Normal modes with boundary dynamics

Houssam Yassin†

Program in Atmospheric and Oceanic Sciences, Princeton University, Princeton, NJ 08544, USA

(Received xx; revised xx; accepted xx)

Three-dimensional geophysical fluids support both internal and boundary-trapped waves. To obtain the normal modes in such fluids we must solve a differential eigenvalue problem for the vertical structure (for simplicity, we only consider horizontally periodic domains). If the boundaries are dynamically inert (e.g., rigid boundaries in the Boussinesq internal wave problem, flat boundaries in the quasigeostrophic Rossby wave problem) the resulting eigenvalue problem typically has a Sturm-Liouville form and the properties of such problems are well-known. However, when restoring forces are also present at the boundaries, then the equations of motion contain a time-derivative in the boundary conditions and this leads to an eigenvalue problem where the eigenvalue appears in the boundary conditions. This article develops the mathematical theory of such problems and explores the novel properties of wave problems with dynamically active boundaries. Physically, the presence of a restoring force at the boundary leads to an additional wave (e.g., surface gravity wave, topographic Rossby wave) and this boundary-trapped wave allows us to expand a larger collection of physical states in terms of these modes. The vertical structure of these waves depends on the horizontal wavevector, they satisfy an indefinite orthogonality relation containing Dirac delta contributions, and they may have jump discontinuities at the boundaries. Additionally, such problems include a δ-sheet formulation analogous to the Bretherton (1966) interpretation of boundary buoyancy gradients as infinitely thin sheets of potential vorticity. We apply the theory to four examples: two Boussinesq problems with a free-surface and two quasigeostrophic problems with a sloping lower boundary.

Key words:

1. Introduction

1.1. *Background*

An important tool in the study of the wave motion near a stable equilibrium is the separation of variables. When applicable, this elementary technique transforms a linear partial differential equation into an ordinary differential eigenvalue problem for each coordinate (e.g., Hillen *et al.* 2012). Upon solving the differential eigenvalue problems, one obtains the normal modes of the physical system. The normal modes are the fundamental wave motions for the given restoring forces, each mode represents an independent degree of freedom in

† Email address for correspondence: hyassin@princeton.edu

which the physical system can oscillate, and any solution of the wave problem may be written as a linear combination of these normal modes.

It is difficult to overstate the utility of this approach. A small selection of theoretical studies of waves making use of modal analysis includes Stokes (2009), Rayleigh (1882), Lamb (1911), Taylor (1936), Lighthill (1969), Rhines (1970), and Gill & Clarke (1974). Moreover, normal modes have found use in the study of non-linear motions as well, for example in Charney (1971), Rhines (1977), Salmon (1978), Fu & Flierl (1980), Ripa (1981), Smith & Vallis (2001), Bouchet & Venaille (2012), Venaille *et al.* (2012), and Scott (2014). In addition, normal modes are used in interpreting oceanographic observations such as in McWilliams (1976), Wunsch (1997), Stammer (1997), Chelton *et al.* (1998), Wang *et al.* (2013), and de La Lama *et al.* (2016) as well as in devising numerical parameterizations for climate models (Ferrari *et al.* 2010). Part of their utility is due to their perceived completeness as well as the putative ability of the gravest modes in describing the vertical structure of ocean dynamics.

In deriving the normal modes, one linearizes the dynamical equations of motion about some equilibrium state. One then encounters linearized restoring forces of two kinds:

1. volume-permeating forces experienced by fluid particles in the interior, and
2. boundary-confined forces only experienced by fluid particles at the boundary.

Examples of volume-permeating forces include the restoring forces resulting from continuous density stratification and continuous volume potential vorticity gradients. These restoring forces respectively result in internal gravity waves (Rayleigh 1882; Sutherland 2010) and Rossby waves (Vallis 2017). Examples of boundary-confined restoring forces include the gravitational force at a free-surface (i.e., at a jump-discontinuity in the background density), forces arising from gradients in surface potential vorticity (e.g., Schneider *et al.* 2003), and the molecular forces giving rise to surface tension. These restoring forces respectively result in surface gravity waves (Sutherland 2010), topographic/thermal waves (Hoskins *et al.* 1985), and capillary waves (Lamb 1975).

We note that the above classification of linear restoring forces differs from the more general classification of forces into “body forces”, “surface forces”, and “line forces” (e.g., Kundu & Cohen 2004, chapter 4.5). Our classification above only pertains to how restoring forces appear in the linearized equations. For instance, although gravity is a body force, the linearized restoring force due to gravity may manifest as both a volume-permeating and a boundary-confined restoring force in linear problems.

Both volume-permeating and boundary-confined restoring forces are of relevance to rotating stratified geophysical flows. In particular, all waves mentioned above are present in the ocean. Indeed, there has been a recent profusion of articles examining normal modes, in oceanographically relevant situations, in which both kinds of restoring forces are present (Tulloch & Smith 2006, 2009; Lapeyre 2009; Scott & Furnival 2012; Smith & Vanneste 2012; Rocha *et al.* 2015; Kelly 2016; LaCasce 2017; de La Lama *et al.* 2016; Brink & Pedlosky 2018; LaCasce & Groeskamp 2020). Many authors (Gill 1982; Smith & Vanneste 2012; Kelly 2016; Brink & Pedlosky 2018) have noted that, in problems involving boundary-confined restoring forces, the resulting modes satisfy a peculiar orthogonality relation. Others (Rocha *et al.* 2015; Kelly 2016) have noted that the term-by-term derivative of a Sturm-Liouville eigenfunction expansion of some function does not generally converge to the derivative of the function when boundary dynamics are present. Furthermore, in the context of quasigeostrophic theory, there has been recent controversy regarding the completeness of the resulting modes.

Volume-permeating forces give rise to a countable infinity of vertical modes—most of which have an oscillatory vertical structure. In contrast, boundary-confined forces give rise to one vertical mode (per dynamically-active boundary) that is boundary-trapped for

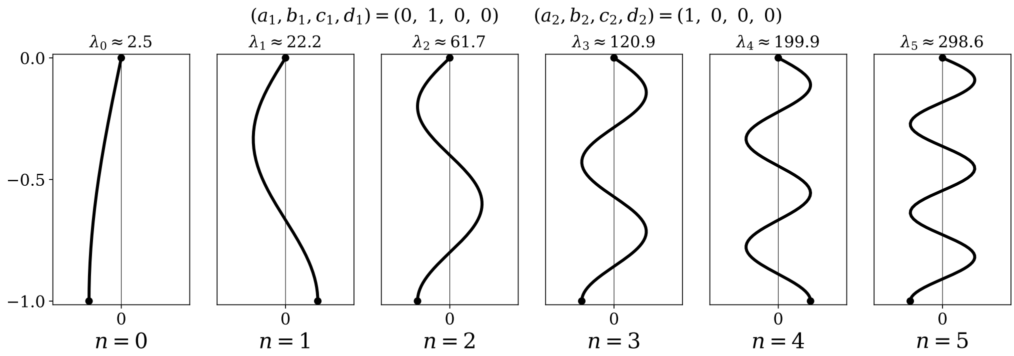

Figure 1: The first six eigenfunctions of the eigenvalue problem (1.1)–(1.3) with $p = 1$, $q = 0$ on the interval $[-1, 0]$, as discussed in §1.2. Boundary parameters in equations (2.2) and (2.3) are given above the figure. Since $c_i = d_i = 0$ for $i \in \{1, 2\}$, then the resulting problem is a regular Sturm-Liouville problem. The eigenfunctions are all continuous and are ordered by their corresponding eigenvalues λ_n, displayed above each eigenfunction. Note, that the nth eigenfunction has n internal zeros in the interval $(-1, 0)$. Numerical solutions to all eigenvalue problem in this article are found using a pseudo-spectral code in `Dedalus` (Burns *et al.* 2020).

sufficiently small horizontal scales. As each mode represents a distinct and independent degree of freedom, the modes arising from volume-permeating forces alone should not be complete—these modes cannot represent every possible physical state—and hence must be supplemented by modes arising from boundary-trapped forces. Variants of this position are taken by Lapeyre (2009), Tulloch & Smith (2009), Scott & Furnival (2012), and Smith & Vanneste (2012).

However, others (Ferrari & Wunsch 2010; Ferrari *et al.* 2010; LaCasce 2012; Rocha *et al.* 2015), citing completeness theorems from Sturm-Liouville theory, argue that any square-integrable function can be represented as a linear combination of eigenfunctions arising from volume-permeating forces. Thus, the modes arising from volume-permeating forces alone are complete. As such, even waves resulting from boundary-confined forces may be thought of as a linear combination of these modes.

This argument turns out to be incorrect. As shown in §2, the addition of boundary-confined restoring forces results in a set of normal modes that is capable of representing a larger collection of physical states. While we discuss topographic Rossby waves and their properties in this article, we address general concerns regarding discrete eigenfunction expansions for quasigeostrophic theory in the companion article Yassin & Griffies (in prep.).

The above literature review illustrates that, although we currently have a well-developed theoretical framework for the analysis of waves resulting from volume-permeating forces, namely that of traditional Sturm-Liouville theory (e.g., Hillen *et al.* 2012; Zettl 2010), we lack an analogous framework for problems involving boundary-confined restoring forces. The aim of the present article is to delineate such a framework, explore some novel aspects of the resulting normal modes, and apply the theory to some geophysically relevant examples. We restrict the discussion to the case of waves in otherwise quiescent fluids for which the theory has a straight-forward application.

1.2. *Summary & outline*

Regular Sturm-Liouville theory

When only volume-permeating forces are present, one typically obtains a regular Sturm-Liouville eigenvalue problem where the eigenvalue λ only appears in the differential equation. That is, one obtains a Sturm-Liouville eigenvalue problem of the form

$$-(p\,\phi')' + q\,\phi = \lambda\, r\,\phi \quad \text{for} \quad z \in (z_1, z_2) \tag{1.1}$$

$$-a_1\,\phi(z_1) + b_1\,(p\,\phi')(z_1) = 0 \tag{1.2}$$

$$-a_2\,\phi(z_2) + b_2\,(p\,\phi')(z_2) = 0. \tag{1.3}$$

where the notation is described in §2. Under suitable restrictions (e.g., Hillen *et al.* 2012), the above eigenvalue problem (1.1)–(1.3) has infinitely many eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ with corresponding eigenfunctions $\{\phi_n\}_{n=0}^{\infty}$. The nth mode has n internal zeros and the eigenfunctions $\{\phi_n\}_{n=0}^{\infty}$ form an orthonormal basis in the Hilbert space of square-integrable functions L^2. In other words, the set $\{\phi_n\}_{n=0}^{\infty}$ is complete in L^2. See figure 1 for an illustration.

If ψ is an arbitrary smooth function, then completeness in L^2 implies that we can write an expansion of the form

$$\sum_{n=0}^{\infty} \langle \psi, \phi_n \rangle_\sigma\, \phi_n, \tag{1.4}$$

that converges to ψ in a mean-square sense [see equation (2.8)], where the inner product $\langle \cdot, \cdot \rangle_\sigma$ is given by equation (2.9). However, for physical problems, the notion of mean-square convergence is not physically relevant. Indeed, if ψ and its series expansion (1.4) are not equal at every point, then the two expressions represent distinct physical states. Instead, for physical problems, we must require

$$\psi(z) = \sum_{n=0}^{\infty} \langle \psi, \phi_n \rangle\, \phi_n(z) \quad \text{and} \quad \psi'(z) = \sum_{n=0}^{\infty} \langle \psi, \phi_n \rangle\, \phi_n'(z) \quad \text{for } z \in [z_1, z_2]. \tag{1.5}$$

A sufficient condition for the uniform (and hence pointwise) convergence of these series is that ψ must satisfy the same boundary conditions as the eigenfunctions ϕ_n; namely, ψ must satisfy equations (1.2) and (1.3). As the eigenfunctions ϕ_n all must satisfy the same boundary conditions [equations (1.2) and (1.3)], eigenfunction expansions in terms of the ϕ_n are constrained at the boundaries of the interval $[z_1, z_2]$.

Eigenvalue problems with λ-dependent boundary conditions

If boundary-confined restoring forces are also present, then the eigenvalue λ will also appear in the appropriate boundary condition [see equations (2.1)–(2.3)] and such problems are outside the scope of traditional Sturm-Liouville theory. We develop the theory of eigenvalue problems with λ-dependent boundary conditions in §2. Some representative eigenfunctions of such problems are illustrated in figure 2 for an eigenvalue problem where both boundary conditions depend on the eigenvalue λ. Perhaps the most conspicuous feature in figure 2 is that the eigenfunctions now have finite jump-discontinuities at the boundaries. A closer look also reveals another striking contrast with Sturm-Liouville eigenfunctions—there are three eigenfunctions with a single zero in figure 2.

In addition, the eigenfunctions form an orthonormal basis in a larger function space than L^2, namely $L^2 \oplus \mathbb{C}^s$, where s is the number of λ-dependent boundaries conditions. In otherwords, the eigenfunctions are complete in the larger space $L^2 \oplus \mathbb{C}^s$. Indeed, the presence of the eigenvalue λ at a boundary condition means that the eigenfunctions no longer all have to satisfy the same boundary condition at the λ-dependent boundary. Thus, each

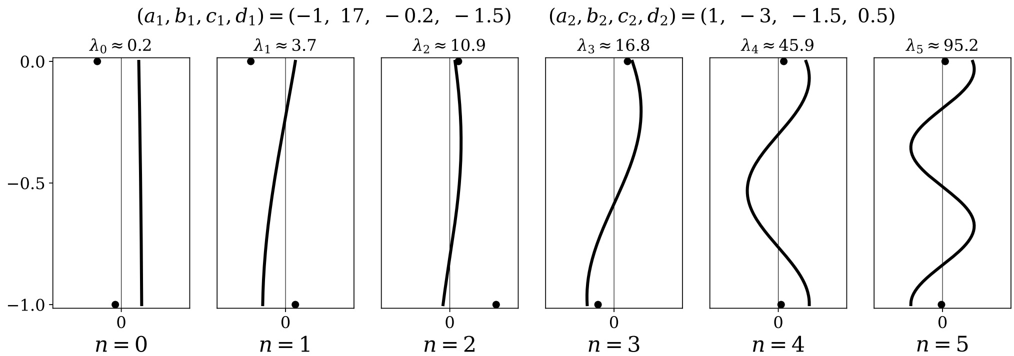

Figure 2: The first six eigenfunctions of the eigenvalue problem (2.1)–(2.3) with $p = 1$, $q = 0$ and the boundary parameters, defined in equations (2.2) and (2.3), are given above the figure. This figure is discussed in §1.2. The black dots represent the function values at the boundaries. Note that the eigenfunctions have finite jump-discontinuities at both boundaries and these boundary jump-discontinuities can only occur when a boundary is λ-dependent. Moreover, note that there are three eigenfunctions ($n = 1, 2, 3$) that each have one internal zero.

λ-dependent boundary condition provides an additional degree of freedom to the collection of eigenfunctions. In terms of the eigenfunction expansion of a smooth function ψ, the additional degrees of freedom manifests as a lack of constraints on ψ at λ-dependent boundaries. The eigenfunction expansion of ψ, along with its term-by-term derivative [e.g., expressions analogous to those in equation (1.5)], now converge uniformly on the interval $[z_1, z_2]$ regardless of what boundary conditions ψ satisfies at λ-dependent boundaries. In fact, we even obtain pointwise convergence to ψ even if ψ has a finite jump-discontinuity at a λ-dependent boundary.

Another intriguing feature is that a finite number of eigenfunctions may satisfy

$$\langle \phi_n, \phi_n \rangle < 0 \tag{1.6}$$

although, in this case, infinitely many modes will still satisfy

$$\langle \phi_n, \phi_n \rangle > 0. \tag{1.7}$$

Indeed, in quasigeostrophic theory, modes satisfying equation (1.6) have an eastward phase speed while modes satisfying equation (1.7) have a westward phase speed. If some eigenfunctions satisfy equation (1.6), then $L^2 \oplus \mathbb{C}^s$ is not a Hilbert space, but instead is a Pontryagin space (see appendix A for a brief introduction to Pontryagin spaces).

In the mathematical literature, the Hilbert space case—where all eigenfunctions satisfy equation (1.7)—is well-known. In this article we extend the mathematical theory to the Pontryagin space case where some eigenfunctions may satisfy equation (1.6). This extension of the theory is necessary for the quasigeostrophic Rossby wave problem. In addition, we reformulate the theory in terms of functions [in contrast to the vector formulations in Fulton (1977) and Smith & Vanneste (2012)] so that the notation is as close as possible to regular Sturm-Liouville theory. Proofs of the various theorems of §2 are given in appendix B.

Physical examples

We provide four geophysically relevant applications of the mathematical theory: two Boussinesq gravity-cappillary wave problems (in §3) and two quasigeostrophic Rossby wave problems (in §4). The additional degree of freedom gained when λ appears in a boundary condition corresponds to the additional wave that appears when a boundary-confined restoring force is present. For the Boussinesq problems, we obtain a surface gravity-capillary wave in addition to the countable infinity of internal gravity waves. For the quasigeostrophic problems, we obtain a topographic Rossby wave in addition to the countable infinity of (internal) Rossby waves. In both problems, the presence of boundary-confined restoring forces renders the vertical structure of the waves dependent on the horizontal wavenumber (although this dependence becomes negligible for higher modes).

Both Boussinesq examples, in §3, are of the normal modes of a continuously stratified fluid with a free-surface. The two examples differ in that the first example (§3.4) is of a non-rotating fluid while the second example (§3.5) is of a rotating fluid on an f-plane. In the second example, we also make the unconventional assumption that the stratification is large enough to suppress rotational effects in the interior but not at the free-surface. Unlike the first example, the eigenfunctions in the second, rotating, example have finite jump-discontinuities at the upper boundary. However, we will see that the actual physical motion corresponds to the continuous *solutions* of the eigenvalue problem. The eigenfunctions are simply convenient mathematical aids that one uses to obtain eigenfunction expansions.

In the quasigeostrophic case, we solve for the normal modes of a quiescent fluid on a β-plane with a sloping lower boundary, a problem that was first considered by Rhines (1970). One can either pose the eigenvalue problem in terms of the streamfunction (§4.4) or in terms of the vertical velocity (§4.5). The streamfunction eigenfunctions are continuous but the vertical velocity eigenfunctions are not. Again, the actual physical motion corresponds to the continuous *solutions* and not the eigenfunction.

Although we do not attempt to do so here, another possible approach to solving these wave problems is the pseudo-energy approach. Both Boussinesq and quasigeostrophic fluids are examples of non-canonical Hamiltonian systems (Shepherd 1990; Morrison 1998). After linearizing a non-canonical Hamiltonian system, the pseudo-energy is the linearized Hamiltonian [see appendix A in Holm *et al.* (1985) as well as section 5 of Shepherd (1990)], and one obtains an eigenvalue problem for which the linear operator is self-adjoint with respect to a pseudo-energy weighted inner product (Vanneste & Shepherd 1999). An application of the spectral theorem then ensures that one obtains real eigenvalues and that the eigenfunctions are orthogonal with respect to the pseudo-energy weighted inner product (Ripa 1981; Held 1985). Indeed, for the quasigeostrophic problem of §4, pseudo-energy orthogonality corresponds to orthogonality with respect to the quadratic form (2.28). However, the advantage of our approach over the pseudo-energy approach is that, not only do we obtain expansion and completeness results that are close analogues to those in Fourier theory, but we also obtain an associated qualitative theory for the eigenfunctions similar to that in Sturm-Liouville theory.

The "δ-function formulation"

In general, whenever boundary-confined restoring forces are present, one can replace the implied inhomogeneous boundary conditions with homogenous boundary conditions at the cost of including Dirac δ contributions to the governing equations. This "δ-function formulation" of the problem is analogous to the Bretherton (1966) interpretation of surface buoyancy gradients in quasigeostrophy as infinitesimally thin layers of volume potential vorticity. We demonstrate this formulation in both the §3 and §4 for the Boussinesq and quasigeostrophic problems respectively. Moreover, we show that the resulting distributions can be expanded in terms of the eigenfunctions. This ability to expand distributions is closely related to the ability of the eigenfunctions to represent functions with finite jump-discontinuities at the λ-dependent boundaries.

2. The eigenvalue problem

In this section, we outline the theory of the differential eigenvalue problem

$$-(p\,\phi')' + q\,\phi = \lambda\, r\,\phi \quad \text{for} \quad z \in (z_1, z_2) \tag{2.1}$$

$$-\left[a_1\,\phi(z_1) - b_1\,(p\,\phi')(z_1)\right] = \lambda\left[c_1\,\phi(z_1) - d_1\,(p\,\phi')(z_1)\right] \tag{2.2}$$

$$-\left[a_2\,\phi(z_2) - b_2\,(p\,\phi')(z_2)\right] = \lambda\left[c_2\,\phi(z_2) - d_2\,(p\,\phi')(z_2)\right] \tag{2.3}$$

where p^{-1}, q, and r are real-valued integrable functions; a_i, b_i, c_i, and d_i are real numbers with $i \in \{1,2\}$; and where $\lambda \in \mathbb{C}$ is the eigenvalue parameter. We further assume that $p > 0$ and $r > 0$, that p and r are twice continuously differentiable, that q is continuous, and that $(a_i, b_i) \neq (0,0)$ for $i \in \{1,2\}$. The system of equations (2.1)–(2.3) is an eigenvalue problem for the eigenvalue $\lambda \in \mathbb{C}$ and differs from a regular Sturm-Liouville problem in that λ appears in the boundary conditions (2.2) and (2.3). That is, setting $c_i = d_i = 0$ recovers the traditional Sturm-Liouville problem (1.1)–(1.3). The presence of λ as part of the boundary condition leads to some fundamentally new mathematical features that are the subject of this section and fundamental to the physics of this study.

