# Students' Resources for the Construction of Scales for Graphing

Cesar Delgado, University of Texas at Austin, 1912 Speedway Stop D5500, Austin, TX 78712, USA, cesar_delgado@austin.utexas.edu
Margaret Lucero, Santa Clara University, 500 El Camino Real, Santa Clara, CA 95053, USA, mlucero@scu.edu

**Abstract:** Graphing is fundamental in the scientific process. Scales are key but little-studied components of graphs. Using a "fine-grained constructivist" perspective, we investigated the resources undergraduate students activate in constructing a scale for difficult data sets ranging over 10 or more orders of magnitude. Following a constant comparison method, we identified resources including: show all elements, zoom in, ratio, halving, and conversion. Students also used their knowledge of bar graphs, histograms, and logarithmic, powers of ten, and linear scales as resources, at times inappropriately. Implications for instruction are outlined.

## Student Resources for Scale Construction

Graphing is a fundamental part of the scientific process. Scales are key but little-researched components of graphs. We investigate the resources that undergraduate students activate in constructing a scale for difficult data sets, adopting a "fine-grained constructivist" perspective (Elby, 2000). We use difficult tasks to preclude students' simply following well-known procedures. Identifying students' resources for representation and building upon them in the classroom should lead to deeper understanding of conventional graphing practices.

## Graphs and Scales

Natural phenomena can often be modeled mathematically; graphs can portray experimental data or the modeled relationship between variables. Graphs effectively describe continuous variation (Lemke, 1998) and succinctly summarize large amounts of data (Latour, 1987). US science standards note a role for graphing in scientific and engineering practices such as "Analyzing and interpreting data" (National Research Council, 2012).

Scales are essential components of graphs, and play an important role in graph interpretation (Leinhardt, Zaslavsky, & Stein, 1990). Scales permit the graphical representation of the magnitude of physical or mathematical quantities, or of the different values of a qualitative variable. Common scales consist of evenly spaced intervals portraying constant additive differences (linear scale), multiplicative differences (logarithmic or powers of ten scale), different values of a nominal- or ordinal-level variable (qualitative scale), or "bins" with ranges of data (histogram scale). In linear and logarithmic scales, the magnitude of data points is represented solely by their location along the scale; such scales are "homogeneous", and they allow the relative magnitude of data points to be directly compared from the graph (Nemirovsky & Tierney, 2001). Homogenous scales follow the convention that equal lengths along the scale represent an equal number of units (Leinhardt et al., 1990). The powers of ten scale is to be taught in high school, and linear, qualitative, and histogram scales in sixth grade or earlier, according to the US Common Core State Standards (NGACBP & CCSSO, 2010).

Research shows that students have difficulty in producing or interpreting conventional graphs (see review by Leinhardt et al., 1990). Difficulties constructing scales include placing quantitative data points on successive, evenly spaced tick marks regardless of values (e.g., Brasell, 1990), or taking a scale as discrete, i.e., considering that there are no points between the labeled points (Leinhardt et al., 1990).

## Theoretical Framework: Fine-Grained Constructivism

Recent research has explored the resources students possess that enable them to understand and engage in representation (e.g., diSessa, Hammer, Sherin, & Kolpakowski, 1991; Elby, 2000; Hammer, Elby, Scherr, & Redish, 2005; Sherin, 2000). This research stresses the importance of building on and reorganizing prior knowledge. It conceives of learners as holding multiple, fine-grained understandings to be called into service as needed, rather than holding coherent but naïve theories that are consistently applied (Elby, 2000; Smith, diSessa, & Roschelle, 1993). Learning, in this "fine-grained constructivist" view (Elby, 2000), consists of knowing when to apply what resource, and of connecting multiple resources into more complete, coherent, and scientifically normative theories. Student ideas are thus seen as potentially productive steps on the way to mature understanding, rather than as "misconceptions" to be replaced. In this view, resources are not intrinsically correct or incorrect, but are activated for an appropriate or inappropriate context (Hammer et al., 2005). Some resources for graphing include *what you see is what you get* (WYSIWYG – an overly literal interpretation of a graph), *stillness* (the lack of motion suggested by a horizontal line on a graph), *constancy* (the idea that something does not change, prompted by a straight line on a graph), and *sudden change* (suggested by inclined segments on a graph) (Elby, 2000). Not yet studied is how students approach constructing scales, and how the resources they employ can then point to instructional strategies that support deeper learning. The following research question guides our study: What resources do undergraduates employ in constructing scales?

