Normal modes with boundary dynamics

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(Received xx; revised xx; accepted xx)

Three-dimensional geophysical fluids support both internal and boundary-trapped waves. To obtain the normal modes in such fluids we must solve a differential eigenvalue problem for the vertical structure (for simplicity, we only consider horizontally periodic domains). If the boundaries are dynamically inert (e.g., rigid boundaries in the Boussinesq internal wave problem, flat boundaries in the quasigeostrophic Rossby wave problem) the resulting eigenvalue problem typically has a Sturm-Liouville form and the properties of such problems are well-known. However, when restoring forces are also present at the boundaries, then the equations of motion contain a time-derivative in the boundary conditions and this leads to an eigenvalue problem where the eigenvalue correspondingly appears in the boundary conditions. This article develops the theory of such problems and explores the properties of wave problems with dynamically-active boundaries. The theory allows us to solve the initial value problem for quasigeostrophic Rossby waves in a region with sloping bottom (we also apply the theory to two Boussinesq problems with a free-surface). For a step-function perturbation at a dynamically-active boundary, we find that the resulting time-evolution consists of waves present in proportion to their projection onto the dynamically-active boundary. If one boundary is dynamically-active, the modes form a basis, not of L^2, but rather of the larger space $L^2 \oplus \mathbb{C}$. The eigenfunctions also have the following properties: several linearly independent eigenfunctions may have an identical number of internal zeros, they satisfy an indefinite orthogonality relation involving boundary contributions, and they may have jump discontinuities at the boundaries.

Key words:

1. Introduction

An important tool in the study of wave motion near a stable equilibrium is the separation of variables. When applicable, this elementary technique transforms a linear partial differential equation into an ordinary differential eigenvalue problem for each coordinate (e.g., Hillen *et al.* 2012). Upon solving the differential eigenvalue problems, one obtains the normal modes of the physical system. The normal modes are the fundamental wave motions for the given restoring forces, each mode represents an independent degree of freedom in which the physical system can oscillate, and any solution of the wave problem may be written as a linear combination of these normal modes.

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To derive the normal modes, we must first linearize the dynamical equations of motion about some equilibrium state. We then encounter linearized restoring forces of two kinds:

1. volume-permeating forces experienced by fluid particles in the interior, and

2. boundary-confined forces only experienced by fluid particles at the boundary.

Examples of volume-permeating forces include the restoring forces resulting from continuous density stratification and continuous volume potential vorticity gradients. These restoring forces respectively result in internal gravity waves (Sutherland 2010) and Rossby waves (Vallis 2017). Examples of boundary-confined restoring forces include the gravitational force at a free-surface (i.e., at a jump discontinuity in the background density), forces arising from gradients in surface potential vorticity (Schneider *et al.* 2003), and the molecular forces giving rise to surface tension. These restoring forces respectively result in surface gravity waves (Sutherland 2010), topographic/thermal waves (Hoskins *et al.* 1985), and capillary waves (Lamb 1975).

In the absence of boundary-confined restoring forces, we can often apply Sturm-Liouville theory (e.g., Hillen *et al.* 2012; Zettl 2010) to the resulting eigenvalue problem. We thus obtain a countable infinity of waves whose vertical structures form a basis of L^2, the space of square-integrable functions (see §2), and, given some initial vertical structure, we know how to solve for the subsequent time-evolution as a linear combination for linearly independent waves. Moreover, a classic result of Sturm-Liouville theory is that the nth mode has n internal zeros.

In the presence of boundary-confined restoring forces, the governing equations have a time-derivative in the boundary conditions. The resulting eigenvalue problem correspondingly contains the eigenvalue parameter in the boundary conditions. Sturm-Liouville theory is inapplicable to such problems.

In this article, we present a general method for solving these problems by delineating a generalization of Sturm-Liouville theory. Some consequences of this theory are the following. There is a countable infinity of waves whose vertical structures form a basis of $L^2 \oplus \mathbb{C}^s$, where s is the number of dynamically-active boundaries; thus, each boundary-trapped wave, in mathematically rigorous sense, provides an additional degree of freedom to the problem. The modes satisfy an orthogonality relation involving boundary terms, the modes may have a negative norm, and the modes may have finite jump discontinuities at dynamically-active boundaries (although the *solutions* are always continuous, see §3.3). When negative norms are possible (as in quasigeostrophic theory), there is a new expression for the Fourier coefficients that one must use to solve initial value problems [see equation (2.23)]. We can also expand boundary step-functions (representing some boundary localized perturbation) as a sum of modes. Moreover, the nth mode may not have n internal zeros; indeed, depending on physical parameters in the problem, two or three linearly independent modes with an identical number of internal zeros may be present.

We apply the theory to three geophysical wave problems. The first is that of a Boussinesq fluid with a free-surface; we find that the nth mode has n internal zeros. The second example is that of a rotating Boussinesq fluid with a free-surface where we assume that the stratification suppresses rotational effects in the interior but not at the upper boundary. We find that there are two linearly independent modes with M internal zeros, where the integer M depends on the ratio of the Coriolis parameter to the horizontal wavenumber, and that the eigenfunctions have a finite jump discontinuity at the upper boundary. The third application is to a quasigeostrophic fluid with a sloping lower boundary. We find that modes with an eastward phase speed have a negative norm whereas modes with a westward phase speed have a positive norm (the sign of the norm has implications for the relative phase of a wave and for series expansions). Moreover, depending on the propagation direction, there can be two linearly independent modes with no internal zeros. For all three examples, we outline the properties of the resulting series expansions and provide the general solution. We also consider the time-evolution resulting from a vertically localized perturbation at a dynamically-active boundary; we idealize such a perturbation as a boundary step-function. The step-function perturbation induces a time-evolution in which the amplitude of each constituent wave is proportional to the projection of that wave onto the boundary.

To our knowledge, most of the above results cannot be found in the literature [however, the gravity wave orthogonality relation has been noted before, e.g., Gill (1982) and Kelly (2016)]. For instance, we provide the only solution to the initial value problem for Rossby waves over topography in the literature [equation (4.20)]. Moreover, many of the properties we discuss arise in practical problems in physical oceanography. The question of whether the quasigeostrophic baroclinic modes are complete is a controversial one (Lapeyre 2009; Ferrari & Wunsch 2010; LaCasce 2012; Scott & Furnival 2012; Smith & Vanneste 2012; Rocha *et al.* 2015). The number of internal zeros of Rossby waves is also a useful quantity in observational physical oceanography [e.g., Clément *et al.* (2014) and de La Lama *et al.* (2016)]. The issue of the differentiability of the resulting series expansions has been raised in various physical applications (Rocha *et al.* 2015; Kelly 2016). In addition, the distinction between L^2 and $L^2 \oplus \mathbb{C}^s$ bases that we present here is useful for equilibrium statistical mechanical calculations where one must decompose fluid motion onto a complete set of modes (Bouchet & Venaille 2012; Venaille *et al.* 2012).

The plan of the article is the following. We formulate the mathematical theory in §2. We then apply the theory to the two Boussinesq wave problems, in §3, and to the quasigeostrophic wave problem, in §4. We consider the time-evolution of a localized perturbation at a dynamically-active boundary in §5. We then conclude in §6.

2. The eigenvalue problem

In this section, we outline the theory of the differential eigenvalue problem

$$-(p\,\phi')' + q\,\phi = \lambda\, r\,\phi \quad \text{for} \quad z \in (z_1, z_2) \tag{2.1}$$

$$-\left[a_1\,\phi(z_1) - b_1\,(p\,\phi')(z_1)\right] = \lambda\left[c_1\,\phi(z_1) - d_1\,(p\,\phi')(z_1)\right] \tag{2.2}$$

$$-\left[a_2\,\phi(z_2) - b_2\,(p\,\phi')(z_2)\right] = \lambda\left[c_2\,\phi(z_2) - d_2\,(p\,\phi')(z_2)\right] \tag{2.3}$$

where p^{-1}, q, and r are real-valued integrable functions; a_i, b_i, c_i, and d_i are real numbers with $i \in \{1, 2\}$; and where $\lambda \in \mathbb{C}$ is the eigenvalue parameter. We further assume that $p > 0$ and $r > 0$, that p and r are twice continuously differentiable, that q is continuous, and that $(a_i, b_i) \neq (0, 0)$ for $i \in \{1, 2\}$. The system of equations (2.1)–(2.3) is an eigenvalue problem for the eigenvalue $\lambda \in \mathbb{C}$ and differs from a regular Sturm-Liouville problem in that λ appears in the boundary conditions (2.2) and (2.3). That is, setting $c_i = d_i = 0$ recovers the traditional Sturm-Liouville problem. The presence of λ as part of the boundary condition leads to some fundamentally new mathematical features that are the subject of this section and fundamental to the physics of this study.

It is useful to define the two boundary parameters

$$D_i = (-1)^{i+1}\,(a_i\,d_i - b_i\,c_i) \quad i = 1, 2. \tag{2.4}$$

Just as the function r acts as a weight for the interval (z_1, z_2) in traditional Sturm-Liouville problems, the constants D_i^{-1} will play analogous roles for the boundaries $z = z_i$ when $D_i \neq 0$.

Outline of the mathematics

The right-definite case, when the $D_i \geqslant 0$ for $i \in \{1, 2\}$, is well-known in the mathematics literature; most of the right-definite results in this section are due to Evans (1970), Walter

(1973), and Fulton (1977). In contrast, the left-definite case, defined below, is much less studied. In this section, we generalize the right-definite results of Fulton (1977) to the left-definite problem as well as provide an intuitive formulation (in terms of functions rather than vectors, for a vector formulation see Fulton 1977) of the eigenvalue problem.

In section §2.1 we state the conditions under which we obtain real eigenvalues and a basis of eigenfunctions. We proceed, in §2.2, to explore the properties of eigenfunctions and eigenfunction expansions. Finally, in §2.3, we discuss oscillation properties of the eigenfunctions. Additional properties of the eigenvalue problem are found in appendix A and a literature review, along with various technical proofs, is found in appendix B.

2.1. *Formulation of the problem*

2.1.1. *The functions space of the problem*

We denote by L^2 the Hilbert space of square-integrable "functions" ϕ on the interval (z_1, z_2) satisfying

$$\int_{z_1}^{z_2} |\phi|^2 \, r \, \mathrm{d}z < \infty. \tag{2.5}$$

To be more precise, the elements of L^2 are not functions but rather equivalence classes of functions (e.g., Reed & Simon 1980, section I.3). Two functions, ϕ and ψ, are equivalent in L^2 (i.e., $\phi = \psi$ in L^2) if they agree in a mean-square sense on $[z_1, z_2]$,

$$\int_{z_1}^{z_2} |\phi(z) - \psi(z)|^2 \, r \, \mathrm{d}z = 0. \tag{2.6}$$

Significantly, we can have $\phi = \psi$ in L^2 but $\phi \neq \psi$ pointwise.

Furthermore, as a Hilbert space, L^2 is endowed with a positive-definite inner product

$$\langle \phi, \psi \rangle_\sigma = \int_{z_1}^{z_2} \phi^* \, \psi \, \mathrm{d}\sigma = \int_{z_1}^{z_2} \phi^* \, \psi \, r \, \mathrm{d}z, \tag{2.7}$$

where the symbol * denotes complex conjugation and the measure σ associated L^2 induces a differential element $\mathrm{d}\sigma = r\, \mathrm{d}z$ (see appendix A). The positive-definiteness is ensured by our assumption that $r > 0$ (i.e., $\langle \phi, \phi \rangle_\sigma > 0$ for $\phi \neq 0$ when $r > 0$).

It is well-known that traditional Sturm-Liouville problems [i.e., equations (2.1)–(2.3) with $c_i = d_i = 0$ for $i = 1, 2$] are eigenvalue problems in some subspace of L^2 (Debnath & Mikusinski 2005). For the more general case of interest here, the eigenvalue problem occurs over a "larger" function space denoted by L^2_μ which we construct in appendix A.