For convenience in the following, it is useful to define the two boundary parameters

$$D_i = (-1)^{i+1}\,(a_i\,d_i - b_i\,c_i) \quad i = 1,2. \tag{2.4}$$

Just as the function r acts as a weight for the interval (z_1, z_2) in traditional Sturm-Liouville problems, the constants D_i^{-1} will play analogous roles for the boundaries $z = z_i$ when $D_i \neq 0$.

Outline of the mathematics and the goals of this section

The right-definite case, when the $D_i > 0$ for $i \in \{1,2\}$, is well-known in the mathematics literature; most of the right-definite results in this section are due to Evans (1970), Walter (1973), and Fulton (1977). In contrast, the left-definite case, defined below, is much less studied. In this section, we generalize the right-definite results of Fulton (1977) to the left-definite problem as well as provide an intuitive formulation (in terms of functions) of the eigenvalue problem.

Our first aim, in §2.1, is to provide sufficient conditions for the reality of the eigenvalues and completeness of the eigenfunctions. A necessary prelude to this aim is the formulation of the function space in which the eigenvalue problem occurs. We then proceed, in §2.2, to explore the properties of the eigenfunctions and the eigenfunction expansions, emphasizing in the process the novel properties that emerge when the spectral parameter λ appears in the boundary conditions. Finally, in §2.3, we discuss oscillation properties of the eigenfunctions.

We provide a literature review of the mathematical problem in appendix B. Furthermore, to aid readability, appendix B contains all technical proofs. In the left-definite case, the underlying function space is no longer a Hilbert space, but rather a Pontryagin space. In appendix A, we provide an introduction to the theory of Pontryagin spaces and prove a result crucial to the extension of Fulton's results to the left-definite problem.

2.1. *Operator formulation*

2.1.1. *The function space of the problem*

We denote by L^2 the Hilbert space of square integrable "functions" ϕ on the interval (z_1, z_2) satisfying

$$\int_{z_1}^{z_2} |\phi|^2\, r\,\mathrm{d}z < \infty. \tag{2.5}$$

To be more precise, the elements of L^2 are not functions, but rather equivalence classes of functions (e.g., Reed & Simon 1980, section I.3). Two functions, ϕ and ψ, are equivalent in

L^2 (i.e., $\phi = \psi$ in L^2) if they agree "almost everywhere" in the interval $[z_1, z_2]$ with respect to the weighted Lebesgue measure σ given by

$$\sigma([a,b]) = \int_a^b r \, \mathrm{d}z \quad \text{where } a, b \in [z_1, z_2]. \tag{2.6}$$

The measure σ induces the differential element

$$\mathrm{d}\sigma(z) = r(z) \, \mathrm{d}z. \tag{2.7}$$

In other words, ϕ and ψ are equal in L^2 if

$$\int_{z_1}^{z_2} |\phi(z) - \psi(z)|^2 \, r \, \mathrm{d}z = 0. \tag{2.8}$$

Significantly, we can have $\phi = \psi$ in L^2 but $\phi \neq \psi$ pointwise.

Furthermore, as a Hilbert space, L^2 is endowed with a positive-definite inner product

$$\langle \phi, \psi \rangle_\sigma = \int_{z_1}^{z_2} \phi^* \, \psi \, \mathrm{d}\sigma = \int_{z_1}^{z_2} \phi^* \, \psi \, r \, \mathrm{d}z. \tag{2.9}$$

where the symbol * denotes complex conjugation. The positive-definiteness is ensured by our assumption that $r > 0$ (i.e., $\langle \phi, \phi \rangle_\sigma > 0$ for $\phi \neq 0$ when $r > 0$).

It is well known that traditional Sturm-Liouville problems [i.e., equations (2.1)–(2.3) with $c_i = d_i = 0$ for $i = 1, 2$] are eigenvalue problems in some dense subspace of L^2. For the more general case of interest here, the eigenvalue problem occurs over a "larger" function space denoted by L^2_μ which we now construct.

First, let the integer $s \in \{0, 1, 2\}$ denote the number of λ-dependent boundary conditions and let S denote the set

$$S = \{ j \mid j \in \{1, 2\} \text{ and } (c_j, d_j) \neq (0, 0) \}. \tag{2.10}$$

S is one of $\emptyset, \{1\}, \{2\}, \{1, 2\}$ and s is the number of elements in the set S.

For $i \in S$, define the pure point measure ν_i by (e.g., Reed & Simon 1980, section I.4, example 2)

$$\nu_i([a,b]) = \begin{cases} D_i^{-1} & \text{if } z_i \in [a,b] \\ 0 & \text{otherwise,} \end{cases} \tag{2.11}$$

where D_i is the combination of boundary condition coefficients given by equation (2.4). The pure point measure ν_i induces the differential element

$$\mathrm{d}\nu_i(z) = D_i^{-1} \, \delta(z - z_i) \, \mathrm{d}z, \tag{2.12}$$

where $\delta(z)$ is the Dirac distribution.

Consider now the space $L^2_{\nu_i}$ of "functions" ϕ satisfying

$$\left| \int_{z_1}^{z_2} |\phi|^2 \, \mathrm{d}\nu_i \right| = \left| D_i^{-1} \right| \int_{z_1}^{z_2} |\phi|^2 \, \delta(z - z_i) \, \mathrm{d}z = \left| D_i^{-1} \right| \, |\phi(z_i)|^2 < \infty. \tag{2.13}$$

Again, elements of $L^2_{\nu_i}$ are not functions, but rather equivalence classes of functions. Two functions, ϕ and ψ, on the interval $[z_1, z_2]$ are equivalent in $L^2_{\nu_i}$ if $\phi(z_i) = \psi(z_i)$. In particular, note that $L^2_{\nu_i}$ is a one-dimensional vector space and is hence isomorphic to the field of complex numbers $\mathbb{C}$

$$L^2_{\nu_i} \cong \mathbb{C}. \tag{2.14}$$

Now define the measure μ by

$$\mu = \sigma + \sum_{i \in S} \nu_i \tag{2.15}$$

with an induced differential element of

$$\mathrm{d}\mu(z) = \left[r(z) + \sum_{i \in S} D_i^{-1} \, \delta(z - z_i) \right] \mathrm{d}z. \tag{2.16}$$

We denote elements of L^2_μ by upper case letters Ψ, where $\Psi(z)$ for $z \in [z_1, z_2]$ is defined by

$$\Psi(z) = \begin{cases} \Psi(z_i) & \text{at } z = z_i, \text{ for } i \in S, \\ \psi(z) & \text{otherwise,} \end{cases} \tag{2.17}$$

where $\Psi(z_i) \in \mathbb{C}$ are constants, for $i \in S$, and the corresponding lower case letter ψ denotes an element of L^2. Two elements Φ and Ψ of L^2_μ are equivalent in L^2_μ if and only if

1. $\Phi(z_i) = \Psi(z_i)$ for $i \in S$, and

2. $\phi(z)$ and $\psi(z)$ are equivalent in L^2 [i.e., as in equation (2.8)].

Here, Φ, as an element of L^2_μ, is defined as in equation (2.17). The primary difference between L^2 and L^2_μ is that L^2_μ can discriminate between functions that disagree at λ-dependent boundaries. While elements of L^2 can be equivalent in L^2 and disagree at the boundaries, two elements of L^2_μ that disagree at the λ-dependent boundaries cannot be equivalent in L^2_μ.

Since the measures σ and ν_i, for $i \in S$, are mutually singular, we have (Reed & Simon 1980, section II.1, example 5)

$$L^2_\mu \cong L^2 \oplus \sum_{i \in S} L^2_{\nu_i} \cong L^2 \oplus \mathbb{C}^s \tag{2.18}$$

from which we see L^2_μ is "larger" by s dimensions, where we recall that s is the number of λ-dependent boundary conditions.

The space L^2_μ is also endowed with an inner product given by

$$\langle \Phi, \Psi \rangle = \int_{z_1}^{z_2} \Phi^* \, \Psi \, \mathrm{d}\mu = \int_{z_1}^{z_2} \Phi^* \, \Psi \, r \, \mathrm{d}z + \sum_{i \in S} D_i^{-1} \, \Phi(z_i)^* \, \Psi(z_i) \tag{2.19}$$

If $D_i > 0$ for $i \in S$ then this inner product is positive-definite and L^2_μ is a Hilbert space. However, this is not the case in general.

Let κ denote the number of negative D_i for $i \in S$ (the possible values are $\kappa = 0, 1, 2$). Then L^2_μ has a κ-dimensional subspace of elements Ψ satisfying

$$\langle \Psi, \Psi \rangle < 0. \tag{2.20}$$

This makes L^2_μ a Pontryagin space of index κ (see appendix A). If $\kappa = 0$ then L^2_μ is again a Hilbert space. Note that, in the present case, L^2_μ also has an infinite-dimensional subspace of elements ψ satisfying

$$\langle \Psi, \Psi \rangle > 0. \tag{2.21}$$

2.1.2. *The eigenvalue problem in L^2_μ*

We now explicitly construct an operator formulation of (2.1)–(2.3) as an eigenvalue problem in the Pontryagin space L^2_μ.

Define the differential operator ℓ acting on a function ϕ by

$$\ell \, \phi = \frac{1}{r} \left[(p \, \phi')' - q \, \phi \right]. \tag{2.22}$$

We also define the following boundary operators for $i \in S$,

$$\mathcal{B}_i\phi = [a_i\,\phi(z_i) - b_i\,(p\,\phi')(z_i)] \tag{2.23}$$
$$C_i\phi = [c_i\,\phi(z_i) - d_i\,(p\,\phi')(z_i)]\,. \tag{2.24}$$

Let Φ be the corresponding element of L^2_μ, as in equation (2.17), with boundary values $\Phi(z_i) = C_i\phi$ for $i \in S$. We then define the operator $\mathcal{L}$, acting on functions Φ, by

$$\mathcal{L}\,\Phi = \begin{cases} -\ell\,\phi & \text{for } z \in (z_1, z_2) \\ -\mathcal{B}_i\,\phi & \text{for } z = z_i \text{ where } i \in S \end{cases} \tag{2.25}$$

with a domain $D(\mathcal{L}) \subset L^2_\mu$ defined by

$$\begin{aligned} D(\mathcal{L}) = \{\Phi \in L^2_\mu \mid \phi \text{ is continuously differentiable, } \ell\,\phi \in L^2,\ \Phi(z_i) = C_i\,\phi \\ \text{for } i \in S \text{ and } \mathcal{B}_i\phi = 0 \text{ for } i \in \{1,2\} \setminus S\}. \end{aligned} \tag{2.26}$$

Recall that S contains indices of the λ-dependent boundary conditions, and therefore, $\{1,2\}\setminus S$ contains the indices of the λ-independent boundary conditions.

Then, on the subspace $D(\mathcal{L})$ of L^2_μ, the eigenvalue problem (2.1)–(2.3) may be written as

$$\mathcal{L}\,\Phi = \lambda\,\Phi. \tag{2.27}$$

It is not difficult to show that $\mathcal{L}$ is a self-adjoint operator in the Pontryagin space L^2_μ (e.g., Russakovskii 1975, 1997).

2.1.3. *Reality and completeness*

There is a natural quadratic form Q induced by the eigenvalue problem (2.1)–(2.3), given by

$$Q(\Phi, \Psi) = \langle \Phi, \mathcal{L}\,\Psi \rangle\,. \tag{2.28}$$

For elements $\Phi, \Psi \in D(\mathcal{L})$, we obtain

$$\begin{aligned} Q(\Phi,\Psi) = \int_{z_1}^{z_2} \left[p\,\phi'^*\,\psi' + q\,\phi^*\,\psi\right]\,\mathrm{d}z + \sum_{i\in\{1,2\}\setminus S} (-1)^{i+1}\,\frac{a_i}{b_i}\,\phi(z_i)^*\,\psi(z_i) \\ - \sum_{i\in S} \frac{1}{D_i} \begin{pmatrix} \psi(z_i) \\ -(p\,\psi')(z_i) \end{pmatrix}^* \cdot \begin{pmatrix} a_i\,c_i & a_i\,d_i \\ a_i\,d_i & b_i\,d_i \end{pmatrix} \begin{pmatrix} \phi(z_i) \\ -(p\,\phi')(z_i) \end{pmatrix} \end{aligned} \tag{2.29}$$

for $b_i \neq 0$ for $i \in \{1,2\} \setminus S$. If $b_i = 0$ for $i \in \{1,2\} \setminus S$ then we replace the term a_i/b_i with zero.

To develop the reality and completeness theorem below, we provide the following definitions.

Definition 2.1 (Right-definite). *The eigenvalue problem (2.1)–(2.3) is said to be right-definite if L^2_μ is a Hilbert space or, equivalently, if*

$$\langle \Phi, \Phi \rangle > 0 \tag{2.30}$$

for all non-zero $\Phi \in L^2_\mu$.

Definition 2.2 (Left-definite). *The eigenvalue problem (2.1)–(2.3) is said to be left-definite if*

$$Q(\Phi, \Phi) \geqslant 0 \tag{2.31}$$

for all $\Phi \in D(\mathcal{L})$.

Rapids articles must not exceed this page length

We remark that right and left-definiteness are not mutually exclusive. Namely, a problem can be neither right- or left-definite; both right- and left-definite; only right-definite; or only left-definite.

The following propositions can then be obtained in a straightforward manner.

Proposition 2.3 (Criterion for Right-definiteness). *The eigenvalue problem (2.1)–(2.3) is right-definite if* $r > 0$ *and* $D_i > 0$ *for* $i \in S$.

Proposition 2.4 (Criterion for Left-definiteness). *The eigenvalue problem (2.1)–(2.3) is left-definite if the following conditions hold:*

(i) the functions p, q *satisfy* $p > 0, q \geqslant 0$,

(ii) for the λ*-dependent boundary conditions, we have*

$$\frac{a_i c_i}{D_i} \leqslant 0, \quad \frac{b_i d_i}{D_i} \leqslant 0, \quad (-1)^i \frac{a_i d_i}{D_i} \geqslant 0 \quad \text{for } i \in S. \tag{2.32}$$

(iii) for the λ*-independent boundary conditions, we have*

$$b_i = 0 \quad \text{or} \quad (-1)^{i+1} \frac{a_i}{b_i} \geqslant 0 \quad \text{if } b_i \neq 0 \quad \text{for } i \in \{1, 2\} \setminus S. \tag{2.33}$$

We remark that, in this article, we always assume that $p > 0$ and $r > 0$.

The reality of the eigenvalues and the completeness of the eigenfunctions in the Pontryagin space L^2_μ is then given by the following theorem.

Theorem 2.5 (Reality and completeness).

Suppose the eigenvalue problem (2.1)–(2.3) is either right-definite or left-definite. Moreover, if the problem is not right-definite, we assume that $\lambda = 0$ *is not an eigenvalue. Then the eigenvalue problem (2.1)–(2.3) has a countable infinity of real simple eigenvalues* λ_n *satisfying*

$$\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \to \infty \tag{2.34}$$

with corresponding eigenfunctions Φ_n*. Furthermore, the set of eigenfunctions* $\{\Phi_n\}_{n=0}^\infty$ *is a complete orthonormal basis for* L^2_μ *satisfying*

$$\langle \Phi_m, \Phi_n \rangle = \pm \delta_{mn}. \tag{2.35}$$

Proof. See appendix B.2. □

Recall that κ denotes the number of negative D_i for $i \in S$. In general, there will be κ eigenfunctions satisfying $\langle \Phi_n, \Phi_n \rangle < 0$ and infinitely many eigenfunctions satisfying $\langle \Phi_n, \Phi_n \rangle > 0$.

2.2. *Properties of the eigenfunctions*

We now investigate the novel properties of the eigenfunctions of the eigenvalue problem (2.1)–(2.3) that emerge when the spectral parameter λ appears in the boundary conditions (2.2) and (2.3). For the remainder of this section, we assume that the eigenvalue problem (2.1)–(2.3) satisfies the requirements of theorem 2.5.

2.2.1. *Eigenfunction expansions*

The eigenvalue problem (2.1)–(2.3) has *eigenfunctions* $\{\Phi_n\}_{n=0}^\infty$ as well as corresponding *solutions* $\{\phi_n\}_{n=0}^\infty$. In other words, while the ϕ_n are the solutions to the differential equation defined by equations (2.1)–(2.3) with $\lambda = \lambda_n$, the eigenfunctions required by the operator formulation of the problem [equation (2.27)] are Φ_n. The functions Φ_n and ϕ_n are related by equation (2.17), with the boundary values $\Phi_n(z_i)$ of Φ_n determined by

$$\Phi_n(z_i) = C_i\, \phi = [c_i\, \phi(z) - d_i\, (p\, \phi')(z)] \quad \text{for } i \in S. \tag{2.36}$$

Thus, while the solutions ϕ_n are continuously differentiable over the closed interval $[z_1, z_2]$, the eigenfunctions Φ_n are continuously differentiable over the open interval (z_1, z_2) but generally have finite jump-discontinuities at the λ-dependent boundaries. The eigenfunctions Φ_n are continuous in the closed interval $[z_1, z_2]$ only if $c_i = 1$ and $d_i = 0$ for $i \in S$. In this case, the eigenfunctions Φ_n coincide with the solutions ϕ_n on the closed interval $[z_1, z_2]$.

Since $\{\Phi_n\}_{n=0}^{\infty}$ is a basis for L^2_μ, then any $\Psi \in L^2_\mu$ may be expanded in terms of the eigenfunctions (see theorem A.7)

$$\Psi = \sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle}\, \Phi_n. \tag{2.37}$$

We emphasize the above equality is an equality in L^2_μ and not a pointwise equality [see the discussion following equation (2.17)].

The following theorem features some of the novel properties of the basis $\{\Phi_n\}_{n=0}^{\infty}$ of L^2_μ. Theorem 2.6 below is a generalization of a theorem first formulated, in the right-definite case, by Walter (1973) and Fulton (1977).

Theorem 2.6 (Eigenfunction expansions). *Let $\{\Phi_n\}_{n=0}^{\infty}$ be the set of eigenfunctions of the eigenvalue problem (2.1)–(2.3). Then the following hold:*

(i) Null series: For $i \in S$, we have

$$0 = D_i^{-1} \sum_{n=0}^{\infty} \frac{1}{\langle \Phi_n, \Phi_n \rangle}\, \Phi_n(z_i)\, \phi_n(z) \tag{2.38}$$

with equality in the sense of L^2.

(ii) Unit series: For $i \in S$, we have

$$1 = D_i^{-1} \sum_{n=0}^{\infty} \frac{1}{\langle \Phi_n, \Phi_n \rangle}\, |\Phi_n(z_i)|^2\,. \tag{2.39}$$

(iii) L^2-expansion: Let $\psi \in L^2$, then

$$\psi = \sum_{n=0}^{\infty} \frac{1}{\langle \Phi_n, \Phi_n \rangle} \left(\int_{z_1}^{z_2} \psi^*\, \phi_n\, r\, \mathrm{d}z \right) \phi_n. \tag{2.40}$$

with equality in the sense of L^2.

(iv) Interior-boundary orthogonality: Let $\psi \in L^2$, then for $i \in S$, we have

$$0 = \sum_{n=0}^{\infty} \frac{1}{\langle \Phi_n, \Phi_n \rangle} \left(\int_{z_1}^{z_2} \psi^*\, \phi_n\, r\, \mathrm{d}z \right) \Phi_n(z_i). \tag{2.41}$$

Proof. For $i \in S$, let Θ_i denote the boundary step-function, given by

$$\Theta_i(z) = \begin{cases} 1 & \text{if } z = z_i \\ 0 & \text{otherwise.} \end{cases} \tag{2.42}$$

Noting that $\langle \Theta_i, \Phi_n \rangle = D_i^{-1} \Phi_n(z_i)$, we expand the boundary step function Θ_i as in equation

(2.37) to obtain

$$\Theta_i = \begin{cases} 1 & \text{if } z = z_i \\ 0 & \text{otherwise} \end{cases} = \sum_{n=0}^{\infty} \frac{\left[D_i^{-1}\,\Phi_n(z_i)\right]}{\langle \Phi_n, \Phi_n \rangle} \begin{cases} \Phi_n(z_i) & \text{if } z = z_i \\ \phi_n(z) & \text{otherwise.} \end{cases} \tag{2.43}$$

The equality at $z = z_i$ yields (2.39) while the L^2 equality gives (2.38).

Now let $\psi \in L^2$ and extend ψ to an element Ψ of L^2_μ, as in equation (2.17), by defining $\Psi(z_i) = 0$ for $i \in S$. Then expanding Ψ as in (2.37)

$$\Psi = \begin{cases} 0 & \text{if } z = z_i \text{ for } i \in S \\ \psi(z) & \text{otherwise} \end{cases} = \sum_{n=0}^{\infty} \frac{\int_{z_1}^{z_2} \psi^*\,\phi_n r \mathrm{d}z}{\langle \Phi_n, \Phi_n \rangle} \begin{cases} \Phi_n(z_i) & \text{if } z = z_i \text{ for } i \in S \\ \phi_n(z) & \text{otherwise.} \end{cases}$$

The equality at $z = z_i$ yields equation (2.41) while the L^2 equality gives equation (2.40). □

As illustrated in the above proof, an important property that distinguishes the basis $\{\Phi_n\}_{n=0}^{\infty}$ of L^2_μ from an L^2 basis is its "sensitivity" to function values at boundary points $z = z_i$ for $i \in S$. The boundary step-function Θ_i, considered as an element of L^2, is equivalent to the zero element of L^2. An expansion of Θ_i in terms of L^2 basis function will consist of a series whose coefficients are all zero. However, when Θ_i is considered an element of L^2_μ (for $i \in S$), then Θ_i is not equivalent to zero, and, as equation (2.43) shows, there is a non-zero series representation of Θ_i in terms of the basis functions of L^2_μ.

