Ecole Polytechnique de Tunisie

Explanation of the algorithm

Prepared by : Houssem Bouazizi

Engineering student in 3ème year EGES

As part of the module: Digital techniques in finance

Academic year: 2020-2021

1. General framework

Option pricing models are mathematical models that use certain variables to calculate the theoretical value of an option.

- Call is an option contract that gives you the right, but not the obligation, to buy the underlying asset at a predetermined price before or on the expiration day.
- A put is an option contract that gives you the right, but not the obligation, to sell the underlying asset at a predetermined price on or before the expiration date.

Options can also be classified according to their exercise period:

- European-style options can only be exercised on the expiry date.
- American-style options may be exercised at any time between the purchase date and the expiry date.

There are two different models for pricing these different types of options:

- Binomial option pricing model (CRR): In this model, we assume that the price of the underlying asset will rise or fall over the period. Given the possible prices of the underlying asset and the strike price of an option, we can calculate the option's payoff in these scenarios, then discount these payoffs and find the option's value to date.
- Black-Scholes-Merton model: This model is based on the following assumptions:
  - Returns on shares are lognormally distributed.
  - The risk-free rate is known and remains constant throughout the term of the option.
  - The share's volatility is known and remains constant over the option term.
  - Transaction costs are omitted from the model
  - Dividends are excluded
  - The options may not be exercised before expiry.

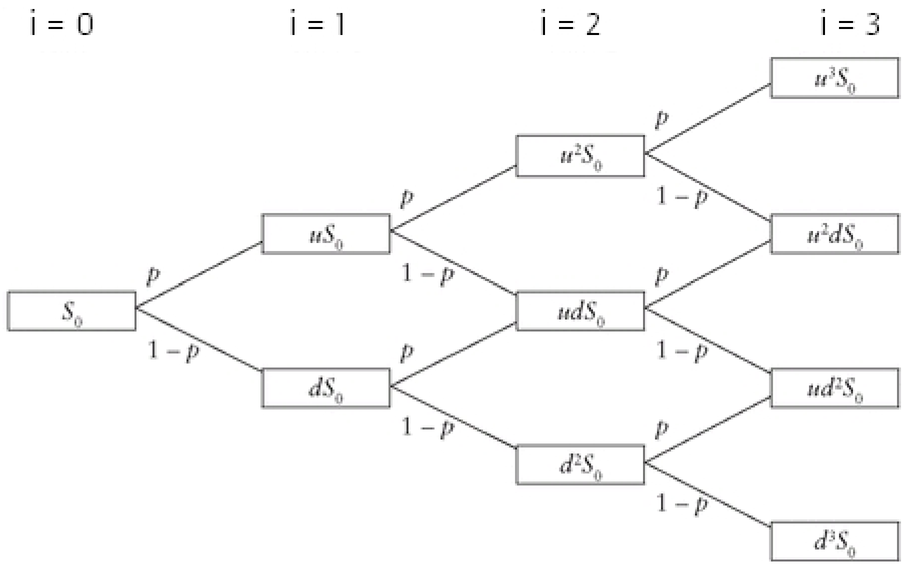
2. Project presentation

The project consists firstly in developing an algorithm for calculating the value of a European and American option with these two Call & Put types, using the CRR model. Secondly, this algorithm consists in showing the convergence of the value of a European Call with that determined by the Black-Scholes model.

3. Theoretical part of the project

- <u>Model CRR</u>

This model can be explained by analyzing the following diagram:



This method uses the following process:

1- Tree creation,
2- Calculation of the option value at the end node of each branch,
3- Progressive calculation of the option value from the previous node, the value of the first node being the option value

- <u>Black-Scholes model</u>

The price of a European call for a spot S0, a maturity T, a strike K, a risk-free rate r and a volatility σ is given by the following formula:

$$C = S_0\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2),$$

avec

$$\mathcal{N}(x) = \left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy.$$

And

$$d_1 = \frac{ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \text{ et } d_2 = \frac{ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