Let the integer $s \in \{0, 1, 2\}$ denote the number of λ-dependent boundary conditions and let S denote the set

$$S = \{ j \mid j \in \{1, 2\} \text{ and } (c_j, d_j) \neq (0, 0) \}. \tag{2.8}$$

S is one of $\emptyset, \{1\}, \{2\}, \{1, 2\}$ and s is the number of elements in the set S. In appendix A, we show that L^2_μ is isomorphic to the space $L^2 \oplus \mathbb{C}^s$ and is thus "larger" than L^2 by s dimensions.

We denote elements of L^2_μ by upper case letters Ψ; we define $\Psi(z)$ for $z \in [z_1, z_2]$ by

$$\Psi(z) = \begin{cases} \Psi(z_i) & \text{at } z = z_i, \text{ for } i \in S, \\ \psi(z) & \text{otherwise}, \end{cases} \tag{2.9}$$

where $\Psi(z_i) \in \mathbb{C}$ are constants, for $i \in S$, and the corresponding lower case letter ψ denotes an element of L^2. Two elements Φ and Ψ of L^2_μ are equivalent in L^2_μ if and only if

1. $\Phi(z_i) = \Psi(z_i)$ for $i \in S$, and

2. $\phi(z)$ and $\psi(z)$ are equivalent in L^2 [i.e., as in equation (2.6)].

Here, Φ, as an element of L^2_μ, is defined as in equation (2.9). The primary difference between L^2 and L^2_μ is that L^2_μ discriminates between functions that disagree at λ-dependent boundaries.

The measure μ associated with L^2_μ (see appendix A) induces a differential element

$$\mathrm{d}\mu(z) = \left[r(z) + \sum_{i\in S} D_i^{-1}\,\delta(z - z_i) \right] \mathrm{d}z, \tag{2.10}$$

where $\delta(z)$ is the Dirac delta. The induced inner product on L^2_μ is

$$\langle \Phi, \Psi \rangle = \int_{z_1}^{z_2} \Phi^*\,\Psi\,\mathrm{d}\mu = \int_{z_1}^{z_2} \Phi^*\,\Psi\,r\,\mathrm{d}z + \sum_{i\in S} D_i^{-1}\,\Phi(z_i)^*\,\Psi(z_i). \tag{2.11}$$

If $D_i > 0$ for $i \in S$ then this inner product is positive-definite and L^2_μ is a Hilbert space. However, this is not the case in general.

Let κ denote the number of negative D_i for $i \in S$ (the possible values are $\kappa = 0, 1, 2$). Then L^2_μ has a κ-dimensional subspace of elements Ψ satisfying

$$\langle \Psi, \Psi \rangle < 0. \tag{2.12}$$

This makes L^2_μ a Pontryagin space of index κ (Bognár 1974). If $\kappa = 0$ then L^2_μ is again a Hilbert space. In the present case, L^2_μ also has an infinite-dimensional subspace of elements ψ satisfying

$$\langle \Psi, \Psi \rangle > 0. \tag{2.13}$$

2.1.2. *Reality and completeness*

In appendix A.2, we reformulate the eigenvalue problem (2.1)–(2.3) as an eigenvalue problem of the form

$$\mathcal{L}\,\Phi = \lambda\,\Phi \tag{2.14}$$

in a subspace of L^2_μ, where $\mathcal{L}$ is a linear operator and Φ an element of L^2_μ. We also define the notions of right- and left-definiteness that are required for the reality and completeness theorem below. The following two propositions can be considered to define right- and left-definiteness for applications of the theory. Both propositions are obtained through straightforward manipulations (see appendix A).

PROPOSITION 2.1 (CRITERION FOR RIGHT-DEFINITENESS). *The eigenvalue problem (2.1)–(2.3) is right-definite if $r > 0$ and $D_i > 0$ for $i \in S$.*

PROPOSITION 2.2 (CRITERION FOR LEFT-DEFINITENESS). *The eigenvalue problem (2.1)–(2.3) is left-definite if the following conditions hold:*

(i) the functions p, q satisfy $p > 0, q \geqslant 0$,

(ii) for the λ-dependent boundary conditions, we have

$$\frac{a_i\,c_i}{D_i} \leqslant 0, \quad \frac{b_i\,d_i}{D_i} \leqslant 0, \quad (-1)^i \frac{a_i\,d_i}{D_i} \geqslant 0 \quad \textit{for } i \in S. \tag{2.15}$$

(iii) for the λ-independent boundary conditions, we have

$$b_i = 0 \quad \textit{or} \quad (-1)^{i+1}\frac{a_i}{b_i} \geqslant 0 \quad \textit{if } b_i \neq 0 \quad \textit{for } i \in \{1,2\} \setminus S. \tag{2.16}$$

The notions of right and left-definiteness are not mutually exclusive. Namely, a problem can be neither right- or left-definite; both right- and left-definite; only right-definite; or only left-definite. In this article, we always assume that $p > 0$ and $r > 0$.

The reality of the eigenvalues and the completeness of the eigenfunctions in the space L^2_μ is given by the following theorem.

THEOREM 2.3 (REALITY AND COMPLETENESS).
Suppose the eigenvalue problem (2.1)–(2.3) is either right-definite or left-definite. Moreover, if the problem is not right-definite, we assume that $\lambda = 0$ is not an eigenvalue. Then the eigenvalue problem (2.1)–(2.3) has a countable infinity of real simple eigenvalues λ_n satisfying

$$\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \to \infty, \tag{2.17}$$

with corresponding eigenfunctions Φ_n. Furthermore, the set of eigenfunctions $\{\Phi_n\}_{n=0}^\infty$ is a complete orthonormal basis for L^2_μ satisfying

$$\langle \Phi_m, \Phi_n \rangle = \pm\delta_{mn}. \tag{2.18}$$

Proof. See appendix B.3. □

Recall that κ denotes the number of negative D_i for $i \in S$. We then have the following corollary of the proof of theorem 2.3.

PROPOSITION 2.4. *Suppose the eigenvalue problem (2.1)–(2.3) is left-definite and that $\lambda = 0$ is not an eigenvalue. Then there are κ negative eigenvalues and their eigenfunctions satisfy*

$$\langle \Phi, \Phi \rangle < 0. \tag{2.19}$$

The remaining eigenvalues are positive and their eigenfunctions satisfy

$$\langle \Phi, \Phi \rangle > 0. \tag{2.20}$$

In other words, proposition 2.4 states that we have the relationship

$$\lambda_n \langle \Phi_n, \Phi_n \rangle > 0 \tag{2.21}$$

for left-definite problems.

2.2. *Properties of the eigenfunctions*

For the remainder of §2, we assume that the eigenvalue problem (2.1)–(2.3) satisfies the requirements of theorem 2.3.

2.2.1. *Eigenfunction expansions*

The eigenvalue problem (2.1)–(2.3) has *eigenfunctions* $\{\Phi_n\}_{n=0}^\infty$ as well as corresponding *solutions* $\{\phi_n\}_{n=0}^\infty$. In other words, while the ϕ_n are the solutions to the differential equation defined by equations (2.1)–(2.3) with $\lambda = \lambda_n$, the eigenfunctions required by the operator formulation of the problem [equation (2.14)] are Φ_n. The functions Φ_n and ϕ_n are related by equation (2.9), with the boundary values $\Phi_n(z_i)$ of Φ_n determined by

$$\Phi_n(z_i) = [c_i\, \phi(z) - d_i\, (p\, \phi')(z)] \quad \text{for } i \in S. \tag{2.22}$$

Thus, while the solutions ϕ_n are continuously differentiable over the closed interval $[z_1, z_2]$, the eigenfunctions Φ_n are continuously differentiable over the open interval (z_1, z_2) but generally have finite jump discontinuities at the λ-dependent boundaries. The eigenfunctions Φ_n are continuous in the closed interval $[z_1, z_2]$ only if $c_i = 1$ and $d_i = 0$ for $i \in S$. In this case, the eigenfunctions Φ_n coincide with the solutions ϕ_n on the closed interval $[z_1, z_2]$.

The boundary conditions of the eigenvalue problem (2.1)–(2.3) are not unique. One can multiply each boundary condition by an arbitrary constant to obtain an equivalent problem. To uniquely specify the eigenfunctions in physical applications, the boundary coefficients $\{a_i, b_i, c_i, d_i\}$ of equations (2.1)–(2.3) must be chosen so that $r\,\mathrm{d}z$ has the same dimensions as $D_i^{-1}\,\delta(z-z_i)\,\mathrm{d}z$ [recall that $\delta(z)$ has the dimension of inverse length]. In the quasigeostrophic problem, we must also invoke continuity and set $c_i = 1$.

Since $\{\Phi_n\}_{n=0}^{\infty}$ is a basis for L^2_μ, then any $\Psi \in L^2_\mu$ may be expanded in terms of the eigenfunctions (Bognár 1974, thereom IV.3.4),

$$\Psi = \sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle}\, \Phi_n. \tag{2.23}$$

We emphasize that the above equality is an equality in L^2_μ and not a pointwise equality [see the discussion following equation (2.9)]. Some properties of L^2_μ expansions are given in appendix A.3.

An important property that distinguishes the basis $\{\Phi_n\}_{n=0}^{\infty}$ of L^2_μ from an L^2 basis is its "sensitivity" to function values at boundary points $z = z_i$ for $i \in S$. See §5 for a physical application.

A natural question is whether the basis $\{\Phi_n\}_{n=0}^{\infty}$ of L^2_μ is also a basis of L^2. Recall that the set $\{\Phi_n\}_{n=0}^{\infty}$ is a basis of L^2 if every element $\psi \in L^2$ can be written *uniquely* in terms of the functions $\{\Phi_n\}_{n=0}^{\infty}$. However, in general, this is not true. If $s > 0$, the L^2_μ basis $\{\Phi_n\}_{n=0}^{\infty}$ is overcomplete in L^2 (Walter 1973; Russakovskii 1997).

2.2.2. *Uniform convergence and term-by-term differentiability*

Along with the eigenfunction expansion (2.23) in terms of the eigenfunctions $\{\Phi_n\}_{n=0}^{\infty}$, we also have the expansion

$$\sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle}\, \phi_n \tag{2.24}$$

in terms of the solutions ϕ_n. The two expansions differ in their behaviour at λ-dependent boundaries, $z = z_i$ for $i \in S$, but are otherwise equal. In particular, the Φ_n eigenfunction expansion (2.23) must converge to $\Psi(z_i)$ at $z = z_i$ for $i \in S$ as this equality is required for Ψ to be equal to the series expansion (2.23) in L^2_μ [see the discussion following equation (2.9)]. Some properties of both expansions are given in appendix A.4. In particular, theorem A.4 shows that the ϕ_n solution series (2.24) does not generally converge to $\Psi(z_i)$ at $z = z_i$.

The following theorem is of central concern for physical applications.

THEOREM 2.5 (UNIFORM CONVERGENCE). *Let ψ be a twice continuously differentiable function on $[z_1, z_2]$ satisfying all λ-independent boundary conditions of the eigenvalue problem (2.1)–(2.3). Define the function Ψ on $z \in [z_1, z_2]$ by*

$$\Psi(z) = \begin{cases} c_i\,\psi(z) - d_i\,(p\,\psi')(z) & \text{at } z = z_i, \text{ for } i \in S, \\ \psi(z) & \text{otherwise.} \end{cases} \tag{2.25}$$

Then

$$\psi(z) = \sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle}\, \phi_n(z) \quad \text{and} \quad \psi'(z) = \sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle}\, \phi_n'(z) \tag{2.26}$$

with both series converging uniformly and absolutely on $[z_1, z_2]$.