A natural question one may ask is whether the basis functions $\{\Phi_n\}_{n=0}^{\infty}$ of L^2_μ are also a basis of L^2. Recall that the set $\{\Phi_n\}_{n=0}^{\infty}$ is a basis of L^2 if every element $\psi \in L^2$ can be written *uniquely* in terms of the functions $\{\Phi_n\}_{n=0}^{\infty}$. However, in general, this is not true. A corollary of theorem 2.6 is that, for $\psi \in L^2$, we have

$$\psi = \sum_{n=0}^{\infty} \frac{1}{\langle \Phi_n, \Phi_n \rangle} \left[\int_{z_1}^{z_2} \psi^*\,\phi_n\,r\,\mathrm{d}z + \sum_{i \in S} D_i^{-1}\,k_i\,\Phi_n(z_i) \right] \phi_n \tag{2.44}$$

where $k_i \in \mathbb{C}$ are arbitrary for $i \in S$, and equality is in the sense of L^2. The constants c_i only modify the the value of the series expansion at the boundary points $z = z_i$ for $i \in S$. Since, in L^2, two functions that differ at a single point are still considered equivalent as elements of L^2 [see equation (2.8)], the above equality holds in L^2 for all $c_i \in \mathbb{C}, i \in S$. Consequently, if $s > 0$, there are too many functions in the set $\{\Phi_n\}_{n=0}^{\infty}$ for it to be a basis of L^2 (see also Walter 1973; Russakovskii 1997).

Corollary 2.7 (Completeness in L^2). *Let s be the number of λ-dependent boundary conditions and $\{\Phi_n\}_{n=0}^{\infty}$ a collection of eigenfunctions for a problem satisfying the requirements of theorem 2.5. If $s > 0$, then the collection of eigenfunctions is complete but not minimal in L^2. There are s redundant eigenfunctions.*

This corollary makes formal the intuition, suggested by the isomorphism (2.18), that L^2_μ is s dimensions "larger" than L^2.

2.2.2. *Uniform convergence and term-by-term differentiability*

Along with the eigenfunction expansion (2.37) in terms of the eigenfunctions $\{\Phi_n\}_{n=0}^{\infty}$, we also have the expansion

$$\sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle}\,\phi_n. \tag{2.45}$$

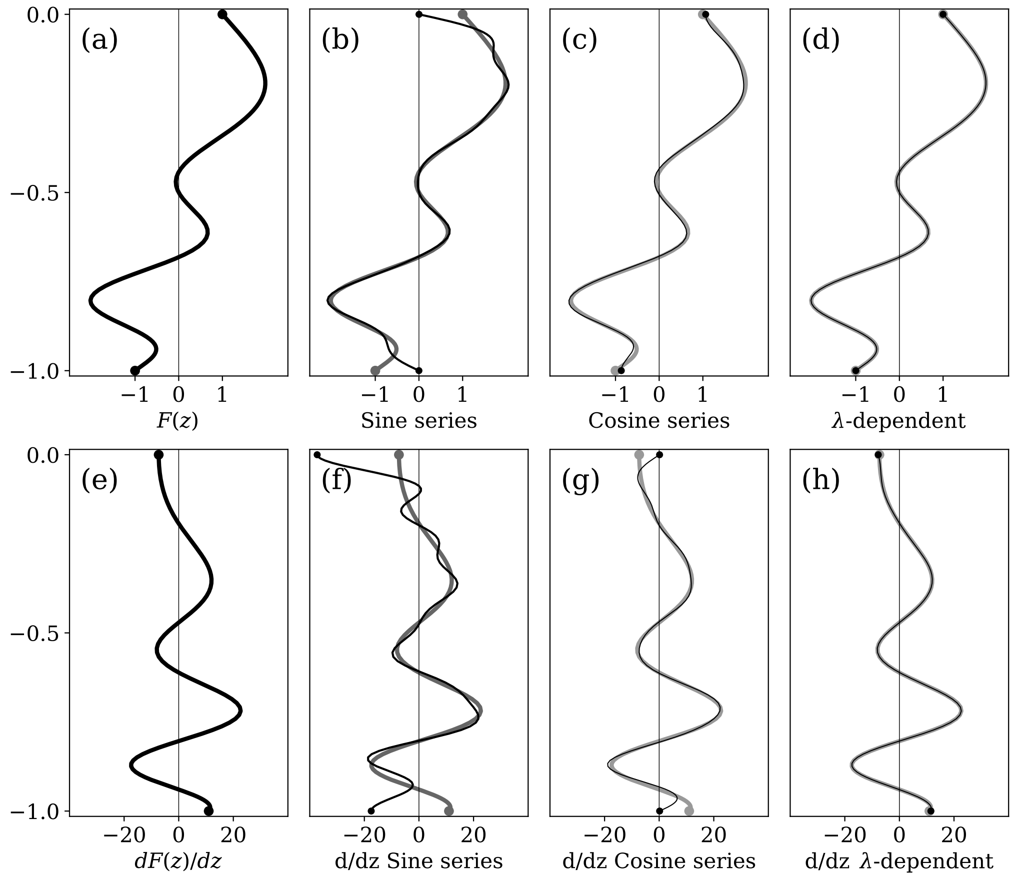

Figure 3: Convergence to a function $F(z) = 1 + 2z + (3/2)\sin(2\pi z)\cos(\pi^2 z^2 + 3)$ for $z \in [-1, 0]$, shown in panel (a), by various eigenfunction expansions of $-\phi'' = \lambda\,\phi$ with fifteen terms, as discussed in §2.2.2. Panel (b) shows the Fourier sine expansion of F. Since the sine eigenfunctions vanish at the boundaries $z = -1, 0$, the series expansion will not converge to F at the boundaries. Panel (c) shows the cosine expansion of F which converges uniformly to F on the closed interval $[-1, 0]$. Panel (d) shows an expansion with boundary coefficients in equations (2.2)–(2.3) given by $(a_1, b_1, c_1, d_1) = (-0.5, -5, 1, 0)$ and $(a_2, b_2, c_2, d_2) = (0.5, -5, 1, 0)$. Since the $c_i = 1$ and $d_i = 0$, then $\Phi_n = \phi_n$ and the series expansions (2.37) and (2.45) coincide. As with the cosine series, the expansion converges uniformly to F on $[-1, 0]$. The derivative of F is shown in panel (e). Panel (f) show the derivative of the sine series expansion; the differentiated series shows poor convergence behaviour since the undifferentiated sine series does not converge uniformly to F on $[z_1, z_2]$. In panel (g), we show the differentiated cosine series. The series does not converge to the derivative F' at the boundaries $z = z_1, z_2$ due to the boundary conditions that the cosine eigenfunctions satisfy. In contrast, in panel (h), the differentiated series obtained from a problem with λ-dependent boundary conditions converges uniformly to the derivative F' on the closed interval $[z_1, z_2]$.

The two expansions differ in their behaviour at λ-dependent boundaries, $z = z_i$ for $i \in S$, but are otherwise equal. In particular, note that the Φ_n expansion (2.37) must converge to $\Psi(z_i)$ at $z = z_i$ for $i \in S$ as this equality is required for Ψ to be equal to the series expansion (2.37) in L^2_μ [see the discussion following equation (2.17)]. However, the theorems below will show that the ϕ_n series (2.45) does not generally converge to $\Psi(z_i)$ at $z = z_i$.

First, we begin with convergence in the interior of the interval, (z_1, z_2). The following theorem states that the Φ_n eigenfunction expansion (2.37) behaves like a regular Fourier series for $z \in (z_1, z_2)$. Let $\psi(z\pm)$ denote the limits from the left ($-$) and right ($+$).

THEOREM 2.8 (EQUICONVERGENCE WITH FOURIER SERIES). *Let $\Psi \in L^2_\mu$ and $z \in (z_1, z_2)$. Then the eigenfunction expansion (2.37) behaves as regards to pointwise convergence just as a Fourier series. In particular, if the Fourier expansion of $\Psi(z)$ converges to $\frac{1}{2}[\Psi(z+) + \Psi(z-)]$, then so does the eigenfunction expansion (2.37).*

Proof. The proof in the case of λ-independent boundary conditions is given by theorem 1.9 in Titchmarsh (1962). Fulton (1977), in studying the right-definite problem, extends the proof to the case with λ-dependent boundary conditions, but the proof applies equally to the left-definite problem. Essentially, the argument is that one can transform the differential equation (2.1) into the form (e.g., Titchmarsh 1962, section 1.14)

$$-\chi'' + Q\,\chi = \lambda\,\chi \tag{2.46}$$

where Q is some continuous function of z. For large enough λ we have

$$-\chi'' \approx \lambda\,\chi \tag{2.47}$$

which is the equation governing Fourier expansions. $\square$

Since the expansions (2.37) and (2.45) in terms of Φ_n and ϕ_n are equal in the interior, then the above theorem applies to the ϕ_n series (2.45) as well. In fact, an identical result holds for traditional Sturm-Liouville expansions (Titchmarsh 1962, theorem 1.9, page 12). It is at the boundaries points, $z = z_1, z_2$, where the novel behaviour of the series expansions (2.37) and (2.45) appears.

For traditional Sturm-Liouville expansions [with eigenfunctions of problem (2.1)-(2.3) with $c_i, d_i = 0$ for $i = 1, 2$], eigenfunction expansions behave like the analogous Fourier series on $[z_1, z_2]$ [page 16 in Titchmarsh (1962) or chapter 1, section 9, in Levitan & Sargsjan (1975)]. In particular, for a twice continuously differentiable function ψ, the eigenfunction expansion of ψ will converge uniformly to ψ on $[z_1, z_2]$ so long as the eigenfunctions ϕ_n do not vanish at the boundaries. If the eigenfunctions vanish at one of the boundaries, then we only obtain uniform convergence if ψ vanishes at the corresponding boundary as well (Brown & Churchill 1993, section 22). Under these conditions, the resulting expansion will be differentiable in the interior of the interval, (z_1, z_2), but not at the boundaries $z = z_1, z_2$ [see chapter 8, section 3, in Levitan & Sargsjan (1975) for the equiconvergence of differentiated Sturm-Liouville series with Fourier series and see section 23 in Brown & Churchill (1993) for the convergence behaviour of differentiated Fourier series].

Returning to the case of eigenfunction expansions for the eigenvalue problem (2.1)-(2.3) with λ-dependent boundaries, the following theorem provides pointwise (as well as uniform, in the case $d_i \neq 0$) convergence conditions for the ϕ_n series (2.45).

THEOREM 2.9 (POINTWISE CONVERGENCE). *Let ψ be a twice continuously differentiable function on the interval $[z_1, z_2]$ satisfying any λ-independent boundary conditions in the eigenvalue problem (2.1)–(2.3). Define the function Ψ on $[z_1, z_2]$ by*

$$\Psi(z) = \begin{cases} \Psi(z_i) & \text{at } z = z_i, \text{ for } i \in S, \\ \psi(z) & \text{otherwise}. \end{cases} \tag{2.48}$$

where $\Psi(z_i)$ are constants for $i \in S$ (the λ-dependent boundaries). Then we have the following.

(i) If $d_i \neq 0$ for $i \in S$, then the ϕ_n series expansion (2.45) converges uniformly to $\psi(z)$ on

the closed interval $[z_1, z_2]$,

$$\sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n(z) = \psi(z). \tag{2.49}$$

Furthermore, for the differentiated series, we have

$$\sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n'(z) = \begin{cases} \left(c_i \, \psi(z_i) - \Psi(z_i)\right) / d_i & \text{at } z = z_i, \text{ for } i \in S \\ \psi'(z) & \text{otherwise.} \end{cases} \tag{2.50}$$

(ii) If $d_i = 0$*, then we have*

$$\sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n = \begin{cases} \Psi(z_i)/c_i & \text{at } z = z_i, \text{ for } i \in S \\ \psi(z) & \text{otherwise.} \end{cases} \tag{2.51}$$

Proof. This theorem is a generalization of corollary 2.1 in Fulton (1977). We provide the extension of the corollary to the left-definite problem in appendix B.3. □

The Φ_n series (2.37) converges to $\Psi(z_i)$ at $z = z_i$ for $i \in S$ (i.e., at λ-dependent boundaries) but otherwise behaves as in theorem 2.9.

Theorem 2.9 only gives differentiability of the series expansion in the case $d_i \neq 0$. The following theorem provides criteria for differentiability in the the case with $d_i = 0$. Moreover, the theorem gives conditions for uniform convergence for the series and its term-by-term derivative. Recall that uniform convergence implies pointwise convergence and generally precludes poor convergence behaviour such as Gibbs phenomena (Kaplan 1993).

Theorem 2.10 (Uniform convergence). *Let* Ψ *be defined as in theorem (2.9), with* ψ *twice continuously differentiable on* $[z_1, z_2]$ *and satisfying all* λ*-independent boundary conditions of the eigenvalue problem (2.1)–(2.3). In addition, suppose that* $\Psi(z_i) = C_i \psi$ *for* $i \in S$*. Then the* ϕ_n *series expansion (2.45) converges uniformly and absolutely to* $\psi(z)$ *on* $[z_1, z_2]$*, and the differentiated series*

$$\sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n' \tag{2.52}$$

converges uniformly and absolutely to ψ' *on* $[z_1, z_2]$.

Proof. See appendix B.3. □

There is a special case of theorem 2.10 that is of particular importance for the physical examples below. Suppose $c_i = 1$ and $d_i = 0$ for $i \in S$. Then $\Phi_n = \phi_n$ and so both the Φ_n series (2.37) and the ϕ_n series (2.45) coincide on $[z_1, z_2]$. By theorem 2.10, given a twice continuously differentiable function $\psi(z)$ on $[z_1, z_2]$ that satisfies any λ-independent boundary conditions, we have

$$\psi(z) = \sum_{n=0}^{\infty} \frac{\langle \psi, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \phi_n(z), \quad \text{and} \quad \psi'(z) = \sum_{n=0}^{\infty} \frac{\langle \psi, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \phi_n'(z) \tag{2.53}$$

with both series converging uniformly on the closed interval $[z_1, z_2]$ regardless of what boundary conditions ψ satisfies at λ-dependent boundaries. In particular, if both boundary conditions are λ-dependent, then the above two expansions converge uniformly on $[z_1, z_2]$ regardless of the boundary conditions ψ satisfies. As mentioned above, for traditional Sturm-

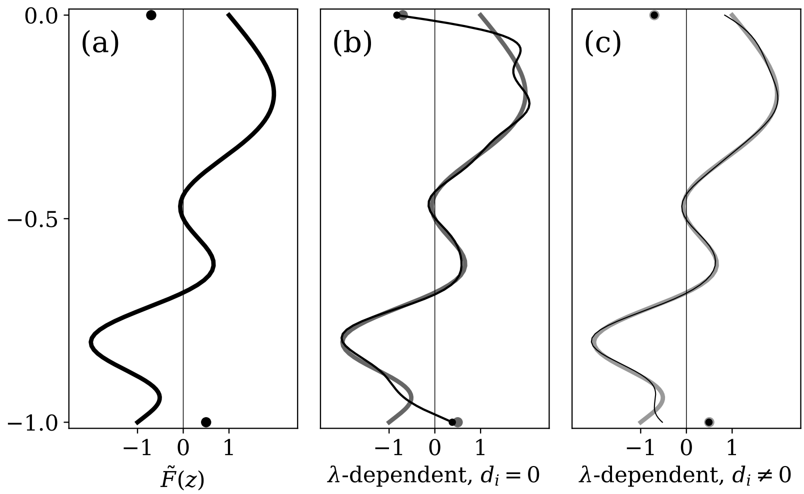

Figure 4: Convergence to a function $\tilde{F}$ with finite jump-discontinuities at the boundaries by two eigenfunction expansions (with λ-dependent boundary conditions) of $-\phi'' = \lambda\,\phi$ with fifteen terms, as discussed in §2.2.2. The function $\tilde{F}(z)$ is defined by $\tilde{F}(z) = F(z)$ for $z \in (z_1, z_2)$ where $F(z)$ is the function defined in figure 3, $F(-1) = 0.5$ at the lower boundary, and $F(0) = -0.7$ at the upper boundary. The function $\tilde{F}$ is shown in panel (a). In panel (b), the boundary coefficients in equations (2.2)–(2.3) are given by $(a_1, b_1, c_1, d_1) = (-0.5, -5, 1, 0)$ and $(a_2, b_2, c_2, d_2) = (0.5, -5, 1, 0)$ as in figure 3. In panel (c), the boundary coefficients are $(a_1, b_1, c_1, d_1) = (-0.5, -5, 1, 0.1)$ and $(a_2, b_2, c_2, d_2) = (0.5, -5, 1, -0.1)$. The Φ_n expansion (2.37) and the ϕ_n expansion (2.45) are not generally equal at the boundaries $z = -1, 0$; this figure shows the Φ_n expansion. The Φ_n series (2.37) converges pointwise to $\tilde{F}$ on $[-1, 0]$, however, the convergence will not be uniform if $d_i = 0$ for $i \in S$, as in panel (b). The boundary values of the Φ_n series (2.37) are shown with a black dot. In panel (b), the eigenfunctions Φ_n are continuous and a large number of terms are required for the series to converge to the discontinuous function $\tilde{F}$. Panel (c) shows that the discontinuous eigenfunction Φ_n have almost converged to the $\tilde{F}$—including at the jump-discontinuities; the black dot in panel (c) overlap with the grey dots, which represent the boundary values of $\tilde{F}$. Although the ϕ_n series (2.45) converges to $\tilde{F}$ in the interior $(-1, 0)$, the ϕ_n series does not generally converge to $\tilde{F}$ at the boundaries but instead converges to the values given in theorem 2.9.

Liouville expansions, an analogous result will hold only if ψ satisfies the same boundary conditions as the eigenfunctions. This special case is illustrated in figure 3.

Another novel property of the Φ_n series expansion (2.37) is that we obtain pointwise convergence to functions that are smooth in the interior of the interval, (z_1, z_2), but have finite jump-discontinuities at λ-dependent boundaries. If $d_i \neq 0$ for $i \in S$, the convergence is even uniform (Fulton 1977, corollary 2.1). Figure 4 illustrates the convergence behaviour for eigenfunctions expansions with λ-dependent boundary conditions in the two cases $d_i = 0$ and $d_i \neq 0$. Note the presence of Gibbs-like oscillations in the case $d_i = 0$ shown in panel (b). Although the Φ_n series (2.37) converges pointwise to the discontinuous function, the ϕ_n series (2.45) converges to the values given in theorem 2.9 at the λ-dependent boundaries.

2.3. *Oscillation theory*

Recall that for regular Sturm-Liouville problems with λ-independent boundary conditions (i.e., equations (2.1)–(2.3) with $s = 0$), we obtain a countable infinity of real simple eigenvalues, λ_n, that may be ordered as

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty \tag{2.54}$$

with associated eigenfunctions ϕ_n. The nth eigenfunction ϕ_n has precisely n internal zeros in the interval (z_1, z_2) so that no two eigenfunctions have the same number of internal zeros.

For the general eigenvalue problem (2.1)–(2.3) with λ-dependent boundary conditions, our discussion thus far suggests that there will be s "additional" eigenfunctions, where s is the number of λ-dependent boundary conditions. Perhaps a reasonable expectation is that some eigenfunctions will now have the same number of internal zeros and that there are at most s of these eigenfunctions.

The above notions indeed turn out to be the correct. The crucial parameter that determines the repeated oscillation numbers is $-b_i/d_i$ for $i \in S$, where b_i and d_i are the boundary coefficients appearing in the boundary conditions (2.2)–(2.3). We attend to the right-definite and left-definite problems in turn.

2.3.1. *Right-definite oscillation theory*

We begin with a result of Linden (1991) in the case where only one boundary condition is λ-dependent.

Lemma 2.11 (Location of double oscillation count).
Suppose the eigenvalue problem (2.1)–(2.3) is right-definite, that $s = 1$, $i \in S$, and that $d_i \neq 0$. Then the eigenfunction Φ_n corresponding to the eigenvalue λ_n will have n internal zeros if $\lambda_n < -b_i/d_i$ and $n - 1$ internal zero if $-b_i/d_i \leqslant \lambda_n$.

What happens if $d_i = 0$? From lemma 2.11 we see that as $d_i \to 0$ then the double oscillation count will occur at progressively larger n. If $d_i = 0$ we expect that double oscillation count to occur at $n = \infty$. This expectation is confirmed by theory of Binding *et al.* (1994).

Lemma 2.12 (No double oscillation count). *Suppose the assumptions of lemma 2.11 but take $d_i = 0$. Then the nth eigenfunction has n internal zeros.*

When both boundary conditions are λ-dependent, the situation is similar. If one of the d_i for $i \in S$ vanishes, the "additional" eigenfunctions occurs at $n = \infty$. Otherwise, the ratio $-b_i/d_i$, and its relation to the eigenvalues, determines how many internal zeros the "additional" eigenfunction has. In this case, we can either have two pairs of eigenfunctions, each pair with a distinct number of internal zeros, or three eigenfunctions all with the same number of internal zeros. We refer the reader to Binding *et al.* (1994) for further discussion.

