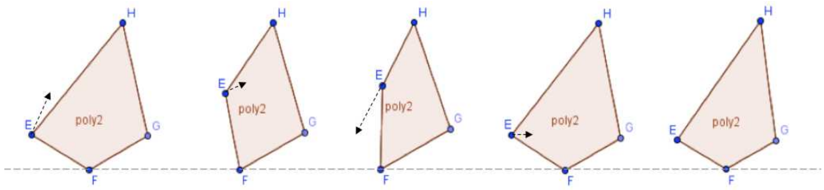

Figure 2. Cornflakes drags point E. Black arrows indicate the direction of the next drag.

After dragging point G, Cornflakes posts two chat messages (lines 6 & 7), announcing that "polygon 2 has all points moving except point g", and notes that point G is also marked with a different color. Cornflakes' account marks G as different from other points based on the claim that all points can move except G, even though she has explicitly moved point G during her last drag. The way she formulates her observations from her drags suggests that she is oriented towards whether a point can be freely moved or not. Cheerios responds to Cornflakes in line 8 by asking if she thinks "it" is restricted or constrained. The indexical "it" can be read as a reference to point G, since this point was the most salient object mentioned in Cornflake's chat messages besides polygon #2. By explicitly mentioning the terms "restricted" and "constrained," Cheerios invokes relevant terminology that encodes specific distinctions among the kinds of points one can make in GeoGebra. Hence, Cheerios' message can be read as an implicit assessment of Cornflake's account in terms of its descriptive adequacy, as well as an attempt to orient Cornflake's proposal towards a more formal account. So, Cheerios' statements can be seen as a *recalibration* move. Both terms seem to index a kind of limitation in the movement of a point based on their dependencies on other points, but the specific distinction encoded in this terminology has not been explicitly specified yet.

Next, Fruitloops posts the message she has been typing while Cheerios' messages appeared in chat, which states that polygon #1 and polygon #2 are almost the same except for the point with the lighter shade (line

10). Fruitloops' account seems to be informed by her observation of Cornflakes' prior dragging moves, and makes a visual reference to the color-coding of each vertex. She then requests the team's permission to try moving the polygon on her own in line 11. In line 12, Fruitloops responds to the question raised by Cheerios, that she is not sure about the restricted/constrained distinction.

Overall, in excerpt 1, the team seems to be oriented towards how the points can be moved around based on the witnessed dragging actions enacted by Cornflakes. The team does not mention more specific dependencies among the vertices, as the drags are rather minimal and hence have not yet hinted at the more complicated structure underlying polygon #2's construction. The team members seem to be oriented towards visually salient features of polygon #2, such as having a point G that is lighter in shade. Based on what is revealed by the drags performed by Cornflakes, the team seems to endorse the interpretation that polygon #2 is very similar to polygon #1 except for the vertex with the light blue shade. Cheerios contributes to the discussion by making the concepts "constrained" and "restricted" relevant to ongoing interaction as a means to categorize vertices. Yet, it is still not clear how these terms should be applied to the problem at hand.

Excerpt 2

Fruitloops takes control of the GeoGebra area and begins to manipulate the shared dynamic drawing (line 15). Figure 3 shows a chronologically ordered series of screen shots from her dragging of point G. The dashed reference lines crossing over point E, which remained stationary while Fruitloops was dragging point G, are provided to aid the comparison of different stages. Fruitloops' drag gradually moves point G in a circular motion, first in a clockwise and then in a counter-clockwise direction, which is followed by several full circles in both directions. As Fruitloops is performing the drag on G, she seems to gradually notice the path that point G is constrained to, which is evidenced in the way she moves the vertex back and forth repeatedly in clockwise and counter-clockwise directions in this episode (Figure 3). Meanwhile, Cheerios requests permission to access the drawing area, which is acknowledged by Cornflakes. However, Fruitloops holds onto her turn in the drawing area, while she is typing what will appear in line 18, which states, "so point g only moves in like a circular motion around point f." Fruitloops' account is a reflection on or noticing of what has been discovered in the dragging. The message specifies the relationship she notices between points G and F without making any reference to more technical terms such as restrictions or constraints, but using only colloquial terms or descriptions. In the next line, Cornflakes agrees. This is followed by further drags of point G around F by Fruitloops, which seem to complement her exposition in line 18 with an enactment of the verbally described movement pattern. These drags also simultaneously verify the proposed relationship, which recalibrates the status of vertex G for the group in this context.