Proof. See appendix B.4. □

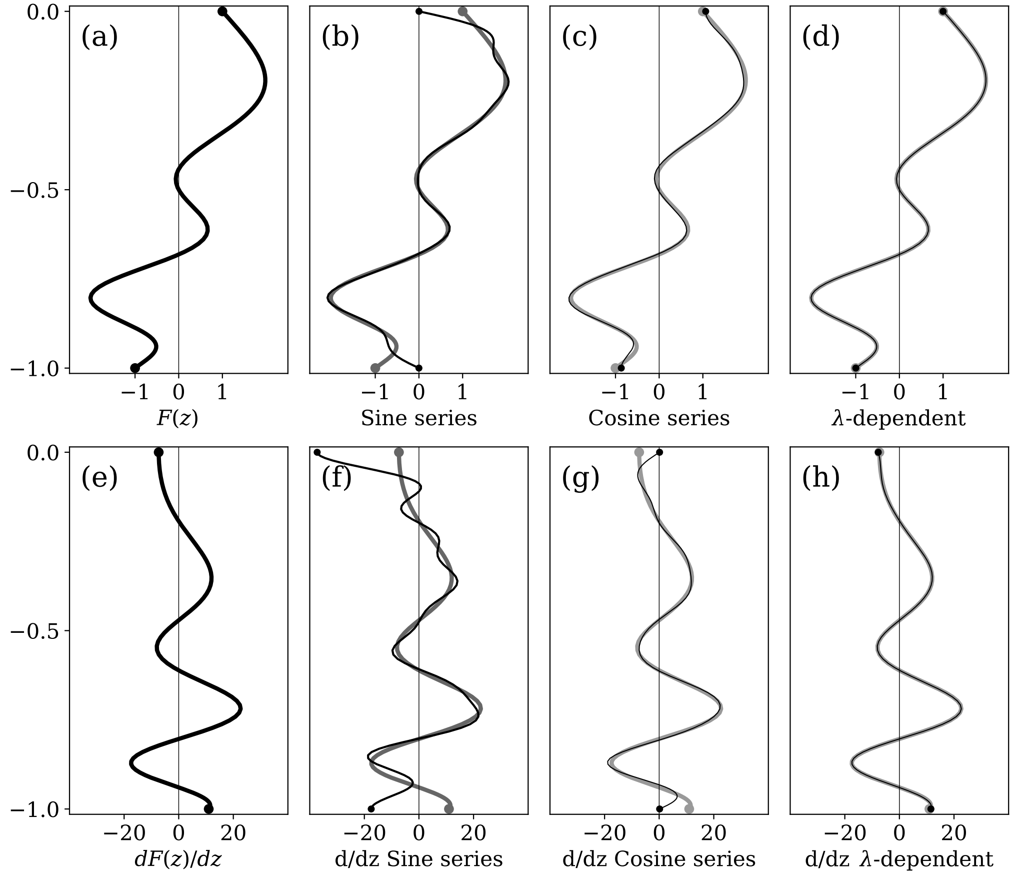

Figure 1: Convergence to a function $F(z) = 1 + 2z + (3/2)\sin(2\pi z)\cos(\pi^2 z^2 + 3)$ for $z \in [-1, 0]$, shown in panel (a), by various eigenfunction expansions of $-\phi'' = \lambda\,\phi$ with fifteen terms, as discussed in §2.2.2. Panel (b) shows the Fourier sine expansion of F. Since the sine eigenfunctions vanish at the boundaries $z = -1, 0$, the series expansion will not converge to F at the boundaries. Panel (c) shows the cosine expansion of F which converges uniformly to F on the closed interval $[-1, 0]$. Panel (d) shows an expansion with boundary coefficients in equations (2.2)–(2.3) given by $(a_1, b_1, c_1, d_1) = (-0.5, -5, 1, 0)$ and $(a_2, b_2, c_2, d_2) = (0.5, -5, 1, 0)$. Since the $c_i = 1$ and $d_i = 0$, then $\Phi_n = \phi_n$ and the series expansions (2.23) and (2.24) coincide. As with the cosine series, the expansion converges uniformly to F on $[-1, 0]$. The derivative of F is shown in panel (e). Panel (f) show the derivative of the sine series expansion; the differentiated series shows poor convergence behaviour since the undifferentiated sine series does not converge uniformly to F on $[z_1, z_2]$. In panel (g), we show the differentiated cosine series. The series does not converge to the derivative F' at the boundaries $z = z_1, z_2$ due to the boundary conditions that the cosine eigenfunctions satisfy. In contrast, in panel (h), the differentiated series obtained from a problem with λ-dependent boundary conditions converges uniformly to the derivative F' on the closed interval $[z_1, z_2]$.

If $c_i = 1$ and $d_i = 0$ for $i \in S$ then we can replace Φ_n by ϕ_n and Ψ by ψ in equation (2.26).

In addition, if both boundary conditions of the eigenvalue problem (2.1)–(2.3) are λ-dependent, then both expansions in equation (2.26) converge uniformly on $[z_1, z_2]$ regardless of the boundary conditions ψ satisfies. As discussed in appendix A.4, for traditional Sturm-Liouville expansions, an analogous result holds only if ψ satisfies the same boundary conditions as the eigenfunctions. Figure 1 contrasts the convergence behaviour of such

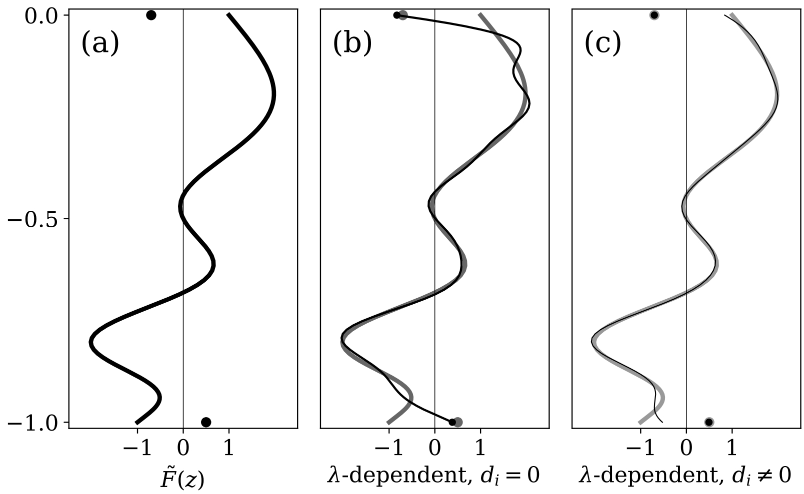

Figure 2: Convergence to a function $\tilde{F}$ with finite jump discontinuities at the boundaries by two eigenfunction expansions (with λ-dependent boundary conditions) of $-\phi'' = \lambda\,\phi$ with fifteen terms, as discussed in §2.2.2. The function $\tilde{F}(z)$ is defined by $\tilde{F}(z) = F(z)$ for $z \in (z_1, z_2)$ where $F(z)$ is the function defined in figure 1, $F(-1) = 0.5$ at the lower boundary, and $F(0) = -0.7$ at the upper boundary. The function $\tilde{F}$ is shown in panel (a). In panel (b), the boundary coefficients in equations (2.2)–(2.3) are given by $(a_1, b_1, c_1, d_1) = (-0.5, -5, 1, 0)$ and $(a_2, b_2, c_2, d_2) = (0.5, -5, 1, 0)$ as in figure 1. In panel (c), the boundary coefficients are $(a_1, b_1, c_1, d_1) = (-0.5, -5, 1, 0.1)$ and $(a_2, b_2, c_2, d_2) = (0.5, -5, 1, -0.1)$. The Φ_n expansion (2.23) and the ϕ_n expansion (2.24) are not generally equal at the boundaries $z = -1, 0$; this figure shows the Φ_n expansion. The Φ_n series (2.23) converges pointwise to $\tilde{F}$ on $[-1, 0]$, however, the convergence will not be uniform if $d_i = 0$ for $i \in S$, as in panel (b). The boundary values of the Φ_n series (2.23) are shown with a black dot. In panel (b), the eigenfunctions Φ_n are continuous and a large number of terms are required for the series to converge to the discontinuous function $\tilde{F}$. Panel (c) shows that the discontinuous eigenfunction Φ_n have almost converged to the $\tilde{F}$—including at the jump discontinuities; the black dot in panel (c) overlap with the grey dots, which represent the boundary values of $\tilde{F}$. Although the ϕ_n series (2.24) converges to $\tilde{F}$ in the interior $(-1, 0)$, the ϕ_n series does not generally converge to $\tilde{F}$ at the boundaries but instead converges to the values given in theorem A.4.

a problem (with continuous eigenfunctions, so $c_i = 1$ and $d_i = 0$ for $i \in S$) with the convergence behaviour of sine and cosine series. All numerical solutions in this article are obtained using a pseudo-spectral code in Dedalus (Burns *et al.* 2020).

Another novel property of the eigenfunction expansions is that we obtain pointwise convergence to functions that are smooth in the interior of the interval, (z_1, z_2), but have finite jump discontinuities at λ-dependent boundaries (see appendix A.4). If $d_i \neq 0$ for $i \in S$, the convergence is even uniform (Fulton 1977, corollary 2.1). Figure 2 illustrates the convergence behaviour for eigenfunction expansions with λ-dependent boundary conditions in the two cases $d_i = 0$ and $d_i \neq 0$. Note the presence of Gibbs-like oscillations in the case $d_i = 0$ shown in panel (b). Although the Φ_n eigenfunction series (2.23) converges pointwise to the discontinuous function, the ϕ_n solution series (2.24) converges to the values given in theorem A.4 at the λ-dependent boundaries. The ability of these series expansions to converge to functions with boundary jump discontinuities is related to their ability to expand distributions in the Bretherton (1966) "δ-function formulation" of a problem. See Yassin & Griffies (submitted) for a quasigeostrophic example.

2.3. *Oscillation theory*

Recall that for regular Sturm-Liouville problems (i.e., equations (2.1)–(2.3) with $c_i = d_i = 0$) we obtain a countable infinity of real simple eigenvalues, λ_n, that may be ordered as

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty, \tag{2.27}$$

with associated eigenfunctions ϕ_n. The nth eigenfunction ϕ_n has n internal zeros in the interval (z_1, z_2) so that no two eigenfunctions have the same number of internal zeros.

However, once the eigenvalue λ appears in the boundary conditions, there may be up to $s + 1$ linearly independent eigenfunctions with the same number of internal zeros. The crucial parameters deciding the number of zeros is $-b_i/d_i$ for $i \in S$, where b_i and d_i are the boundary coefficients appearing in the boundary conditions (2.2)–(2.3). The following lemma outlines the possibilities when only one boundary condition is λ-dependent.

Lemma 2.6 (Location of double oscillation count).
Suppose that $s = 1$, $i \in S$, and let κ be the number of negative D_i for the eigenvalue problem (2.1)–(2.3). We have the following possibilities.

(i) Right-definite, $d_i \neq 0$: The eigenfunction Φ_n corresponding to the eigenvalue λ_n has n internal zeros if $\lambda_n < -b_i/d_i$ and $n - 1$ internal zero if $-b_i/d_i \leqslant \lambda_n$.

(ii) Right-definite, $d_i = 0$: The nth eigenfunction has n internal zeros.

(iii) Left-definite: If $\kappa = 0$ then all eigenvalues are positive, the problem is right-definite, and either (i) or (ii) applies. Otherwise, if $\kappa = 1$, then the eigenvalues may be ordered as

$$\lambda_0 < 0 < \lambda_1 < \lambda_2 < \cdots \to \infty. \tag{2.28}$$

Both eigenfunctions Φ_0 and Φ_1 have no internal zeros. The remaining eigenfunctions Φ_n, for $n > 1$, have $n - 1$ internal zeros.

Proof. Parts (i), (ii) and (ii) are due to Linden (1991), Binding *et al.* (1994), and Binding & Browne (1999), respectively. □

When both boundary conditions are λ-dependent, the situation is similar. See Binding *et al.* (1994) and (Binding & Browne 1999) for further discussion.