2.3.2. *Left-definite oscillation theory*

We begin by stating a useful property of left-definite problems. Recall that κ is the number of negative D_i for $i \in S$.

Proposition 2.13. *Suppose the eigenvalue problem (2.1)–(2.3) is left-definite and that $\lambda = 0$ is not an eigenvalue. Then there are κ negative eigenvalues and their eigenfunctions satisfy*

$$\langle \Phi, \Phi \rangle < 0. \tag{2.55}$$

The remaining eigenvalues are positive and their eigenfunctions satisfy

$$\langle \Phi, \Phi \rangle > 0. \tag{2.56}$$

Proof. This property is a consequence of theorem A.17. □

In other words, proposition 2.13 states that we have the relationship

$$\lambda_n \langle \Phi_n, \Phi_n \rangle > 0 \tag{2.57}$$

for left-definite problems.

Before proceeding, we note that the left-definite oscillation theory in this section is due to Binding & Browne (1999) who address the general left-definite problem without the restriction that r must be positive. When r is restricted to be positive, one only has to note that the eigenvalues must then be bounded from below and the results of this section follow. See Zettl (2010) for a discussion of regular left-definite Sturm-Liouville problems (with λ-independent boundary conditions).

In the case of one λ-dependent boundary condition, we have the following result (Binding & Browne 1999).

LEMMA 2.14 (LOCATION OF LEFT-DEFINITE DOUBLE OSCILLATION).
Suppose the eigenvalue problem (2.1)–(2.3) is left-definite, that $\lambda = 0$ is not an eigenvalue, that $s = 1$, and $i \in S$. If $\kappa = 0$ then all eigenvalues are positive, the problem is right-definite, and one of lemma 2.11 or lemma 2.12 applies. Otherwise, if $\kappa = 1$, then the eigenvalues may be ordered as

$$\lambda_0 < 0 < \lambda_1 < \lambda_2 < \cdots < \to \infty \tag{2.58}$$

Both eigenfunctions Φ_0 and Φ_1 have no internal zeros. The remaining eigenfunctions Φ_n, for $n > 1$, have $n - 1$ internal zeros.

Note that, in the left-definite case, we still obtain two eigenfunctions with the same number of internal zeros if $d_i = 0$ for $i \in S$.

For two λ-dependent boundaries, we refer the reader to Binding & Browne (1999). We only describe some features relevent for the physical examples here and in Yassin & Griffies (in prep.). Generally speaking, the results of the right-definite theory of §2.3.1 holds for the boundary with $D_i > 0$ so that, if $d_i = 0$ and $D_i > 0$, one of the "additional" eigenfunctions will occur at $n = \infty$. If both $D_1, D_2 < 0$ (i.e., $\kappa = 2$) then we obtain the two additional eigenfunctions that either have no internal zeros or one internal zero. In particular, when both $D_1, D_2 < 0$ and $d_1 = d_2 = 0$, the sequence of internal zeros for the eigenfunctions $\{\Phi_n\}_{n=0}^{\infty}$ takes the form (Binding & Browne 1999, case 6.1)

$$1,\ 0,\ 0,\ 1,\ 2,\ 3, \ldots \tag{2.59}$$

3. Boussinesq gravity-capillary waves

Consider a rotating Boussinesq fluid on an f-plane with a reference Boussinesq density of ρ_0. The fluid is subject to a constant gravitational acceleration g in the downwards, $-\hat{\boldsymbol{z}}$, direction, and to a surface tension T (with dimensions of force per unit length, which equals to mass per square time, see Lamb 1975) at its upper boundary. The upper boundary of the fluid, given by $z = \eta$, is a free surface defined by the function $\eta(\boldsymbol{x}, t)$, where $\boldsymbol{x} = \hat{\boldsymbol{x}}\, x + \hat{\boldsymbol{y}}\, y$ is the horizontal position vector. The lower boundary of the fluid is a flat rigid surface given by $z = -H$. The fluid region is periodic in both horizontal directions $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$.

3.1. *Linear equations of motion*

The governing equations for infinitesimal perturbations about a background state of no motion, characterized by a prescribed background density of $\rho_B = \rho_B(z)$, are

$$\partial_t^2 \nabla^2 w + f_0^2\, \partial_z^2 w + N^2\, \nabla_z^2 w = 0 \quad \text{for } z \in (-H, 0) \tag{3.1}$$

$$w = 0 \quad \text{for } z = -H \tag{3.2}$$

$$-\partial_t^2 \partial_z w - f_0^2\, \partial_z w + g_b\, \nabla_z^2 w - \tau\, \nabla_z^4 w = 0 \quad \text{for } z = 0 \tag{3.3}$$

where w is the vertical velocity, f_0 is the constant value of the Coriolis frequency, the prescribed buoyancy frequency N^2 is given by

$$N^2(z) = -\frac{g}{\rho_0} \frac{d\rho_B(z)}{dz} \quad \text{for } z \in (-H, 0), \tag{3.4}$$

the acceleration g_b is the effective gravitational acceleration at the upper boundary

$$g_b = -\frac{g}{\rho_0} \left[\rho_a - \rho_B(0-)\right] \tag{3.5}$$

where ρ_a is the density of the overlying fluid, and the parameter τ is given by

$$\tau = \frac{T}{\rho_0} \tag{3.6}$$

where T is the surface tension. The three-dimensional Laplacian is denoted $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$, the horizontal Laplacian is denoted by $\nabla_z^2 = \partial_x^2 + \partial_y^2$, and the horizontal biharmonic operator is given by $\nabla_z^4 = \nabla_z^2\, \nabla_z^2$. See equation (1.37) in Dingemans (1997) for the surface tension term in (3.3). The remaining terms in equation (3.1)–(3.3) are standard. Consistent with our assumption that $\eta(\boldsymbol{x}, t)$ is small, we evaluate the upper boundary condition at $z = 0$ rather than $z = \eta(\boldsymbol{x}, t)$ in equation (3.3).

3.2. *A δ-sheet formulation*

Equations (3.1)–(3.3) consist of a linear partial differential equation (3.1) with one homogeneous boundary condition (3.2) and one inhomogeneous boundary condition (3.3). One can reformulate equations (3.1)–(3.3) as a problem with only homogenous boundary conditions provided one adds an infinitesimally thin sheet of $\partial_z^2 w$ at the upper boundary $z = 0$. This formulation is analogous to the interpretation, in quasigeostrophic theory, of boundary buoyancy gradients as an infinitesimally thin sheet of volume potential vorticity at the boundary (see Bretherton 1966; Schneider *et al.* 2003).

In this formulation, $\partial_z w$ is discontinuous at the upper boundary. We impose the homogenous boundary condition

$$\partial_z w|_{z=0} = 0 \tag{3.7}$$

at the upper boundary. Recall that the derivative of a step-function is a Dirac distribution, e.g.,

$$\frac{d\Theta_2(z)}{dz} = \Theta_2\Big|_{z_2-}^{z_2} \delta(z - z_2) = \delta(z - z_2) \tag{3.8}$$

where the boundary step-function Θ_2 at $z_2 = 0$ is defined by equation (2.42). If we let $(\partial_z^2 w)_{\text{dist}}$ be the derivative over the semi-closed interval $(-H, 0]$, then there will be a δ-contribution at $z = 0$ due to the discontinuity in $\partial_z w$,

$$(\partial_z^2 w)_{\text{dist}} = \partial_z^2 w - \partial_z w\big|_{z=0-} \delta(z) \quad \text{for } z \in (-H, 0]. \tag{3.9}$$

We define $(\nabla^2 w)_{\text{dist}}$ in a similar manner,

$$(\nabla^2 w)_{\text{dist}} = \nabla_z^2 w + (\partial_z^2 w)_{\text{dist}} = \nabla^2 w - \partial_z w\big|_{z=0-}\,\delta(z) \quad \text{for } z \in (-H, 0]. \tag{3.10}$$

The subscript "dist" indicates that $(\partial_z^2 w)_{\text{dist}}$ and $(\nabla^2 w)_{\text{dist}}$ are now distributions due to the Dirac δ contributions in their definitions.

The background density $\rho_B(z)$ has a jump-discontinuity at the upper boundary. At $z = 0$, the density jumps from $\rho_B(0-)$ to some constant $\rho_B(0) = \rho_a$ that characterizes the density of the overlying fluid. Then

$$\left[\frac{d\rho_B(z)}{dz}\right]_{\text{dist}} = \frac{d\rho_B(z)}{dz} + \rho_B\Big|_{0-}^{0}\,\delta(z) \tag{3.11}$$

$$= \frac{d\rho_B(z)}{dz} + (\rho_a - \rho_B|_{0-})\,\delta(z) \quad \text{for } z \in (-H, 0]. \tag{3.12}$$

Then the buoyancy frequency distribution N^2_{dist} is given by

$$N^2_{\text{dist}}(z) = -\frac{g}{\rho_0}\left[\frac{d\rho_B(z)}{dz}\right]_{\text{dist}} = N^2(z) + g_b\,\delta(z) \quad \text{for } z \in (-H, 0]. \tag{3.13}$$

We can now rewrite both the equation for the interior (3.1) and the inhomogeneous upper boundary condition (3.3) as a single time-evolution equation

$$\partial_t^2\left(\nabla^2 w\right)_{\text{dist}} + f_0^2\,(\partial_z^2 w)_{\text{dist}} + N^2_{\text{dist}}\,\nabla_z^2 w - \tau\,\nabla_z^4 w\,\delta(z) = 0 \quad \text{for } z \in (-H, 0]. \tag{3.14}$$

Equation (3.14), along with the homogenous boundary conditions (3.2) and (3.7), are equivalent to equation (3.1) with the homogenous boundary condition (3.2) and the inhomogeneous boundary condition (3.3).

Equation (3.14) illustrates two distinct ways boundary-confined restoring forces can arise. Boundary-confined restoring forces can arise due to some discontinuity present in a body force such as buoyancy—as in N^2_{dist}. Alternatively, boundary-confined restoring forces can arise due forces that are only present at the boundaries of the fluid such as surface tension.

3.3. *The Boussinesq eigenvalue problem*

We assume wave solutions of the form

$$w(\boldsymbol{x}, z, t) = \hat{w}(z)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{3.15}$$

where $\boldsymbol{k} = \hat{\boldsymbol{x}}\,k_1 + \hat{\boldsymbol{y}}\,k_2$ is the horizontal wavevector and ω is the angular frequency. Substituting the wave solution (3.15) into the governing equations (3.1)–(3.3) yields the following differential eigenvalue problem for the eigenvalue ω^2,

$$(f_0^2 - \omega^2)\,\hat{w}'' + (\omega^2 - N^2)\,k^2\,\hat{w} = 0 \quad \text{for } z \in (-H, 0), \tag{3.16}$$

$$\hat{w} = 0 \quad \text{for } z = -H, \tag{3.17}$$

$$(f_0^2 - \omega^2)\,\hat{w}' + (g_b + \tau\,k^2)\,k^2\,\hat{w} = 0 \quad \text{for } z = 0, \tag{3.18}$$

where the wavenumber $k = |\boldsymbol{k}|$ is the length of the wavevector $\boldsymbol{k}$. In general, the eigenvalue problem (3.16)–(3.18) is not of the form given by equations (2.1)–(2.3). However, there are four special cases when the eigenvalue problem (3.16)–(3.18) does take the form given by equations (2.1)–(2.3):

1. Non-rotating fluid: $f_0 = 0$.

2. Hydrostatic fluid: $\omega^2 \ll N^2$.

3. Rotating upper surface: When the Coriolis frequency f_0 may be neglected in the interior but not at the upper boundary.

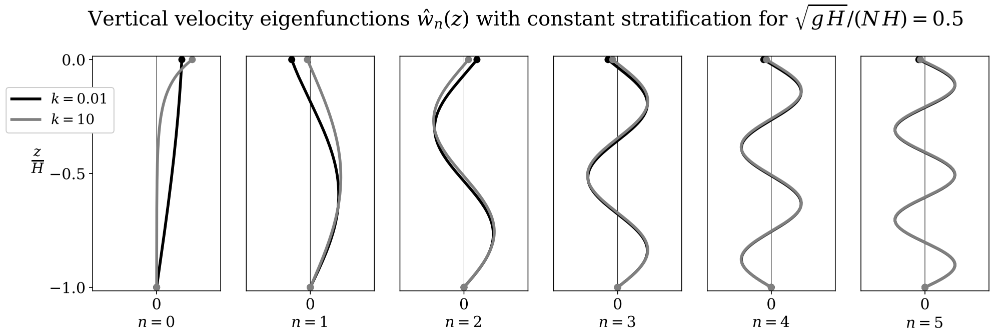

Figure 5: The vertical velocity eigenfunctions $\hat{W}_n = \hat{w}_n$ of the non-rotating Boussinesq eigenvalue problem (3.19)–(3.21) for two distinct wavenumbers with constant stratification, as discussed in §3.4. For both wavenumbers, the nth eigenfunction has n internal zeros as in regular Sturm-Liouville theory. The zeroth mode ($n = 0$) corresponds to a surface gravity wave and is trapped to the upper boundary for large horizontal wavenumbers. In contrast to the internal wave problem with a rigid lid, the modes $\hat{w}_n$ now depend on the horizontal wavenumber k through the boundary condition (3.21), however, this dependence is weak for $n \gg 1$, as can be observed in this figure; for $n > 2$, the modes for $k = 0.01$ (in black) and for $k = 10$ (in grey) nearly coincide. The horizontal wavenumbers k are non-dimensionalized by H.

4. When N^2 is constant.

We will now examine the case *1* of a non-rotating fluid and case *3* of a rotating upper boundary. The eigenfunctions in these two cases display qualitatively distinct properties. The analysis for the other two cases is similar.

3.4. *Non-rotating Boussinesq fluid*

When $f_0 = 0$, we rewrite the eigenvalue problem (3.16)–(3.18) as

$$-\hat{w}'' + k^2\,\hat{w} = \sigma^{-2}\,N^2\,\hat{w} \quad \text{for } z \in (-H, 0) \tag{3.19}$$

$$\hat{w} = 0 \quad \text{for } z = -H \tag{3.20}$$

$$(g_b + \tau\,k^2)^{-1}\hat{w}' = \sigma^{-2}\,\hat{w} \quad \text{for } z = 0 \tag{3.21}$$

where $\sigma = \omega/k$ is the phase speed. Equations (3.19)–(3.21) are an eigenvalue problem for the eigenvalue $\lambda = \sigma^{-2}$.

Definiteness & the underlying function space

Equations (3.19)–(3.21) form an eigenvalue problem with one λ-dependent boundary condition, namely, the upper boundary condition (3.21). One can verify that the eigenvalue problem (3.19)–(3.21) is right-definite using proposition 2.3 and left-definite using proposition 2.4. The underlying function space of the problem is

$$L^2_\mu \cong L^2 \oplus \mathbb{C}. \tag{3.22}$$

We write $\hat{W}_n$ for the eigenfunctions and $\hat{w}_n$ for the solutions of equation (3.19)–(3.21) [see the paragraph containing equation (2.36)]. The eigenfunctions $\hat{W}_n$ are related to the solutions $\hat{w}_n$ by (2.17) with boundary values $\hat{W}_n(0)$ given by equation (2.36). However, since $c_2 = 1$ and $d_2 = 0$ in equation (3.21) [compare with equations (2.1)–(2.3)] then $\hat{W}_n = \hat{w}_n$ on the closed interval $[-H, 0]$; thus, the solutions w_n are also the eigenfunctions.

By theorem 2.5, the eigenfunctions $\{\hat{w}_n\}_{n=0}^{\infty}$ form an orthonormal basis of L^2_μ. Note that orthonormality is defined with respect to the inner product given by equation (2.19). For functions φ and ϕ, the inner product is

$$\langle \varphi, \phi \rangle = \int_{-H}^{0} \varphi\, \phi \,\mathrm{d}\mu \tag{3.23}$$

$$= \int_{-H}^{0} \varphi\, \phi\, N^2 \,\mathrm{d}z + (g_b + \tau\, k^2)\varphi(0)\, \phi(0) \tag{3.24}$$

where the induced differential element $\mathrm{d}\mu$ is obtained from equation (2.16) and equation (2.4),

$$\mathrm{d}\mu(z) = \left[N^2_{\mathrm{dist}}(z) + \tau\, k^2\, \delta(z)\right] \mathrm{d}z = \left[N^2(z) + \left(g_b + \tau\, k^2\right) \delta(z)\right] \mathrm{d}z. \tag{3.25}$$

Orthonormality is then given by

$$N_0^4\, H^3\, \delta_{mn} = \langle \hat{w}_m, \hat{w}_n \rangle \tag{3.26}$$

where N_0 is a characteristic value of N^2. We have introduced the factor $N_0^4\, H^3$ on the left-hand side of equation (3.26) so that the Kronecker delta δ_{mn} is non-dimensional.

Since the eigenvalue problem (3.19)–(3.21) is right-definite then L^2_μ, with the inner product (3.23), is a Hilbert space. That is, all eigenfunctions $\hat{w}_n$ satisfy

$$\langle \hat{w}_n, \hat{w}_n \rangle > 0. \tag{3.27}$$

Left-definiteness, along with proposition 2.13, ensures that all eigenvalues $\lambda_n = \sigma_n^{-2}$ are positive. Indeed, the phase speeds σ_n satisfy

$$\sigma_0^2 > \sigma_1^2 > \cdots > \sigma_n^2 > \cdots \to 0. \tag{3.28}$$

Properties of the eigenfunctions

Since $c_2 = 1$ and $d_2 = 0$ then, from equation (2.36), all eigenfunctions $\hat{W}_n = \hat{w}_n$ are continuous—that is, none of the eigenfunctions have finite jump-discontinuities at the upper boundary. See figure 5 for an illustration of the first six eigenfunctions. By lemma 2.11, since $d_2 = 0$, the nth eigenfunction $\hat{w}_n$ has n internal zeros in the interval $(-H, 0)$.

By corollary 2.7, the eigenfunctions $\{\hat{w}_n\}_{n=0}^{\infty}$ are complete in L^2 but do not form a basis in L^2. This overcompleteness in L^2 is due to the additional eigenfunction present in the L^2_μ basis. The presence of a free-surface provides an additional degree of freedom over the usual rigid-lid L^2 basis of internal wave eigenfunctions. Indeed, the $n = 0$ wave in figure 5 corresponds to a surface gravity wave, while the remaining modes can be considered internal gravity waves (although with some surface motion). Thus, the mathematical notion that there is an additional mode when a boundary condition is λ-dependent is given a concrete physical expression in this problem with the presence of a surface gravity wave.

Expansion properties

Given a twice continuously differentiable function $\chi(z)$ satisfying $\chi(-H) = 0$, then from theorem 2.10, we have

$$\chi(z) = \sum_{n=0}^{\infty} \langle \chi, \hat{w}_n \rangle\; \hat{w}_n(z), \quad \text{and} \quad \chi'(z) = \sum_{n=0}^{\infty} \langle \chi, \hat{w}_n \rangle\; w_n'(z) \tag{3.29}$$

with both series converging uniformly on $[-H, 0]$ (note that χ is not required to satisfy any particular boundary condition at $z = 0$). If χ is the vertical structure at time $t = 0$

$$\hat{w}(z, t = 0) = \chi(z) \tag{3.30}$$

then the general solution of the wave problem (at some wavenumber $\boldsymbol{k}$) is

$$\hat{w}(z,t) = \sum_{n=0}^{\infty} \langle \chi, \hat{w}_n \rangle \, w_n(z) \, \mathrm{e}^{-\mathrm{i}\sigma_n k t}. \tag{3.31}$$

Moreover, we can expand the distribution $(\partial_z^2 w)_{\mathrm{dist}}$ of equation (3.9) in terms of the modes $\{w_n\}_{n=0}^{\infty}$. For wave solutions of the form (3.15), the distribution (3.9) has the form

$$(\hat{w}'')_{\mathrm{dist}} = \hat{w}'' - \hat{w}'|_{z=0^-} \, \delta(z). \tag{3.32}$$

For a vertical structure χ, the expansion of $(\chi'')_{\mathrm{dist}}$—defined as in equation (3.32)—is

$$(\chi'')_{\mathrm{dist}}(z) = \sum_{n=0}^{\infty} \langle \chi, \hat{w}_n \rangle \, \hat{w}_n(z) \left\{ \left[k^2 - \frac{N^2(z)}{\sigma_n^2} \right] - \left[\frac{g_b + \tau \, k^2}{\sigma_n^2} \right] \delta(z) \right\} \tag{3.33}$$

The ability of the modes $\{\hat{w}_n\}$ to expand distributions of the form (3.32) is closely related to their ability to represent functions with finite jump-discontinuities at the boundaries.

Uniqueness of the eigenvalue problem (3.19)–(3.21)

It may seem that the choice $c_2 = 1$ is arbitrary—if one multiplies the upper boundary condition (3.21) by $(g_b + \tau \, k^2)$, we obtain the equivalent boundary condition

$$\hat{w}' = \sigma^{-2} \, (g_b + \tau \, k^2) \, \hat{w} \quad \text{for } z = 0. \tag{3.34}$$

Now $c_2 = (g_b + \tau \, k^2)$ and so, by equation (2.36), all eigenfunctions $\hat{W}_n$ have a finite jump-discontinuity at the upper boundary and $\hat{W}_n \neq \hat{w}_n$. However, the alternative upper boundary condition (3.34) is not physical since the differential element $\mathrm{d}\mu(z) = [N^2 + (g_b + \tau \, k^2)^{-1} \, \delta(z)] \, \mathrm{d}z$ associated with the boundary condition (3.34) does not have consistent physical dimensions. Thus, we see that the boundary coefficients $\{a_i, b_i, c_i, d_i\}$ of equations (2.1)–(2.3) must be chosen so that $r \, \mathrm{d}z$ has the same dimensions as $D_i^{-1} \, \delta(z - z_i) \, \mathrm{d}z$ [recall that $\delta(z)$ has the dimension of inverse length].