14. ***fruitloops** (3:22:42): takes control of the drawing area*
15. ***fruitloops** (3:23:04): drags Point G (Figure 3)*
16. **cheerios** (3:23:18): ok can i try
17. **cornflakes** (3:23:22): sure
18. **fruitloops** (3:23:23): so point g only moves in like a circular motion around point f
19. **cornflakes** (3:23:35): @fruitloops yea
20. ***fruitloops** (3:23:50): drags Point G*

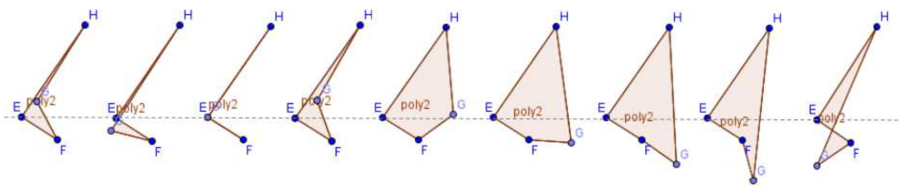

Figure 3. Chronologically ordered screenshots taken while Fruitloops is dragging vertex G.

In this episode, through her rigorous drags of vertex G, Fruitloops uncovers an important property of polygon #2: that vertex G always follows a specific circular path around point F. As soon as she gains control of the drawing area, Fruitloops starts dragging the point with the lighter shade. Fruitloops' initial drags on G seem rather exploratory, which gradually becomes more orderly and purposeful as she notices the constraint imposed on G. While Fruitloops is communicating this noticing to her teammates, she coordinates her actions across both chat and drawing areas in such a way that her verbal description in chat can be read in relation to her ongoing enactment on the shared drawing. In short, the sequential organization of Fruitloops' actions across both interaction spaces made her point witnessable by her teammates. This instance highlights another important aspect of dragging in dynamic geometry. The progression of drags from exploratory trials to purposeful demonstrations/tests serves both as a public display of an evolving understanding and as a resource for communicating abstract visuo-spatial relationships that may be difficult to articulate in text. Thus, drags also have a social role in this context, as demonstrable actions embodying specific conjectures about dependencies among geometric objects.

Excerpt 3:
Following Fruitloops' demonstration, Cheerios asks about the difference between the terms "constrained" and "restricted" in line 23. Cornflakes states that "constrained is limited function," which provides some specificity for one of the terms. In the meantime, Fruitloops continues to drag vertices of polygon #2. She first drags point H slightly to the bottom. No other point seems to be affected by this move. Next, she begins to drag point E (Figure 4). Point E is gradually moved away and towards point F, which simultaneously moves point G out and towards point F. Fruitloops carefully and slowly drags E around F, which suggests that she is oriented towards the relationship among E, G and F triggered by the dragging action on E.

21. ***fruitloops** (3:24:09): drags Point H (Figure 4)*
22. ***fruitloops** (3:24:15): drags Point E (Figure 4)*
23. **cheerios** (3:24:16): what si the difference between constrained and restricted
24. ***fruitloops** (3:24:17): drags Point G (Figure 4)*
25. ***fruitloops** (3:24:21): drags Point E (Figure 4)*
26. **cheerios** (3:24:24): is*
27. **cornflakes** (3:24:41): constrained is limited function
28. **fruitloops** (3:24:46): also when you move e, g moves away or closer to f
29. **fruitloops** (3:25:08): so i think g it definitly constrained
30. ***fruitloops** (3:25:12): drags Point H*
31. **cornflakes** (3:25:13): yes
32. **cornflakes** (3:25:19): i think that too
33. **cheerios** (3:25:25): why though
34. ***fruitloops** (3:25:31): drags Point,F*
35. **fruitloops** (3:25:59): and g moves whenever you move point e and f but it doesnt move when you move h
36. **cheerios** (3:26:20): okay
37. ***fruitloops** (3:26:31): releases control of the drawing area*