## Methods

Our participants are 64 undergraduates at a major public research university in the Midwestern USA, enrolled in an interdisciplinary Geoscience/History course. The students were a cross sample of the university both by year (10% freshmen, 29% sophomores, 25% juniors, 35% seniors) and by major (19% STEM, 52% social sciences, 13% other, 15% undeclared)(Delgado, 2014). All enrolled students participated in the study. We expected that the diversity by year and major would provide insight into a broad array of students' resources.

Students completed identical paper-and-pencil instruments during the first and next-to-last discussion sections. The tasks analyzed here involved constructing one scale for the age of six events and one for the size of six objects, along a 25-cm horizontal line, so that their positions represented their magnitude. The 20 minutes allotted allowed all students to finish the assessment. The time data points were given as years ago (ya), and included: Big Bang (14 billion ya), emergence of life on Earth (3.8 billion ya), emergence of hominids (7 million ya), emergence of Homo Sapiens (600,000 ya), origins of writing (6000 ya), and first human moon landing (at the time, 39 ya). The size data points were given in nanometers (nm) and meters (m) and included: diameter of an atom (0.1 nm), diameter of an adenovirus (100 nm), width of a human hair (0.0001 m), diameter of a dime (0.018 m), height of an elephant (3 m), and length of a football field (100 m). The large range of values was intended to preclude the routine construction of a linear scale, in order to better reveal student resources. Six students were interviewed after completing the tasks for the second time; they were shown their scales and asked to discuss them. The students interviewed were a convenience sample - those who volunteered.

Coding for identification of resources was done using a constant comparison method (Strauss & Corbin, 1998). The data were analyzed in a series of iterative stages. After student interviews were transcribed, both authors independently analyzed them using open coding to identify emergent themes in each student's stated approach to the scale construction task, while also scrutinizing the interviewed students' scales. We identified many common themes among the transcripts and respective scales and came to a consensus by discussion. The interviews and scales then went through another round of coding to refine the resource categories. Both authors discussed any differences among the categories and reached consensus as to their definition and classification. In the final analytic stage, one author developed narrative descriptions for each participant's use of resources, while the other author searched for representative quotations from the transcripts. We then studied the scales of the students who were not interviewed in order to infer other potential resources, using an iterative coding process similar to the one for the student interviews. The findings from scales alone (without interview data) are more inferential and will require additional research to confirm or refine them.

## Results and Discussion

### Student #11: Show All Elements

This student's four scales were similar. She said they were linear scales, but she did not follow the convention of marking evenly-spaced tick marks and labeling them with numerical values. On a conventional linear scale, the four smallest data points should be indistinguishably close to the origin. She instead spaced the four data points out (see Fig. 1). She said, "So I realized that I probably might have been spacing it out just for space for writing", because "you can only go so small", even though she knew that the data points "should all be at that point right there". She felt the need to clearly portray all data points, even at the cost of accuracy. We call this the *show all elements* resource. It may stem from drawing, where one portrays every important element and usually avoids drawing one object on top of another. This resource is problematic in this case but useful in others. For instance, selecting a range for a scale that just encompasses the data (e.g., 0-25 for data ranging from 2-24) permits the visualization of finer differences between data points than a larger range (e.g., 0-100 or 0-1000). The appropriateness of the *show all elements* resource thus depends on context.

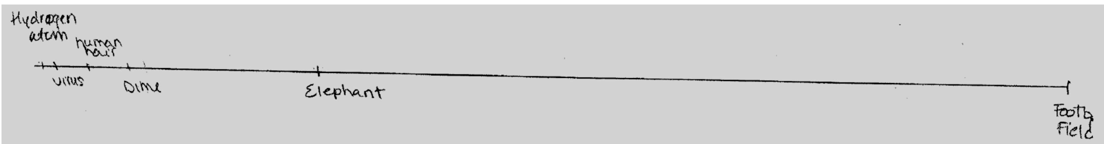


Figure 1. Student #11's scale for size, beginning of course.

### Student #46: Linear Scale, Zoom In, Ratio, Conversion

This student activated different resources for time and size. He explicitly stated that his scale for time was linear. His scale conformed to the convention of using evenly-spaced intervals to show equal numbers of units (Leinhardt et al., 1990) - see Figure 2. The conventional linear scale can become a resource in and of itself, composed initially of simpler resources that are coherent and useful for a given context. Over time and through repeated successful use, they become coordinated and constitute a resource as a whole (Hammer et al., 2005). The *linear scale* resource is neither correct nor incorrect *per se* (Hammer et al., 2005). The student realized the limitation of the linear scale for this data set: "I realized I would never be able to cram…". He created a second, coordinated scale, explaining, “This is me trying to compensate for the fact that I didn’t have any space for one billion years”. Yet, the extra scale did not solve the problem: “I realized I needed to zoom in much, much further because I still needed to get to 7 million years… If I wanted to do it on a linear scale, yeah, I would have had to keep zooming and zooming and zooming”. We term this resource *zoom in.* It may stem from experiences with microscopes, cameras, and adjusting the view in navigation, word processing, and multimedia presentation software. Zooming in is useful for this data set, but would be superfluous with small ranges of data.