3. Boussinesq gravity-capillary waves

Consider a rotating Boussinesq fluid on an f-plane with a reference Boussinesq density of ρ_0. The fluid is subject to a constant gravitational acceleration g in the downwards, $-\hat{\boldsymbol{z}}$, direction, and to a surface tension T (with dimensions of force per unit length, see Lamb 1975) at its upper boundary. The upper boundary of the fluid, given by $z = \eta$, is a free-surface defined by the function $\eta(\boldsymbol{x}, t)$, where $\boldsymbol{x} = \hat{\boldsymbol{x}}\, x + \hat{\boldsymbol{y}}\, y$ is the horizontal position vector. The lower boundary of the fluid is a flat rigid surface given by $z = -H$. The fluid region is periodic in both horizontal directions $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$.

3.1. *Linear equations of motion*

The governing equations for infinitesimal perturbations about a background state of no motion, characterized by a prescribed background density of $\rho_B = \rho_B(z)$, are

$$\partial_t^2 \nabla^2 w + f_0^2\, \partial_z^2 w + N^2\, \nabla_z^2 w = 0 \quad \text{for } z \in (-H, 0) \tag{3.1}$$

$$w = 0 \quad \text{for } z = -H \tag{3.2}$$

$$-\partial_t^2 \partial_z w - f_0^2\, \partial_z w + g_b\, \nabla_z^2 w - \tau\, \nabla_z^4 w = 0 \quad \text{for } z = 0, \tag{3.3}$$

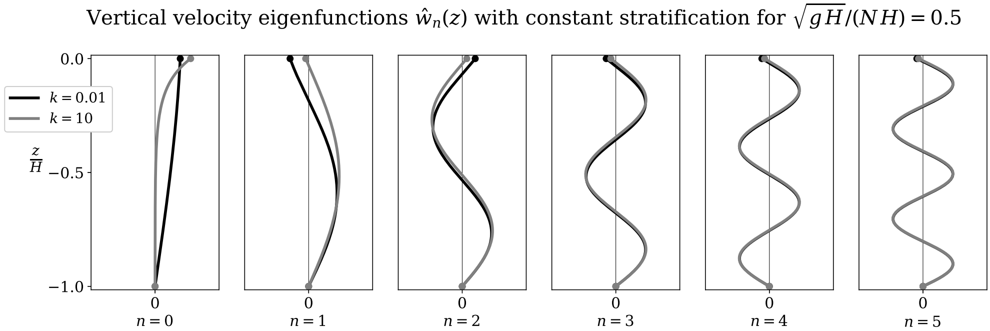

Figure 3: The vertical velocity eigenfunctions $\hat{W}_n = \hat{w}_n$ of the non-rotating Boussinesq eigenvalue problem (3.8)–(3.10) for two distinct wavenumbers with constant stratification, as discussed in §3.2. For both wavenumbers, the nth eigenfunction has n internal zeros as in regular Sturm-Liouville theory. The zeroth mode ($n = 0$) corresponds to a surface gravity wave and is trapped to the upper boundary for large horizontal wavenumbers. In contrast to the internal wave problem with a rigid lid, the modes $\hat{w}_n$ now depend on the horizontal wavenumber k through the boundary condition (3.10), however, this dependence is weak for $n \gg 1$, as can be observed in this figure; for $n > 2$, the modes for $k = 0.01$ (in black) and for $k = 10$ (in grey) nearly coincide. The horizontal wavenumbers k are non-dimensionalized by H.

where w is the vertical velocity, f_0 is the constant value of the Coriolis frequency, the prescribed buoyancy frequency N^2 is given by

$$N^2(z) = -\frac{g}{\rho_0}\frac{d\rho_B(z)}{dz}, \tag{3.4}$$

the acceleration g_b is the effective gravitational acceleration at the upper boundary

$$g_b = -\frac{g}{\rho_0}\left[\rho_a - \rho_B(0-)\right] \tag{3.5}$$

where ρ_a is the density of the overlying fluid, and the parameter τ is given by

$$\tau = \frac{T}{\rho_0} \tag{3.6}$$

where T is the surface tension. The three-dimensional Laplacian is denoted $\nabla^2 = \partial_x^2+\partial_y^2+\partial_z^2$, the horizontal Laplacian is denoted by $\nabla_z^2 = \partial_x^2+\partial_y^2$, and the horizontal biharmonic operator is given by $\nabla_z^4 = \nabla_z^2\,\nabla_z^2$. See equation (1.37) in Dingemans (1997) for the surface tension term in (3.3). The remaining terms in equation (3.1)–(3.3) are standard (Gill 1982). Consistent with our assumption that $\eta(\boldsymbol{x}, t)$ is small, we evaluate the upper boundary condition at $z = 0$ in equation (3.3).

3.2. *Non-rotating Boussinesq fluid*

We assume wave solutions of the form

$$w(\boldsymbol{x}, z, t) = \hat{w}(z)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} \tag{3.7}$$

where $\boldsymbol{k} = \hat{\boldsymbol{x}}\, k_x + \hat{\boldsymbol{y}}\, k_y$ is the horizontal wavevector and ω is the angular frequency. Substituting the wave solution (3.7) into equations (3.1)–(3.3) and setting $f_0 = 0$ yields

$$-\hat{w}'' + k^2\, \hat{w} = \sigma^{-2}\, N^2\, \hat{w} \quad \text{for } z \in (-H, 0) \tag{3.8}$$

$$\hat{w} = 0 \quad \text{for } z = -H \tag{3.9}$$

$$(g_b + \tau\, k^2)^{-1} \hat{w}' = \sigma^{-2}\, \hat{w} \quad \text{for } z = 0, \tag{3.10}$$

where $\sigma = \omega/k$ is the phase speed and $k = |\boldsymbol{k}|$ is the horizontal wavenumber. Equations (3.8)–(3.10) are an eigenvalue problem for the eigenvalue $\lambda = \sigma^{-2}$.

Definiteness & the underlying function space

Equations (3.8)–(3.10) form an eigenvalue problem with one λ-dependent boundary condition, namely, the upper boundary condition (3.10). The underlying function space is then

$$L^2_\mu \cong L^2 \oplus \mathbb{C}. \tag{3.11}$$

We write $\hat{W}_n$ for the eigenfunctions and $\hat{w}_n$ for the solutions of the eigenvalue problem (3.8)–(3.10) [see the paragraph containing equation (2.22)]. The eigenfunctions $\hat{W}_n$ are related to the solutions $\hat{w}_n$ by equation (2.9) with boundary values $\hat{W}_n(0)$ given by equation (2.22). However, since $c_2 = 1$ and $d_2 = 0$ in equation (3.10) [compare with equations (2.1)–(2.3)] then $\hat{W}_n = \hat{w}_n$ on the closed interval $[-H, 0]$; thus, the solutions w_n are also the eigenfunctions.

By theorem 2.3, the eigenfunctions $\{\hat{w}_n\}_{n=0}^{\infty}$ form an orthonormal basis of L^2_μ. For functions φ and ϕ, the inner product is

$$\langle \varphi, \phi \rangle = \frac{1}{N_0^2\, H} \left[\int_{-H}^{0} \varphi\, \phi\, N^2\, \mathrm{d}z + (g_b + \tau\, k^2) \varphi(0)\, \phi(0) \right] \tag{3.12}$$

obtained from equations (2.11) and equation (2.4); we have introduced the factor $1/(N_0^2\, H)$ in the above expression for dimensional consistency in eigenfunction expansions (N_0^2 is a typical value of N^2). Orthonormality is then given by

$$\delta_{mn} = \langle \hat{w}_m, \hat{w}_n \rangle \tag{3.13}$$

and we have chosen the solutions $\hat{w}_n$ to be non-dimensional (so the Kronecker delta is non-dimensional as well).

One verifies that the eigenvalue problem (3.8)–(3.10) is right-definite using proposition 2.1 and left-definite using proposition 2.2. Right-definiteness implies that L^2_μ, with the inner product (3.12), is a Hilbert space. That is, all eigenfunctions $\hat{w}_n$ satisfy

$$\langle \hat{w}_n, \hat{w}_n \rangle > 0. \tag{3.14}$$

Left-definiteness, along with proposition 2.4, ensures that all eigenvalues $\lambda_n = \sigma_n^{-2}$ are positive. Indeed, the phase speeds σ_n satisfy

$$\sigma_0^2 > \sigma_1^2 > \cdots > \sigma_n^2 > \cdots \to 0. \tag{3.15}$$

Properties of the eigenfunctions

By lemma 2.6, the nth eigenfunction $\hat{w}_n$ has n internal zeros in the interval $(-H, 0)$. See figure 3 for an illustration of the first six eigenfunctions.

The eigenfunctions $\{\hat{w}_n\}_{n=0}^{\infty}$ are complete in L^2 but do not form a basis in L^2; in fact, the basis is overcomplete in L^2. The presence of a free-surface provides an additional degree of freedom over the usual rigid-lid L^2 basis of internal wave eigenfunctions. Indeed, the

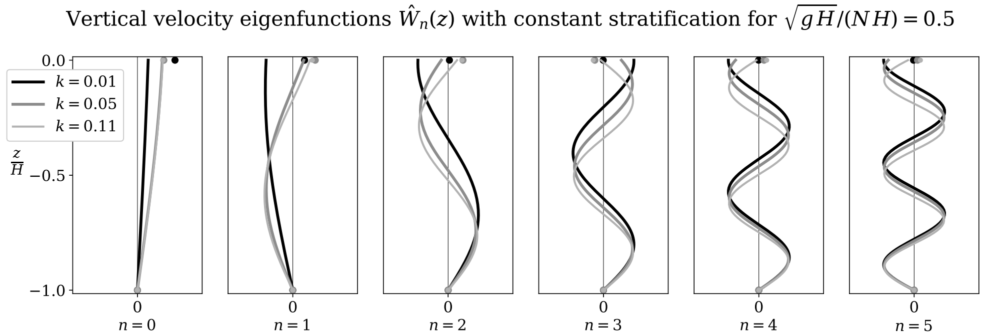

Figure 4: The vertical velocity eigenfunctions $\hat{W}_n$ of a Boussinesq fluid with a rotating upper boundary—eigenvalue problem (3.19)–(3.21). This figure is discussed in §3.3. The wavenumbers k in the figure are non-dimensionalized by the depth H. The dots represent the values of the eigenfunctions at the boundaries. Note that the eigenfunctions have a finite jump discontinuity at $z = 0$. For $kH = 0.01$ (given by the black line) there are two modes with no internal zeros. As k increases, we obtain two modes with one internal zero (at $kH = 0.05$, the thick grey line) and then two modes with three internal zeros (at $kH = 0.11$, the thin grey line).

$n = 0$ wave in figure 3 corresponds to a surface gravity wave, while the remaining modes are internal gravity waves (with some surface motion).

Expansion properties

Given a twice continuously differentiable function $\chi(z)$ satisfying $\chi(-H) = 0$, then, from theorem 2.5, we have

$$\chi(z) = \sum_{n=0}^{\infty} \langle \chi, \hat{w}_n \rangle \, \hat{w}_n(z) \quad \text{and} \quad \chi'(z) = \sum_{n=0}^{\infty} \langle \chi, \hat{w}_n \rangle \, w_n'(z), \tag{3.16}$$

with both series converging uniformly on $[-H, 0]$ (note that χ is not required to satisfy any particular boundary condition at $z = 0$). If χ is the vertical structure at time $t = 0$ (and at some wavevector $\boldsymbol{k}$) then the subsequent time-evolution is given by

$$w(\boldsymbol{x}, z, t) = \sum_{n=0}^{\infty} \langle \chi, \hat{w}_n \rangle \, w_n(z) \, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \sigma_n k t)}. \tag{3.17}$$

3.3. *A Boussinesq fluid with a rotating upper boundary*

Let N_0^2 be a typical value of $N^2(z)$. Consider the situation where $f_0^2/N_0^2 \ll 1$ but

$$\frac{g_b + \tau k^2}{f_0^2 H} \sim O(1). \tag{3.18}$$

Accordingly, we may neglect the Coriolis parameter in the interior equation (3.1) but not at the upper boundary condition (3.3). Substituting the wave solution (3.7) into equations (3.1)–(3.3) yields

$$-\hat{w}'' + k^2 \hat{w} = \sigma^{-2} N^2 \hat{w} \quad \text{for } z \in (-H, 0) \tag{3.19}$$

$$\hat{w} = 0 \quad \text{for } z = -H \tag{3.20}$$

$$(g_b + \tau k^2)^{-1} \hat{w}' = \sigma^{-2} \left[\hat{w} + \frac{f_0^2}{k^2} (g_b + \tau k^2)^{-1} \hat{w}' \right] \quad \text{for } z = 0, \tag{3.21}$$

where $\sigma = \omega/k$ is the phase speed. Equations (3.19)–(3.21) form an eigenvalue problem for the eigenvalue $\lambda = \sigma^{-2}$.