3.5. *A Boussinesq fluid with a rotating upper boundary*

Let N_0^2 be a typical value of $N^2(z)$. Consider the situation where $f_0^2/N_0^2 \ll 1$ but

$$\frac{g_b + \tau \, k^2}{f_0^2 \, H} \sim O(1). \tag{3.35}$$

Accordingly we may neglect the Coriolis parameter in the interior equation (3.16) but not at the upper boundary condition (3.18). The Boussinesq eigenvalue problem (3.16)–(3.18) reduces to

$$-\hat{w}'' + k^2 \, \hat{w} = \sigma^{-2} \, N^2 \, \hat{w} \quad \text{for } z \in (-H, 0) \tag{3.36}$$

$$\hat{w} = 0 \quad \text{for } z = -H \tag{3.37}$$

$$(g_b + \tau \, k^2)^{-1} \hat{w}' = \sigma^{-2} \left[\hat{w} + \frac{f_0^2}{k^2} (g_b + \tau \, k^2)^{-1} \hat{w}' \right] \quad \text{for } z = 0 \tag{3.38}$$

where $\sigma = \omega/k$ is the phase speed. Equations (3.36)–(3.38) form an eigenvalue problem for the eigenvalue $\lambda = \sigma^{-2}$.

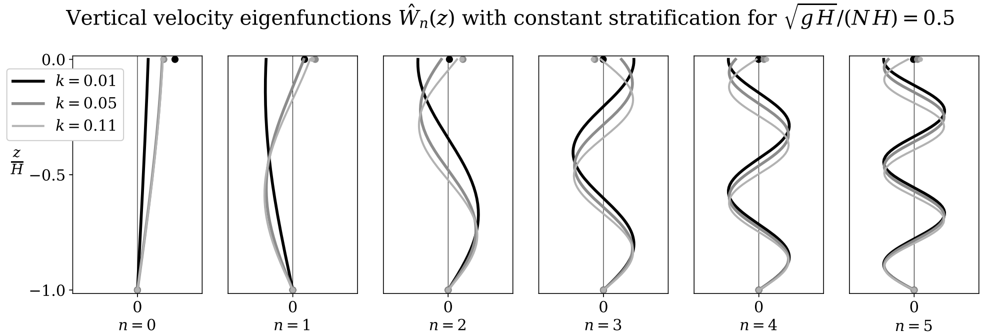

Figure 6: The vertical velocity eigenfunctions $\hat{W}_n$ of a Boussinesq fluid with a rotating upper boundary—eigenvalue problem (3.36)–(3.38). This figure is discussed in §3.5. The wavenumbers k in the figure are non-dimensionalized by the depth H. The dots represent the values of the eigenfunctions at the boundaries. Note that the eigenfunctions have a finite jump-discontinuity at $z = 0$. For $kH = 0.01$ (given by the black line) there are two modes with no internal zeros. As k increases, we obtain two modes with one internal zero (at $kH = 0.05$, the thick grey line) and then two modes with three internal zeros (at $kH = 0.11$, the thin grey line).

Geophysical relevance

Using values of $f_0 = 10^{-4}$ s^{-1} and $g_b = 10$ m s^{-2}, then equation (3.35) gives $g_b/(f_0^2 H) \sim O(1)$ when $H = 10^7$ m—a depth larger than Earth's radius. This estimate shows that the gravitational free-surface contribution is geophysically irrelevant.

For the surface tension component τk^2, the surface tension T of water is $T = 10^{-3}$ kg s^{-2} giving a value $\tau = T/\rho_0 = 10^{-6}$ m^3 s^{-2} where we a density of $\rho_0 = 10^3$ kg m^{-3} is used. Then equation (3.35) gives $(\tau k^2)/(f_0^2 H) \sim O(1)$ when

$$H k^{-2} \sim 10^2 \text{ m}^3. \tag{3.39}$$

For $H = 10^3$ m (e.g., approximate depth of the ocean), then we must have a horizontal length-scale of $k^{-1} \sim 0.32$ m. Alternatively, for a horizontal length-scale $k^{-1} = 10$ m we obtain a depth of $H \sim 1$ m. Thus, the above scale analysis indicates that the eigenvalue problem (3.36)–(3.38) may be relevant at some geophysical scales.

Definiteness & the underlying function space

As in the previous non-rotating problem (3.19)–(3.21), the eigenvalue problem (3.36)–(3.38) is both right-definite and left-definite, as can be verified through propositions 2.3 and 2.4. Since only one boundary condition is λ-dependent, the underlying functions space is again given by equation (3.22) and, by right-definiteness, the function space (3.22), equipped with the inner product (3.23), is a Hilbert space. Thus, all eigenfunctions $\hat{W}_n$ satisfy

$$\left\langle \hat{W}_m, \hat{W}_n \right\rangle > 0 \tag{3.40}$$

where the inner product is given by equation (3.23). By theorem 2.5, all eigenvalues $\lambda_n = \sigma_n^{-2}$ are real and the corresponding eigenfunctions $\{\hat{W}_n\}_{n=0}^{\infty}$ form an orthonormal basis of the Hilbert space L^2_μ. By proposition 2.13, all eigenvalues $\lambda_n = \sigma_n^{-2}$ are positive and satisfy equation (3.28).

Boundary jump-discontinuity of the eigenfunctions

The main difference between the previous non-rotating problem (3.19)–(3.21) and the above problem (3.36)–(3.38) is that, in the present problem, if $f_0 \neq 0$ then $d_2 \neq 0$ [see equation

(2.3)]. Thus, by equation (2.36), the eigenfunctions $\hat{W}_n$, in general, have a jump-discontinuity at the upper boundary $z = 0$ (see figure 6) and so are not equal to the solutions $\hat{w}_n$. The eigenfunction $\hat{W}_n$ are defined by $\hat{W}_n(z) = \hat{w}_n(z)$ for $z \in [-H, 0)$ and

$$\hat{W}_n(0) = \hat{w}_n(0) + \frac{f_0^2}{k^2}(g_b + \tau\, k^2)^{-1}\, \hat{w}_n'(0) \tag{3.41}$$

where the value of the eigenfunction $\hat{W}_n$ at the upper boundary $z = 0$ is given by (2.36). Consequently, the eigenfunctions $\hat{W}_n$ generally have a jump-discontinuity at the upper boundary,

$$\hat{W}_n\Big|_{z=0-}^{z=0} = \frac{f_0^2}{k^2}(g_b + \tau\, k^2)^{-1}\hat{w}_n'(0) = \frac{1}{\frac{f_0^2}{k^2}\,\sigma_n^2 - 1}\,\hat{w}_n(0) \tag{3.42}$$

for $\sigma_n \neq k^2/f_0^2$, where equation (3.38) is used in the last-equality. For large n, we have $\lambda_n = \sigma_n^{-2} \to \infty$ so that

$$\hat{W}_n\Big|_{z=0-}^{z=0} \approx -\hat{w}_n(0) \qquad \text{for } n \text{ sufficiently large,} \tag{3.43}$$

which implies

$$\hat{W}_n(0) \approx 0 \qquad \text{for } n \text{ sufficiently large} \tag{3.44}$$

as can be seen in figure 6.

Physical motion in the vertical velocity modes is given by the solutions $\hat{w}_n$ which are continuous over the closed interval $[-H, 0]$. The jump-discontinuity in the eigenfunctions $\hat{W}_n$ does not correspond to any physical motion; instead, the eigenfunctions $\hat{W}_n$ are convenient mathematical aids used to obtain eigenfunction expansions in the function space L^2_μ.

Number of internal zeros of the eigenfunctions

Another consequence of $d_2 \neq 0$ is that by, lemma 2.11, there are two distinct solutions $\hat{w}_M$ and $\hat{w}_{M+1}$ with the same number of internal zeros (i.e., M) in the interval $(-H, 0)$. Noting that

$$-\frac{b_2}{d_2} = \frac{k^2}{f_0^2} \tag{3.45}$$

the integer M is determined by

$$\sigma_0^2 > \sigma_1^2 > \cdots > \sigma_M^2 > \frac{f_0^2}{k^2} \geqslant \sigma_{M+1}^2 > \cdots > 0. \tag{3.46}$$

A smaller f_0 or a larger k implies a larger M and hence that $\hat{w}_M$ and $\hat{w}_{M+1}$ have a larger number of internal zeros, as shown in figure 6.

Expansion properties

As in the eigenvalue problem for a non-rotating Boussinesq fluid (3.19)–(3.21), corollary 2.7 tells us that the eigenfunctions $\{\hat{w}_n\}_{n=0}^\infty$ are complete in L^2 but the eigenfunctions do not form a basis in L^2 (the modes are overcomplete in L^2). There is now one "additional" eigenfunction—the surface gravity-capillary mode ($n = 0$ in figure 6).

Given a twice continuously differentiable function $\chi(z)$ satisfying $\chi(-H) = 0$, we define the discontinuous function $X(z)$ by

$$X(z) = \begin{cases} \chi(z) & \text{for } z \in [-H, 0) \\ \chi(0) + \frac{f_0^2}{k^2}\left(g_b + \tau\, k^2\right)^{-1} \chi'(0) & \text{for } z = 0 \end{cases} \tag{3.47}$$

that is, we define X as in theorem 2.10 so that $X(0) = C_2\chi$. Then, by theorem 2.10, we have the expansions

$$\chi(z) = \sum_{n=0}^{\infty} \left\langle X, \hat{W}_n \right\rangle \hat{w}_n(z), \quad \text{and} \quad \chi'(z) = \sum_{n=0}^{\infty} \left\langle X, \hat{W}_n \right\rangle w_n'(z). \tag{3.48}$$

Moreover, if $\chi(z)$ is the vertical structure at $t = 0$,

$$\hat{w}(z, t = 0) = \chi(z), \tag{3.49}$$

then the time-evolution of the vertical structure $\chi(z)$ is

$$\hat{w}(z, t) = \sum_{n=0}^{\infty} \left\langle X, \hat{W}_n \right\rangle \hat{w}_n(z)\, \mathrm{e}^{-\mathrm{i}\sigma_n k t}. \tag{3.50}$$

Finally, as for the non-rotating problem, we can expand the distribution $(\chi'')_{\text{dist}}$, defined as in (3.32), as

$$(\chi'')_{\text{dist}}(z) = \sum_{n=0}^{\infty} \left\langle X, \hat{W}_n \right\rangle \hat{w}_n(z) \left\{ \left[k^2 - \frac{N^2(z)}{\sigma_n^2} \right] - \left[\frac{g_b + \tau\, k^2}{\sigma_n^2 - f_0^2/k^2} \right] \delta(z) \right\} \tag{3.51}$$

when $\sigma_n^2 \neq f_0^2/k^2$ for $n = 0, 1, 2, \ldots$

4. Quasigeostrophic waves

4.1. *Equations of motion*

The quasigeostrophic equations are an asymptotic regime of the Boussinesq equations that emerge in the limit of rapid rotation and strong stratification. We write the geostrophic horizontal velocity as

$$\boldsymbol{u} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_z \psi \tag{4.1}$$

where ψ is the geostrophic streamfunction and $\boldsymbol{\nabla}_z = \hat{\boldsymbol{x}}\, \partial_x + \hat{\boldsymbol{y}}\, \partial_y$ is the horizontal gradient. The quasigeostrophic equations then consist of a time-evolution equation for the vertical component of the vorticity $\zeta = \nabla_z^2 \psi$,

$$\partial_t \zeta + \hat{\boldsymbol{z}} \cdot [\boldsymbol{\nabla}_z \psi \times \boldsymbol{\nabla}_z (\zeta + f)] = f_0\, \partial_z w \tag{4.2}$$

and a time-evolution equation for the buoyancy $b = f_0\, \partial_z \psi$,

$$\partial_t b + \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_z \psi \times \boldsymbol{\nabla}_z b) + N^2\, w = 0, \tag{4.3}$$

where w is the vertical velocity, $N^2(z)$ is the buoyancy frequency, and the Coriolis frequency f is a linear function of the horizontal position vector $\boldsymbol{x} = (x, y)$ with $f(\boldsymbol{0}) = f_0$.

The vorticity equation (4.2) and buoyancy equation (4.3) can be combined to form a material conservation equation for the volume potential vorticity q,

$$\partial_t q + \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_z \psi \times \boldsymbol{\nabla}_z q) = 0 \tag{4.4}$$

where the volume potential vorticity is given by

$$q = f + \nabla_z^2 \psi + \partial_z \left(S^{-1}\, \partial_z \psi \right) \tag{4.5}$$

with $S = N^2/f_0^2$. See Vallis (2017) for more details.

We consider a fluid region that is periodic in both horizontal directions $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. The fluid region is bounded above and below by rigid boundaries that deviate infinitesimally from being flat. That is, the upper and lower boundaries are located at

$$z = z_i + h_i \approx z_i \quad i \in \{1, 2\} \tag{4.6}$$

where h_i is infinitesimally small topography, $i = 1$ is the lower boundary, and $i = 2$ is the upper boundary. The time-evolution equations at the boundaries are obtained by setting $w = 0$ at the boundaries and using the buoyancy equation (4.3),

$$\partial_t r_i + \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_z \psi \times \boldsymbol{\nabla}_z r_i) = 0 \quad \text{for } i \in \{1, 2\} \tag{4.7}$$

where the surface potential vorticity r_i is given by

$$r_i = (-1)^{i+1} \left[f_0\, h_i + \left(S^{-1}\, \partial_z \psi \right) \big|_{z=z_i} \right] \quad \text{for } i \in \{1, 2\}. \tag{4.8}$$

Equations (4.4) and (4.7) form a closed system describing the evolution of a quasi-geostrophic fluid.

4.2. *A δ-sheet formulation*

The δ-sheet formulation for quasigeostrophic theory is due to Bretherton (1966)—see also section 5 in Hoskins *et al.* (1985) and Schneider *et al.* (2003). This formulation derives from the observation that the surface potential vorticities r_i can be thought of as infinitesimally thin sheets of volume potential vorticity q. We then define a potential vorticity distribution q_{dist} given by

$$q_{\text{dist}} = q + \sum_{i=1}^{2} r_i\, \delta(z - z_i). \tag{4.9}$$

The evolution equation for q_{dist}

$$\partial_t q_{\text{dist}} + \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_z \psi \times \boldsymbol{\nabla}_z q_{\text{dist}}) = 0 \tag{4.10}$$

with the homogeneous boundary conditions

$$\partial_z \psi \big|_{z_1} = \partial_z \psi \big|_{z_2} = 0 \tag{4.11}$$

offer an equivalent description of quasigeostrophic dynamics.

4.3. *Linear equations*

Linearizing equations (4.4) and (4.7) about a quiescent background state with a background volume potential vorticity q_B satisfying

$$\boldsymbol{\nabla}_z q_B = \boldsymbol{\nabla}_z f \tag{4.12}$$

and background surface potential vorticity $r_{i\,B}$ satisfying

$$\boldsymbol{\nabla}_z r_{i\,B} = (-1)^{i+1} f_0\, \boldsymbol{\nabla}_z h_i \quad \text{for } i \in \{1, 2\}, \tag{4.13}$$

leads to

$$\partial_t \left[\nabla_z^2 \psi + \partial_z \left(S^{-1}\, \partial_z \psi \right) \right] + \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_z \psi \times \boldsymbol{\nabla}_z f) = 0 \tag{4.14}$$

$$\partial_t \left(S^{-1}\, \partial_z \psi \right) |_{z_i} + \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_z \psi \times f_0\, \boldsymbol{\nabla}_z h_i) = 0 \quad \text{for } i \in \{1, 2\}. \tag{4.15}$$

We assume f, h_1, and h_2 are linear functions of the horizontal position vector $\boldsymbol{x}$. Equation (4.14) and (4.15) are the linear equations governing infinitesimal perturbations to a resting quasigeostrophic fluid on a β-plane with (infinitesimally) linearly sloping upper and lower boundaries. Typically $\boldsymbol{\nabla}_z f = \beta\,\hat{\boldsymbol{y}}$ so that

$$\hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_z \psi \times \boldsymbol{\nabla}_z f) = \beta\,\partial_x \psi \tag{4.16}$$

but we will continue writing $\boldsymbol{\nabla}_z f$ to maintain the close analogy between $\boldsymbol{\nabla}_z f$ and $f_0 \boldsymbol{\nabla}_z h_i$.

4.4. *The streamfunction eigenvalue problem*

We assume wave solutions of the form

$$\psi(\boldsymbol{x}, z, t) = \hat{\psi}(z)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{4.17}$$

where $\boldsymbol{k} = \hat{\boldsymbol{x}}\, k_x + \hat{\boldsymbol{y}}\, k_y$ is the horizontal wavenumber and ω is the angular frequency.

Before proceeding, it is useful to define the following. First, we define the β parameter by

$$\beta = |\boldsymbol{\nabla}_z f| \tag{4.18}$$

which agrees with the usual definition when $\boldsymbol{\nabla}_z f = \beta\,\hat{\boldsymbol{y}}$ and $\beta > 0$. We denote by $\Delta\theta_f$ the angle between the horizontal wavevector $\boldsymbol{k}$ and the gradient of Coriolis parameter $\boldsymbol{\nabla}_z f$,

$$\sin(\Delta\theta_f) = \frac{1}{k\,\beta}\, \hat{\boldsymbol{z}} \cdot (\boldsymbol{k} \times \boldsymbol{\nabla}_z f)\,. \tag{4.19}$$

Positive angles are measured counter-clockwise relative to $\boldsymbol{k}$. Thus, $\Delta\theta_f > 0$ indicates that $\boldsymbol{k}$ points to the right of $\boldsymbol{\nabla}_z f$, while $\Delta\theta_f < 0$ indicates that $\boldsymbol{k}$ points to the left of $\boldsymbol{\nabla}_z f$.

We define the topographic parameters α_i by

$$\alpha_i = |f_0\, \boldsymbol{\nabla}_z h_i|\,. \tag{4.20}$$

In analogy with $\Delta\theta_f$, we define the two angles $\Delta\theta_i$ for $i \in \{1, 2\}$ by

$$\sin(\Delta\theta_i) = \frac{1}{k\,\alpha_i}\, \hat{\boldsymbol{z}} \cdot (\boldsymbol{k} \times f_0 \boldsymbol{\nabla}_z h_i) \tag{4.21}$$

with a similar interpretation assigned to $\Delta\theta_i > 0$ and $\Delta\theta_i < 0$.

Now substitute the wave solution (4.17) into the linear quasigeostrophic equations (4.14) and (4.15); if we assume that $\alpha_i \sin(\Delta\theta_i) \neq 0$ for $i \in \{1, 2\}$, $\omega \neq 0$, as well as $k \neq 0$, then we obtain

$$-(S^{-1}\,\hat{\psi}')' + k^2\,\hat{\psi} = \lambda\,\hat{\psi} \quad \text{for } z \in (z_1, z_2) \tag{4.22}$$

$$-\frac{\beta}{\alpha_i}\,\frac{\sin(\Delta\theta_f)}{\sin(\Delta\theta_i)}\, S^{-1}\,\hat{\psi}' = \lambda\,\psi \quad \text{for } z = z_i \text{ where } i \in \{1, 2\} \tag{4.23}$$

where we have defined the eigenvalue λ by

$$\lambda = -\frac{k\,\beta\,\sin(\Delta\theta_f)}{\omega}. \tag{4.24}$$

If $\alpha_i \sin(\Delta\theta_i) = 0$, for $i \in \{1, 2\}$, then the corresponding boundary condition is replaced with

$$S^{-1}\,\hat{\psi}' = 0 \quad \text{for } z = z_i. \tag{4.25}$$

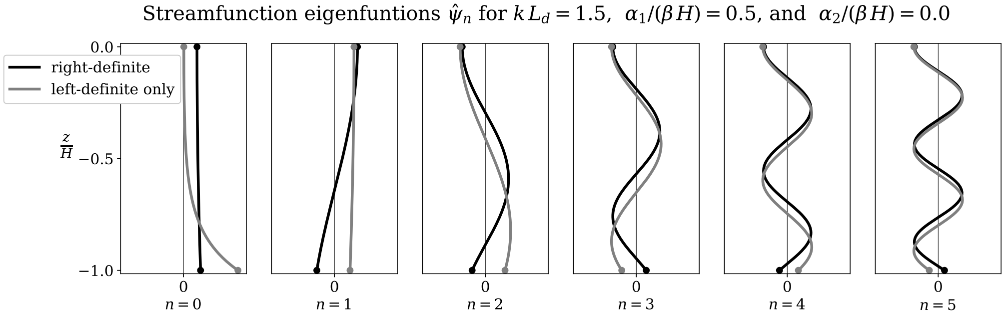

Figure 7: The streamfunction eigenfunctions $\hat{\psi}_n$ of the quasigeostrophic eigenvalue problem with a sloping bottom from §4.4. Two cases are shown. The first is with $\Delta\theta_f = -90°$ and $\Delta\theta_1 = -30°$ and is both right-definite and left-definite. The second is with $\Delta\theta_f = -45°$ and $\Delta\theta_1 = 15°$ and is only left-definite. In the right-definite case the nth eigenfunction has n internal zero whereas in the left-definite only case there are two eigenfunctions ($n = 0, 1$) with no internal zeros.