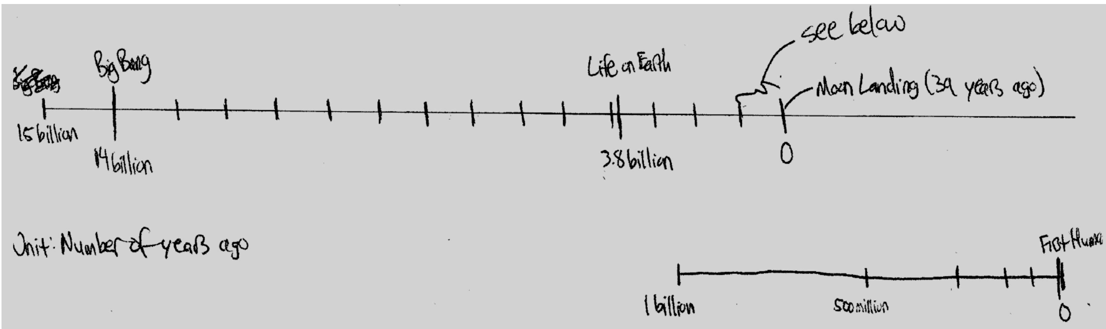


Figure 2. Student #46’s scale for time, beginning of course.

The student instead used a *ratio* resource for the size data, focusing on relative size differences between adjacent data points. This approach should in principle result in a scale with spacing identical to a logarithmic scale, although in this student’s scale the hair-dime difference is overly large (see Fig. 3). He said:

> I think when I was doing this, I was thinking this [atom-virus] is 1,000 times bigger and this [virus-hair] is another 1,000 times bigger. I left a bigger gap here [hair-dime] trying to compensate the bigger gap. See this jump here [elephant-football field]? This is only 30 times. In reality it should be much much smaller, but I was having trouble illustrating that.

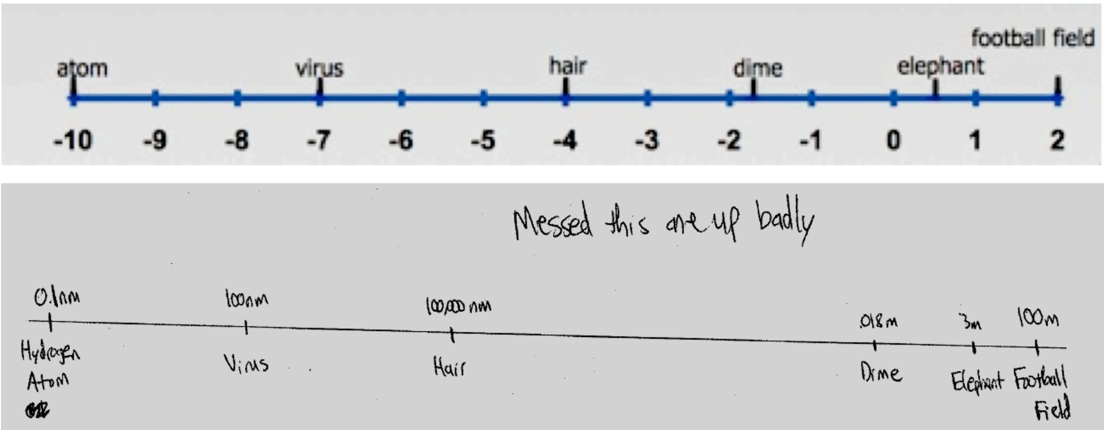


Figure 3. Student #46’s scale for size, beginning of course (bottom), and conventional logarithmic scale (top).

This student used nanometers for the smaller objects and meters for the larger ones (see Fig. 3). He noted that “It was probably easier for me to grasp this as 100 nanometers than… 0.0001 [sic] meters… It’s all about putting it in terms, for me, that are easy to understand.” We identified this as a *conversion* resource. This resource may stem from academic experiences with unit conversions within or across systems, from everyday experiences purchasing food (e.g., ounce to pound conversions) or thinking about time (e.g., changing 120 minutes to 2 hours), or both. This resource is useful for large ranges, but not for small ranges of data.

A striking feature about student #46’s scales is his activation of a linear resource for the time data but a ratio resource for size, despite the fact that both data sets had similar ranges and posed similar difficulties: “I wasn’t thinking of time as an idea of relative scale. I was thinking of time specifically as a linear, like you would see on a graph… or a time plot…I never would have considered doing time as a relative thing.”