Definiteness & the underlying function space

As in the previous case, the eigenvalue problem is both right-definite and left-definite, the underlying function space L^2_μ is given by equation (3.11), and the appropriate inner product is equation (3.12). By right-definiteness, the space L^2_μ, equipped with the inner product (3.12), is a Hilbert space; thus, all eigenfunctions $\hat{W}_n$ satisfy

$$\left\langle \hat{W}_m, \hat{W}_n \right\rangle > 0. \tag{3.22}$$

By theorem 2.3, all eigenvalues $\lambda_n = \sigma_n^{-2}$ are real and the corresponding eigenfunctions $\{\hat{W}_n\}_{n=0}^{\infty}$ form an orthonormal basis of the Hilbert space L^2_μ. By proposition 2.4, all eigenvalues $\lambda_n = \sigma_n^{-2}$ are positive and satisfy equation (3.15).

Boundary jump discontinuity of the eigenfunctions

The main difference between the previous non-rotating problem (3.8)–(3.10) and the above problem (3.19)–(3.21) is that, in the present problem, if $f_0 \neq 0$ then $d_2 \neq 0$ [see equation (2.3)]. Thus, by equation (2.22), the eigenfunctions $\hat{W}_n$ generally have a jump discontinuity at the upper boundary $z = 0$ (see figure 4) and so are not equal to the solutions $\hat{w}_n$. The eigenfunctions $\hat{W}_n$ are defined by $\hat{W}_n(z) = \hat{w}_n(z)$ for $z \in [-H, 0)$ and

$$\hat{W}_n(0) = \hat{w}_n(0) + \frac{f_0^2}{k^2}(g_b + \tau\, k^2)^{-1}\, \hat{w}_n'(0) \tag{3.23}$$

[see equation (2.22)]. It is not difficult to show that

$$\hat{W}_n(0) \approx 0 \qquad \text{for } n \text{ sufficiently large,} \tag{3.24}$$

as can be seen in figure 4.

Physical motion is given by the solutions $\hat{w}_n$ which are continuous over the closed interval $[-H, 0]$. The jump discontinuity in the eigenfunctions $\hat{W}_n$ does not correspond to any physical motion; instead, the eigenfunctions $\hat{W}_n$ are convenient mathematical aids used to obtain eigenfunction expansions in the function space L^2_μ.

Number of internal zeros of the eigenfunctions

Another consequence of $d_2 \neq 0$ is that by, lemma 2.6, there are two distinct solutions $\hat{w}_M$ and $\hat{w}_{M+1}$ with the same number of internal zeros (i.e., M) in the interval $(-H, 0)$. Noting that

$$-\frac{b_2}{d_2} = \frac{k^2}{f_0^2} \tag{3.25}$$

the integer M is determined by

$$\sigma_0^2 > \sigma_1^2 > \cdots > \sigma_M^2 > \frac{f_0^2}{k^2} \geqslant \sigma_{M+1}^2 > \cdots > 0. \tag{3.26}$$

A smaller f_0 or a larger k implies a larger M and hence that $\hat{w}_M$ and $\hat{w}_{M+1}$ have a larger number of internal zeros, as shown in figure 4.

Expansion properties

As in the previous problem, the eigenfunctions are complete in L^2_μ but overcomplete in L^2 due to the additional surface gravity-capillary wave.

Given a twice continuously differentiable function $\chi(z)$ satisfying $\chi(-H) = 0$, we define the discontinuous function $X(z)$ by

$$X(z) = \begin{cases} \chi(z) & \text{for } z \in [-H, 0) \\ \chi(0) + \frac{f_0^2}{k^2} \left(g_b + \tau\, k^2\right)^{-1} \chi'(0) & \text{for } z = 0 \end{cases} \tag{3.27}$$

as in theorem 2.5. Then, by theorem 2.5, we have the expansions

$$\chi(z) = \sum_{n=0}^{\infty} \left\langle X, \hat{W}_n \right\rangle \hat{w}_n(z) \quad \text{and} \quad \chi'(z) = \sum_{n=0}^{\infty} \left\langle X, \hat{W}_n \right\rangle w_n'(z). \tag{3.28}$$

Moreover, if $\chi(z)$ is the vertical structure at $t = 0$ (and at some wavevector $\boldsymbol{k}$) then the subsequent time-evolution is given by

$$w(\boldsymbol{x}, z, t) = \sum_{n=0}^{\infty} \left\langle X, \hat{W}_n \right\rangle \hat{w}_n(z)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \sigma_n k t)}. \tag{3.29}$$

4. Quasigeostrophic waves

4.1. *Linear equations*

Linearizing the quasigeostrophic equations about a quiescent background state with an infinitesimally sloping lower boundary, at $z = -H$, and a rigid flat upper boundary, at $z = 0$, renders

$$\partial_t \left[\nabla_z^2 \psi + \partial_z \left(S^{-1}\, \partial_z \psi\right)\right] + \hat{\boldsymbol{z}} \cdot \left(\boldsymbol{\nabla}_z \psi \times \boldsymbol{\nabla}_z f\right) = 0 \quad \text{for } z \in (-H, 0) \tag{4.1}$$

$$\partial_t \left(S^{-1}\, \partial_z \psi\right) + \hat{\boldsymbol{z}} \cdot \left(\boldsymbol{\nabla}_z \psi \times f_0\, \boldsymbol{\nabla}_z h\right) = 0 \quad \text{for } z = -H \tag{4.2}$$

$$\partial_t \left(S^{-1}\, \partial_z \psi\right) = 0 \quad \text{for } z = 0. \tag{4.3}$$

See Rhines (1970), Charney & Flierl (1981), Straub (1994), or Yassin & Griffies (submitted) for details. The streamfunction ψ is defined through $\boldsymbol{u} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_z \psi$ where $\boldsymbol{u}$ is the horizontal velocity and $\boldsymbol{\nabla}_z = \hat{\boldsymbol{x}}\, \partial_x + \hat{\boldsymbol{y}}\, \partial_y$ is the horizontal Laplacian. The stratification parameter S is given by

$$S(z) = \frac{N^2(z)}{f_0^2}, \tag{4.4}$$

where N^2 is the buoyancy frequency and f_0 is the reference Coriolis parameter. The latitude dependent Coriolis parameter f is defined by

$$f(y) = f_0 + \beta\, y. \tag{4.5}$$

Finally, $h(\boldsymbol{x})$ is the height of the topography at the lower boundary and is a linear function of the horizontal position vector $\boldsymbol{x}$. Consistent with quasigeostrophic theory, we assume that topography h is small and so we evaluate the lower boundary condition at $z = -H$ in equation (4.2).

4.2. *The streamfunction eigenvalue problem*

We assume wave solutions of the form

$$\psi(\boldsymbol{x}, z, t) = \hat{\psi}(z)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{4.6}$$

where $\boldsymbol{k} = \hat{\boldsymbol{x}}\, k_x + \hat{\boldsymbol{y}}\, k_y$ is the horizontal wavevector and ω is the angular frequency.

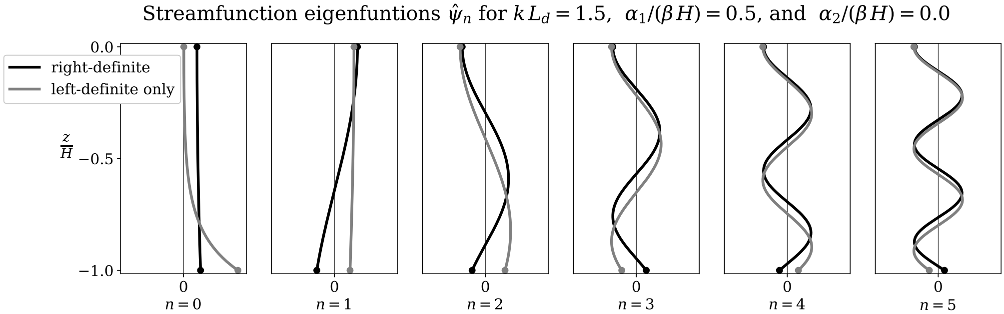

Figure 5: The streamfunction eigenfunctions $\hat{\psi}_n$ of the quasigeostrophic eigenvalue problem with a sloping bottom from §4.2. Two cases are shown. The first is with $\Delta\theta_f = -90°$ and $\Delta\theta_1 = -30°$ and is both right-definite and left-definite. The second is with $\Delta\theta_f = -45°$ and $\Delta\theta_1 = 15°$ and is only left-definite. In the right-definite case the nth eigenfunction has n internal zero whereas in the left-definite only case there are two eigenfunctions ($n = 0, 1$) with no internal zeros.

We denote by $\Delta\theta_f$ the angle between the horizontal wavevector $\boldsymbol{k}$ and the gradient of Coriolis parameter $\boldsymbol{\nabla}_z f$,

$$\sin(\Delta\theta_f) = \frac{1}{k\,\beta}\,\hat{z}\cdot(\boldsymbol{k}\times\boldsymbol{\nabla}_z f)\,, \tag{4.7}$$

where $k = |\boldsymbol{k}|$ is the horizontal wavenumber. Positive angles are measured counter-clockwise relative to $\boldsymbol{k}$. Thus, $\Delta\theta_f > 0$ indicates that $\boldsymbol{k}$ points to the right of $\boldsymbol{\nabla}_z f$ while $\Delta\theta_f < 0$ indicates that $\boldsymbol{k}$ points to the left of $\boldsymbol{\nabla}_z f$.

We define the topographic parameter α by

$$\alpha = |f_0\,\boldsymbol{\nabla}_z h|\,. \tag{4.8}$$

In analogy with $\Delta\theta_f$, we define the angle $\Delta\theta_h$ by

$$\sin(\Delta\theta_h) = \frac{1}{k\,\alpha}\,\hat{z}\cdot(\boldsymbol{k}\times f_0\boldsymbol{\nabla}_z h) \tag{4.9}$$

with a similar interpretation assigned to $\Delta\theta_h > 0$ and $\Delta\theta_h < 0$.

Substituting the wave solution (4.6) into the linear quasigeostrophic equations (4.1)–(4.3) and assuming that $\alpha\,\sin(\Delta\theta_h) \neq 0$, $\omega \neq 0$, and $k \neq 0$, we obtain

$$-(S^{-1}\,\hat{\psi}')' + k^2\,\hat{\psi} = \lambda\,\hat{\psi} \quad \text{for } z \in (-H, 0) \tag{4.10}$$

$$-\frac{\beta}{\alpha}\,\frac{\sin(\Delta\theta_f)}{\sin(\Delta\theta_h)}\,S^{-1}\,\hat{\psi}' = \lambda\,\psi \quad \text{for } z = -H \tag{4.11}$$

$$S^{-1}\,\psi' = 0 \quad \text{for } z = 0, \tag{4.12}$$

where we have defined the eigenvalue λ by

$$\lambda = -\frac{k\,\beta\,\sin(\Delta\theta_f)}{\omega}. \tag{4.13}$$

Since $k \neq 0$ then $\lambda = 0$ is not an eigenvalue.