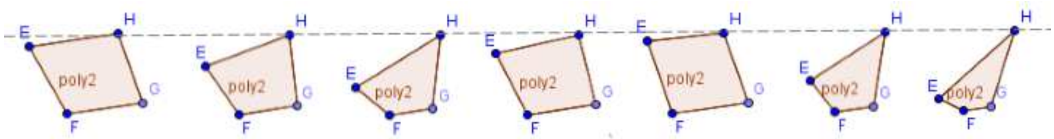

Figure 4. Fruitloops drags vertex E, first towards F, then away from F and then towards it again.

Fruitloops then posts in line 28 that "also when you move e, g moves away or closer to f." This can be read as a verbalization of the recent drag on E, which demonstrates the relationship among points E, G and F, where dragging E affects the position of G with respect to F. Similar to the previous instances, the verbal announcement of the noticed properties immediately follow the dragging actions. In line 29, Fruitloops elaborates on the prior message by proposing that G must be constrained. This statement also responds to the ongoing discussion of the distinction between constraints and restrictions, by proposing point G as an instance of a constrained object. In other words, the posting indexes G as a particular instance of a constrained point. In lines 31 and 32, Cornflakes concurs with Fruitloops' observation. In line 33 Cheerios posts a message wondering why the proposed relationship holds, which problematizes for the whole team the underlying cause of the relationship proposed by Fruitloops. In line 35, Fruitloops summarizes her observations after performing another drag on F. She states that point G moves whenever points E and F are moved, but G does not move when H is moved.

In this episode, Fruitloops identifies additional key relationships among points E, F, G and H through her systematic dragging of these points. She is oriented toward observing how each point is influenced by her drags on other points. Through initially explorative and progressively deliberate drags, Fruitloops notices that G's position is influenced by moving E or F, but not H. However, her statements do not specify the nature of those relationships in terms of concepts such as lengths or angles yet. The verbal accounts are primarily characterizations of visual effects triggered by drags of different points. Based on the relationships identified between E, F and G, Fruitloops proposes that G must be constrained, which provides further specificity to (i.e., a recalibration of) what is referred to by the term "constraint" by proposing G as an exemplar (lines 28, 29).

Excerpt 4

In line 38 Fruitloops takes up the prior discussion of the distinction between constraint and restriction. Fruitloops suggests that G is constrained because it can be moved, but the function is limited (i.e., limited to move on a circle around F). Then Fruitloops posts a question asking for the definition of "dependant" (sic) in line 40. This concept is mentioned in the task description, which asks the team to identify the dependencies in each polygon. About a minute later, a chat message from Cheerios appears, stating the need for the other line or point: otherwise "it wont work." In line 43, Cornflakes agrees and states that some points depend on each other. The definitions are rather implicit and ambiguous at this point, but the concept of dependence gradually attains its meaning as a kind of connection between two or more objects in this exchange.

38. **fruitloops** (3:26:42): @ cheerios. i think its constrained because it moves but the function is limited
39. **cheerios** (3:27:36): oh i see
40. **fruitloops** (3:27:37): what is the definition of dependant
41. **cheerios** (3:28:52): u need the other line or point otherwise it wont work
42. **fruitloops** (3:28:54): do you guys have any idea of how this was made?
43. **cornflakes** (3:29:15): yeah some points are dependent on others
44. **cornflakes** (3:29:43): maybe some invisible circles and the shapes could be dependent on thos circles
45. **cheerios** (3:30:02): yea maybe like the triangles
46. **fruitloops** (3:30:20): maybe because point g only moves in a circular motion around point f
47. **cornflakes** (3:30:35): but why?
48. **fruitloops** (3:30:55): i think it has to do with how it was constructed
49. **cheerios** (3:31:03): i agree
50. **cornflakes** (3:31:29): YES
51. **fruitloops** (3:31:44): cause eremember how before in the other topic we would sometimes use circles to construct stuff and then hide the circles? well maybe thiis quad was made using a circle
52. **cornflakes** (3:31:58): yeah and one of the points was on the circle
53. **cheerios** (3:32:38): yeah that makes sense remember when we made the triangle the same thing happened
54. **cornflakes** (3:32:43): yes
55. **fruitloops** (3:33:10): but i dont really know how it could have been made?
56. *__fruitloops__ (3:33:48): releases control of the drawing area*
57. **cheerios** (3:34:14): maybe they used another shape instead of circles
58. **fruitloops** (3:34:17): do you thinkk point e is the same distance away from f as g?
59. *__fruitloops__ (3:34:25): takes control of the drawing area*
60. *__fruitloops__ (3:34:26-3:35:02): drags Point G (Figure 5)*