4.4.1. *Sloping lower boundary and flat upper boundary*

For a flat upper boundary (i.e., $\alpha_2 = 0$), the appropriate eigenvalue problem consists of equations (4.22), (4.23) with $i = 1$, and (4.25) with $i = 2$. That is, we have

$$-(S^{-1}\hat{\psi}')' + k^2\hat{\psi} = \lambda\hat{\psi} \quad \text{for } z \in (-H, 0) \tag{4.26}$$

$$-\frac{\beta}{\alpha_i}\frac{\sin(\Delta\theta_f)}{\sin(\Delta\theta_i)} S^{-1}\hat{\psi}' = \lambda\psi \quad \text{for } z = -H \tag{4.27}$$

$$S^{-1}\psi' = 0 \quad \text{for } z = 0. \tag{4.28}$$

Since $k \neq 0$ then $\lambda = 0$ is not an eigenvalue.

Definiteness & the underlying function space

The eigenvalue problem has one λ-dependent boundary condition and so the underlying function space is

$$L^2_\mu \cong L^2 \oplus \mathbb{C}. \tag{4.29}$$

The differential element $\mathrm{d}\mu$ is obtained by using equations (2.16) and (2.4),

$$\mathrm{d}\mu(z) = \left[1 + \frac{\alpha_1}{\beta}\frac{\sin(\Delta\theta_i)}{\sin(\Delta\theta_f)}\delta(z+H)\right]\mathrm{d}z, \tag{4.30}$$

implying an inner product of the form

$$\langle\varphi, \phi\rangle = \int_{-H}^{0} \varphi\,\phi\,\mathrm{d}\mu \tag{4.31}$$

$$= \int_{-H}^{0} \varphi\,\phi\,\mathrm{d}z + \frac{\alpha_1}{\beta}\frac{\sin(\Delta\theta_i)}{\sin(\Delta\theta_f)}\varphi(-H)\,\phi(-H), \tag{4.32}$$

where $\mathrm{d}\mu$ is given by equation (4.30).

By proposition 2.3, the problem is right-definite for horizontal wavevectors $\boldsymbol{k}$ satisfying

$$\frac{\sin(\Delta\theta_i)}{\sin(\Delta\theta_f)} > 0 \tag{4.33}$$

and, in such cases, L^2_μ equipped with the inner product (2.19) is a Hilbert space. However, L^2_μ

is not a Hilbert space for all wavevectors $\boldsymbol{k}$. By proposition 2.4, the problem is left-definite for all wavevectors $\boldsymbol{k}$ and so L^2_μ, equipped with the inner product (2.19), is generally a Pontryagin space.

We write $\hat{\Psi}_n$ for the eigenfunctions and $\hat{\psi}_n$ for the solutions of equations (4.26)–(4.28). The eigenfunctions $\hat{\Psi}_n$ are related to the solutions $\hat{\psi}_n$ by (2.17) with boundary values $\hat{\Psi}_n(0)$ given by equation (2.36). However, since $c_1 = 1$ and $d_1 = 0$ in equation (4.23) [compare with equations (2.1)–(2.3)] then $\hat{\Psi}_n = \hat{\psi}_n$ on the closed interval $[-H, 0]$. Thus, the solutions ψ_n are also the eigenfunctions.

With theorem 2.5, we deduce that all eigenvalues λ_n are real and the corresponding eigenfunctions $\{\hat{\psi}_n\}_{n=0}^{\infty}$ form an orthonormal basis for L^2_μ. Orthonormality is defined with respect to the inner product given by equation (4.31). Orthonormality of the eigenfunctions then takes the form

$$\pm\,\delta_{mn} = \frac{1}{H}\left\langle \hat{\psi}_m, \hat{\psi}_n \right\rangle \tag{4.34}$$

where we have taken the eigenfunctions ψ_m and ψ_n to be non-dimensional.

Properties of the eigenfunctions

Since $c_1 = 1$ and $d_1 = 0$ then, by equation (2.36), all eigenfunctions $\hat{\psi}_n$ are continuous—none of the eigenfunctions have finite jump-discontinuities at the lower boundary. Furthermore, by lemma 2.14, the number of internal zeros of the eigenfunctions $\{\hat{\psi}_n\}_{n=0}^{\infty}$ depends on the definiteness of the problem (see figure 7) :

1. if the problem is right-definite then the nth eigenfunction has n internal zeros,

2. if the problem is only left-definite then both ψ_0 and ψ_1 have no internal zeros; the remaining eigenfunctions ψ_n, for $n > 1$, have $n - 1$ internal zeros.

As the problem is left-definite for all wavevectors $\boldsymbol{k}$, we can use proposition 2.13 to determine the sign of the eigenvalues. Proposition 2.13 informs us that

$$\lambda_n \left\langle \hat{\psi}_n, \hat{\psi}_n \right\rangle > 0. \tag{4.35}$$

In the first case, when the problem is right-definite, all eigenvalues are positive and all eigenfunctions $\hat{\psi}_n$ satisfy $\left\langle \hat{\psi}_n, \hat{\psi}_n \right\rangle > 0$. In the second case, when the problem is only left-definite, then there is one negative eigenvalue λ_0 and the corresponding eigenfunction $\hat{\psi}_0$ satisfies $\left\langle \hat{\psi}_0, \hat{\psi}_0 \right\rangle < 0$. The remaining eigenvalues are positive and their corresponding eigenfunctions satisfy $\left\langle \hat{\psi}_n, \hat{\psi}_n \right\rangle > 0$. In fact, from equation (4.24), we see that waves with $\left\langle \hat{\psi}_n, \hat{\psi}_n \right\rangle > 0$ have westward phase speeds $\omega_n/k < 0$ while waves with $\left\langle \hat{\psi}_n, \hat{\psi}_n \right\rangle < 0$ have eastward phase speeds $\omega_n/k > 0$.

4.4.2. *Expansion properties*

By corollary 2.7 the eigenfunctions $\{\hat{\psi}_n\}_{n=0}^{\infty}$ are complete in L^2 but do not form a basis in L^2. There is one "redundant" eigenfunction. Physically, the additional eigenfunction corresponds to the topographic Rossby waves ($n = 0$ in figure 7).

Given a twice continuously differentiable function $\phi(z)$ satisfying $\phi'(0) = 0$, then from theorem 2.10, we have

$$\phi(z) = \sum_{n=0}^{\infty} \left\langle \phi, \hat{\psi}_n \right\rangle \hat{\psi}_n(z), \quad \text{and} \quad \phi'(z) = \sum_{n=0}^{\infty} \left\langle \phi, \hat{\psi}_n \right\rangle \hat{\psi}_n(z) \tag{4.36}$$

with both series converging uniformly on $[-H, 0]$ (note that ϕ is not required to satisfy any boundary condition at $z = -H$). If at some wavevector $\boldsymbol{k}$ the vertical structure at time $t = 0$ is given by ϕ,

$$\hat{\psi}(z, t = 0) = \phi(z) \tag{4.37}$$

then the time-evolution is given by

$$\hat{\psi}(z,t) = \sum_{n=0}^{\infty} \left\langle \phi, \hat{\psi}_n \right\rangle \hat{\psi}_n(z)\, \mathrm{e}^{-i\omega_n t} \tag{4.38}$$

where the angular frequency ω_n is given by equation (4.24).

For wave solutions of the form (4.17), the potential vorticity distribution (4.9) is

$$\hat{q}_{\mathrm{dist}}(z) = \hat{q}(z) + \hat{r}_1\, \delta(z+H) \tag{4.39}$$

where $\hat{q}$ and $\hat{r}_1$, for a function ϕ, are given by

$$\hat{q} = -k^2\,\phi + \left(S^{-1}\,\phi'\right)' \quad \text{and} \quad \hat{r}_1 = \left(S^{-1}\,\phi'\right)\Big|_{z=-H}. \tag{4.40}$$

Note that r_2 must be zero in this problem since the upper boundary is flat. We can then expand the distribution q_{dist} in terms of the eigenfunctions,

$$\hat{q}_{\mathrm{dist}}(z) = -\sum_{n=0}^{\infty} \lambda_n \left\langle \phi, \hat{\psi}_n \right\rangle \hat{\psi}_n(z) \left[1 + \frac{\alpha_1}{\beta}\,\frac{\sin\left(\Delta\theta_i\right)}{\sin\left(\Delta\theta_f\right)}\,\delta(z+H)\right]. \tag{4.41}$$

4.5. *The vertical velocity eigenvalue problem*

An equivalent eigenvalue problem for the quasigeostrophic normal modes can be obtained in terms of the vertical velocity. One obtains the vertical velocity eigenvalue problem by solving for the vertical velocity $\hat{w}$ and its derivative $\hat{w}'$ in equations (4.2) and (4.3). For a flat upper boundary ($\alpha_2 = 0$), the vertical velocity eigenvalue problem is given by

$$-\hat{w}'' + k^2\, S\, \hat{w} = \lambda\, S\, \hat{w} \quad \text{for } z \in (-H, 0) \tag{4.42}$$

$$k^2\,\hat{w} = \lambda \left[\hat{w} - \frac{\alpha_1}{\beta}\,\frac{\sin\left(\Delta\theta_1\right)}{\sin\left(\Delta\theta_f\right)}\,\hat{w}'\right] \quad \text{for } z = -H, \tag{4.43}$$

$$\hat{w} = 0 \quad \text{for } z = 0. \tag{4.44}$$

where the eigenvalue λ is given by

$$\lambda = -\frac{k}{\omega}\,\beta\,\sin\left(\Delta\theta_f\right). \tag{4.45}$$

Definiteness & the underlying function space

The eigenvalue problem (4.42)–(4.44) has one λ-dependent boundary condition and so the underlying function space is

$$L^2_\mu = L^2 \oplus \mathbb{C}. \tag{4.46}$$

The differential element $\mathrm{d}\mu$ is obtained by using equations (2.16) and (2.4),

$$\mathrm{d}\mu(z) = \left[S(z) - \frac{1}{k^2}\,\frac{\beta}{\alpha_1}\,\frac{\sin\left(\Delta\theta_f\right)}{\sin\left(\Delta\theta_1\right)}\,\delta(z+H)\right]\mathrm{d}z. \tag{4.47}$$

The corresponding inner product (2.19), for functions φ and ϕ, is given by

$$\langle \varphi, \phi \rangle = \int_{-H}^{0} \varphi\,\phi\,\mathrm{d}\mu \tag{4.48}$$

$$= \int_{-H}^{0} \varphi\,\phi\,S\,\mathrm{d}z - \frac{1}{k^2}\,\frac{\beta}{\alpha_1}\,\frac{\sin\left(\Delta\theta_f\right)}{\sin\left(\Delta\theta_1\right)}\,\varphi(-H)\,\phi(-H). \tag{4.49}$$

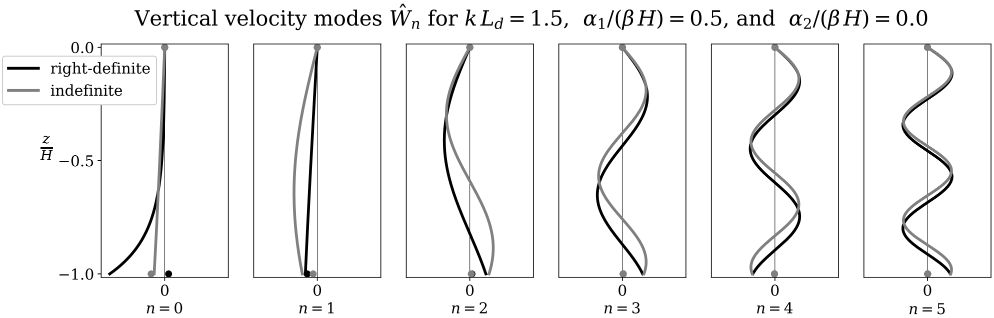

Figure 8: The vertical eigenfunctions $\hat{W}_n$ of the quasigeostrophic eigenvalue problem with a sloping bottom from §4.5. The black and grey dots represent the values of the eigenfunctions at the boundaries. Note that the eigenfunctions have a finite jump-discontinuity at the lower boundary. Two cases are shown. The first is right-definite with $\Delta\theta_f = -45°$ and $\Delta\theta_1 = 15°$. The second is indefinite with $\Delta\theta_f = -90°$ and $\Delta\theta_1 = -30°$. In both cases, the first two eigenfunctions ($n = 0, 1$) have no internal zeros.

The eigenvalue problem (4.42)–(4.44) is right-definite for horizontal wavevectors $\boldsymbol{k}$ satisfying

$$\frac{\sin(\Delta\theta_1)}{\sin(\Delta\theta_f)} < 0. \tag{4.50}$$

The problem, however, is not left-definite for any wavevector $\boldsymbol{k}$. When the problem is right-definite, the theory of §2 applies: if equation (4.50) holds then the eigenvalues are real and the corresponding eigenfunctions form a basis for L^2_μ. But when (4.50) is not satisfied, we are not able to make these conclusions based on the theory of §2.

We can, however, make progress by exploiting the relationship between the vertical velocity eigenvalue problem (4.42)–(4.44) and the streamfunction eigenvalue problem (4.26)–(4.28). Due to equations (4.2) and (4.3), both problems have identical eigenvalues and their solutions are related by

$$\hat{\psi}_n = -\mathrm{i}\left[\frac{f_0/\omega_n}{k^2 + \frac{k}{\omega_n}\beta\sin(\Delta\theta_f)}\right]\hat{w}_n' \quad \text{and} \quad \hat{w}_n = \mathrm{i}\left[\frac{f_0\,\omega_n}{N^2}\right]\hat{\psi}_n' \quad \text{for } z \in (z_1, z_2). \tag{4.51}$$

Thus the eigenvalues of the vertical velocity eigenvalue problem (4.42)-(4.43) are real and simple even when the problem is indefinite. Furthermore, due to the simplicity of the eigenvalues, no generalized eigenfunctions can arise (see appendix A.2). Each eigenvalue λ_n has precisely one eigenfunction and the set of all eigenfunctions form a basis for L^2_μ.

Boundary jump-discontinuities of the eigenfunctions

Since $d_1 \neq 0$ [compare equations (4.42)–(4.44) with equations (2.1)–(2.3)], then the eigenfunctions $\hat{W}_n$ are not equal to the solutions $\hat{w}_n$ of equations (4.42)–(4.44). The eigenfunctions $\hat{W}_n$ are related to the solutions by (2.17) with the boundary value $\hat{W}_n(-H)$ determined by equation (2.36) with $i = 1$. Explicitly, the eigenfunctions $\hat{W}_n$ are defined by $\hat{W}_n(z) = \hat{w}_n(z)$ for $z \in (-H, 0]$ and

$$\hat{W}_n(-H) = \hat{w}_n(-H) - \frac{\alpha_1}{\beta}\frac{\sin(\Delta\theta_1)}{\sin(\Delta\theta_f)}\hat{w}_n'(-H). \tag{4.52}$$

Thus, the eigenfunctions $\hat{W}_n$ are generally discontinuous at the lower boundary, as can be observed in figure 8.

The discontinuity in the vertical velocity at the lower boundary implied by equation (4.52) is

$$\hat{W}_n\Big|_{-H}^{-H+} = \frac{\alpha_1}{\beta}\,\frac{\sin(\Delta\theta_1)}{\sin(\Delta\theta_f)}\,\hat{w}_n'(-H) = \left(\frac{\lambda_n - k^2}{\lambda_n}\right)\hat{w}_n(-H). \tag{4.53}$$

where the lower boundary condition (4.43) is used in the last equality. For sufficiently large n, we have

$$\hat{W}_n\Big|_{z_1}^{z_1+} \approx \hat{w}_n(z_1+) \tag{4.54}$$

so that the vertical velocity jumps from a value of $\hat{W}_n(-H+)$ right above the bottom boundary to $\hat{W}_n(-H) \approx 0$ at the bottom—as can be observed in figure 8.

Number of internal zeros of the eigenfunctions

Another consequence of $d_1 \neq 0$ is that, in the right-definite case, there are two distinct eigenfunctions $\hat{w}_M$ and $\hat{w}_{M+1}$ that have the same number of internal zeros in the interval (z_1, z_2). Observing that

$$-\frac{b_1}{d_1} = -k^2 < 0 \tag{4.55}$$

we conclude, using lemma 2.11, that $M = 0$ and that, in the right-definite case, there are two eigenfunctions, $\hat{w}_0$ and $\hat{w}_1$, with no internal zeros. When the problem is not right-definite, we cannot apply the theory of §2.3. In figure 8, we see that in both the right-definite and indefinite case we have two eigenfunctions with no internal zeros.

Expansion properties

By corollary 2.7, the eigenfunctions $\{\hat{w}_n\}_{n=0}^{\infty}$ are complete in L^2 but do not form a basis of L^2. There is one "redundant" eigenfunction. There is an additional degree of freedom over the usual vertical velocity L^2 basis of quasigeostrophy; this additional degree of freedom corresponds to the topographic Rossby wave.

Given a twice continuously differentiable function $\chi(z)$ satisfying $\chi'(0) = 0$, we define the discontinuous function $X(z)$ by

$$X(z) = \begin{cases} \chi(z) & \text{for } z \in (-H, 0] \\ \hat{\chi}(-H) - \frac{\alpha_1}{\beta}\frac{\sin(\Delta\theta_1)}{\sin(\Delta\theta_f)}\,\hat{\chi}'(-H) & \text{for } z = -H, \end{cases} \tag{4.56}$$

that is, we define X as in theorem 2.10 so that $X(-H) = C_1\chi$. Then, by theorem 2.10, we have the expansions

$$\chi(z) = \sum_{n=0}^{\infty}\left\langle X, \hat{W}_n\right\rangle \hat{w}_n(z), \quad \text{and } \chi'(z) = \sum_{n=0}^{\infty}\left\langle X, \hat{W}_n\right\rangle \hat{w}_n'(z) \tag{4.57}$$

with both series converging uniformly on the closed interval $[-H, 0]$ (note that χ is not required to satisfy any boundary conditions at $z = -H$). Moreover, if at some wavevector $\boldsymbol{k}$ the vertical structure at $t = 0$ is $\chi(z)$,

$$\hat{w}(z, t = 0) = \chi(z), \tag{4.58}$$

then the time-evolution of the vertical structure $\chi(z)$ is

$$\hat{w}(z, t) = \sum_{n=0}^{\infty}\left\langle X, \hat{W}_n\right\rangle \hat{w}_n(z)\,\mathrm{e}^{-\mathrm{i}\omega_n t} \tag{4.59}$$

where the angular frequency ω_n is given by equation (4.24).

5. Summary and conclusions

We have developed a framework for the analysis of three-dimensional wave problems with dynamically active boundaries (i.e., boundaries where time derivatives appear in the boundary conditions). The resulting waves have vertical structures that depend on the wavevector $\boldsymbol{k}$: For Boussinesq gravity waves, the dependence is only through the wavenumber k whereas the dependence for quasigeostrophic Rossby waves is on both the wavenumber k and the propagation direction $\boldsymbol{k}/k$. Moreover, the vertical structures of the waves are complete in a space larger than L^2, namely, they are complete in $L^2_\mu \cong L^2 \oplus \mathbb{C}^s$ where s is the number of dynamically active boundaries (and the number of boundary-trapped waves). Essentially, each dynamically active boundary contributes an additional boundary-trapped wave and hence an additional degree of freedom to the problem. Mathematically, the presence of boundary-trapped waves allows us to expand a larger collection of functions (with a uniformly convergent series) in terms of the modes. Furthermore, the resulting series are term-by-term differentiable and the differentiated series converges uniformly. For Boussinesq gravity waves with a free-surface, we are now able to expand functions that do not vanish at the upper boundary. For quasigeostrophic Rossby waves with a sloping lower boundary, the vertical derivative of the streamfunction no longer has to vanish at the lower boundary. In fact, the normal modes have the intriguing property converging pointwise to functions with finite jump discontinuities at the boundaries. We also found that problems with dynamically active boundaries have a "δ-sheet" formulation analogous to the Bretherton (1966) formulation in quasigeostrophic theory. In this formulation, various distributions arise, and these distributions can be represented in terms of the modes as well.

Normal mode decompositions of quasigeostrophic motion play an important role in physical oceanography (e.g., Wunsch 1997; Lapeyre 2009; LaCasce 2017). In the companion article Yassin & Griffies (in prep.), we explore the implications of the mathematical framework developed in this article to normal mode decompositions in bounded quasigeostrophic fluids. Another applications is to the extension of equilibrium statistical mechanical calculations (e.g., Bouchet & Venaille 2012; Venaille *et al.* 2012) to three-dimensional systems with dynamically active boundaries. Finally, a natural application of the mathematical framework here is to the development of weakly non-linear wave turbulence theories (e.g., Fu & Flierl 1980; Smith & Vallis 2001; Scott 2014) in systems with both internal and boundary-trapped waves.

Acknowledgements. I sincerely thank Stephen Griffies for helpful comments and insights offered throughout this project.

Funding. This report was prepared by Houssam Yassin under award NA18OAR4320123 from the National Oceanic and Atmospheric Administration, U.S. Department of Commerce. The statements, findings, conclusions, and recommendations are those of the author and do not necessarily reflect the views of the National Oceanic and Atmospheric Administration, or the U.S. Department of Commerce.

Declaration of interests. The author reports no conflict of interest.