Definiteness & the underlying function space

The eigenvalue problem has one λ-dependent boundary condition and so the underlying function space is

$$L^2_\mu \cong L^2 \oplus \mathbb{C}. \tag{4.14}$$

The appropriate inner product is obtained from equations (2.11) and (2.4)

$$\langle \varphi, \phi \rangle = \frac{1}{H} \left[\int_{-H}^{0} \varphi \, \phi \, \mathrm{d}z + \frac{\alpha}{\beta} \frac{\sin(\Delta\theta_h)}{\sin(\Delta\theta_f)} \varphi(-H) \, \phi(-H) \right] \tag{4.15}$$

where we have introduced the factor $1/H$ for dimensional consistency in eigenfunction expansions. By proposition 2.1, the problem is right-definite for horizontal wavevectors $\boldsymbol{k}$ satisfying

$$\frac{\sin(\Delta\theta_h)}{\sin(\Delta\theta_f)} > 0 \tag{4.16}$$

and, in such cases, L^2_μ equipped with the inner product (2.11) is a Hilbert space. However, L^2_μ is not a Hilbert space for all wavevectors $\boldsymbol{k}$. By proposition 2.2, the problem is left-definite for all wavevectors $\boldsymbol{k}$ and so L^2_μ, equipped with the inner product (2.11), is generally a Pontryagin space.

We write $\hat{\Psi}_n$ for the eigenfunctions and $\hat{\psi}_n$ for the solutions of equations (4.10)–(4.12). The eigenfunctions $\hat{\Psi}_n$ are related to the solutions $\hat{\psi}_n$ by (2.9) with boundary values $\hat{\Psi}_n(0)$ given by equation (2.22). However, since $c_1 = 1$ and $d_1 = 0$ in equation (4.11) [compare with equations (2.1)–(2.3)] then $\hat{\Psi}_n = \hat{\psi}_n$ on the closed interval $[-H, 0]$. Thus, the solutions ψ_n are also the eigenfunctions.

With theorem 2.3, we deduce that all eigenvalues λ_n are real and the corresponding eigenfunctions $\{\hat{\psi}_n\}_{n=0}^{\infty}$ form an orthonormal basis for L^2_μ. Orthonormality is defined with respect to the inner product given by equation (4.15) and takes the form

$$\pm\, \delta_{mn} = \left\langle \hat{\psi}_m, \hat{\psi}_n \right\rangle \tag{4.17}$$

where we have taken the eigenfunctions $\hat{\psi}_m$ and $\hat{\psi}_n$ to be non-dimensional.

Properties of the eigenfunctions

By lemma 2.6, the number of internal zeros of the eigenfunctions $\{\hat{\psi}_n\}_{n=0}^{\infty}$ depends on the propagation direction and hence [by equation (4.16)] on the definiteness of the problem (see figure 5):

1. if the problem is right-definite then the nth eigenfunction has n internal zeros,

2. if the problem is not right-definite then both ψ_0 and ψ_1 have no internal zeros; the remaining eigenfunctions ψ_n, for $n > 1$, have $n - 1$ internal zeros.

As the problem is left-definite for all wavevectors $\boldsymbol{k}$, we can use proposition 2.4 to determine the sign of the eigenvalues. Proposition 2.4 informs us that

$$\lambda_n \left\langle \hat{\psi}_n, \hat{\psi}_n \right\rangle > 0. \tag{4.18}$$

In the first case, when the problem is right-definite, all eigenvalues are positive and all eigenfunctions $\hat{\psi}_n$ satisfy $\left\langle \hat{\psi}_n, \hat{\psi}_n \right\rangle > 0$. In the second case, when the problem is only left-definite, then there is one negative eigenvalue λ_0 and the corresponding eigenfunction $\hat{\psi}_0$ satisfies $\left\langle \hat{\psi}_0, \hat{\psi}_0 \right\rangle < 0$. The remaining eigenvalues are positive and their corresponding eigenfunctions satisfy $\left\langle \hat{\psi}_n, \hat{\psi}_n \right\rangle > 0$. In fact, from equation (4.13), we see that waves with $\left\langle \hat{\psi}_n, \hat{\psi}_n \right\rangle > 0$ have westward phase speeds $\omega_n/k < 0$ while waves with $\left\langle \hat{\psi}_n, \hat{\psi}_n \right\rangle < 0$ have eastward phase speeds $\omega_n/k > 0$.

Expansion properties

The eigenfunctions $\{\hat{\psi}_n\}_{n=0}^{\infty}$ are complete in L^2_μ but overcomplete in L^2. Physically, there is now an additional eigenfunction corresponding to a topographic Rossby wave ($n = 0$ in figure 5).

Given a twice continuously differentiable function $\phi(z)$ satisfying $\phi'(0) = 0$, then from theorem 2.5, we have

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\langle \phi, \hat{\psi}_n \rangle}{\langle \hat{\psi}_n, \hat{\psi}_n \rangle} \hat{\psi}_n(z) \quad \text{and} \quad \phi'(z) = \sum_{n=0}^{\infty} \frac{\langle \phi, \hat{\psi}_n \rangle}{\langle \hat{\psi}_n, \hat{\psi}_n \rangle} \hat{\psi}_n'(z), \tag{4.19}$$

with both series converging uniformly on $[-H, 0]$ (note that ϕ is not required to satisfy any particular boundary condition at $z = -H$). If the vertical structure at time $t = 0$ (and at some wavevector $\boldsymbol{k}$) is given by ϕ, then the subsequent time-evolution is given by

$$\psi(\boldsymbol{x}, z, t) = \sum_{n=0}^{\infty} \frac{\langle \phi, \hat{\psi}_n \rangle}{\langle \hat{\psi}_n, \hat{\psi}_n \rangle} \hat{\psi}_n(z)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_n t)}, \tag{4.20}$$

where the angular frequency ω_n is given by equation (4.13).

5. A localized perturbation at the boundary

We now consider a localized perturbation at a dynamically-active boundary; we idealize such a perturbation by a boundary step-function Θ_i (for $i \in S$) given by

$$\Theta_i(z) = \begin{cases} 1 & \text{if } z = z_i \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Using equation (2.23), the series expansion of Θ_i is found to be

$$\Theta_i = \frac{1}{D_i} \sum_{n=0}^{\infty} \frac{\Phi_n(z_i)}{\langle \Phi_n, \Phi_n \rangle} \Phi_n(z). \tag{5.2}$$

For the non-rotating Boussinesq problem of §3.2, a step-function perturbation with amplitude w_0 (at some wavevector $\boldsymbol{k}$) yields the time-evolution

$$w(\boldsymbol{x}, z, t) = w_0 \left(\frac{g_b + \tau\, k^2}{N_0^2\, H} \right) \sum_{n=0}^{\infty} \hat{w}_n(0)\, \hat{w}_n(z)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \sigma_n k t)}. \tag{5.3}$$

Analogously, for the quasigeostrophic problem of §4.2, a step-function perturbation with amplitude ψ_0 (at some wavevector $\boldsymbol{k}$) yields the time-evolution

$$\psi(\boldsymbol{x}, z, t) = \psi_0 \left[\frac{\alpha\, \sin(\Delta\theta_h)}{H\, \beta\, \sin(\Delta\theta_f)} \right] \sum_{n=0}^{\infty} \frac{\hat{\psi}_n(-H)}{\langle \hat{\psi}_n, \hat{\psi}_n \rangle} \hat{\psi}_n(z)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_n t)}. \tag{5.4}$$

That both the above series converge to a step-function at $t = 0$ (and $\boldsymbol{x} = \boldsymbol{0}$) is confirmed by theorem A.3 along with theorem 2 in Fulton (1977).

We thus see that a step-function perturbation induces wave motion with an amplitude that is proportional to the boundary-confined restoring force (at wavevector $\boldsymbol{k}$). Moreover, the amplitude of each constituent wave in the resulting motion is proportional to the projection of that wave onto the dynamically-active boundary.

6. Summary and conclusions

We have developed a mathematical framework for the analysis of three-dimensional wave problems with dynamically-active boundaries (i.e., boundaries where time derivatives appear in the boundary conditions). The resulting waves have vertical structures that depend on the wavevector $\boldsymbol{k}$: For Boussinesq gravity waves, the dependence is only through the wavenumber k whereas the dependence for quasigeostrophic Rossby waves is on both the wavenumber k and the propagation direction $\boldsymbol{k}/k$. Moreover, the vertical structures of the waves are complete in a space larger than L^2, namely, they are complete in $L^2_\mu \cong L^2 \oplus \mathbb{C}^s$ where s is the number of dynamically active boundaries (and the number of boundary-trapped waves). Essentially, each dynamically active boundary contributes an additional boundary-trapped wave and hence an additional degree of freedom to the problem. Mathematically, the presence of boundary-trapped waves allows us to expand a larger collection of functions (with a uniformly convergent series) in terms of the modes. The resulting series are term-by-term differentiable and the differentiated series converges uniformly. In fact, the normal modes have the intriguing property converging pointwise to functions with finite jump discontinuities at the boundaries, a property related to their ability to expand distributions in the Bretherton (1966) "δ-function formulation" of a physical problem. By considering a step-function perturbation at a dynamically-active boundary, we find that the subsequent time-evolution consists of waves whose amplitude is proportional to their projection at the dynamically-active boundary. Within the mathematical formulation is a qualitative oscillation theory relating the number of internal zeros of the eigenfunctions to physical quantities; indeed, for the quasigeostrophic problem, the number of zeros of the topographic Rossby wave depends on the propagation direction while, for the rotating Boussinesq problem, the ratio of the Coriolis parameter to the horizontal wavenumber determines at which integer M we obtain two modes with M zeros.

Normal mode decompositions of quasigeostrophic motion play an important role in physical oceanography (e.g., Wunsch 1997; Lapeyre 2009; LaCasce 2017). In the companion article Yassin & Griffies (submitted), we explore the implications of the mathematical framework developed in this article to normal mode decompositions in bounded quasigeostrophic fluids. Another applications is to the extension of equilibrium statistical mechanical calculations (e.g., Bouchet & Venaille 2012; Venaille *et al.* 2012) to three-dimensional systems with dynamically-active boundaries. Finally, the mathematical framework developed here is useful for the development of weakly non-linear wave turbulence theories (e.g., Fu & Flierl 1980; Smith & Vallis 2001; Scott 2014) in systems with both internal and boundary-trapped waves.

Acknowledgements. I sincerely thank Stephen Griffies for helpful comments and insights offered throughout this project.

Funding. This report was prepared by Houssam Yassin under award NA18OAR4320123 from the National Oceanic and Atmospheric Administration, U.S. Department of Commerce. The statements, findings, conclusions, and recommendations are those of the author and do not necessarily reflect the views of the National Oceanic and Atmospheric Administration, or the U.S. Department of Commerce.

Declaration of interests. The author reports no conflict of interest.

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Appendix A. Additional properties of the eigenvalue problem

A.1. *Construction of L^2_μ*

First, define the weighted Lebesgue measure σ by

$$\sigma([a, b]) = \int_a^b r \, \mathrm{d}z \quad \text{where } a, b \in [z_1, z_2]. \tag{A 1}$$

The measure σ induces the differential element

$$\mathrm{d}\sigma(z) = r(z)\, \mathrm{d}z \tag{A 2}$$

and is the measure associated with L^2 [see equations (2.6) and (2.7)].

Now, for $i \in S$ [see equation (2.8)], define the pure point measure ν_i by (e.g., Reed & Simon 1980, section I.4, example 2)

$$\nu_i([a,b]) = \begin{cases} D_i^{-1} & \text{if } z_i \in [a,b] \\ 0 & \text{otherwise,} \end{cases} \tag{A 3}$$

where D_i is the combination of boundary condition coefficients given by equation (2.4). The pure point measure ν_i induces the differential element

$$\mathrm{d}\nu_i(z) = D_i^{-1}\, \delta(z - z_i)\, \mathrm{d}z, \tag{A 4}$$

where $\delta(z)$ is the Dirac distribution.