## Student #17: Powers of Ten, Ratio

This student drew a conventional powers-of-ten scale for size, at the beginning of the course (see Figure 4). Just like with the conventional linear scale, a *powers-of-ten* scale can be a resource in and of itself. However, the student had serious misgivings about his graph: “I remember when I did this it didn’t fully represent how… It’s just misleading. I mean, the numbers are right, it just makes it that if someone who doesn’t understand the logarithmic of it… Its just going to look like these are, I don't know, the same." He instead drew a proportional scale drawing on the *ratio* resource on the end-of-course scale. This student's unease with a scale that followed conventions for a power of ten scale suggests that some students who learn to draw these may not fully understand them. Supporting the use of a ratio resource – which some students activated spontaneously – may improve instruction of power of ten scales so that deeper understanding is engendered.

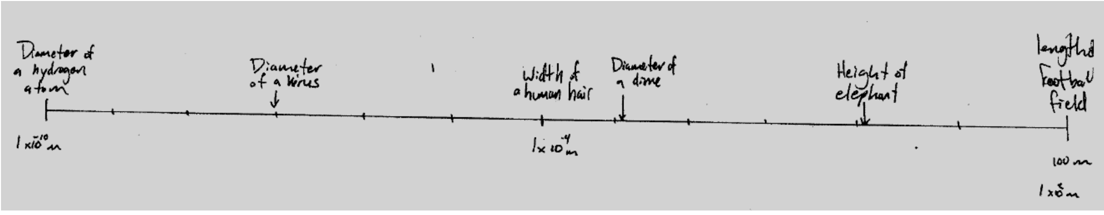


Figure 4. Student #17's scale for size, beginning of course

## Student #34: Powers of Ten, Conversion

This student produced conventional powers-of-ten scales explicitly labeled as logarithmic for all four tasks. On the end-of-course scale for size, he used the SI prefixes nano-, micro-, milli-, and kilo-, in combination with the numbers 1, 10, and 100 only (and 0.1 nm, at the smallest end of the scale). Thus, this student used the *conversion* and *powers of ten* resources. Unlike student #17, he had few misgivings about the logarithmic scale. However, like student #17, he also noted that size was easier to think about in terms of proportion than time.

Swarat and colleagues (Swarat, Light, Park, & Drane, 2011) consider that the use of different units for the large and small ends of a scale are indicative of a "fragmented" scale, the least advanced type of scale they characterized. However, thinking of a large range of size data in terms of various units, or even inventing new units – "unitizing" - may be a powerful and appropriate strategy, one used by experts (Tretter, Jones, Andre, Negishi, & Minogue, 2006). The resources of *conversion* and *powers of ten* are useful in this context but would be cumbersome and of little value for small data ranges, which can easily be represented with a linear scale.

## Student #48: Halving, Show all Elements

Student #48 produced linear scales for all four tasks. However, he did not use evenly-spaced tick marks to define intervals (see Figure 5). On the first scales, he had no tick marks at all (and yet the spacing of data corresponded closely to a linear scale). On the end of course scales, he used *halving* iteratively to construct his scale: "I tried to put zero at one end and the biggest thing that we had… and just tried to divide it up into similar units. … I'd go down the middle and go like 'that's half, so that's 50, 25, 12.5, 6.'" According to Confrey et al., halving and doubling are "operation primitives", or fundamental understandings (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009). The scale was homogeneous even though the tick marks were unevenly spaced. This student also felt the need to *show all elements*: "I tried to distinguish which ones I thought were bigger and which ones smaller. I don't know if their actual position is represented."

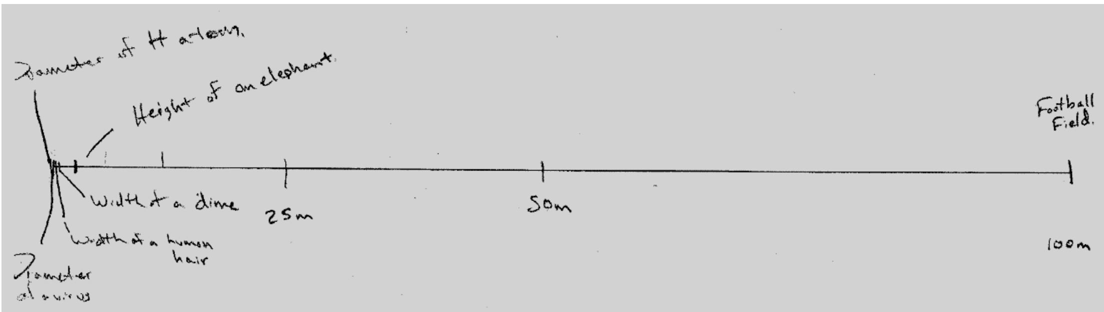


Figure 5. Student #48's scale for size, end of course.