Author ORCID. H. Yassin, https://orcid.org/0000-0003-1742-745X

Appendix A. Pontryagin spaces

An introduction to the theory of Pontryagin spaces can be found in Iohvidov & Krein (1960) as well as in the monograph of Bognár (1974). Another resource is the monograph of Azizov & Iokhvidov (1989) on linear operators in indefinite inner product spaces.

The purpose of this appendix is to provide a brief introduction to the theory of Pontryagin spaces and review the facts required to prove a spectral theorem for positive compact operators in Pontryagin spaces (theorem A.17). The proof of theorem 2.5 is given in appendix B.2 and crucially makes use of theorem A.17.

Unless otherwise stated, we follow the terminology and definitions of Bognár (1974).

A.1. *Indefinite inner product spaces*

Definition A.1 (Inner product space). *Let Π be a vector space over the field of complex numbers $\mathbb{C}$ and let $\langle\cdot,\cdot\rangle : \Pi\times\Pi \to \mathbb{C}$ be a hermitian sesquilinear form on Π. The pair $(\Pi, \langle\cdot,\cdot\rangle)$ is said to be an inner product space.*

In the above definition, the inner product $\langle\cdot,\cdot\rangle$ is not assumed to be definite. We can then classify the elements of Π into three classes. Let $\phi \in \Pi$, if $\langle\phi,\phi\rangle > 0$ then ϕ is said to *positive*, if $\langle\phi,\phi\rangle < 0$ then ϕ is said to be *negative*, and if $\langle\phi,\phi\rangle = 0$ then ϕ is said to be *neutral* or *isotropic*.

Definition A.2 (Indefinite inner product space). *If the inner product space $(\Pi, \langle\cdot,\cdot\rangle)$ contains both positive and negative elements, we say that the inner product $\langle\cdot,\cdot\rangle$ is indefinite on Π and that $(\Pi, \langle\cdot,\cdot\rangle)$ is an indefinite inner product space.*

An important property of indefinite inner product spaces is the following (Bognár 1974, lemma I.2.1).

Lemma A.3. *Every indefinite inner product space contains non-zero neutral elements.*

A subspace $E \subset \Pi$ is said to be *degenerate* if there exists a non-zero element $\phi \in E$ that is orthogonal to every element of E. Otherwise, the space E is said to be *non-degenerate*.

If the inner product is positive-definite on a subspace of $E \subset \Pi$, the subspace E is said to be *positive-definite*. Likewise, if the inner product is negative-definite on a subspace of $E \subset \Pi$, the subspace E is said to *negative-definite*.

We formulate the definition of a Pontryagin space as in Azizov & Iokhvidov (1981) by first defining the more general construct of a Krein space.

Definition A.4 (Krein space). *Let $(\Pi, \langle\cdot,\cdot\rangle)$ be an inner product space. Suppose that the space Π admits a decomposition into a direct sum*

$$\Pi = \Pi^+ \oplus \Pi^- \tag{A 1}$$

of orthogonal subspaces such that both $(\Pi^+, +\langle\cdot,\cdot\rangle)$ and $(\Pi^-, -\langle\cdot,\cdot\rangle)$ are Hilbert spaces. Then $(\Pi, \langle\cdot,\cdot\rangle)$ is said to be a Krein space.

Essentially, a Krein space is a non-degenerate, decomposable, and complete inner product space.

Given a Krein space $(\Pi, \langle\cdot,\cdot\rangle)$ with an orthogonal decomposition $\Pi = \Pi^+ \oplus \Pi^-$, let $\kappa = \dim \Pi^-$. The number κ is known as the *rank of indefiniteness* of the Krein space Π.

Definition A.5 (Pontryagin space). *If the Krein space $(\Pi, \langle\cdot,\cdot\rangle)$ has a finite rank of indefiniteness κ, then we say that $(\Pi, \langle\cdot,\cdot\rangle)$ is a Pontryagin space of index κ and we denote the space by Π_κ.*

We note that if the inner product $\langle\cdot,\cdot\rangle$ is positive-definite, then $\kappa = 0$ and Π_0 is therefore a Hilbert space.

From the definition of a Krein space, the subspaces $(\Pi^\pm, \pm\langle\cdot,\cdot\rangle)$ are assumed to be complete under the induced norms $\|\phi\|_\pm = \sqrt{\pm\langle\phi,\phi\rangle}$. It follows that the “natural" norm on a Krein space is given by

$$\|\phi\|^2 = \langle\phi_+,\phi_+\rangle - \langle\phi_-,\phi_-\rangle \tag{A 2}$$

where $\phi = \phi_+ + \phi_-$ with $\phi_\pm \in \Pi^\pm$. In fact, one can always associate with a Krein space a corresponding Hilbert space $(\Pi, \langle\cdot,\cdot\rangle_+)$ where the positive-definite inner product $\langle\cdot,\cdot\rangle_+$ is given by

$$\langle\phi,\psi\rangle_+ = \langle\phi_+,\psi_+\rangle - \langle\phi_-,\psi_-\rangle\,, \quad \phi,\psi \in \Pi \tag{A 3}$$

where $\phi = \phi_+ + \phi_-$ and $\psi = \psi_+ + \psi_-$, with $\phi_\pm, \psi_\pm \in \Pi^\pm$ (Azizov & Iokhvidov 1981).

Importantly, in the case of a Pontryagin space Π_κ, this implies that the Pontryagin space $(\Pi_\kappa, \langle\cdot,\cdot\rangle)$ is topologically equivalent to the Hilbert space $(\Pi_\kappa, \langle\cdot,\cdot\rangle_+)$ (Iohvidov & Krein 1960, section 1.2). Therefore, topological properties of the Hilbert space $(\Pi_\kappa, \langle\cdot,\cdot\rangle_+)$, such as sequence convergence and compactness, are naturally conveyed to the Pontryagin space $(\Pi_\kappa, \langle\cdot,\cdot\rangle)$.

The following property is quite intuitive (Bognár 1974, theorem IX.1.4).

Theorem A.6 (Dense subspaces). *Every dense subspace of the Pontryagin space Π_κ contains a κ-dimensional negative-definite subspace.*

An orthonormal system $\{e_n\}_{n=0}^\infty \subset \Pi_\kappa$ is a collection of elements satisfying

$$\langle e_m, e_n\rangle = \pm\,\delta_{mn} \tag{A 4}$$

The following theorem gives some properties of orthonormal systems in Pontryagin spaces (Bognár 1974, thereom IV.3.4).

Theorem A.7 (Orthonormal systems in Pontryagin spaces).
Let $\{e_n\}_{n=0}^\infty$ be a complete orthonormal system in the Pontryagin space Π_κ. Then the following statements hold.

(i) Π_κ admits an orthogonal decomposition $\Pi_\kappa = \Pi^+ \oplus \Pi^-$ such that all positive elements e_n in $\{e_n\}_{n=0}^\infty$ belong to Π^+ and form a complete orthonormal system in Π^+ whereas all negative elements e_n of $\{e_n\}_{n=0}^\kappa$ belong to Π^- and form a complete orthonormal system in Π^-.

(ii) The norm of an element $\phi \in \Pi_\kappa$ is given by

$$\|\phi\|^2 = \langle\phi,\phi\rangle_+ = \sum_{n=0}^\infty |\langle\phi, e_n\rangle|^2 \tag{A 5}$$

(iii) For every element $\phi \in \Pi_\kappa$, we have the expansion

$$\phi = \sum_{n=0}^\infty \frac{\langle\phi, e_n\rangle}{\langle e_n, e_n\rangle}\, e_n. \tag{A 6}$$

A.2. *Linear operators in Pontryagin spaces*

We now discuss properties of linear operators in a Pontryagin space Π_κ. First, we review some basic definitions from spectral theory.

In what follows, we denote by $\mathcal{I}$ the identity operator on Π_κ. Given an operator $\mathcal{A}$, we denote its domain by $D(\mathcal{A})$. We assume all bounded operators have $D(\mathcal{A}) = \Pi_\kappa$.

Definition A.8. *Let $\mathcal{A}$ be a linear operator on Π_κ.*

(i) If $\mathcal{A} - \lambda\mathcal{I}$ is not invertible for some $\lambda \in \mathbb{C}$, we say λ is an eigenvalue *of $\mathcal{A}$. The* eigenspace *E_λ associated with the eigenvalue λ is given by*

$$E_\lambda = \{\phi \in \Pi_\kappa \mid (\mathcal{A} - \lambda\,\mathcal{I})\phi = 0\}. \tag{A 7}$$

Non-zero elements of E_λ are the eigenvectors *of $\mathcal{A}$ belonging to the eigenvalue λ. The dimension of the eigenspace is known as the* geometric multiplicity *of the eigenvalue λ.*

(ii) A non-zero element $\phi \in \Pi_\kappa$ is a generalized eigenvector *of the operator $\mathcal{A}$ belonging to the eigenvalue λ if there exists a positive integer n such that $\phi \in D(A^n)$ and $(\mathcal{A}-\lambda\mathcal{I})^n\phi = 0$.*

(iii) The generalized eigenspace *G_λ is the span of all generalized eigenvectors belonging to the eigenvalue λ. The dimension of the G_λ is known as the* algebraic multiplicity *of the eigenvalue λ.*

(iv) The elements $\phi_0, \phi_1, \ldots, \phi_{p-1}$ in Π_κ are a Jordan chain of length *p belonging to the eigenvalue λ if $\phi_0 \neq 0$ and*

$$\mathcal{A}\,\phi_j = \lambda\,\phi_j + \phi_{j-1} \quad j = 0, 1, \ldots, p-1 \tag{A 8}$$

where we define $\phi_{-1} = 0$. The elements in a Jordan chain are linearly independent and belong to G_λ.

The geometric multiplicity of an eigenvalue λ is the number of linearly independent eigenvectors belonging to λ. It is only when the geometric multiplicity is not equal to the algebraic multiplicity that generalized eigenvectors, and hence non-trivial Jordan chains, can arise. In the finite-dimensional case, this corresponds to the situation when a matrix operator is non-diagonalizable. This leads to the following definition.

Definition A.9 (Semi-simple eigenvalue). *An eigenvalue is* semi-simple *if $E_\lambda = G_\lambda$ so that every generalized eigenvector is an eigenvector. Equivalently, an eigenvalue is semi-simple when the geometric multiplicity is equal to the algebraic multiplicity.*

The notion of bounded, self-adjoint, and compact operators carry over identically from Hilbert spaces.

The following is an important characteristic of self-adjoint operators in a Pontryagin space (Bognár 1974, lemma II.3.8).

Lemma A.10 (Eigenspaces of a self-adjoint operator).
Let λ be an eigenvalue of a self-adjoint operator $\mathcal{A}$ in Pontryagin space Π_κ. If the eigenspace E_λ is definite, then the eigenvalue is real and semi-simple.

In a Hilbert space, a self-adjoint operator has real semi-simple eigenvalues. This useful property, alas, no longer holds in a Pontryagin space (see theorems IX.4.6 to IX.4.9 in Bognár 1974).

Theorem A.11 (Spectrum of a self-adjoint operator).
Let $\mathcal{A}$ be a self-adjoint operator in a Pontryagin space Π_κ. Then

(i) the non-real spectrum of $\mathcal{A}$ consists of mutually distinct pairs of eigenvalues $\lambda_1, \ldots, \lambda_s$ and $\lambda_1{}^*, \ldots, \lambda_s{}^*$ such that

$$\sum_{j=1}^{s} \dim G_{\lambda_j} \leqslant \kappa \tag{A 9}$$

(ii) the number of non-semi-simple eigenvalue cannot exceed κ.

(iii) the length of a Jordan chain corresponding to a real eigenvalue cannot be greater than $2\kappa + 1$.

Furthermore, the span of all of the generalized eigenspaces is not necessarily Π_κ (Azizov & Iokhvidov 1989, lemma 4.2.14).

Lemma A.12 (Completeness of generalized eigenspaces).
Let $\mathcal{A}$ be a bounded self-adjoint operator in a Pontryagin space Π_κ with a spectrum having no more than a countable set of accumulation points. Then the span of the generalized eigenspaces is dense in Π_κ if and only if the eigenspace E_0 is non-degenerate.

It is perhaps demoralizing that Pontryagin space self-adjoint operators no longer possess the desirable properties that their analogues in Hilbert spaces do. However, all is not lost. Upon some further restrictions on self-adjoint operators, some of these properties can be recovered.

Definition A.13 (Positive operator). *A self-adjoint operator $\mathcal{A}$ in the Pontryagin space Π_κ is said to be* positive *if $\langle \mathcal{A}x, x\rangle \geqslant 0$ for every $x \in D(\mathcal{A})$, and* strictly positive *if $\langle \mathcal{A}x, x\rangle > 0$ for every non-zero $x \in D(\mathcal{A})$.*

It is important to note that the identity operator $\mathcal{I}$ of the Pontryagin space Π_κ is only a positive operator when Π_κ is a Hilbert space (i.e., when $\kappa = 0$).

The importance of positive operators is due to the following two properties (see theorems VII.1.2 and VII.1.3 in Bognár 1974)

Lemma A.14 (Definite eigenspace property).
Let $\mathcal{A}$ be a positive operator in the Pontryagin space Π_κ. If λ is positive eigenvalue of $\mathcal{A}$ then the corresponding eigenspace is positive-definite. Likewise, if λ is a negative eigenvalue of $\mathcal{A}$ then the corresponding eigenspace is negative-definite.

Theorem A.15. *The spectrum of a bounded positive operator in a Pontryagin space is real.*

We remark that an unbounded positive operator need not have a real spectrum.

The eigenvalues of a bounded positive operator, although real, are not necessarily semi-simple. We have not yet found a class of operators analogous to self-adjoint operators in a Hilbert space. However, we are close.

Lemma A.16 (Countable eigenvalues). *The set of distinct non-zero eigenvalues of a positive compact operator in a Pontryagin space is either finite or countable with zero as an accumulation point. Further, the eigenspaces corresponding to non-zero eigenvalues are finite-dimensional.*

Proof. The proof of this theorem for Hilbert spaces is given by theorems 4.1.18 and 4.9.19 in Debnath & Mikusinski (2005). The proof uses the fact that (a) compact operators map weakly convergent sequences to convergent sequences and (b) eigenvectors corresponding to distinct real eigenvalues are orthogonal.

The first fact (a) is a general property of compact operators in Banach spaces (Reed & Simon 1980, theorem VI.11). The second fact (b) can easily be shown to hold in a Pontryagin space. Noting that, by theorem A.15, a positive compact operator may only have real eigenvalues, the result follows. □

The following theorem shows that the closest analogue to self-adjoint operators in Hilbert spaces are the positive compact operators.

Theorem A.17 (Positive compact operators).
Let $\mathcal{A}$ be a positive compact operator in a Pontryagin space Π_κ and suppose that $\lambda = 0$ is not an eigenvalue. Then all eigenvalues are real and the corresponding eigenvectors form an orthonormal basis for Π_κ. There are precisely κ negative eigenvalues and the remaining eigenvalues are positive. Moreover, positive eigenvalues have positive eigenvectors and negative eigenvalues have negative eigenvectors.

Proof. First, note that by theorem A.15 the eigenvalues are all real. Since $\lambda = 0$ is not an eigenvalue, then by lemma A.14, all eigenspaces are definite and hence, by lemma A.10, all eigenvalues are semi-simple.

Since $\mathcal{A}$ is a compact operator (see lemma A.16) and $\lambda = 0$ is not an eigenvalue, then, by lemma A.12, the span of the generalized eigenspaces is dense in Π_κ [see also Langer & Schneider (1991) and Schneider & Vonhoff (1999)]. Since all eigenvalues are semi-simple, then all generalized eigenvectors are eigenvectors and so the span of the eigenvectors is dense in Π_κ. Orthogonality of eigenvectors can be shown as in a Hilbert space.

Let λ be an eigenvalue and ϕ the corresponding eigenvector. By the positivity of $\mathcal{A}$, we have

$$\langle \mathcal{A}\,\phi, \phi\rangle = \lambda\,\langle \phi, \phi\rangle \geqslant 0. \tag{A 10}$$

Since all eigenspaces are definite, it follows that positive eigenvectors must correspond to positive eigenvalues and negative eigenvectors must correspond to negative eigenvalues.

Finally, by theorem A.6, any dense subset of Π_κ must contain a negative-definite κ dimensional subspace. Consequently, there are κ negative eigenvectors and hence κ negative eigenvalues. □

Appendix B. Further mathematical discussion

B.1. *Literature survey*

There is an extensive literature associated with the eigenvalue problem (2.1)–(2.3) with λ-dependent boundary conditions (see Schäfke & Schneider 1966; Fulton 1977, and citations within). The S-hermitian theory of Schäfke & Schneider (1965, 1966, 1968) establishes a framework with which one can study matrix differential eigenvalue problems—with λ-dependent boundary conditions—of the form

$$F\boldsymbol{y} = \lambda G\boldsymbol{y}. \tag{B 1}$$

where F and G are hermitian differential-boundary operators. If the bilinear form induced by F or G is “definite", that is, if

$$\langle \boldsymbol{y}_1, F\boldsymbol{y}_2\rangle \geqslant 0, \quad \text{or} \quad \langle \boldsymbol{y}_1, G\boldsymbol{y}_2\rangle \geqslant 0 \tag{B 2}$$

for a suitable collection of vector functions $\boldsymbol{y}_1, \boldsymbol{y}_2$ and where $\langle\cdot,\cdot\rangle$ is an appropriate inner product, then the problem is respectively called left-definite or right-definite. Under such definiteness conditions only real eigenvalues are possible. See Schneider (1974) for an explicit application to when the eigenvalue problem (2.1)–(2.3) is right-definite. We state the precise conditions for left- and right-definiteness for the eigenvalue problem (2.1)–(2.3) in §2. Here, it suffices to say that such definiteness conditions impose restrictions on the possible values of the coefficients a_i, b_i, c_i, d_i for $i \in \{1, 2\}$ that appear in the boundary conditions (2.2) and (2.3).

The right-definite problem

The theory in the right-definite case is well-known. A measure-theoretic treatment is given in Evans (1970). Walter (1973) obtains eigenfunction expansion theorems by reformulating (2.1)–(2.3) as an eigenvalue problem for a self-adjoint operator in a Hilbert space. Fulton (1977) applies the residue calculus techniques of Titchmarsh (1962) to the right-definite problem and, in the process, extends some well-known properties of Fourier series to eigenfunction expansions associated with (2.1)–(2.3). In addition, Fulton (1977) shows the physical relevance of the (2.1)-(2.3) with $d_i \neq 0$ in the context of a problem in heat transfer. A recent Hilbert space approach to the right-definite problem, in the context of obtaining a projection basis for quasigeostrophic dynamics, is given in Smith & Vanneste (2012). However, in general, the quasigeostrophic wave problem is left-definite.

The indefinite problem

Much is also known about the general, indefinite, problem. The earliest application of Pontryagin space techniques to the problem is due to Russakovskii (1975) who examines a more general problem with λ-polynomial boundary conditions. Langer & Schneider (1991) apply the S-hermitian formalism of Schäfke & Schneider (1965, 1966, 1968) to the indefinite problem. Such studies on the indefinite problem have determined that, in general, the eigenvalue problem may have a finite number of non-real eigenvalues and a finite number of eigenvalues may be associated with non-trivial Jordan chains (see definition A.8). The latter implies the incompleteness of the eigenfunctions in the underlying function space.

The left-definite problem

To summarize, the above articles establish that, in the right-definite case, the eigenvalues of equations (2.1)–(2.3) are real and the eigenfunctions form a basis for the underlying function space. However, no such result is available for the left-definite problem. One can use the S-hermitian theory of Schäfke & Schneider (1965, 1966, 1968) to show that eigenvalues are real (see also Langer & Schneider 1991) but completeness results of the sort we seek are inaccessible with such an approach. Instead, as we show in this article, the eigenvalue problem is naturally formulated in a Pontryagin space, and, in such a setting, one can prove, in the left-definite case, that the eigenvalues are real and that the eigenfunctions form a basis for the underlying function space. We prove this result, stated in theorem 2.5, in appendix B.2.

With these completeness results, we may apply the residue calculus techniques of Titchmarsh (1962) to extend the results of Fulton (1977) to the left-definite problem. Indeed, Fulton (1977) uses a combination of Hilbert space methods as well as the residue calculus techniques to prove various convergence results for the right-definite problem. However, only theorem 1 of Fulton (1977) makes use of Hilbert space methods. If we extend Fulton's theorem 1 to the left-definite problem, then all the results of Fulton (1977) will apply equally to the left-definite problem. A left-definite analogue of theorem 1 of Fulton (1977), along with its proof, is given in appendix B.3.

B.2. *Proof of theorem 2.5*

Proof. The proof for the left-definite case is essentially the standard proof (e.g., Debnath & Mikusinski 2005, section 5.10) with theorem A.17 substituting for the Hilbert-Schmidt theorem. We give a general outline nonetheless.

First, it is well-known that $\mathcal{L}$ is self-adjoint in L^2_μ (e.g. Russakovskii 1975, 1997) and this can be verified explicitly. Since $\lambda = 0$ is not an eigenvalue, then the inverse operator $\mathcal{L}^{-1}$ exists and is an integral operator on L^2_μ given by

$$(\mathcal{L}^{-1}\,\phi)(z) = \int_{z_1}^{z_2} G(z,t)\,\phi(t)\,\mathrm{d}\mu(t) \tag{B 3}$$

where $G(z,t)$ is a symmetric kernel—continuous in the interior of the interval (z_1, z_2) but possibly having finite jump-discontinuities at the boundary points z_1 and z_2. For an explicit construction, see section 4 in Walter (1973), Fulton (1977), and Hinton (1979). The operator $\mathcal{L}^{-1}$ is self-adjoint and compact with $D(\mathcal{L}^{-1}) = \Pi_\kappa$ (Walter 1973). The eigenvalue problem for $\mathcal{L}$, equation (2.27), is then equivalent to

$$\mathcal{L}^{-1}\,\phi = \lambda^{-1}\,\phi \tag{B 4}$$

and both problems have the same eigenfunctions.