In line 42, Fruitloops asks the team if they have any idea how the polygon was made. In line 44, Cornflakes proposes that there may be invisible circles accounting for the dependencies they have uncovered. In line 45, Cheerios agrees and states that this situation is “like triangles.” Cheerios seems to be referring to the team’s past constructions during previous sessions, where they used circles to make equilateral and inscribed triangles. In line 46, Fruitloops endorses the possibility of a hidden circle, based on the observation that G is only moving in circles around point F. In line 51, Fruitloops elaborates further by reminding other members about a past exercise where they used a circle as part of a larger construction and then hid it from view by making it invisible in GeoGebra. In line 57, Cheerios proposes the possibility that the point may even be constrained to an object other than a circle.

In line 58, Fruitloops solicits other members’ assessment about the observations that points E and G are equally distant from point F. Next, she drags point G on the GeoGebra board, making circles around point F. Snapshots from Fruitloops’ dragging actions are given in Figure 5. Fruitloops slows down when point G gets near point E, and drags it back and forth as the two points coincide with each other. This drag seems to explore the possibility that EF and FG have the same length. This is the first instance where a group member mentioned distance as a way to characterize a dependency among a set of points.

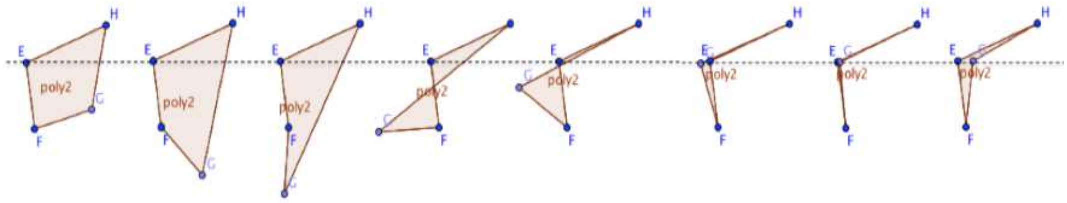

Figure 5. Fruitloops’ drag on vertex G. G is seen as moving in a circle around F. Fruitloops slows down when G is about to move near vertex E.

In this episode, the team starts to reflect on the relationships uncovered between the points, and the terminology that should be used to characterize those relationships. The team begins to develop conjectures about possible ways polygon #2 might have been constructed. At this time, they relate the observed behavior to their prior experiences using circles in earlier sessions. The team seems to agree on the idea/conjecture that at least G would be constrained on such a circle. The notion of *dependence* makes its first appearance in the chat, gradually becoming a resource for describing how the polygon might have been constructed. In this episode, the team makes another key observation regarding the underlying structure of polygon #2: that the edges EF and FG have the same length. Fruitloops’ drags of point E around F led her to realize that her drags influence the length of EF and FG in similar ways, where E and G can even be collapsed onto the same point.

Towards the end of their discussion of polygon #2, the team discusses how the dependencies they had discovered could be implemented in GeoGebra. The possibility of using an invisible circle for constraining G and the use of the compass tool to define two line segments of the same length are mentioned as possible steps in the construction. The team also discusses the order of the construction steps that might have been used to produce polygon #2. The proposed steps of the construction can be considered as an informal proof account, explaining why the polygon has the discovered properties. However, the team disagrees about which point would be plotted first, and cannot account for the joint relationship between points E, F and G, which precludes them from proceeding further in their joint inquiry.