## Student #63: Ratio, Logarithmic Scale, WYSIWYG

This student created all four scales employing a logarithmic resource but using "ballpark" estimations with rounded numbers. However, she found the logarithmic scale troubling despite having used it in a physics class:

> Student: Oh for some reason I thought you had to make it what do you call it… Spatially accurate, like to scale. So you'd have this one actually be 100 times longer than this one.
> Interviewer: That would be a linear scale.
> Student: So you're saying a logarithmic scale doesn't even take in to account whether or not it's like, spatially accurate? ... Yeah that's how the physics graph worked 'cause we got rid of the ten and just used the exponents. But then you get a really distorted graph.

She expected that length along the scale would be proportional to the size represented, which is a characteristic of linear scales and also of drawings. In interpreting her own scale, she activated the WYSIWYG resource and was dissatisfied when her scale did not display the transparency she desired. The tendency to represent space as space is a “boundary” that students need to cross in order to produce novel representations and to understand conventional ones (Sherin, 2000). The logarithmic scale does not represent space in a straightforward manner. The teaching of logarithmic graphs in school will need to acknowledge students’ preference for and experience with more naturalistic linear representations, and help students contextualize when each is more useful.

## Resources Inferred From Scales of Other Students

In this section, we present other possible resources that we infer from scales alone. These are more inferential than the ones identified from interviews and will require additional research to confirm and better understand.

### Qualitative Scales or Ordering

Prior research has reported students placing data points at successive evenly-spaced tick marks without regards to the values (e.g., Brasell, 1990), and we encountered some – see Figure 6. This may stem from a *qualitative scale* resource (as learned from bar graphs) for quantitative data. The qualitative scale resource is appropriate for ordinal- or nominal-level data in a bar graph, but sacrifices the relative size information of the quantitative data. It may be that *ordering* is the resource activated here, rather than resources related to bar graphs.

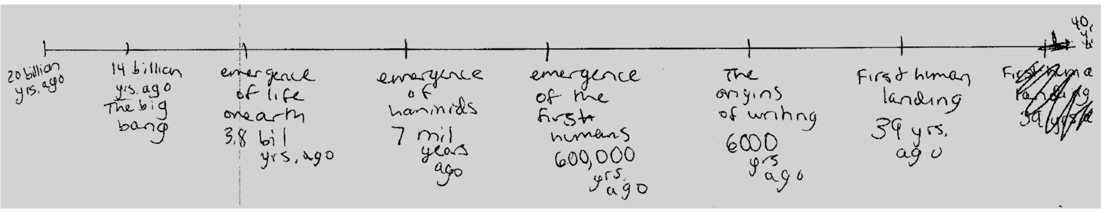


Figure 6. Qualitative scale (Student #1, beginning of course)

### Bins

One student created a scale with *bins* or intervals (e.g., 100-1000 years ago). This scale is appropriate for histograms; however, the actual value of quantitative data points is lost. See Figure 7.

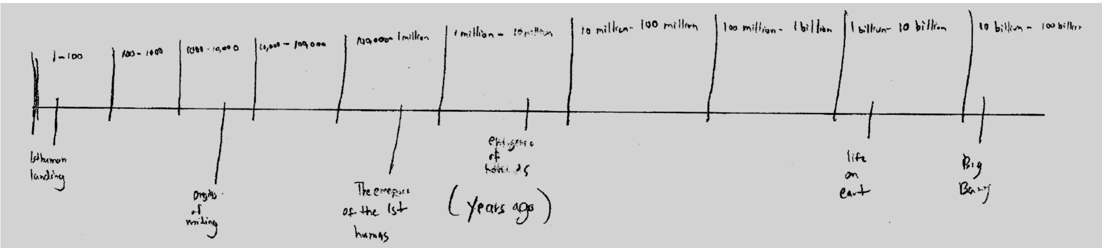


Figure 7. A scale using bins (intervals) (Student #18, end of course)

### Scale Break

Some students used *scale break* symbols to indicate that their scales did not include the full range of values. See Figure 8. While the tick marks were evenly spaced in this scale, it is not a qualitative scale because the scale breaks imply a break in the values, which are quantitative. While this resource does not actually solve the problem of representing a wide range of values, it does signal that problem explicitly.