- <u>Convergence CRR-BS</u>

The price of a European call is given by :

$$
\begin{aligned}
C(n) &= e^{-rT}\mathbb{E}(S_T-K)_+ \\
&= e^{-rT}\sum_{j=0}^{n} C_j^n p^j\,(1-p)^{n-j}(S_0u^jd^{n-j}-K)_+ \\
&= e^{-rT}\sum_{j=k}^{n} C_j^n p^j\,(1-p)^{n-j}(S_0u^jd^{n-j}-K), \qquad k=\min\{j\in\mathbb{N};\ S_0u^jd^{n-j}>K\} \\
&= S_0e^{-rT}\sum_{j=k}^{n} C_j^n\,(up)^j[d(1-p)]^{n-j}-Ke^{-rT}\sum_{j=k}^{n} C_j^n p^j(1-p)^{n-j} \\
&\qquad\qquad \text{avec} \qquad k=\min\{j\in\mathbb{N};\ j\geq \frac{\ln(K/S_0d^n)}{\ln(u/d)}\} \\
&= S_0\sum_{j=k}^{n} C_j^n q^j\,(1-q)^{n-j}-Ke^{-rT}\sum_{j=k}^{n} C_j^n(p)^j(1-p)^{n-j}, \qquad q=upe^{-r\delta t} \\
&= S_0\Phi(k,n,q)-Ke^{-rT}\Phi(k,n,p), \qquad \text{avec} \qquad \Phi(k,n,p)=\sum_{j=k}^{n} C_j^n p^j\,(1-p)^{n-j}.
\end{aligned}
$$

We verify that 1-q=d(1-p)Ret that 0<q<1 . This follows from the relation S=R(pSu-(1-p)Sd) from the fact that 0<p<1 and also from the inequalities 0<d<R<u<1 . We have lim Φ (k,n,q)=N(d1) and lim Φ (k,n,p)=N(d2). Φ (k,n,q) is the sum of a random variable that follows the binomial distribution. The sum tends to N when n tends to infinity.

4. Explanation of the algorithm

- **Importing modules**

The modules used for this algorithm are: **math, numpy, matplotlib** for graphics and **scipy** for statistical calculations.

- **Option pricing**
  - `Binomial_tree` function for binomial tree preparation

    In this step, I've tried to prepare the binomial tree, which takes two values each time, either upper value (with probability p) or down value (with probability q). At each stage of the tree, the node diffuses into two nodes: one takes u*St and the other d*St, where St represents the spot price at a given period. The process is repeated for a predetermined number of periods.

  - Function `CRR_option_price` for pricing an option This function takes as parameters :
    - `option_type:` to define whether the option is European or American
    - `call_put:` to define whether the option is a call or a put
    - `St_Tree:` the result of the preparation function
    - `T,n,S,volatility,r, K:` parameters used to determine the value of an option: maturity, number of periods, spot, interest rate and the strike

- `u, d, p, q`: parameters also returned by the binomial tree preparation function

The first step in developing this function is to calculate the tradeoff according to the type of call or put option and assign these tradeoffs (parameter: `optionTree`) to the nodes of the binomial tree. Then, depending on the type of EUR/US option, we calculate the spots for each period using a backward algorithm and according to the `optionTree` parameter and the option's pricing parameters. Finally, the option's theoretical value is the initial node of the tree: `optionTree[0,0]`

- Request for input of the various pricing parameters and the type of option to be priced.
- Option value display

- **Convergence CRR-BS**

- Choice of a wide range of values for n: number of periods (for this algorithm I chose 150 periods)
- Calculate the theoretical value for each n and put them in a list `L`
- Elaboration of the `black_Scholes_pricing` function: a function that calculates the value of a European call: `option_price` according to the formula mentioned in the theoretical part.
- This function calculates the option value for 150 periods and encapsulates them in an `L_BS` list.
- We plot the curve representing the different values of `L and L_BS as a` function of the number of periods.

➡ An analysis of the resulting graph shows that for n=150 the option value obtained by the Black-Scholes model converges to that obtained by the CRR model.

- Finally, to better observe this convergence, we plot the percentage difference between the option value obtained by the Black-Scholes model and that obtained by the CRR model.

➡ By analyzing the curve, the percentage difference converges to 0 for n tends to infinity.