Discussion

Previous literature characterizing dragging moves has primarily focused on how an *individual* learner’s cognitive processes are shaped through interaction with dynamic representations, without emphasizing the social and practical significance of such actions. In this paper, we underlined the social-interactional implications of dragging actions in a *collaborative* CSCL problem-solving context. The excerpts analyzed in this paper present a detailed view of the lived work of joint reasoning performed by a team of students while they were working together to discover the geometric properties of a given dynamic polygon. The team went through a sequence of sense-making steps, including dragging, noticing, stating in chat, bridging to past meaningful experiences of intersubjective shared understanding, and using technical terms like dependency. Our analysis of the excerpts suggest that through an interactive process of *calibrating and recalibrating their indexical references* (Zemel & Koschmann, 2013) to the evolving visual configurations witnessed during different dragging performances, the team members were able to collectively notice several key dependencies among constituent elements, describe them in colloquial/semi-formal terms and produce conjectures for the underlying causes of those dependencies.

The progression of dragging performances from exploratory trials to purposeful demonstrations serves both as a public display of an evolving understanding, and as a resource for noticing and communicating abstract visuo-spatial relationships that may be difficult to describe and follow in textual communication. In the excerpts analyzed above, the availability of the intermediary stages of dragging actions made the reasoning that goes with the unfolding dragging activity witnessable by the group. The witnessed unfolding of visual changes served as an indexical ground which (a) gave sense to subsequent utterances that refer to the noticed regularities, and (b) provided further specificity to technical terms that distinguish relevant geometric relationships such as constraints and dependencies by enacting them in the dynamic figure. Moreover, the emerging purposefulness of the drags was made evident with verbal glosses following an episode of dragging, which accounted for what was there to be noticed. Hence, actions in both interaction spaces mutually elaborate each other, where (a) drags highlight key relationships and eliminate the need to verbalize every complex detail, while (b) verbal accounts direct others' attention to relevant parts of the figure where the regularities can be located.

The analyzed excerpts also suggest that not all drags are equally effective for noticing key geometric properties. This point is supported by a comparison of the dragging performances of Cornflakes and Fruitloops, and the subsequent proposals the team members had made in the discussion following those drags. Initial drags by Cornflakes led to the conjecture that polygon #2 is very similar to #1 (whose vertices had no dependencies), except for the vertex that was marked with a different color. Only after Fruitloops took over and performed more strategic drags, did the team realize that there was more to the underlying geometric structure of polygon #2. In particular, the team noticed the following regularities: (a) G moves around F in a circle and when G is moved no other vertex moves, (b) when H is moved, no other vertex moves, (c) G moves when F is moved, (d) G moves when E is moved, and (f) E and G are always equidistant from F.

The analysis of the team's work in dragging its figures shows how collaborative learning about the nature of geometric dependencies develops gradually through hands-on exploration guided by challenging tasks in a computer-supported environment. In the remaining part of their chat session, which is not covered in the above excerpts, the team members continued to explore similar polygons by taking turns dragging. The dragging strategies developed in the excerpts above were appropriated by other members during those explorations. This exemplifies the gradual transformation of one member's public display of dragging-mediated reasoning into a shared practice of geometric reasoning for the team. Through calibration and recalibration of indexical references that refer to the discovered properties of the shared dynamic drawing, team members gradually made sense of key geometry concepts as they were enacted by dragging actions on shared figures. The affordances of the VMT environment for making the results of intermediary stages of drags available for all participants and the way participants coordinated such actions with their chat messages were consequential for collaborative meaning making online. For this reason, a key design requirement to support collaborative learning in CSCL settings should be the inclusion of mechanisms that help participants effectively coordinate representational affordances, especially in contexts like geometry, where diagrams and concepts need to be closely aligned with each other. Likewise, an important part of CSCL methodology should include the analysis of discourse and actions as referential components of intersubjective meaning making.

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