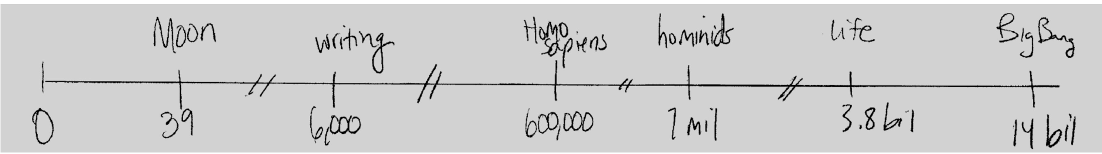


Figure 8. A scale with scale breaks (Student #10, beginning of course).

### Writing

One student created a scale with no tick marks. The scale included the name of each event followed by its age, in a sequential manner. See Figure 9. We speculate that the resource he activated was *writing*. This resource would be appropriate (with added punctuation) in a prose account of the data, but not to construct a scale.

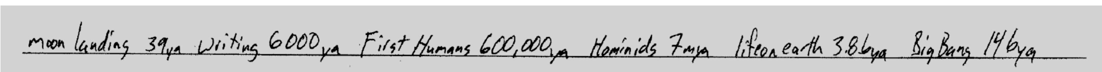


Figure 9. A scale that may reflect the use of a writing resource (Student #29, beginning of course).

### Grouping
Some students had an idiosyncratic placement of the data points, with the elephant alone in the middle of the graph. See Figure 10. This "fragmented" scale (Swarat et al., 2011) may stem from a *grouping* resource. The groups might correspond to objects many times smaller than a human, many times larger, and roughly human size, as the human body is a fundamental landmark for size (Tretter et al., 2006). Grouping or classifying is an important conceptual resource, but results in the loss of much information when building a quantitative scale.

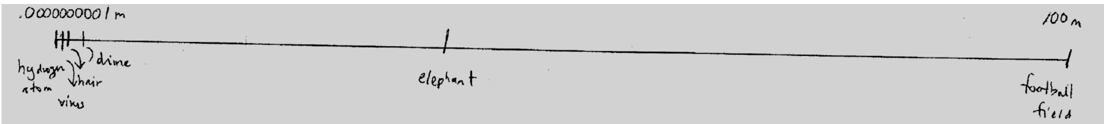


Figure 10. A scale that may reflect a grouping resource (Student #7, beginning of course).

## General Discussion
We can conceptually divide the resources we identified into several classes. Drawing has already been shown to be an important source of resources (diSessa, 2004); the *WYSIWYG* (Elby, 2000) resource identified here for scales, and the *show all elements* resources likely derive from drawing. Experiences with graphing also provide resources, some appropriate for this data (*logarithmic, powers of ten*), and some less appropriate or useful (*bins*, *qualitative scales*, *linear scales*). The next class of consists of resources that are appropriate for making sense of the wide range of data, but that would be cumbersome and superfluous with smaller ranges of data: *zoom in, ratio, conversion,* and *scale breaks*. Additionally, some resources employed fundamental knowledge: the operation primitive of *halving* (Confrey et al., 2009), *writing, grouping,* and possibly *ordering*.

The resources students employ, even if they lead to unconventional or suboptimal scales, are ideas on which to build. Additionally, the approximations to conventional representations that students invent, based on their resources, reveal potential pathways of learning. In the next section we discuss specific educational implications of our findings, which extend as far back as elementary school. Further research will be required to study the impact of these suggestions. Limitations of this study include the small number of students interviewed, the reduced racial or ethnic diversity (most students are non-Hispanic Whites), and the fact that the sample was filtered by ability (as these were students at a highly selective university).

# Implications

## Drawing, Measuring, and Homogeneous Spaces
Even among undergraduates, creating a homogeneous scale was far from universal. Instruction thus needs to create a greater awareness of homogeneous spaces. Drawing might be a useful resource because drawings of a plane orthogonal to the viewer (e.g., a bird's eye view) are homogeneous (if one disregards perspective). The idea that equal lengths represent an equal number of units is used in an implicit manner in drawing; instructional activities could make it explicit through the use of scale factors, beginning with simple ones such as a 12:1 scale where a distance of one foot in the real world is portrayed as one inch on the drawing. Such activities might also build understanding of conversion and ratios. Another early experience with homogeneous scales is the use of rulers. Activities have been developed in which students use collaborative exploration to reinvent measurement, in the elementary grades; these involve reflecting about the constancy of unit size, the need to place units end-to-end without gaps or overlaps, and the idea that measurements do not always consist of a whole number of units (Lehrer, 2003). Such activities may provide resources for graphing quantitative data.