Consider now the space $L^2_{\nu_i}$ of "functions" ϕ satisfying

$$\left| \int_{z_1}^{z_2} |\phi|^2 \, \mathrm{d}\nu_i \right| = \left| D_i^{-1} \right| \int_{z_1}^{z_2} |\phi|^2 \, \delta(z - z_i)\, \mathrm{d}z = \left| D_i^{-1} \right| \, |\phi(z_i)|^2 < \infty. \tag{A 5}$$

Elements of $L^2_{\nu_i}$ are not functions, but rather equivalence classes of functions. Two functions, ϕ and ψ, on the interval $[z_1, z_2]$ are equivalent in $L^2_{\nu_i}$ if $\phi(z_i) = \psi(z_i)$. In particular, $L^2_{\nu_i}$ is a one-dimensional vector space and is hence isomorphic to the field of complex numbers $\mathbb{C}$

$$L^2_{\nu_i} \cong \mathbb{C}. \tag{A 6}$$

Now define the measure μ by

$$\mu = \sigma + \sum_{i \in S} \nu_i \tag{A 7}$$

with an induced differential element of

$$\mathrm{d}\mu(z) = \left[r(z) + \sum_{i \in S} D_i^{-1}\, \delta(z - z_i) \right] \mathrm{d}z. \tag{A 8}$$

Then L^2_μ is the space of equivalence classes of functions that are square-integrable with respect to the measure μ.

Since the measures σ and ν_i, for $i \in S$, are mutually singular, we have (Reed & Simon 1980, section II.1, example 5)

$$L^2_\mu \cong L^2 \oplus \sum_{i \in S} L^2_{\nu_i} \cong L^2 \oplus \mathbb{C}^s \tag{A 9}$$

from which we see that L^2_μ is "larger" by s dimensions.

A.2. *The eigenvalue problem in L^2_μ*

We construct here an operator formulation of (2.1)–(2.3) as an eigenvalue problem in the Pontryagin space L^2_μ.

Define the differential operator ℓ acting on a function ϕ by

$$\ell\, \phi = \frac{1}{r} \left[(p\, \phi')' - q\, \phi \right]. \tag{A 10}$$

We also define the following boundary operators for $i \in S$,

$$\mathcal{B}_i \phi = \left[a_i\, \phi(z_i) - b_i\, (p\, \phi')(z_i) \right] \tag{A 11}$$
$$\mathcal{C}_i \phi = \left[c_i\, \phi(z_i) - d_i\, (p\, \phi')(z_i) \right]. \tag{A 12}$$

Let Φ be an element of L^2_μ, as in equation (2.9), with boundary values $\Phi(z_i) = C_i\phi$ for $i \in S$ and equal to ϕ elsewhere. We then define the operator $\mathcal{L}$, acting on functions Φ, by

$$\mathcal{L}\,\Phi = \begin{cases} -\ell\,\phi & \text{for } z \in (z_1, z_2) \\ -\mathcal{B}_i\,\phi & \text{for } z = z_i \text{ where } i \in S \end{cases} \tag{A 13}$$

with a domain $D(\mathcal{L}) \subset L^2_\mu$ defined by

$$\begin{aligned} D(\mathcal{L}) = \{\Phi \in L^2_\mu \mid \phi \text{ is continuously differentiable, } \ell\,\phi \in L^2,\ \Phi(z_i) = C_i\,\phi \\ \text{for } i \in S \text{ and } \mathcal{B}_i\phi = 0 \text{ for } i \in \{1,2\} \setminus S\}. \end{aligned} \tag{A 14}$$

Recall that S contains indices of the λ-dependent boundary conditions, and therefore, $\{1,2\}\setminus S$ contains the indices of the λ-independent boundary conditions.

Then, on the subspace $D(\mathcal{L})$ of L^2_μ, the eigenvalue problem (2.1)–(2.3) may be written as

$$\mathcal{L}\,\Phi = \lambda\,\Phi. \tag{A 15}$$

As shown in Russakovskii (1975, 1997), $\mathcal{L}$ is a self-adjoint operator in the space L^2_μ.

There is a natural quadratic form Q, induced by the eigenvalue problem (2.1)–(2.3), given by

$$Q(\Phi, \Psi) = \langle \Phi, \mathcal{L}\,\Psi\rangle\,. \tag{A 16}$$

For elements $\Phi, \Psi \in D(\mathcal{L})$, we obtain

$$\begin{aligned} Q(\Phi,\Psi) = \int_{z_1}^{z_2} \left[p\,\phi'^*\,\psi' + q\,\phi^*\,\psi\right]\,\mathrm{d}z + \sum_{i\in\{1,2\}\setminus S} (-1)^{i+1}\,\frac{a_i}{b_i}\,\phi(z_i)^*\,\psi(z_i) \\ -\sum_{i\in S}\frac{1}{D_i}\begin{pmatrix}\psi(z_i)\\ -(p\,\psi')(z_i)\end{pmatrix}^* \cdot \begin{pmatrix} a_i\,c_i & a_i\,d_i \\ a_i\,d_i & b_i\,d_i\end{pmatrix}\begin{pmatrix}\phi(z_i)\\ -(p\,\phi')(z_i)\end{pmatrix} \end{aligned} \tag{A 17}$$

for $b_i \neq 0$ for $i \in \{1,2\} \setminus S$. If $b_i = 0$ for $i \in \{1,2\} \setminus S$ then we replace the term a_i/b_i with zero.

To develop the reality and completeness theorem 2.3, we provide the following definitions.

Definition A.1 (Right-definite). *The eigenvalue problem* (2.1)–(2.3) *is said to be right-definite if L^2_μ is a Hilbert space or, equivalently, if*

$$\langle \Phi, \Phi\rangle > 0 \tag{A 18}$$

for all non-zero $\Phi \in L^2_\mu$.

Definition A.2 (Left-definite). *The eigenvalue problem* (2.1)–(2.3) *is said to be left-definite if*

$$Q(\Phi, \Phi) \geqslant 0 \tag{A 19}$$

for all $\Phi \in D(\mathcal{L})$.

One can then prove propositions 2.1 and 2.2 through straightforward manipulations.

A.3. *Properties of eigenfunction expansions*

The following theorem features some of the novel properties of the basis $\{\Phi_n\}_{n=0}^\infty$ of L^2_μ. Theorem A.3 below is a generalization of a theorem first formulated, in the right-definite case, by Walter (1973) and Fulton (1977).

Theorem A.3 (Eigenfunction expansions). *Let $\{\Phi_n\}_{n=0}^\infty$ be the set of eigenfunctions of the eigenvalue problem* (2.1)–(2.3)*. Then the following properties hold.*

(i) Null series: For $i \in S$, we have

$$0 = D_i^{-1} \sum_{n=0}^{\infty} \frac{1}{\langle \Phi_n, \Phi_n \rangle} \, \Phi_n(z_i) \, \phi_n(z) \tag{A 20}$$

with equality in the sense of L^2.

(ii) Unit series: For $i \in S$, we have

$$1 = D_i^{-1} \sum_{n=0}^{\infty} \frac{1}{\langle \Phi_n, \Phi_n \rangle} \, |\Phi_n(z_i)|^2 \, . \tag{A 21}$$

(iii) L^2-expansion: Let $\psi \in L^2$, then

$$\psi = \sum_{n=0}^{\infty} \frac{1}{\langle \Phi_n, \Phi_n \rangle} \left(\int_{z_1}^{z_2} \psi^* \, \phi_n \, r \, \mathrm{d}z \right) \phi_n. \tag{A 22}$$

with equality in the sense of L^2.

(iv) Interior-boundary orthogonality: Let $\psi \in L^2$, then for $i \in S$, we have

$$0 = \sum_{n=0}^{\infty} \frac{1}{\langle \Phi_n, \Phi_n \rangle} \left(\int_{z_1}^{z_2} \psi^* \, \phi_n \, r \, \mathrm{d}z \right) \Phi_n(z_i). \tag{A 23}$$

Proof. The proof is similar to the proof of corollary 1.1 in Fulton (1977). □

A.4. *Pointwise convergence and Sturm-Liouville series*

Theorem 3 in Fulton (1977) states that the Φ_n series expansion (2.23) behaves like a Fourier series in the interior of the interval (z_1, z_2) (see appendix B for why this theorem applies in the left-definite case). Since the expansions (2.23) and (2.24) in terms of Φ_n and ϕ_n are equal in the interior, then the above theorem applies to the ϕ_n series (2.24) as well. It is at the boundaries points, $z = z_1, z_2$, where the novel behaviour of the series expansions (2.23) and (2.24) appears.

For traditional Sturm-Liouville expansions [with eigenfunctions of problem (2.1)-(2.3) with $c_i, d_i = 0$ for $i = 1, 2$], eigenfunction expansions behave like the analogous Fourier series on $[z_1, z_2]$ [page 16 in Titchmarsh (1962) or chapter 1, section 9, in Levitan & Sargsjan (1975)]. In particular, for a twice continuously differentiable function ψ, the eigenfunction expansion of ψ converges uniformly to ψ on $[z_1, z_2]$ so long as the eigenfunctions ϕ_n do not vanish at the boundaries. If the eigenfunctions vanish at one of the boundaries, then we only obtain uniform convergence if ψ vanishes at the corresponding boundary as well (Brown & Churchill 1993, section 22). Under these conditions, the resulting expansion will be differentiable in the interior of the interval, (z_1, z_2), but not at the boundaries $z = z_1, z_2$ [see chapter 8, section 3, in Levitan & Sargsjan (1975) for the equiconvergence of differentiated Sturm-Liouville series with Fourier series and see section 23 in Brown & Churchill (1993) for the convergence behaviour of differentiated Fourier series].

Returning to the case of eigenfunction expansions for the eigenvalue problem (2.1)–(2.3) with λ-dependent boundaries, the following theorem provides pointwise (as well as uniform, in the case $d_i \neq 0$) convergence conditions for the ϕ_n series (2.24).

Theorem A.4 (Pointwise convergence). *Let ψ be a twice continuously differentiable function on the interval $[z_1, z_2]$ satisfying any λ-independent boundary conditions in the* *eigenvalue problem (2.1)–(2.3). Define the function Ψ on $[z_1, z_2]$ by*

$$\Psi(z) = \begin{cases} \Psi(z_i) & \text{at } z = z_i, \text{ for } i \in S, \\ \psi(z) & \text{otherwise}. \end{cases} \tag{A 24}$$

where $\Psi(z_i)$ are constants for $i \in S$ (the λ-dependent boundaries). Then we have the following.

(i) If $d_i \neq 0$ for $i \in S$, then the ϕ_n series expansion (2.24) converges uniformly to $\psi(z)$ on the closed interval $[z_1, z_2]$,

$$\sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n(z) = \psi(z). \tag{A 25}$$

Furthermore, for the differentiated series, we have

$$\sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n'(z) = \begin{cases} \left(c_i \psi(z_i) - \Psi(z_i)\right)/d_i & \text{at } z = z_i, \text{ for } i \in S \\ \psi'(z) & \text{otherwise}. \end{cases} \tag{A 26}$$

(ii) If $d_i = 0$, then we have

$$\sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n = \begin{cases} \Psi(z_i)/c_i & \text{at } z = z_i, \text{ for } i \in S \\ \psi(z) & \text{otherwise}. \end{cases} \tag{A 27}$$

Proof. This theorem is a generalization of corollary 2.1 in Fulton (1977). We provide the extension of the corollary to the left-definite problem in appendix B.4. □

The Φ_n series (2.23) converges to $\Psi(z_i)$ at $z = z_i$ for $i \in S$ (i.e., at λ-dependent boundaries) but otherwise behaves as in theorem A.4.

Appendix B. Literature survey and mathematical proofs

B.1. *Literature survey*

There is an extensive literature associated with the eigenvalue problem (2.1)–(2.3) with λ-dependent boundary conditions (see Schäfke & Schneider 1966; Fulton 1977, and citations within). One can use the S-hermitian theory of Schäfke & Schneider (1965, 1966, 1968) to show that one obtains real eigenvalues when the problem is either right-definite or left-definite (see §2) but completeness results in L^2_μ are unavailable in this theory.