Since $\mathcal{L}$ is self-adjoint, then the condition of left-definiteness is equivalent to the positivity of $\mathcal{L}$. Positivity is preserved by inversion so that $\mathcal{L}^{-1}$ is also positive. The operator $\mathcal{L}^{-1}$ is therefore a positive compact operator and so satisfies the requirements of theorem A.17. Application of theorem A.17 to $\mathcal{L}^{-1}$ then assures that all eigenvalues λ_n are real, the eigenfunctions form an orthonormal basis for L^2_μ, and the sequence of eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ is countable and bounded from below.

The claim that the eigenvalues are simple is verified in Binding & Browne (1999) for the left-definite problem. Alternatively, an argument similar to that of Fulton (1977) and (Titchmarsh 1962, page 12) can be made to prove the simplicity of the eigenvalues. □

B.3. *Extending Fulton (1977) to the left-definite problem*

The following is a left-definite analogue of theorem 1 in Fulton (1977). The proof is almost identical to the right-definite case (Fulton 1977; Hinton 1979) with minor modifications. Essentially, since $\langle \Psi, \Psi \rangle$ can be negative, we must replace these terms in the inequalities below with the induced Hilbert space inner product $\langle \Psi, \Psi \rangle_+$ given by equation (A 3). Our L^2_μ Green's functions G corresponds to $\tilde{G}$ in Hinton (1979).

Theorem B.1 (A left-definite extension of Fulton's theorem 1).
Let $\Psi \in L^2_\mu$ be defined on the interval $[z_1, z_2]$ by

$$\Psi(z) = \begin{cases} \Psi(z_i) & \text{at } z = z_i, \text{ for } i \in S, \\ \psi(z) & \text{otherwise}. \end{cases} \tag{B 5}$$

where $\psi \in L^2$ and $\Psi(z_i)$ are constants for $i \in S$. The eigenfunctions Φ_n are defined similarly (see §2).

(i) Parseval formula: For $\Psi \in L^2_\mu$, we have

$$\langle \Psi, \Psi \rangle = \sum_{n=0}^{\infty} \frac{|\langle \Psi, \Phi_n \rangle|^2}{\langle \Phi_n, \Phi_n \rangle}. \tag{B 6}$$

(ii) For $\Psi \in D(\mathcal{L})$, we have

$$\Psi = \sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \Phi_n. \tag{B 7}$$

with equality in the sense of L^2_μ. Moreover, we have

$$\psi = \sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n \tag{B 8}$$

which converges uniformly and absolutely for $z \in [z_1, z_2]$ and may be differentiated term-by-term, with the differentiated series converging uniformly and absolutely to ψ' for $z \in [z_1, z_2]$. The boundaries series

$$\Psi(z_i) = \sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \Phi_n(z_i) \tag{B 9}$$

for $i \in S$ is absolutely convergent.

Proof. The Parseval formula (B 6) is a consequence of the completeness of the eigenfunctions $\{\Phi_n\}_{n=0}^{\infty}$ in L^2_μ, given by theorem 2.5, and theorem A.7. Similarly, the expansion (B 7) is also due to completeness of the eigenfunctions.

We first prove that the series (B 8) converges uniformly and absolutely for $z \in [z_1, z_2]$. We begin with the identity

$$\phi_n(z) = (\lambda - \lambda_n) \langle G(z, \cdot, \lambda), \Phi_n \rangle \tag{B 10}$$

where $\lambda \in \mathbb{C}$ is not an eigenvalue of $\mathcal{L}$, and G is the L^2_μ Green's function (see equation (8) in Hinton 1979). Then

$$\sum_{n=0}^{\infty} \lambda_n \frac{|\phi_n|^2}{|\lambda - \lambda_n|^2} = \sum_{n=0}^{\infty} \lambda_n \, |\langle G(z, \cdot, \lambda), \Phi_n \rangle|^2 \leqslant \langle G(z, \cdot, \lambda), \mathcal{L} G(z, \cdot, \lambda) \rangle_+ \leqslant B_1(\lambda) \tag{B 11}$$

where $\langle \cdot, \cdot \rangle_+$ is the induced Hilbert space inner product given by equation (A 3) and $B_1(\lambda)$ is a z independent upper bound (equation 9 in Hinton 1979). In addition, since $\Psi \in D(\mathcal{L})$, then $\langle \mathcal{L}\Psi, \mathcal{L}\Psi \rangle_+ < \infty$. Thus, we obtain

$$\sum_n \lambda_n^2 \, |\langle \Psi, \Phi_n \rangle|^2 = \langle \mathcal{L}\Psi, \mathcal{L}\Psi \rangle_+ < \infty. \tag{B 12}$$

The uniform and absolute convergence of (B 8) follows from

$$\sum_{n=0}^{\infty} \left| \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n \right| = \sum_{n=0}^{\infty} \left| \left(\frac{\phi_n}{\lambda - \lambda_n} \right) (\lambda - \lambda_n) \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \right| \tag{B 13}$$

$$\leqslant \sqrt{\left(\sum_{n=0}^{\infty} \left| \frac{\phi_n}{\lambda - \lambda_n} \right|^2 \right) \left(\sum_{n=0}^{\infty} |\lambda - \lambda_n|^2 \, |\langle \Psi, \Phi_n \rangle|^2 \right)} \tag{B 14}$$

along with equations (B 11) and (B 12). The absolute convergence of the boundary series (B 9) follows as well.

To show that the series (B 8) is term-by-term differentiable, it is sufficient to show that the differentiated series

$$\psi' = \sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n' \tag{B 15}$$

converges uniformly for $z \in [z_1, z_2]$ (Kaplan 1993, section 6.14, theorem 33).

From equation (B 10), we have

$$\frac{\phi_n'}{\lambda - \lambda_n} = \frac{d}{dz} \langle G(z, \cdot, \lambda), \Phi_n \rangle = \langle \partial_z G(z, \cdot, \lambda), \Phi_n \rangle \, . \tag{B 16}$$

Since $\partial_z G(z, \cdot, \lambda) \in L^2_\mu$, we obtain

$$\sum_{n=0}^{\infty} \left| \frac{\phi_n'}{\lambda - \lambda_n} \right|^2 = \langle G(z, \cdot, \lambda), G(z, \cdot, \lambda) \rangle_+ \leqslant B_2(\lambda) \tag{B 17}$$

where $B_2(\lambda)$ is a bound independent of z (see equation 10 in Hinton 1979). Then

$$\sum_{n=0}^{\infty} \left| \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n' \right| = \sum_{n=0}^{\infty} \left| \left(\frac{\phi_n'}{\lambda - \lambda_n} \right) (\lambda - \lambda_n) \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \right| \tag{B 18}$$

$$\leqslant \sqrt{\left(\sum_{n=0}^{\infty} \left| \frac{\phi_n'}{\lambda - \lambda_n} \right|^2 \right) \left(\sum_{n=0}^{\infty} |\lambda - \lambda_n|^2 \, |\langle \Psi, \Phi_n \rangle|^2 \right)}, \tag{B 19}$$

which, along with equations (B 12) and (B 17), implies the uniform and absolute convergence of the differentiated series (B 15) for $z \in [z_1, z_2]$. □

REFERENCES

Azizov, T. Ya. & Iokhvidov, I. S. 1981 Linear operators in spaces with indefinite metric and their applications. *J Sov. Math.* **15**, 438–490.

Azizov, T. Ya. & Iokhvidov, I. S. 1989 *Linear operators in spaces with an indefinite metric*, p. 304. Wiley, (Translated by E. R. Dawson).

Binding, P. A. & Browne, Patrick J. 1999 Left definite Sturm-Liouville problems with eigenparameter dependent boundary conditions. *Differential and Integral Equations* **12**, 167–182.

Binding, P. A., Browne, P. J. & Seddighi, K. 1994 Sturm–Liouville problems with eigenparameter dependent boundary conditions. *Proc. Edinburgh Math. Soc.* **37**, 57–72.

Bognár, J. 1974 *Indefinite Inner Product Spaces*, p. 223. Springer-Verlag.

Bouchet, Freddy & Venaille, Antoine 2012 Statistical mechanics of two-dimensional and geophysical flows. *Phys. Rep.* **515**, 227–295.

Bretherton, F. P. 1966 Critical layer instability in baroclinic flows. *Q. J. R. Meteorol. Soc.* **92**, 325–334.

Brink, K. H. & Pedlosky, J. 2018 Rossby Waves with Continuous Stratification and Bottom Friction. *J. Phys. Oceanogr.* **48**, 2209–2219.

Brown, James Ward & Churchill, Ruel V. 1993 *Fourier Series and Boundary Value Problems*, 5th edn., p. 348. McGraw-Hill.

Burns, Keaton J., Vasil, Geoffrey M., Oishi, Jeffrey S., Lecoanet, Daniel & Brown, Benjamin P. 2020 Dedalus: A flexible framework for numerical simulations with spectral methods. *Phys. Rev. Res.* **2**.

Charney, Jule G. 1971 Geostrophic Turbulence. *J. Atmos. Sci.* **28**, 1087–1095.

Chelton, Dudley B., deSzoeke, Roland A., Schlax, Michael G., El Naggar, Karim & Siwertz, Nicolas 1998 Geographical Variability of the First Baroclinic Rossby Radius of Deformation. *J. Phys. Oceanogr.* **28**, 433–460.

Debnath, Lokenath & Mikusinski, Piotr 2005 *Introduction to Hilbert Spaces with Applications*, 3rd edn., p. 580. Academic Press.

Dingemans, Maarten W. 1997 *Water Wave Propagation Over Uneven Bottoms*, *Advanced Series on Ocean Engineering*, vol. 1, p. 500. Singapore: World Scientific Publishing.

Evans, W. D. 1970 A non-self-adjoint differentila operator in $l^2[a, b)$. *Q. J. of Math.* **21**, 371–383.

Ferrari, Raffaele, Griffies, Stephen M., Nurser, A. J.George & Vallis, Geoffrey K. 2010 A boundary-value problem for the parameterized mesoscale eddy transport. *Ocean Modelling* **32**, 143–156.

Ferrari, Raffaele & Wunsch, Carl 2010 The distribution of eddy kinetic and potential energies in the global ocean. *Tellus A* **62**, 92–108.

Fu, Lee-Lueng & Flierl, Glenn R. 1980 Nonlinear energy and enstrophy transfers in a realistically stratified ocean. *Dyn. Atmos. Oceans* **4**, 219–246.

Fulton, Charles T. 1977 Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions. *Proc. Royal Soc. Edinburgh Sec. A: Math.* **77**, 293–308.

Gill, Adrian E. 1982 *Atmosphere-ocean dynamics*, p. 662. San Diego: Acad. Press.

Gill, A. E. & Clarke, A. J. 1974 Wind-induced upwelling, coastal currents and sea-level changes. *Deep Sea Res. Oceanogr. Abstr.* **21**, 325–345.

Held, Isaac M. 1985 Pseudomomentum and the Orthogonality of Modes in Shear Flows. *J. Atmos. Sci.* **42**, 2280–2288.

Hillen, T, Leonard, I. E. & Van Roessel, H. 2012 *Partial Differential Equations: Theory and Completely Solved Problems*, p. 678. Hoboken, New Jersey: Wiley.

Hinton, Don B. 1979 An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary conditions. *Q. J. Math.* **30**, 33–42.

Holm, Darryl D., Marsden, Jerrold E., Ratiu, Tudor & Weinstein, Alan 1985 Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.* **123**, 1–116.

Hoskins, B. J., McIntyre, M. E. & Robertson, A. W. 1985 On the use and significance of isentropic potential vorticity maps. *Q. J. R. Meteorol. Soc.* **111**, 877–946.

Iohvidov, I. S. & Krein, M. G. 1960 Spectral theory of operators in spaces with an indefinite metric. I. In *Eleven Papers on Analysis*, *American Mathematical Society Translations: Series 2*, vol. 13, pp. 105–175. American Mathematical Society.

Kaplan, Wilfred 1993 *Advanced Calculus*, 4th edn., p. 746. Addison-Wesley Publishing Company.

Kelly, Samuel M. 2016 The Vertical Mode Decomposition of Surface and Internal Tides in the Presence of a Free Surface and Arbitrary Topography. *J. Phys. Oceanogr.* **46**, 3777–3788.

Kundu, Pijush K. & Cohen, Ira M. 2004 *Fluid Mechanics*, 3rd edn., p. 759. Elsevier.

de La Lama, Marta Sanchez, LaCasce, J. H. & Fuhr, Helle Kristine 2016 The vertical structure of ocean eddies. *Dynamics and Statistics of the Climate System* **1**.

LaCasce, J. H. 2012 Surface Quasigeostrophic Solutions and Baroclinic Modes with Exponential Stratification. *J. Phys. Oceanogr.* **42**, 569–580.

LaCasce, J. H. 2017 The Prevalence of Oceanic Surface Modes. *Geophysical Research Letters* **44**, 11097–11105.

LaCasce, J. H. & Groeskamp, Sjoerd 2020 Baroclinic Modes over Rough Bathymetry and the Surface Deformation Radius. *J. Phys. Oceanogr.* **50**, 2835–2847.

Lamb, Horace 1911 On atmospheric oscillations. *Proc. Royal Soc. London. A* **84** (574), 551–572.

Lamb, Horace 1975 *Hydrodynamics*, 6th edn., p. 738. Cambridge University Press.

Langer, Heinz & Schneider, Albert 1991 On spectral properties of regular quasidefinite pencils $f - \lambda g$. *Results in Mathematics* **19**, 89–109.

Lapeyre, Guillaume 2009 What Vertical Mode Does the Altimeter Reflect? On the Decomposition in Baroclinic Modes and on a Surface-Trapped Mode. *J. Phys. Oceanogr.* **39**, 2857–2874.

Levitan, B. M. & Sargsjan, I. S. 1975 *Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators*, *Translations of Mathematical Monographs*, vol. 39, p. 525. American Mathematical Society.

Lighthill, Michael James 1969 Dynamic response of the Indian Ocean to onset of the Southwest Monsoon. *Philos. Trans. R. Soc., A* **265**, 45–92.

Linden, Hansjörg 1991 Leighton's bounds for Sturm-Liouville eigenvalues with eigenvalue parameter in the boundary conditions. *J. Math. Analys. Applic.* **156**, 444–456.

McWilliams, James C. 1976 Maps from the Mid-Ocean Dynamics Experiment: Part I. Geostrophic Streamfunction. *J. Phys. Oceanogr.* **6**, 810–827.

Morrison, P. J. 1998 Hamiltonian description of the ideal fluid. *Rev. Mod. Phys.* **70**, 467–521.

Rayleigh 1882 Investigation of the Character of the Equilibrium of an Incompressible Heavy Fluid of Variable Density. *Proceedings of the London Mathematical Society* **s1-14**, 170–177.

Reed, Michael & Simon, Barry 1980 *Methods of modern mathematical physics*, , vol. 1, p. 400. Academic Press.

Rhines, P. B. 1970 Edge-, bottom-, and Rossby waves in a rotating stratified fluid. *Geophys. Astrophys. Fluid Dyn.* **1**, 273–302.

Rhines, P. B. 1977 The dynamics of unsteady currents. In *Marine Modeling* (ed. J. J. O'Brien, Edward D. Goldberg & I. N. McCave), *The Sea*, vol. 6, p. 129. Interscience Publishers.

Ripa, P. 1981 On the theory of nonlinear wave-wave interactions among geophysical waves. *J. Fluid Mech.* **103**, 87–115.

Rocha, Cesar B., Young, William R. & Grooms, Ian 2015 On Galerkin Approximations of the Surface Active Quasigeostrophic Equations. *J. Phys. Oceanogr.* **46**, 125–139.

Russakovskii, E. M. 1975 Operator treatment of boundary problems with spectral parameters entering via polynomials in the boundary conditions. *Func. Anal. Applic.* **9**, 358–359.

Russakovskii, E. M. 1997 Matrix boundary value problems with eigenvalue dependent boundary conditions (the linear case). In *Topics in Interpolation Theory* (ed. H. Dym, V. Katsnelson, B. Fritzsche & B. Kirstein), pp. 453–462. Birkhäuser.

Salmon, Rick 1978 Two-layer quasi-geostrophic turbulence in a simple special case. *Geophys. Astrophys. Fluid Dyn.* **10**, 25–52.

Schäfke, F. W. & Schneider, A. 1965 S-hermitesche Rand-Eigenwertprobleme. I. *Mathematische Annalen* **162**, 9–26.

Schäfke, F. W. & Schneider, A. 1966 S-hermitesche Rand-Eigenwertprobleme. II. *Mathematische Annalen* **165**, 236–260.

Schäfke, F. W. & Schneider, A. 1968 S-hermitesche Rand-Eigenwertproblem. III. *Mathematische Annalen* **177**, 67–94.

Schneider, Albert 1974 A note on eigenvalue problems with eigenvalue parameter in the boundary conditions. *Mathematische Zeitschrift* **136**, 163–167.

Schneider, Albert & Vonhoff, Rolf 1999 Indefinite Boundary Eigenvalue Problems in a Pontrjagin Space Setting. *Results in Mathematics* **35**, 325–354.

Schneider, Tapio, Held, Isaac M. & Garner, Stephen T. 2003 Boundary Effects in Potential Vorticity Dynamics. *J. Atmos. Sci.* **60**, 1024–1040.

Scott, Julian F. 2014 Wave turbulence in a rotating channel. *J. Fluid Mech.* **741**, 316–349.

Scott, Robert B. & Furnival, Darran G. 2012 Assessment of Traditional and New Eigenfunction Bases Applied to Extrapolation of Surface Geostrophic Current Time Series to Below the Surface in an Idealized Primitive Equation Simulation. *J. Phys. Oceanogr.* **42**, 165–178.

Shepherd, Theodore G. 1990 Symmetries, Conservation Laws, and Hamiltonian Structure in Geophysical Fluid Dynamics. In *Advances in Geophysics* (ed. Renata Dmowska & Barry Saltzman), , vol. 32, pp. 287–338. Elsevier.

Smith, K. Shafer & Vallis, Geoffrey K. 2001 The Scales and Equilibration of Midocean Eddies: Freely Evolving Flow. *J. Phys. Oceanogr.* **31**, 554–571.

Smith, K. Shafer & Vanneste, Jacques 2012 A Surface-Aware Projection Basis for Quasigeostrophic Flow. *J. Phys. Oceanogr.* **43**, 548–562.

Stammer, Detlef 1997 Global Characteristics of Ocean Variability Estimated from Regional TOPEX/POSEIDON Altimeter Measurements. *J. Phys. Oceanogr.* **27**, 1743–1769.

Stokes, George Gabriel 2009 On the Theory of Oscillatory Waves. In *Mathematical and Physical Papers*, pp. 197–229. Cambridge: Cambridge University Press.

Sutherland, Bruce R. 2010 *Internal Gravity Waves*, p. 377. Cambridge: Cambridge University Press.

Taylor, Geoffrey Ingram 1936 The oscillations of the atmosphere. *Proc. R. Soc. London, Ser. A* **156**, 318–326.

Titchmarsh, E. C. 1962 *Eigenfunction expansions associated with second-order differential equations*, 2nd edn., , vol. 1, p. 203. Clarendon Press.

Tulloch, R. & Smith, K. S. 2006 A theory for the atmospheric energy spectrum: Depth-limited temperature anomalies at the tropopause. *Proc. Natl. Acad. Sci. U.S.A.* **103**, 14690–14694.

Tulloch, Ross & Smith, K. Shafer 2009 Quasigeostrophic Turbulence with Explicit Surface Dynamics: Application to the Atmospheric Energy Spectrum. *J. Atmos. Sci.* **66**, 450–467.

Vallis, Geoffrey K. 2017 *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-scale Circulation*, 2nd edn., p. 946. Cambridge University Press.

Vanneste, J. & Shepherd, T. G. 1999 On wave action and phase in the non–canonical Hamiltonian formulation. *Proc. R. Soc. London, Ser. A* **455**, 3–21.

Venaille, A., Vallis, G. K. & Griffies, S. M. 2012 The catalytic role of the beta effect in barotropization processes. *J. Fluid Mech.* **709**, 490–515.

Walter, Johann 1973 Regular eigenvalue problems with eigenvalue parameter in the boundary condition. *Mathematische Zeitschrift* **133**, 301–312.

Wang, Jinbo, Flierl, Glenn R., LaCasce, Joseph H., McClean, Julie L. & Mahadevan, Amala 2013 Reconstructing the Ocean's Interior from Surface Data. *J. Phys. Oceanogr.* **43**, 1611–1626.

Wunsch, Carl 1997 The Vertical Partition of Oceanic Horizontal Kinetic Energy. *J. Phys. Oceanogr.* **27**, 1770–1794.

Yassin, Houssam & Griffies, Stephen M. in prep. On the Discrete Normal Modes of Quasigeostrophic Theory. *J. Phys. Oceanogr.* .

Zettl, Anton 2010 *Sturm-Liouville Theory*, *Mathematical Surveys and Monographs*, vol. 121, p. 328. American Mathematical Society.