## Different Types of Scales and Meta-Level Knowledge of Graphing
Students need to learn to construct and interpret linear, logarithmic, powers of ten, histogram, and qualitative scales, but they also need to acquire the meta-level knowledge surrounding each scale. Instruction may not typically be building this type of knowledge. Students may learn how to graph without linking to their "metarepresentational competence" - their existing understandings of what representations are for, how to create them, and how to evaluate them critically (diSessa & Sherin, 2000). They may be able to produce a fairly conventional scale, yet not truly understand what they have produced. One meta-level understanding is that the choice of representation is impacted by the use it will be given (diSessa et al., 1991). For our data sets, a linear representation might be effective in showing that the Big Bang occurred an extremely long time ago relative to human experience, but is less effective in showing the large relative differences among the more recent events. Logarithmic scales have pros and cons as well: they allow all six data points to be effectively represented, but placing data points that are not exact powers of ten is difficult, there is no way to place zero on the scale, etc. Scales produced using bins, grouping, ordering, or a qualitative approach likewise have advantages and disadvantages. When each type of graph is introduced, meta-level discussion about its idiosyncrasies, conventions, affordances, and constraints is essential so that students can consider when each is appropriate.

### Resources for Large Ranges of Data

Students tapped many resources to work with the vast ranges of time and space: through analogy with instruments and software (zoom in), by using proportional reasoning (ratio), by using more familiar numerals (conversion), or by sidestepping the complications of constructing a homogenous scale (scale breaks). The use of similar open-ended tasks with younger students may reveal additional resources that help them make sense of conventional scales and thus graphing. The use of more than one unit on a single scale is unconventional but might scaffold understanding when using large ranges of values. Units can serve as a powerful cognitive tool (Delgado, 2010). However, it would be important to then advance to normative, single-unit scales.

### Show all Elements, Ratio, and Halving to Scaffold Learning of Logarithmic Scales

For students to deeply engage in thinking about complex and counterintuitive logarithmic scales, a need to know must be established. The show all elements resource, along with a data set that spans many orders of magnitude, can establish a need to know, since a linear scale fails to distinguish among smaller data points. Instruction of logarithmic scales that builds on students' resources may lead to better understanding than students #17 and #63 displayed. Several students spontaneously used iterative halving to generate unconventional but homogeneous linear scales. This suggests the possibility of systematically using this resource for both linear *and* logarithmic scales. Linear halving (see figure 5) could be paired with multiplicative halving in which *evenly-spaced* tick marks are labeled with iteratively halved data. This would invite discussion of the nature of additive and multiplicative reasoning, scaffolding an understanding of linear vs. logarithmic scales. Since halving is thought to be an operation primitive (Confrey et al., 2009), base-2 logarithmic scales might provide an intuitive way to build conceptual understanding of base-ten logarithms. Other students used a ratio resource. Activities where students develop their own scales based on ratios between adjacent data points (as students #17, 46, and 63 did) could build conceptual understanding of the logarithmic scale. Spatial scale is better suited to ratio reasoning than temporal scale, according to our participants, so size data could be used before time data. A data set with data points that are exact powers of ten and differ by either one or two powers of ten might be useful. Once students realize that 100-fold steps should all be represented by the same length, as should 10-fold steps by a single (different) length, they can explore the size of the two steps. The crucial insight that a 100-fold difference should occupy twice the length as a 10-fold difference could be catalyzed by adding a data point at the geometric midpoint of points with a 100-fold difference. Another potentially powerful insight about the logarithmic scale is that every successive tick mark covers 90% of the remaining range. This idea could be scaffolded by iterative halving, where each tick mark covers half of the remaining range.

### Zooming and Logarithmic Scales

Student #46 zoomed into the last billion years of his time scale, but realized he needed to zoom in again and again. Repeatedly zooming in to the smallest *10%* of each scale, and making each scale the same length, could lay the foundations for understanding the logarithmic scale, where each smaller interval covers 10% of the previous. The course textbook (Christian, 2005) featured a series of scales, but they appeared at the beginning of each section rather than together. Juxtaposing successive magnifications may be more effective.

### Powers of Ten, Logarithmic Scales, and Homogeneous Spaces

Powers-of-ten scales are not homogeneous. For instance, between 1 and 10 there is a difference of 9 and between 10 and 100 a difference of 90, but each difference is represented by the same distance along the scale. A logarithmic scale transforms those values to 0, 1, and 2, constituting a homogenous space. However, the placement of data points is identical in both. Comparing and contrasting logarithmic and powers of ten scales in high school science or mathematics classes would allow an opportunity to reflect on homogeneous spaces.