The right-definite theory is well-known (Evans 1970; Walter 1973; Fulton 1977). In particular, Fulton (1977) applies the residue calculus techniques of Titchmarsh (1962) to the right-definite problem and, in the process, extends some well-known properties of Fourier series to eigenfunction expansions associated with (2.1)–(2.3). A recent Hilbert space approach to the right-definite problem, in the context of obtaining a projection basis for quasigeostrophic dynamics, is given by Smith & Vanneste (2012).

The left-definite problem is less examined. As we show in this article, the eigenvalue problem is naturally formulated in a Pontryagin space, and, in such a setting, one can prove, in the left-definite case, that the eigenvalues are real and that the eigenfunctions form a basis for the underlying function space. We prove this result, stated in theorem 2.3, in appendix B.3.

With these completeness results, we may apply the residue calculus techniques of Titchmarsh (1962) to extend the results of Fulton (1977) to the left-definite problem. Indeed, Fulton (1977) uses a combination of Hilbert space methods as well as residue calculus techniques to prove various convergence results for the right-definite problem. However, only theorem 1 of Fulton (1977) makes use of Hilbert space methods. If we extend Fulton's theorem 1 to the left-definite problem, then all the results of Fulton (1977) will apply equally to the left-definite problem. A left-definite analogue of theorem 1 of Fulton (1977), along with its proof, is given in appendix B.4.

B.2. *A Pontryagin space theorem*

A Pontryagin space Π_κ, for a finite non-negative integer κ, is a Hilbert space with a κ-dimensional subspace of elements satisfying

$$\langle \phi, \phi \rangle < 0. \tag{B 1}$$

An introduction to the theory of Pontryagin spaces can be found in Iohvidov & Krein (1960) as well as in the monograph of Bognár (1974). Another resource is the monograph of Azizov & Iokhvidov (1989) on linear operators in indefinite inner product spaces.

Pontryagin spaces admit a decomposition

$$\Pi_\kappa = \Pi^+ \oplus \Pi^- \tag{B 2}$$

into orthogonal subspaces $(\Pi^+, +\langle\cdot,\cdot\rangle)$ and $(\Pi^-, -\langle\cdot,\cdot\rangle)$. Moreover, one can associate with a Pontryagin space $(\Pi_\kappa, \langle\cdot,\cdot\rangle)$ a corresponding Hilbert space $(\Pi, \langle\cdot,\cdot\rangle_+)$ where the positive-definite inner product $\langle\cdot,\cdot\rangle_+$ is defined by

$$\langle \phi, \psi \rangle_+ = \langle \phi_+, \psi_+ \rangle - \langle \phi_-, \psi_- \rangle, \quad \phi, \psi \in \Pi, \tag{B 3}$$

where $\phi = \phi_+ + \phi_-$ and $\psi = \psi_+ + \psi_-$, with $\phi_\pm, \psi_\pm \in \Pi^\pm$ (Azizov & Iokhvidov 1981).

As a prerequisite to proving theorem 2.3, we require the following.

Theorem B.1 (Positive compact Pontryagin space operators).
Let $\mathcal{A}$ be a positive compact operator in a Pontryagin space Π_κ and suppose that $\lambda = 0$ is not an eigenvalue. Then all eigenvalues are real and the corresponding eigenvectors form an orthonormal basis for Π_κ. There are precisely κ negative eigenvalues and the remaining eigenvalues are positive. Moreover, positive eigenvalues have positive eigenvectors and negative eigenvalues have negative eigenvectors.

Proof. By theorem VII.1.3 in Bognár (1974) the eigenvalues are all real. Moreover, since $\lambda = 0$ is not an eigenvalue, then all eigenspaces are definite (Bognár 1974, theorem VII.1.2) and hence all eigenvalues are semi-simple (Bognár 1974, lemma II.3.8).

Since $\mathcal{A}$ is a compact operator and $\lambda = 0$ is not an eigenvalue, then the span of the generalized eigenspaces is dense in Π_κ (Azizov & Iokhvidov 1989, lemma 4.2.14). Since all eigenvalues are semi-simple, then all generalized eigenvectors are eigenvectors and so the span of the eigenvectors is dense in Π_κ. Orthogonality of eigenvectors can be shown as in a Hilbert space.

Let λ be an eigenvalue and ϕ the corresponding eigenvector. By the positivity of $\mathcal{A}$, we have

$$\langle \mathcal{A}\phi, \phi \rangle = \lambda \langle \phi, \phi \rangle \geqslant 0. \tag{B 4}$$

Since all eigenspaces are definite, it follows that positive eigenvectors must correspond to positive eigenvalues and negative eigenvectors must correspond to negative eigenvalues.

Finally, by theorem IX.1.4 in Bognár (1974), any dense subset of Π_κ must contain a negative-definite κ dimensional subspace. Consequently, there are κ negative eigenvectors and hence κ negative eigenvalues. □

B.3. *Proof of theorem 2.3*

Proof. The proof for the left-definite case is essentially the standard proof (e.g., Debnath & Mikusinski 2005, section 5.10) with theorem B.1 substituting for the Hilbert-Schmidt theorem. We give a general outline nonetheless.

First, it is well-known that $\mathcal{L}$ is self-adjoint in L^2_μ (e.g., Russakovskii 1975, 1997). Since $\lambda = 0$ is not an eigenvalue, then the inverse operator $\mathcal{L}^{-1}$ exists and is an integral operator on L^2_μ. For an explicit construction, see section 4 in Walter (1973), Fulton (1977), and Hinton (1979). The eigenvalue problem for $\mathcal{L}$, equation (2.14), is then equivalent to

$$\mathcal{L}^{-1}\,\phi = \lambda^{-1}\,\phi \tag{B 5}$$

and both problems have the same eigenfunctions.

The operator $\mathcal{L}^{-1}$ is a positive compact operator and so satisfies the requirements of theorem B.1. Application of theorem B.1 to $\mathcal{L}^{-1}$ then assures that all eigenvalues λ_n are real, the eigenfunctions form an orthonormal basis for L^2_μ, and the sequence of eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ is countable and bounded from below.

The claim that the eigenvalues are simple is verified in Binding & Browne (1999) for the left-definite problem. Alternatively, an argument similar to that of Fulton (1977) and (Titchmarsh 1962, page 12) can be made to prove the simplicity of the eigenvalues. □

B.4. *Extending Fulton (1977) to the left-definite problem*

The following is a left-definite analogue of theorem 1 in Fulton (1977). The proof is almost identical to the right-definite case (Fulton 1977; Hinton 1979) with minor modifications. Essentially, since $\langle\Psi,\Psi\rangle$ can be negative, we must replace these terms in the inequalities below with the induced Hilbert space inner product $\langle\Psi,\Psi\rangle_+$ given by equation (B 3). Our L^2_μ Green's functions G corresponds to $\tilde{G}$ in Hinton (1979).

Theorem B.2 (A left-definite extension of Fulton's theorem 1). *Let $\Psi \in L^2_\mu$ be defined on the interval $[z_1, z_2]$ by*

$$\Psi(z) = \begin{cases} \Psi(z_i) & \text{at } z = z_i, \text{ for } i \in S, \\ \psi(z) & \text{otherwise}, \end{cases} \tag{B 6}$$

where $\psi \in L^2$ and $\Psi(z_i)$ are constants for $i \in S$. The eigenfunctions Φ_n are defined similarly (see §2).

(i) Parseval formula: For $\Psi \in L^2_\mu$, we have

$$\langle\Psi,\Psi\rangle = \sum_{n=0}^{\infty} \frac{|\langle\Psi,\Phi_n\rangle|^2}{\langle\Phi_n,\Phi_n\rangle}. \tag{B 7}$$

(ii) For $\Psi \in D(\mathcal{L})$, we have

$$\Psi = \sum_{n=0}^{\infty} \frac{\langle\Psi,\Phi_n\rangle}{\langle\Phi_n,\Phi_n\rangle}\Phi_n. \tag{B 8}$$

with equality in the sense of L^2_μ. Moreover, we have

$$\psi = \sum_{n=0}^{\infty} \frac{\langle\Psi,\Phi_n\rangle}{\langle\Phi_n,\Phi_n\rangle}\phi_n, \tag{B 9}$$

which converges uniformly and absolutely for $z \in [z_1, z_2]$ and may be differentiated term-by-

term, with the differentiated series converging uniformly and absolutely to ψ' for $z \in [z_1, z_2]$. The boundaries series

$$\Psi(z_i) = \sum_{n=0}^{\infty} \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \Phi_n(z_i), \tag{B 10}$$

for $i \in S$, is absolutely convergent.

Proof. The Parseval formula (B 7) is a consequence of the completeness of the eigenfunctions $\{\Phi_n\}_{n=0}^{\infty}$ in L^2_μ, given by theorem 2.3, and theorem IV.3.4 in Bognár (1974). Similarly, the expansion (B 8) is also due to completeness of the eigenfunctions.

We first prove that the series (B 9) converges uniformly and absolutely for $z \in [z_1, z_2]$. We begin with the identity

$$\phi_n(z) = (\lambda - \lambda_n) \, \langle G(z, \cdot, \lambda), \Phi_n \rangle \tag{B 11}$$

where $\lambda \in \mathbb{C}$ is not an eigenvalue of $\mathcal{L}$, and G is the L^2_μ Green's function [see equation (8) in Hinton (1979)]. Then

$$\sum_{n=0}^{\infty} \lambda_n \frac{|\phi_n|^2}{|\lambda - \lambda_n|^2} = \sum_{n=0}^{\infty} \lambda_n \, |\langle G(z, \cdot, \lambda), \Phi_n \rangle|^2 \leqslant \langle G(z, \cdot, \lambda), \mathcal{L} G(z, \cdot, \lambda) \rangle_+ \leqslant B_1(\lambda) \tag{B 12}$$

where $\langle \cdot, \cdot \rangle_+$ is the induced Hilbert space inner product given by equation (B 3) and $B_1(\lambda)$ is a z independent upper bound (equation 9 in Hinton 1979). In addition, since $\Psi \in D(\mathcal{L})$, then $\langle \mathcal{L}\Psi, \mathcal{L}\Psi \rangle_+ < \infty$. Thus, we obtain

$$\sum_n \lambda_n^2 \, |\langle \Psi, \Phi_n \rangle|^2 = \langle \mathcal{L}\Psi, \mathcal{L}\Psi \rangle_+ < \infty. \tag{B 13}$$

The uniform and absolute convergence of (B 9) follows from

$$\sum_{n=0}^{\infty} \left| \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \phi_n \right| = \sum_{n=0}^{\infty} \left| \left(\frac{\phi_n}{\lambda - \lambda_n} \right) (\lambda - \lambda_n) \, \frac{\langle \Psi, \Phi_n \rangle}{\langle \Phi_n, \Phi_n \rangle} \right| \tag{B 14}$$

$$\leqslant \sqrt{ \left(\sum_{n=0}^{\infty} \left| \frac{\phi_n}{\lambda - \lambda_n} \right|^2 \right) \left(\sum_{n=0}^{\infty} |\lambda - \lambda_n|^2 \, |\langle \Psi, \Phi_n \rangle|^2 \right) } \tag{B 15}$$

along with equations (B 12) and (B 13). The absolute convergence of the boundary series (B 10) follows as well.

To show that the series (B 9) is term-by-term differentiable, it is sufficient to show that the differentiated series converges uniformly for $z \in [z_1, z_2]$ (Kaplan 1993, section 6.14, theorem 33). The proof of the unform convergence of the differentiated series follows from the identity (Hinton 1979)

$$\frac{\phi_n'}{\lambda - \lambda_n} = \frac{\mathrm{d}}{\mathrm{d}z} \langle G(z, \cdot, \lambda), \Phi_n \rangle = \langle \partial_z G(z, \cdot, \lambda), \Phi_n \rangle \, . \tag{B 16}$$

and a similar argument.

□

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