## Conclusion

This study identifies some resources that are activated when students are tasked with constructing a scale. These resources are cued according to context – for instance, the time data was less likely to activate the ratio resource than size data. The power of the fine-grained constructivist perspective is that unconventional student ideas are not seen as "misconceptions" to be replaced, but as the application of a resource that is useful to students in a different setting. How teachers respond to student ideas is essential. Rather than dismissing them as wrong or seeing them as manifestations of a naïve theory, teachers can use students' ideas to help them build better-organized, more broadly-applicable ideas. As teachers become more aware of these resources, they can better plan instructional activities that allow students to construct more sophisticated understandings and explanations. If we first acknowledge students' resources for representation and build upon them in the classroom, they should develop a deeper understanding of the powerful and diverse standardized representations scientists and mathematicians have developed to represent magnitude and relationships between variables.

## References


Brasell, H. M. (1990). Graphs, graphing, and graphers. In M. B. Rowe (Ed.), *The process of knowing*. Washington, DC: National Science Teachers Association.

Christian, D. (2005). *Maps of time: An introduction to big history*. Berkeley: University of California Press.

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In *PME Conference Proceedings* (Vol. 2, pp. 345–352). Thessaloniki, Greece.

Delgado, C. (2014). Collective landmarks for deep time: A new tool for evolution. *Journal of Biological Education*. Advance online publication. doi: 10.1080/00219266.2013.849280.

Delgado, C. (2010). Units of length: a notational system for conceptual understanding of size and scale. In *Proceedings of the 9th International Conference of the Learning Sciences - Volume 2* (pp. 362–363). Chicago, Illinois: International Society of the Learning Sciences.

diSessa, A. A. (2004). Metarepresentation: Native competence and targets for instruction. *Cognition & Instruction*, *22*(3), 293–331.

diSessa, A. A., Hammer, D., Sherin, B., & Kolpakowski, T. (1991). Inventing graphing: Meta-representational expertise in children. *Journal of Mathematical Behavior*, *10*(2), 117–160.

diSessa, A. A., & Sherin, B. (2000). Meta-representation: An introduction. *Journal of Mathematical Behavior*, *19*(4), 385–398.

Elby, A. (2000). What students' learning of representations tells us about constructivism. *Journal of Mathematical Behavior*, *19*(4), 481–502.

Hammer, D., Elby, A., Scherr, R. A., & Redish, E. F. (2005). Resources, framing , and transfer. In J. Mestre (Ed.), *Transfer of learning from a modern multidisciplinary perspective* (pp. 89–119). Greenwich, CT: Information Age Publishing.

Latour, B. (1987). *Science in action: How to follow scientists and engineers through society.* Cambridge, MA: Harvard University Press.

Lehrer, R. (2003). Developing understanding of measurement. In J. Kilpatrick, W. G. Martin, & D. Schrifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 179–192). Reston, VA: National Council of Teachers of Mathematics.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, *60*(1), 1–64.

Lemke, J. L. (1998). Multiplying meaning: Visual and verbal semiotics in scientific text. In J. R. Martin & R. Veel (Eds.), *Reading science: Critical and functional perspectives on discourses of science* (pp. 87–113). London: Routledge.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and Council of Chief State School Officers.

Nemirovsky, R., & Tierney, C. (2001). Children creating ways to represent changing situations: On the development of homogeneous spaces. *Education Studies in Mathematics*, *45*(1), 67–102.

National Research Council. (2012). *A framework for K-12 science education: Practices, crosscutting concepts, and core ideas*. Washington, DC: National Academies Press.

Sherin, B. (2000). How students invent representations of motion: A genetic account. *Journal of Mathematical Behavior*, *19*(4), 399–441.

Smith, J. P., diSessa, A. A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *Journal of the Learning Sciences*, *3*(2), 115–163.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Procedures and techniques for developing grounded theory.* (2nd ed.). Thousand Oaks, CA: Sage.

Swarat, S., Light, G., Park, E. J., & Drane, D. (2011). A typology of undergraduate students' conceptions of size and scale: Identifying and characterizing conceptual variation. *Journal of Research in Science Teaching*, *48*(5), 512–533. doi:10.1002/tea.20403

Tretter, T. R., Jones, M. G., Andre, T., Negishi, A., & Minogue, J. (2006). Conceptual boundaries and distances: Students' and experts' concepts of the scale of scientific phenomena. *Journal of Research in Science Teaching*, *43*(3), 282–319. doi:10.1002/tea.20123


## Acknowledgments


This study was supported by University of Michigan Rackham Merit and School of Education Scholar's Award fellowships to the first author. We wish to express our thanks to the instructor and students of the course. Our appreciation to Leema Berland, Jill Marshall, Brian Fortney, and anonymous reviewers for very helpful comments on this and earlier drafts of this paper.